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NUMERICAL EVOLUTION OF GENERAL RELATIVISTIC VOIDS^a

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Abstract

In this paper, we study the evolution of a relativistic, superhorizon-sized void embedded in a Friedmann-Robertson-Walker universe. We numerically solve the spherically symmetric general relativistic equations in comoving, synchronous coordinates. Initially, the fluid inside the void is taken to be homogeneous and nonexpanding. In a radiation-dominated universe, we find that radiation diffuses into the void at approximately the speed of light as a strong shock—the void collapses. We also find the surprising result that the cosmic collapse time (the 1st-crossing time) is much smaller than previously thought, because it depends not only on the radius of the void, but also on the ratio of the temperature inside the void to that outside. If the ratio of the initial void radius to the outside Hubble radius is less than the ratio of the outside temperature to that inside, then the collapse occurs in less than the outside Hubble time. Thus, superhorizon-sized relativistic voids may thermalize and homogenize relatively quickly. These new simulations revise the current picture of superhorizon-sized void evolution after first-order inflation.

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I. Introduction

The Big Bang model predicts the evolution of a homogeneous and isotropic universe. The confirmation of its predictions (e.g. the $3K$ Planck spectrum of the microwave background, the primordial abundances of ${}^4\text{He}$, ${}^3\text{He}$, D , and ${}^7\text{Li}$ from Big Bang Nucleosynthesis, and the number of light neutrino species) is stunning. One is led to question *why* the universe can be described so well by a homogeneous and isotropic model.

Inflation can provide the answer to this question. It occurs when a scalar field σ has non-zero potential energy $V(\sigma)$ which dominates the energy density of the universe.^{1,2} The key ingredients to all models are that the universe expands superluminally during inflation and that there is massive entropy generation afterward. If the universe increases at least 10^{27} times its original size, the flatness and horizon problems are solved; the universe is dynamically driven to homogeneity and isotropy. The most interesting class of models is first-order inflation. Here, the scalar field is trapped in the false vacuum state of a strongly first-order potential. Bubbles of true vacuum are nucleated at different spacetime points, and the end of inflation occurs when the universe is filled with true-vacuum bubbles of varying sizes. The original model, Guth’s “old inflation”,¹ does not work because it fails to percolate (fill) the universe with true-vacuum bubbles. More recent models of first-order inflation which modify the gravitational or particle sector (e.g. “extended inflation”³) are promising as early-universe inflationary scenarios. Because the universe expands as a power-law in time rather than exponentially, percolation is guaranteed to occur.

The end of inflation occurs when true-vacuum bubbles of different sizes fill all of space. A confusing mess of scalar field dynamics then occurs as the bubbles collide.⁴ Because all the energy is contained in the bubble walls, reheat occurs when the σ -field gradient energy is converted into locally thermal radiation. After reheat, the standard homogeneous and isotropic Big Bang model describes the evolution of that part of the universe that had contained horizon-sized bubbles, since these would have been thermalized during reheat. The very large superhorizon-sized bubbles (which were nucleated early on during inflation), however, have traditionally been a problem.⁵ After the σ -field in the bubble wall decays to relativistic particles, a nearly empty void is formed. Since these voids are much larger than the Hubble radius outside the void, it has been thought that the inside of the largest superhorizon-sized voids would remain empty until after recombination, thus producing unobserved temperature fluctuations of order unity in the microwave background.^{6,7,8} The longevity of these voids follows from assuming that a superhorizon-sized void expands conformally with spacetime.^{6,8} The earliest time at which thermalization could occur would then be when the Hubble radius outside the void is of order the size of the void, since this is the expected 1st-crossing time (i.e. the time for photons originally in the void wall to reach the origin). If the void has comoving size r_0 , this occurs in time $\Delta t = t - t_i \simeq H_{\text{out}}^{-1}(t_i)(c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i))^2$, where the initial void radius is $R_{\text{wall}}(t_i) \equiv r_0 a(t_i)$, $cH_{\text{out}}^{-1}(t_i)$ is the Hubble radius outside the void, $a(t)$ is the cosmic scale factor and t_i is the cosmic time after reheating. Other authors suggest that the void would fill in with radiation, but this is estimated to occur on similar time scales.⁹ Because of this “big bubble problem”, first-order inflation models are fine-tuned in order to keep the production of large bubbles at a minimum.^{6,8,9}

Motivated by the first-order (e.g. extended) inflation “big-bubble problem”, we present numerical studies of the evolution of a superhorizon-sized general relativistic void embedded in a Friedmann-Robertson-Walker universe. Although pressureless and thin-shell superhorizon-sized voids have been studied in the past,^{10,11} this is the first study of general relativistic voids with pressure and of arbitrary size and void wall structure. We emphasize that our results are not dependent on any particular inflation model. These new simulations show that opposite sides of a superhorizon-sized relativistic void interact

in a *very* short cosmic time, thereby suggesting that these voids can also thermalize and homogenize on short time scales.

In Section II, we discuss the general relativistic spherically symmetric metric in Lagrangian gauge and synchronous coordinates, and present the equations to be solved numerically. In Section III, the initial conditions, boundary conditions and numerical techniques used are described. In addition, remarks about the deceleration of a void wall are made. In Section IV, we derive the surprising result that the 1st-crossing time of a superhorizon-sized general relativistic void can be vanishingly short. Section V contains numerical solutions for pressureless dust and comparisons to exact Tolman-Bondi solutions. A test of the code for the relativistic Friedmann-Robertson-Walker solution is presented in Section VI. The numerical evolution of nonrelativistic, special relativistic, and general relativistic voids with pressure is examined in Section VII. Here it is found that a relativistic superhorizon-sized void collapses in the form of a strong shock moving at the speed of light. Thus, the collapse time is approximately the 1st-crossing time. Finally, Section VIII contains a discussion of the results.

II. Spherically Symmetric General Relativistic Fluids

A. The (Lagrangian) Metric

The most general spherically symmetric metric is¹²

$$ds^2 = c^2 l(t, r) dt^2 + a(t, r) dr dt + h(t, r) dr^2 + k(t, r) d\Omega^2, \quad (2.1)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\psi^2$ and c is the speed of light. To this metric we can apply general transformations of the type $t = f_1(t', r')$ and $r = f_2(t', r')$ without altering the spherical symmetry. If we perform the necessary transformations to eliminate the $dr dt$ term, then Eq. (2.1) can be written as

$$ds^2 = -c^2 \Phi^2(t, r) dt^2 + \Lambda^2(t, r) dr^2 + R^2(t, r) d\Omega^2, \quad (2.2)$$

where we have dropped the primes, and where we require our coordinates to be comoving with the fluid. At time t , the metric function $R(t, r)$ is the Eulerian distance that a fluid shell labeled by r is located from the center of coordinates. More precisely, $2\pi R(t, r)$ is the spacelike circumference of a sphere centered on the origin which contains all particles with comoving coordinate r . We have thus chosen the Lagrangian gauge with synchronous coordinates (Gaussian normal coordinates). Transformations of the form $\tilde{t} = f(t)$ and $\tilde{r} = g(r)$ can still be made. This metric was first used to study the general relativistic collapse of stars to black holes or neutron stars during supernovae.^{13,14}

It is very important in numerical general relativity to carefully choose the appropriate gauge and coordinates to best match the physical problem to be studied. We have thus specifically chosen the Lagrangian gauge and synchronous coordinates to evolve a general relativistic void embedded in an expanding Friedmann-Robertson-Walker (FRW) universe. In the Lagrangian gauge, we gain maximal coverage of the fluid in a numerical scheme. This is important since the void is embedded in an expanding universe, so that we would continuously lose mass shells in a Eulerian scheme. In addition, the final results are much more easily relatable to our own approximately FRW homogeneous and isotropic universe. We note that asynchronous coordinates could instead be used with the Lagrangian gauge. However, here time and space are mixed up so that the region outside a void is no longer spatially homogeneous—the necessary initial conditions are not obvious and would need to be determined using comoving synchronous coordinates to initially set up and evolve the void. Asynchronous slicing of space-time (e.g. polar slicing¹⁵) is usually used numerically to study the collapse of a star to a black hole in a flat non-expanding universe. These coordinates are necessary to study the physically

interesting mass zones outside the apparent horizon after a mass shell has crossed this horizon. (In synchronous coordinates, once a mass shell crosses the apparent horizon, numerical integration stops so that the evolution of the mass shells outside this horizon cannot be studied). Since we are evolving an *underdense* region here and do not expect apparent horizons to form, we do not need to resort to these coordinates.

Because we wish to embed a void in a FRW universe, we now relate the metric functions from Eqn(2.2) to those from the familiar FRW metric:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2/c^2} + r^2 d\Omega^2 \right), \quad (2.3)$$

where $a(t)$ is the cosmic scale factor, and k is -1 , 0 or 1 for negative, zero and positive spatial curvature, respectively. The solutions describe a universe which is homogeneous and isotropic on each time slice.² For the FRW metric then, $\Phi = 1$, $\Lambda = a(t)/\sqrt{1 - kc^{-2}r^2}$ and the Eulerian distance is $R(t) = ra(t)$.

In this paper, the particles are assumed to be everywhere in local thermal equilibrium, so that we can describe them as a fluid. The stress-energy tensor for a viscous fluid with energy density ρ and pressure p (measured in the frame of the fluid) is ^b

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + p P^{\alpha\beta} - 2\eta \sigma^{\alpha\beta} - 3\zeta \Theta P^{\alpha\beta} \quad (2.4)$$

where u^α is the fluid 4-velocity, $P^{\alpha\beta} = u^\alpha u^\beta + g^{\alpha\beta}$ is the projection operator, $\sigma^{\alpha\beta} = 1/2 (\nabla_\mu u^\alpha P^{\mu\beta} + \nabla_\mu u^\beta P^{\mu\alpha}) - 1/3 \Theta P^{\alpha\beta}$ is the shear viscosity tensor, $\Theta = \nabla_\mu u^\mu$ is the fluid expansion coefficient, and η and ζ are arbitrary functions of r and t . If $\eta = \zeta = 0$, the fluid is non-viscous. For Eqn(2.2) with comoving fluid 4-velocity $u^\alpha = (-\Phi^{-1}, 0, 0, 0)$, $\sigma_r^r = \beta$ and $\sigma_\theta^\theta = \sigma_\psi^\psi = -1/2 \beta$, where $\beta \equiv 2/3 (\dot{\Lambda}/\Lambda - U/R)$, and $\Theta = \dot{\Lambda}/\Lambda + 2U/R$. (For the FRW metric, $\beta = 0$ and $\Theta = 3\dot{a}/a$). The stress tensor is diagonal, with components $T_t^t = -\rho$, $T_r^r = (p - \zeta\Theta) - 2\eta\beta$, and $T_\theta^\theta = T_\psi^\psi = (p - \zeta\Theta) + \eta\beta$. We will employ a scalar artificial viscosity, Q , so that $\eta = 0$ and $\zeta\Theta = -Q$, where Q will be significantly non-zero only in areas of steep “velocity” gradients, as in a shock. This is essential for stabilizing numerical shocks, as is well known.¹⁶ This viscosity will dissipate enough energy on small scales so that the numerical solution approaches the exact solution in the limit that the grid spacing approaches zero.

B. Fluids composed of massless particles

If a fluid consists of (effectively) massless, locally thermalized particles, we can relate p to ρ through the equation of state $p = p(\rho)$. For photons (or particles with mass μ having local temperatures $T \gg \mu$), $p = \rho/3$.^{2,17} Setting $\mathcal{R}_\alpha^\beta - g_\alpha^\beta \mathcal{R} = 8\pi G_N T_\alpha^\beta$, where \mathcal{R}_α^β is the Ricci tensor and G_N is Newton’s constant, and using the conservation equations $\nabla_\mu T^{\mu\alpha} = 0$, five independent equations are found: $G_0^0 = T_0^0$, $G_1^1 = T_1^1$, $G_0^1 = T_0^1$, $T^{0\mu}_{;\mu} = 0$, and $T^{1\mu}_{;\mu} = 0$. We define

$$\Gamma \equiv R'/\Lambda, \quad (2.5)$$

$$M' \equiv 4\pi c^{-2} \rho R^2 R' \quad (2.6)$$

and $U \equiv \Phi^{-1}(\partial R/\partial t)$, where the prime denotes differentiation with respect to r . The general relativistic equations can be written

$$\dot{R} = \Phi U \quad (2.7)$$

^bWe set $c = 1$ for the rest of this subsection

$$\dot{U} = -\Phi \left(\frac{G_N M}{R^2} + \frac{4\pi G_N (p+Q)R}{c^2} \right) - \frac{c^2 \Gamma^2 \Phi (p+Q)'}{(\rho+p+Q)R'} \quad (2.8)$$

$$\dot{M} = -4\pi(p+Q)R^2 \Phi U / c^2 \quad (2.9)$$

$$\dot{\rho} = -\Phi(\rho+p+Q) \frac{(R^2 U)'}{R^2 R'} \quad (2.10)$$

$$\Phi' = -\Phi \frac{(p+Q)'}{\rho+p+Q} \quad (2.11)$$

$$\Gamma^2 \equiv 1 + (U/c)^2 - 2G_N M / (Rc^2), \quad (2.12)$$

where the dot denotes differentiation with respect to t , and where we have included an alternate definition for Γ . There is also the auxiliary equation: $\dot{\Lambda} = \Lambda \Phi U' / R$.

The quantity $U \equiv \dot{R} / \Phi$ describes a particle's ‘‘velocity’’ as measured in the frame of the fluid, since for $dr = d\theta = d\psi = 0$, the infinitesimal proper time that each observer measures is $d\tau = \sqrt{-ds^2} = c\Phi dt$. For $G_N = 0$, if a particle has velocity v , then $\Gamma = 1/\sqrt{1-(v/c)^2}$ and $U = \Gamma v$ (see Eqn (C.9)); Γ and U represent the two non-trivial components of the 4-velocity of the fluid. If $\Gamma \gg 1$, then $U/c \simeq \Gamma$ and the fluid is moving at relativistic velocities relative to a stationary observer.

We can rewrite Eqn (2.6) in terms of the proper volume element $d^3V = 4\pi R^2 \Lambda dr = 4\pi R^2 R' dr / \Gamma$. The ‘‘mass-energy’’ function M then becomes

$$M(t, r) = c^{-2} \int d^3V \Gamma \rho. \quad (2.13)$$

For $G_N = 0$, because the volume along the radial direction is Lorentz-contracted by the factor $1/\sqrt{1-(v/c)^2} = \Gamma$ and ρ is the energy density measured in the frame of the fluid, $\Gamma\rho$ is just the energy density of a fluid parcel as measured by a stationary observer. Therefore, $M(r, t)$ is just the total ‘‘mass-energy’’ contained within comoving coordinate r at time t .

To better understand these equations, we relate them to the FRW equations. Recall the FRW results following Eqn (2.3): $\Phi = 1$, $R = ra$ and $\Lambda = a/\sqrt{1-kr^2/c^2}$. From Eqn(2.7), the ‘‘velocity’’ is $U = \dot{R} = r\dot{a}$. Since $R' = a(t)$, $\Gamma = \sqrt{1-kr^2/c^2}$ from Eqn(2.5). For a spatially flat ($k = 0$) FRW model then, $\Gamma = 1$ even though the fluid at R is moving away from the origin with velocity U . Thus in general relativity, Γ is not the relativistic gamma-factor of the fluid with respect to the origin. Using the fact that $M = 4\pi c^{-2} \rho R^3 / 3$, we see that Eqn (2.12) is just Friedmann's equation, $\dot{H}^2 = (\dot{a}/a)^2 = 8\pi G_N c^{-2} \rho / 3 - kc^2/a^2$, with $H \equiv \Phi^{-1} \dot{a}/a = U/R$.

In a spatially flat FRW universe then, $U = c^{-1} R \sqrt{8\pi G_N \rho / 3} = \sqrt{2G_N M / R}$. If we set $U \equiv U_{\text{GRAV}} + U_{\text{PEC}}$, where $U_{\text{GRAV}} \equiv 2G_N M / R$ is the gravitational velocity and U_{PEC} is the peculiar velocity, Γ can be expressed in terms of these velocities: $\Gamma^2 = 1 + (U_{\text{PEC}}/c)^2 + 2U_{\text{PEC}}U_{\text{GRAV}}/c^2$.

For a non-viscous fluid with $p = (\gamma - 1)\rho$, Eqn(2.11) (the conservation of momentum equation, $T^{\mu}_{1;\mu} = 0$) can be integrated exactly to give $\Phi(t, r_2) = \Phi(t, r_1) [\rho(t, r_1) / \rho(t, r_2)]^{(\gamma-1)/\gamma}$. We are interested in evolving a void embedded in a FRW homogeneous and isotropic universe, so that $p' = \rho' = 0$ outside the void. We define Φ_{out} and ρ_{out} to be the spatially constant values of Φ and ρ outside the void. (In what follows, the

subscripts “in” and “out” represent the spatially constant values inside and outside the evolving void region, respectively). Taking $\gamma = 4/3$, we find

$$\Phi = \Phi_{\text{out}} \left(\frac{\rho_{\text{out}}}{\rho} \right)^{1/4}. \quad (2.14)$$

The fact that this equation can be solved exactly is important for calculating the 1st-crossing time for general relativistic voids, as will be discussed in Section IV.

To gain some physical insight into Φ , we calculate the potential energy of a fluid distribution. A comoving observer has $dr = d\theta = d\psi = 0$, so that the proper acceleration measured by this observer is $a_r = \Phi^{-1}\dot{U}$. Since the force is radial, we can write $a_r = -\partial\phi/\partial R$, where ϕ is the potential. Using Eqn(2.8) and integrating, we can write the potential as the following sum: $\phi(t, R) = \phi_{\text{GRAV}}(t, R) + \phi_{\text{FLUID}}(t, R)$, where

$$\phi_{\text{GRAV}} = 4\pi G_N \int_0^R [M/R^2 + c^{-2}pR] dR \quad (2.15)$$

$$\phi_{\text{FLUID}} = \int_0^R c^2 \Gamma^2 dp / (\rho + p) \quad (2.16)$$

for $Q = 0$. For a void with $p = \rho/3$, we obtain the usual gravitational potential $\phi_{\text{GRAV}} \simeq 4\pi G_N \rho_{\text{in}} R^2/3 = G_N M/R$ inside the void, and $\phi_{\text{GRAV}} \simeq 4\pi G_N \rho_{\text{out}} R^2/3 \simeq G_N M/R$ outside. If in addition we choose $\Gamma(t_i, R) = 1$ initially, then $\phi_{\text{FLUID}}(t_i) = \ln[\rho(t_i, R)/\rho_{\text{in}}(t_i)]^{1/4} = -\ln[\Phi(t_i, R)/\Phi_{\text{in}}(t_i)]$. (Note that the contribution to the fluid potential is zero inside, but (potentially much) greater than zero outside the void. Thus it can substantially increase the already large potential outside the void). Therefore, Φ is proportional to the exponential of the fluid potential, ϕ_{FLUID} .

C. Fluids composed of massive particles

Suppose instead we consider a fluid which consists of particles of mass μ and with arbitrary temperature T . Then, the total energy density of the fluid is the mass energy density plus the internal energy density. Denoting the (proper) mass density (μ times the number density) by $n(t, r)$ and the internal energy per unit mass by $\epsilon(t, r)$,

$$\rho = c^2 n(1 + \epsilon/c^2). \quad (2.17)$$

Here we have traded one unknown for two because in general the pressure depends not only on ρ but also on n . For a fluid composed of relativistic (nonrelativistic) particles, $\epsilon/c^2 > 1$ ($\epsilon/c^2 < 1$). If the fluid obeys the perfect gas law, then its pressure is

$$p = (\gamma - 1)n\epsilon. \quad (2.18)$$

For a highly relativistic species with $\gamma = 4/3$, the energy density is three times the pressure: $\rho = n\epsilon = p/(\gamma - 1) = 3p$, whereas for a highly nonrelativistic fluid, the energy density is much larger than the pressure: $\rho \simeq nc^2 = p/[(\gamma - 1)\epsilon/c^2] \gg p$. We assume that the total number of particles per comoving volume is constant: $\nabla_\mu(nu^\mu) = 0$.¹⁹ Using Eqn(2.5), this can be integrated to give

$$f(r) = 4\pi n R^2 R' / \Gamma \quad (2.19)$$

$$\equiv r^2 \quad (2.20)$$

where $f(r)$ is an arbitrary function depending only on the coordinate r . Specifying $f(r)$ completely fixes the arbitrariness of the metric functions under transformations in r (as

discussed after Eqn (2.2)). This particular definition for f is necessary in order to write the difference schemes in a geometrical way that allows shocks and explosions to be numerically stable at the origin.¹⁶

We can now rewrite the full set of general relativistic equations ((2.7)-(2.12)) as^{13,14}

$$\dot{R} = \Phi U \quad (2.21)$$

$$\dot{U} = -\Phi \left(\frac{G_N M}{R^2} + \frac{4\pi G_N (p+Q)R}{c^2} \right) - \frac{4\pi \Gamma \Phi R^2 (p+Q)'}{w r^2} \quad (2.22)$$

$$\dot{M} = -4\pi (p+Q) R^2 \Phi U / c^2 \quad (2.23)$$

$$\dot{n} = -\frac{n \Phi (R^2 U)'}{R^2 R'} \quad (2.24)$$

$$\dot{\epsilon} = -\frac{4\pi \Phi (p+Q) (R^2 U)'}{\Gamma r^2} \quad (2.25)$$

$$\Phi' = -\Phi \frac{(p+Q)'}{n w c^2}, \quad (2.26)$$

where Γ is given by Eqn(2.12) and $w \equiv 1 + (\epsilon + p/n)/c^2$ is the relativistic enthalpy. Equations (2.21)-(2.26) (along with the definitions for Γ and w given in the previous sentence) are the set used in the numerical code.^c

When the kinetic energy of each particle is much less than its mass energy $\epsilon/c^2 \ll 1$ (or $T/\mu \ll 1$), we obtain the nonrelativistic Lagrangian fluid equations.²¹ (They can also be obtained by setting $c^2 \rightarrow \infty$ in Eqns (2.21)-(2.26)). For future reference, in this limit $\Phi \rightarrow 1$, $\Gamma \rightarrow 1$, and $w \rightarrow 1$ so that $U = \dot{R}$ is the fluid velocity, $M(t, r) = r^3/3$ is the total mass within r and $n = r^2/(R^2 R')$ is the mass density.

The artificial viscosity used here is given by Equation (C.21), and is generalized from the expression used by previous workers:^{22,14}

$$\begin{aligned} Q &= k^2 n (1 + \epsilon/(\Gamma c^2)) (U')^2 dr^2 / \Gamma \quad \text{for } U' < 0 \\ Q &= 0 \quad \text{otherwise.} \end{aligned} \quad (2.27)$$

Consider the behavior of n and p in the limit that the entropy, $S(t, r)$, within r is conserved. Using the thermodynamic relation $TdS = d(\rho/n) + pd(1/n)$, we find $\dot{\epsilon} = p\dot{n}/n^2$. Then, the pressure and internal energy for a shell labeled by r are related to their initial values by $p \propto n^\gamma$ and $\epsilon \propto p^{(\gamma-1)/\gamma}$. In the ultrarelativistic limit with $\gamma = 4/3$, $n \propto \rho^{3/4}$ and $\epsilon \propto \rho^{1/4}$. In addition, if the fluid is a ploytrope (i.e. isentropic), $S' = 0$. Using $TdS = d(\rho/n) + pd(1/n)$ again, we find the familiar result²⁰

$$\frac{\epsilon(t, r_1)}{\epsilon(t, r_2)} = \left(\frac{n(t, r_1)}{n(t, r_2)} \right)^{\gamma-1} = \left(\frac{p(t, r_1)}{p(t, r_2)} \right)^{(\gamma-1)/\gamma}. \quad (2.28)$$

For ultrarelativistic fluids, $\rho = n\epsilon$ and therefore $\epsilon \propto \rho^{(\gamma-1)/\gamma}$ and $n \propto \rho^{1/\gamma}$. If $\gamma = 4/3$, $\epsilon \propto \rho^{1/4}$ and $n \propto \rho^{3/4}$. This is the property of a relativistic fluid, since then $\rho \propto T^4$ and $n \propto T^3$.

^cFor technical difficulties in the general relativistic case, we determine M via the \dot{M} equation rather than the M' equation. In addition, we determine n via the \dot{n} equation rather than through the analytical solution $n = \Gamma r^2/(R^2 R')$, because too much intrinsic viscosity is introduced for special relativistic voids otherwise.

D. Fluid Deceleration in the Void Wall

In this section we show that if the outward peculiar velocity of the wall of a general relativistic void is very large, then the deceleration of this wall can be enormous. This “damping force” is responsible for slowing down and collapsing a superhorizon-sized void. Without this fluid force, the void would expand (not collapse) from gravitational forces (see V).

In section II-B, we saw that for special relativistic fluids ($G_N = 0$), U is the net outward momentum per particle mass μ . Similarly, if in the general relativistic case $U^2 \gg U_{\text{GRAV}}^2 = 2G_N M/R$, the gravitational attraction inward is negligible and the dynamics will be dominated by special relativistic effects; U can be loosely interpreted as the peculiar momentum per particle per mass. If $\Gamma \gg 1$ and $U > 0$ in the wall of a void, the wall moves outward with momentum much greater than the gravitational attraction inward. We now investigate what happens to this wall. From Eqn (2.22), the “conservation of momentum” equation is

$$nw\Phi^{-1}\dot{U} = -nw \left(\frac{G_N M}{R^2} + \frac{4\pi G_N (p+Q)R}{c^2} \right) - \Gamma^2 \frac{(p+Q)'}{R'}. \quad (2.29)$$

The functions Γ , Φ , n , p , Q , M , w and R' are always positive. If $U > 0$, then at the inner edge of the void wall where $(p+Q)' > 0$, \dot{U} will be negative—the fluid there is decelerated. This deceleration is due to both gravitational and fluid forces. We examine the fluid force contribution only. Using Eqns (2.12), (2.6) and (2.22) we find that

$$\dot{\Gamma} = -\Gamma U \Phi \frac{(p+Q)'}{nwc^2 R'}. \quad (2.30)$$

Again, if $(p+Q)' > 0$ and $U > 0$, $\dot{\Gamma}$ will be negative and the fluid particles there lose their energy per particle mass.

Suppose the wall has a very large outward momentum so that $U/c \gg \sqrt{2G_N M/(Rc^2)} > 1$. Then $\Gamma \simeq U/c$. Setting $p = \rho/3$ and $Q = 0$ as for a relativistic, non-viscous fluid,

$$\dot{\Gamma} = -\Gamma^2 \frac{\Phi \rho'}{4\rho R'} \simeq -\Gamma^2 \frac{\rho_{\text{out}}^{1/4} \rho'}{4 \rho^{5/4} R'}, \quad (2.31)$$

where we have used the solution for Φ from Eqn (2.14). We consider the deceleration of fluid shells at the inner edge of a steep void wall. We can write $\rho'/R' \simeq \Delta\rho_{\text{wall}}/\Delta R_{\text{wall}}$, where $\Delta\rho_{\text{wall}}$ is the difference in the energy density over the width of the wall, and ΔR_{wall} is the thickness of the wall. Then, since $\Delta\rho_{\text{wall}} \simeq \rho_{\text{max}}$, where ρ_{max} is the maximum wall energy density, we can approximate $\dot{\Gamma}$ by

$$\dot{\Gamma} \simeq -\frac{\Gamma^2}{4} \left(\frac{\rho_{\text{out}} \rho_{\text{max}}^4}{\rho^5} \right)^{1/4} \Delta R_{\text{wall}}^{-1}. \quad (2.32)$$

Initially, except for the first factor of Γ^2 , all factors on the right hand side of the previous equation are independent of Γ . Thus, $\dot{\Gamma} \propto -\Gamma^2$, which can be a very large damping factor! The second factor is proportional to $\rho_{\text{out}} \rho_{\text{max}}^4 / \rho^5 \leq (\rho_{\text{out}}/\rho)^5$, so for the mass shells on the innermost part of the wall (i.e. those shells with the smallest values of ρ/ρ_{out}) this factor can be enormous. The third factor is $\Delta R_{\text{wall}}^{-1}$, so the thinner the wall, the faster it will slow down. If the second and third factors change slowly enough

with time, then the slow-down time for the wall is roughly independent of its initial value of Γ_0 , since $-\int_{\Gamma_0}^1 d\Gamma/\Gamma^2 \simeq 1 \gtrsim (\rho_{\text{out}}/\rho)^5 \Delta R_{\text{wall}}^{-1} \Delta t/4$ for $\Gamma_0 \gg 1$. This could be an extremely important result, and would imply that the initial peculiar wall velocity could never be large enough to cause a void to expand for an arbitrarily long time. However, because the second and third factors in Eqn (2.32) will change with time and depend on Γ , this is only a crude guess.

As a concluding remark, we note that the wall of a void formed during first-order inflation has an enormous outward peculiar velocity. This enormous velocity has been thought to cause a void to expand “indefinitely”. However, with such a large deceleration of the void wall, it will slow down in a finite (and possibly small) amount of time. A future paper will explore this numerically.³⁴

III. Initial Conditions, Boundary Conditions, and Numerical Techniques

A. Initial Conditions

The grid used in this code is initially equally spaced: $\Delta R(t_i) \equiv R(t_i)_{j+1} - R(t_i)_j = \text{constant}$, where the subscript j denotes the spatial grid point number and ranges from $j \in [0, j_B]$, where j_B is its value at the outer boundary. (Note from Eqns (2.19) and (2.20) that in general, $\Delta r \neq \text{constant}$ then). The 0th and 1st-grid points are located at $R_0 = -\Delta R(t_i)/2$ and $R_1 = \Delta R(t_i)/2$ respectively. Thus, $R_j(t_i) = R_1(t_i) + (j-1)\Delta R(t_i)$.

The initial conditions for the functions at t_i are determined as follows. The viscosity is set to zero: $Q(t_i, R) = 0$. The energy density $\rho(t_i, R)$ (or the “mass-energy” $M(t_i, R)$), the “velocity” $U(t_i, R)$ (or $\Gamma(t_i, R)$) and the specific internal energy, $\epsilon(t_i, R)$, are chosen as functions of the radius $R(t_i)$. (If M is specified initially instead of ρ , we determine ρ via Eqn. (2.6)). For the cases run in this paper however, we will initially specify the fluid to be a polytrope (constant entropy on the initial time slice (II-C)) and $\epsilon/c^2 \gg 1$ or $\epsilon/c^2 \ll 1$. Using the relations found at the end of Section II-C, $\epsilon(t_i, R)$ is determined once the internal energy $\epsilon(t_i, R_B)$ is specified at the outer boundary ($R_B \equiv R(t_i)_{j_B}$):

$$\begin{aligned} \epsilon(t_i, R) &= \epsilon(t_i, R_B) \left[\rho(t_i, R)/\rho(t_i, R_B) \right]^{(\gamma-1)/\gamma} \quad \text{for } \epsilon/c^2 \gg 1 \\ \epsilon(t_i, R) &= \epsilon(t_i, R_B) \left[\rho(t_i, R)/\rho(t_i, R_B) \right]^{(\gamma-1)} \quad \text{for } \epsilon/c^2 \ll 1 \end{aligned} \quad (3.1)$$

We then determine n and p by $n(t_i, R) = \rho(t_i, R)/(1 + \epsilon(t_i, R))$ and $p(t_i, R) = (\gamma - 1)n(t_i, R)\epsilon(t_i, R)$ respectively. Next, we find r (and M if it is not initially specified) by integrating outward from $r = 0$ using the 4th-order Runge-Kutta method:

$$\begin{aligned} r &= \left(3 \int_0^{R(t_i, r)} 4\pi n R^2 dR/\Gamma \right)^{1/3} \\ &\left(\text{and } M(t_i, R) = \int_0^R n(1 + \epsilon/c^2) R^2 dR \right), \end{aligned} \quad (3.2)$$

where we have omitted the (t_i, R) 's for clarity. Finally, Φ is found by integrating Eqn (2.26) inwards from the outer boundary (again using the 4th-order Runge-Kutta method) once $\Phi(t_i, R_B)$ is specified.

We are interested in evolving a non-expanding, empty void embedded in a FRW universe. We will therefore initially set the energy density outside the void ($\rho_{\text{out}} \equiv \rho(t_i, R_B)$) to be constant and equal to the spatially flat FRW value. Then, $H_{\text{out}}^2(t_i) \equiv H^2(t_i, R_B) = (\xi/t_i)^2 = 8\pi G_N c^{-2} \rho_{\text{out}}(t_i)/3$, where $a_{\text{out}}(t) = a(t_i)(t/t_i)^\xi$ is the cosmic scale factor and cH_{out}^{-1} is the Hubble radius outside the void. (Note that for the $k = 0$ FRW universe, $r = (4\pi n(t_i))^{1/3} R(t_i, r)$, or $a(t_i) = (4\pi n(t_i))^{-1/3}$). For $p = \rho/3$ and $p = 0$,

$\xi = 1/2$ and $\xi = 2/3$ respectively. We can quantify the initial size of a void by measuring its radius relative to the horizon size outside the void initially: $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) = c^{-1}\xi R_{\text{wall}}(t_i)/t_i$, where $R_{\text{wall}}(t_i)$ is the void “radius”. Following past convention², we loosely equate the Hubble radius with the horizon in the phrase “superhorizon-sized”. (Horizon in this context is not to be confused with the particle horizon.) A void is defined to be superhorizon-sized if $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) > 1$ and subhorizon-sized if $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) < 1$. In addition, $\Gamma_{\text{out}}(t_i) = 1$ (see II-B) (or $U = \sqrt{2G_N M/R}$) and $\Phi_{\text{out}}(t_i) \equiv 1$ (see II-A).

In this paper, we consider voids which are initially either compensated or uncompensated in energy density and which have the following distributions. The “mass-energy” function for compensated voids is defined to be

$$M(t_i, R) = .5\rho_{\text{out}}(t_i)[(1 + \tanh x) + \alpha(1 - \tanh x)]R^3(t_i)/3, \quad (3.3)$$

where $x \equiv (R(t_i) - R_{\text{wall}}(t_i))/\Delta R_{\text{wall}}(t_i)$, $\Delta R_{\text{wall}}(t_i)$ is the wall thickness and α is a specified constant less than or equal to 1. Because M reaches its spatially flat FRW value outside the void, the energy density missing from the void has been put in the wall. For uncompensated voids, the energy density is instead initially specified. It is

$$\rho(t_i, R) = .5\rho_{\text{out}}(t_i)[(1 + \tanh x) + \alpha(1 - \tanh x)]. \quad (3.4)$$

Here, the energy density missing from the void has not been put into the wall. Therefore, the region outside a compensated void will always be a spatially flat ($k = 0$) FRW universe, whereas the region outside an uncompensated void is a negative spatially curved ($k < 0$) FRW universe.

The inside of the void is chosen to be homogeneous initially. Since we want it to be nonexpanding in the limit that it is empty ($\rho \rightarrow 0$), we choose the inside of the void to be a spatially flat ($k = 0$) FRW “mini” universe. We reason as follows. Friedmann’s equation inside the void is $H_{\text{in}}^2 \simeq 8\pi G_N c^{-2} \rho_{\text{in}}/3 - kc^{-2}r^2/R^2$ (see II-B), where the subscript ‘in’ denotes quantities inside the void, $R = ra_{\text{in}}$ and $H_{\text{in}} \equiv U_{\text{in}}/R = \Phi_{\text{in}}^{-1}\dot{a}_{\text{in}}/a_{\text{in}}$. In the limit that $\rho \rightarrow 0$, the inside of the void is non-expanding ($\dot{a}_{\text{in}} = 0$) only if $k = 0$. Since we cannot numerically choose $\rho_{\text{in}} = 0$ (because $\Phi_{\text{in}} = \infty$ from Eqn(2.14)), we would like the inside of the void to not expand on time scales that the outside region expands in. This is satisfied if $\rho_{\text{in}}/\rho_{\text{out}} \ll 1$ because the Hubble time inside the void is much larger than that outside the void— $H_{\text{in}}^{-1}/H_{\text{out}}^{-1} = \sqrt{\rho_{\text{out}}/\rho_{\text{in}}}$. (As a check, Eqns. (2.22) and (2.25) show that as $U \rightarrow 0$, $\epsilon \rightarrow 0$ and $p \rightarrow 0$, then $\dot{U} \rightarrow 0$ and $\dot{\epsilon} \rightarrow 0$). Because the void is approximately homogeneous, the “mass-energy” inside the void is $M(t_i, R) \simeq c^{-2}\rho_{\text{in}}R^3/3$. Finally, we set $\Gamma_{\text{in}}(t_i) = 1$ inside the void since $\Gamma = \sqrt{1 - kr^2/c^2}$. The “velocity is then $U = \sqrt{2G_N M/R} \simeq c^{-1}R\sqrt{8\pi G_N \rho/3}$.

We will only consider two types of initial velocity profiles in this paper. The first is $U = U_{\text{GRAV}} = \sqrt{2G_N M/R}$ (or $\Gamma(t_i, R) = 1$). This specifies that the outward “velocity” of each particle is just large enough to compensate for the inward gravitational attraction (i.e. the peculiar velocity, U_{PEC} , is zero). The second is

$$U(t_i, R) = c^{-1}R\sqrt{8\pi G_N \rho/3}. \quad (3.5)$$

For $R \ll R_{\text{wall}}(t_i) - \Delta R_{\text{wall}}(t_i)$ and $R \gg R_{\text{wall}}(t_i) + \Delta R_{\text{wall}}(t_i)$, $\Gamma \simeq 1$. In the wall region, however, $\Gamma > 1$. This corresponds to an initial net outward “peculiar momentum” of the particles in the wall.

B. Boundary Conditions

We set the outer boundary conditions for all times to be those for a homogeneous fluid. This specification works well in practice as long as the action is taking place away from this boundary. Thus we set $n' = \epsilon' = p' = 0$ and $Q = 0$ at $j = j_B$, where j_B is the grid point number for the outermost comoving coordinate. In addition, we set $\Phi_{j_B}(t) \equiv 1$, its FRW value (see Eqn (2.3)). Note that specifying $\Phi_{j_B}(t)$ completely eliminates the arbitrariness of the time coordinate, as discussed after Eqn (2.2). The present definition sets t to be the FRW cosmic time outside the void. Thus if an initially inhomogeneous fluid becomes homogeneous, then from Eqn(2.26), $\Phi(t, r) = 1$ everywhere, and $t = \text{constant}$ hypersurfaces correspond to $t = \text{constant}$ FRW homogeneous and isotropic hypersurfaces.

It is possible to determine the outer boundary conditions for all t by solving the equations with $Q = p' = 0$. One is then left with two 1st-order ordinary differential equations to solve. The boundary conditions determined this way, however, give larger errors than the ones shown below, and therefore were not used. We instead integrate

$$\begin{aligned}\dot{R} &= U \\ \dot{U} &= -\left(G_N M/R^2 + 4\pi G_N p R/c^2\right) \\ \dot{M} &= -4\pi p R^2 U/c^2.\end{aligned}\tag{3.6}$$

using the MacCormack method. We then determine n , ϵ and p from Eqns (2.28) and Eqn (2.18) by setting $n_{j_B} = n_{j_B-1}$, $\epsilon_{j_B} = \epsilon_{j_B}(t_i) [n_{j_B}(t)/n_{j_B}(t_i)]^{\gamma-1}$ and $p_{j_B} = (\gamma - 1)n_{j_B}\epsilon_{j_B}$.

Finally, reflecting boundary conditions are used at the inner boundary: $R_0^n = -R_1^n$, $U_0^n = -U_1^n$, $p_0^n = p_1^n$, $n_0^n = n_1^n$, $\epsilon_0^n = \epsilon_1^n$ and $Q_0^n = Q_1^n$, where the superscript n refers to values on the n^{th} time slice.

C. Numerical Integration

The \dot{U} , $\dot{\epsilon}$, \dot{R} , \dot{n} and \dot{M} equations are integrated using the 2-step MacCormack predictor-corrector method.²³ In Appendix B, we give the exact form for the difference equations which allows inbound shocks to rebound off the origin. To illustrate the MacCormack method, we show the predictor and corrector steps for \dot{n} as an example. We continue to use the convention that n_j^i is the value of n on the j^{th} spatial grid point and at the i^{th} time step. Suppose we know all quantities on the i^{th} time slice. We would like to determine them on the $(i+1)^{\text{th}}$ time slice. First we predict the new quantities (with forward differencing) using the functional values on the i^{th} time slice:

$$n_p^{i+1} = n_j^i - \Delta t n_j^i \Phi_j^i \frac{R_{j+1}^{i2} U_{j+1}^i - R_j^{i2} U_j^i}{R_j^{i2} (R_{j+1}^i - R_j^i)}.\tag{3.7}$$

After using similar methods to obtain U_p^{i+1} , R_p^{i+1} , M_p^{i+1} and ϵ_p^{i+1} (and setting $p_p^{i+1} = (\gamma - 1)n_{p_j}^{i+1}\epsilon_{p_j}^{i+1}$) for all j , we integrate the Φ' equation inwards from $j = j_B$ using the 4th-order Runge-Kutta method with linear interpolations to determine Φ_p^{i+1} for all j . We integrate again, using the predicted values obtained above (with backward differencing), and then average these with the previously predicted values:

$$n_j^{i+1} = .5 \left(n_p^{i+1} + n_j^i - \Delta t n_{p_j}^i \Phi_{p_j}^i \frac{R_{p_j}^{i2} U_{p_j}^i - R_{p_{j-1}}^{i2} U_{p_{j-1}}^i}{R_{p_j}^{i2} (R_{p_j}^i - R_{p_{j-1}}^i)} \right).\tag{3.8}$$

We obtain the other values in a similar way, and then integrate again to find Φ_j^{i+1} .

As is well known,¹⁶ it is important to choose small enough time steps Δt to satisfy the Courant condition. This condition requires Δt to be smaller than the time taken for sound to cross from any one grid point to the next. The speed of sound for relativistic fluids is $c_s = \sqrt{(\partial p / \partial \rho)_S}$.^{13,14} Using the fact that $TdS = d(\rho/n) + pd(1/n)$, we find

$$c_s = \sqrt{\frac{\gamma p}{nw}}. \quad (3.9)$$

As $c^2 \rightarrow \infty$, $w = 1$ and we obtain the usual expression for the speed of sound in nonrelativistic fluids. Note that for $\epsilon/c^2 \gg 1$, $c_s = c\sqrt{\gamma - 1} = .57c$ for perfect fluids with $\gamma = 4/3$. The Courant condition requires the proper time per proper distance to be equal to C/c_s , where C is the Courant number and is approximately $1/3$ to $1/2$ for strong shocks. Since the infinitesimal proper time and proper distance are $\Phi \Delta t$ and $\Lambda \Delta r = \Delta R / \Gamma$ respectively, the Courant condition becomes $\Delta t_i^n = C (R_{i+1}^n - R_i^n) / (\Gamma_i^n \Phi_i^n (c_s)_i^n)$.¹⁴ In addition, when $G_N \neq 0$, regions of the universe expand and contract. We require the time steps to be small enough for the functions n , ϵ , and M to change sufficiently slowly. We therefore set $\Delta t_i^n = \bar{f} (l_i^n / \dot{l}_i^n)$ to be the maximum time step allowed for $l = n$, ϵ and M , where \bar{f} is a constant less than one.

After the n^{th} corrector step, Δt_i^n is calculated for all i , and Δt^{n+1} is set to the smallest value obtained:

$$(\Delta t)^{n+1} = \min \left(\Delta t_{\text{max}}, C \frac{R_{i+1}^n - R_i^n}{\Gamma_i^n \Phi_i^n (c_s)_i^n}, \left[\bar{f} \frac{n_i^n}{\dot{n}_i^n}, \bar{f} \frac{\epsilon_i^n}{\dot{\epsilon}_i^n}, \bar{f} \frac{M_i^n}{\dot{M}_i^n} \right]_{\text{if } G_N \neq 0} \right), \quad (3.10)$$

where Δt_{max} is a specified upper bound, if desired.²⁵

We apply a convergence test to the code for test problems where analytic solutions are available. The relative error in q , where q denotes any quantity, is defined to be

$$e_i = |q_i - \tilde{q}(r_i)| / \tilde{q}(r_i), \quad (3.11)$$

where $\tilde{q}(r_i)$ is the exact solution and q_i is the numerical solution. We obtain a global measure of the error by defining

$$L_1 = \frac{1}{N} \sum_{i=1}^N e_i, \quad (3.12)$$

where N is the total number of grid points. This error is proportional to the grid spacing to some power: $L_1 \propto \Delta R^s$, where s is the convergence rate. If $s \simeq 2$, the code is second-order, as desired. These tools have been used previously to test codes in other applications.²⁶

IV. 1st-Crossing Time for Relativistic Voids

In this section, we first review the standard lore for the evolution of superhorizon-sized voids, and then calculate the 1st-crossing time (the cosmic time taken for a photon initially at the inner edge of the void wall to reach the origin) in this picture. We then calculate the *actual* 1st-crossing time. If at the 1st-crossing time the fluid were approximately homogeneous and isotropic, distortions from the original void would be negligible and a (nearly) FRW universe would result everywhere.

It has been suggested that superhorizon-sized voids formed from first-order inflation would conformally expand with spacetime during the radiation-dominated period.^{6,8} Thus, the size of a void at time t would be $R = r_0 a(t)$, where r_0 is the comoving coordinate of the void and $a(t)$ is the cosmic scale factor. There are several justifications to support this belief. First, small density perturbations conformally expand with spacetime. Second, vacuum bubbles conformally expand during inflation after an initial short growing period. Third, the time taken for the void wall to slow down is expected to be enormous because the initial outward momentum of the void wall is enormous. However, this reasoning is not enough to conclude that a superhorizon-sized void conformally expands with spacetime. First, a void is not a small perturbation in spacetime. Thus linear results can not be applied to the description of a void. Second, it is the negative pressure that causes vacuum bubbles to expand; a similar configuration having positive pressure would instead accelerate inward.^d Third, although part of the wall may still move out, the deceleration of the inner void wall is extremely large (see II-D), so that the void can still collapse (see VII).

If a superhorizon-sized void were to conformally expand in spacetime, then the earliest time at which thermalization and homogenization can occur is when the horizon is of order the size of the void, since this is the expected 1st-crossing time. If the void has comoving size r_0 and the outside Hubble radius (that outside the void) is $cH_{\text{out}}^{-1}(t_i)$, this occurs when $r_0 a(t) = H_{\text{out}}^{-1}(t)$. For evolution during the radiation-dominated epoch, $H_{\text{out}}^{-1}(t) = 2t$ and $p = \rho/3$ so that the time is of order $\Delta t_c \equiv t - t_i \simeq H_{\text{out}}^{-1}(t_i)(c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i))^2$. If the void is much larger than the Hubble radius outside the void ($c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) \gg 1$), then the 1st-crossing time is very large: $\Delta t_c/H_{\text{out}}^{-1}(t_i) \gg 1$, independent of the “emptiness” of the void. Other authors suggest that the void would fill in with radiation.⁹ If spacetime at the void wall continues to expand as $a(t)$ when the fluid diffuses into the void, the comoving radius of the wall is roughly $r \simeq r_0 - c \int_{t_i}^t dt'/a(t)$ or

$$ra(t_i) \simeq R_{\text{wall}}(t_i) - c \int_{t_i}^t dt' \sqrt{t_i/t'} \quad (4.1)$$

in a radiation-dominated universe. The 1st-crossing time (i.e. the earliest thermalization time) then, is when $r \simeq 0$, or when $\Delta t_c \simeq .5H_{\text{out}}^{-1}(t_i)(c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i))^2$, which is roughly the same as the time for the horizon to “engulf” a conformally expanding void.^e Because $R = ra(t)$, the radius of the void would be given by $R \simeq (R_{\text{wall}}(t_i) - cH_{\text{out}}^{-1}(t_i)\sqrt{t/t_i})\sqrt{t/t_i}$, which for nearly all of the time is $R \simeq R_{\text{wall}}(t_i)\sqrt{t/t_i}$; the void conformally stretches with spacetime. At time $t \simeq H_{\text{out}}^{-1}(t_i)(c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i))^2$, the void radius decreases quite rapidly to zero. Thus, although the qualitative void evolution is quite different in these two pictures, the quantitative 1st-crossing times are not because in both the void comoves with spacetime. The important point is that the earliest possible thermalization time in both pictures (i.e. the 1st-crossing time) is thought to be

$$\Delta t_c \simeq H_{\text{out}}^{-1}(t_i)(c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i))^2. \quad (4.2)$$

^dIn the bubble wall, $G_N = 0$, $|p'| > 0$ and $p < 0$.¹ Using Eqn(2.22), we see that the acceleration is positive, so that the bubble wall moves outward during inflation. But for normal positive pressure with $p' > 0$, the acceleration is *negative*, so that the wall must accelerate inward.

^eWe thank Michael Turner for this explanation.

We will show in this section that the actual 1st-crossing time is remarkably shorter than Eqn(4.2). In doing so we will show that Eqn(4.1) is fundamentally flawed.

Consider two radially propagating photons A and B. Photon A starts at the inner edge of the void wall and propagates inward, and photon B begins at the outer edge of the void wall and moves outward. We would like to calculate the distance each travels in time $\Delta t = t - t_i$, where t_i is the initial time. Using Eqns (2.2) and (2.5), the infinitesimal coordinate distance traveled by a photon in time dt is $dr = cdt\Phi\Gamma/R'$. Define $\rho_{\text{in}}(t)$ and $\rho_{\text{out}}(t)$ to be the energy densities inside and outside the *evolving* wall region of the void. (Thus the subscripts “in” and “out” refer only to the undisturbed fluid). Initially, $\rho'_{\text{in}}(t_i) = \rho'_{\text{out}}(t_i) = 0$, and $\Gamma(t_i, R) = 1$. We also consider non-viscous, relativistic fluids, so that $p = \rho/3$ and $Q(t, r) = 0$.

We will first calculate the distances photons A and B travel in special relativistic voids ($G_N = 0$). From Eqn. (2.22), the fluid acceleration is zero inside and outside the void: $\dot{U}_{\text{in}} = \dot{U}_{\text{out}} = 0$. Therefore, $R(t, r) = R(t_i, r)$ inside and outside the void, so that the infinitesimal distance traveled by photons A and B in time dt is $dR = c\Phi\Gamma dt$. From Eqn. (2.10), we see that ρ_{out} and ρ_{in} are constant in time. Since $\Phi_{\text{out}} = 1$, we find that $\Phi_{\text{in}}(t, r) = \Phi_{\text{in}}(t_i, r) = (\rho_{\text{out}}(t_i)/\rho_{\text{in}}(t_i))^{1/4}$ from Eqn. (2.14). In addition, since $\dot{\Gamma} \propto p'$ (Eqn. (2.30)), $\Gamma_{\text{in}}(t) = \Gamma_{\text{out}}(t) = 1$. Therefore in time $\Delta t \equiv t - t_i$, photon B travels outward the distance $\Delta R_B(t) \equiv R_B(t) - R_{\text{wall}}(t_i)$ given by

$$\Delta R_B = c\Phi_{\text{out}}\Gamma_{\text{out}}\Delta t = c\Delta t, \quad (4.3)$$

while photon A travels inward the distance $\Delta R_A(t) \equiv R_{\text{wall}}(t_i) - R_A(t)$ given by

$$\Delta R_A(t) = c\Phi_{\text{in}}\Gamma_{\text{in}}\Delta t = c[\rho_{\text{out}}(t_i)/\rho_{\text{in}}(t_i)]^{1/4} \Delta t = c T_{\text{out}}(t_i)/T_{\text{in}}(t_i) \Delta t > c\Delta t. \quad (4.4)$$

Thus the emptier the void, the farther photon A moves relative to photon B! (It is important to note that this is a strictly relativistic effect; for nonrelativistic voids (II-C), $\Phi(t, r) \simeq 1$ inside and outside this void so that the distance traveled by photons A and B are approximately the same: $\Delta R_A \simeq c\Delta t = \Delta R_B$). Define Δt_c to be the 1st-crossing time (the time taken for photon A to reach the origin). Then from Eqn. (4.4) with $\Delta R_A = R_{\text{wall}}(t_i)$, the 1st-crossing time is

$$\Delta t_c = c^{-1} R_{\text{wall}}(t_i) [\rho_{\text{in}}(t_i)/\rho_{\text{out}}(t_i)]^{1/4}. \quad (4.5)$$

Since $\rho_{\text{out}}(t_i) \geq \rho_{\text{in}}(t_i)$, Δt_c ranges from $R_{\text{wall}}(t_i)/c$ to zero. Therefore in the limit that $\rho_{\text{in}}(t_i)/\rho_{\text{out}}(t_i) \rightarrow 0$, a photon will reach the origin in *zero* cosmic time. However, calling a region a “fluid” if it is empty ($\rho_{\text{in}} = 0$) is incorrect. Suffice it to say that the 1st-crossing time can be arbitrarily small.

We note that if Eulerian, synchronous coordinates were used with the metric $ds^2 = -c^2 dT^2 + dR^2 + R^2 d\Omega^2$, the distance traveled by photon A or B in time ΔT would be the same: $\Delta R = c\Delta T$. Thus the time ΔT_c for photon A to reach the origin is $\Delta T_c = c^{-1} R_{\text{wall}}(t_i)$. (We can also see this using Lagrangian coordinates, since the infinitesimal proper time measured by a comoving observer inside the void is $d\tau = \Phi_{\text{in}} dt = cdT$, and therefore the time as measured by this observer for photon A to reach the origin is $\Delta\tau_c = c\Phi_{\text{in}}\Delta t_c = R_{\text{wall}}(t_i)$). Thus, the quick 1st-crossing time is due to the choice of comoving, synchronous coordinates, a choice we do not have for superhorizon-sized general relativistic voids ($G_N \neq 0$) embedded in a FRW universe.^f

^f If the spatially flat FRW metric is transformed to the Eulerian gauge with synchronous coordinates, a coordinate singularity at the Hubble radius (see Appendix A). Thus this gauge and coordinate choice cannot be used to describe the evolution of superhorizon-sized general relativistic voids.

We now calculate the distance traveled by photons A and B in the same cosmic time for a general relativistic void. Again, the infinitesimal coordinate distance a photon travels in time dt is $dr = c\Phi\Gamma dt/R'$. Because the pressure outside the void redshifts due to Hubble expansion, $\Phi_{\text{in}}(t)$ decreases in time from Eqn(2.14) (since $\Phi_{\text{out}}(t) = 1$), and Δt_c consequently increases. We first calculate the distance photon B travels. Outside the void, $\Gamma_{\text{out}}(t) = 1$, and $R = ra(t)$ so that $R' = a(t_i)\sqrt{t/t_i}$.²⁴ The comoving distance photon B has traveled at time t is $\Delta r = ct_i/a(t_i)(\sqrt{t/t_i} - 1)$, so that the distance photon B travels is $\Delta R_B \equiv R_B - R_{\text{wall}}(t_i) = (\sqrt{t/t_i} - 1)[R_{\text{wall}}(t_i) + 1/2 cH_{\text{out}}^{-1}(t_i)\sqrt{t/t_i}]$.

We now calculate the location of photon A. Assume that the energy density inside the void changes negligibly: $\rho_{\text{in}}(t) = \text{constant}$. (We will address this approximation in a moment). Then $\Phi_{\text{in}}(t) = (\rho_{\text{out}}(t)/\rho_{\text{in}}(t_i))^{1/4}$. Because the energy density outside the void is redshifted as $\rho \propto 1/t^2$ (see VI),²⁴ $\Phi_{\text{in}}(t) \simeq \Phi_{\text{in}}(t_i)\sqrt{t_i/t}$. In addition, since $\dot{\Gamma} \propto p'$ (from Eqn. (2.30)), $\Gamma_{\text{in}}(t) = \Gamma_{\text{out}}(t) = 1$. Integrating, we find that the cosmic time taken for photon A to travel the distance $\Delta R_A \equiv R_{\text{wall}}(t_i) - R_A$ is

$$\Delta t \equiv t - t_i = c^{-1}\Delta R_A \left(\frac{\rho_{\text{in}}(t_i)}{\rho_{\text{out}}(t_i)}\right)^{1/4} \left[1 + \frac{c^{-1}\Delta R_A}{2H_{\text{out}}^{-1}(t_i)} \left(\frac{\rho_{\text{in}}(t_i)}{\rho_{\text{out}}(t_i)}\right)^{1/4}\right], \quad (4.6)$$

and the location of photon A as a function of time is

$$R_A = R_{\text{wall}}(t_i) - c\Phi_{\text{in}}(t_i) \int_{t_i}^t dt' \sqrt{t_i/t'} \quad (4.7)$$

$$= R_{\text{wall}}(t_i) - c\Phi_{\text{in}}(t_i)H_{\text{out}}^{-1}(t_i) \left[\sqrt{t/t_i} - 1\right]. \quad (4.8)$$

As in the special relativistic case, $\Delta R_A(t) > \Delta R_B(t)$, so that photon A travels farther than photon B in the same amount of cosmic time. The 1st-crossing time relative to the initial outside Hubble time is then ($\Delta R_A = R_{\text{wall}}(t_i)$)

$$\frac{\Delta t_c}{H_{\text{out}}^{-1}(t_i)} = \frac{c^{-1}R_{\text{wall}}(t_i)}{H_{\text{out}}^{-1}(t_i)} \left(\frac{\rho_{\text{in}}(t_i)}{\rho_{\text{out}}(t_i)}\right)^{1/4} \left[1 + \frac{c^{-1}R_{\text{wall}}(t_i)}{2H_{\text{out}}^{-1}(t_i)} \left(\frac{\rho_{\text{in}}(t_i)}{\rho_{\text{out}}(t_i)}\right)^{1/4}\right]. \quad (4.9)$$

In addition, if the more stringent condition $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) < \Phi_{\text{in}}(t_i)$ holds, then

$$\frac{\Delta t_c}{H_{\text{out}}^{-1}(t_i)} \lesssim 1 \quad \text{when} \quad \frac{c^{-1}R_{\text{wall}}(t_i)}{H_{\text{out}}^{-1}(t_i)} < \left(\frac{\rho_{\text{out}}(t_i)}{\rho_{\text{in}}(t_i)}\right)^{1/4} \quad (4.10)$$

—the minimum thermalization time (i.e. the 1st-crossing time) is less than the initial Hubble time outside the void! (Note that the general and special relativistic results (Eqns(4.9) and (4.5)) are equal in this case, since the energy density outside the void is constant during this time).

Before discussing further implications, we find the condition for which Eqn(4.6) is satisfied; it was derived under the assumption that $\rho_{\text{in}}(t) \simeq \rho_{\text{in}}(t_i)$. Using Eqn(2.10), inside the void $|\dot{\rho}/\rho| = 4(R^2\dot{R})/(3R^2R') = 4\dot{R}/R$ (since $R = ra(t)$) so that if $\rho_{\text{in}} \simeq \text{constant}$, then $R \simeq \text{constant}$ (in time) inside the void. Because $U^2 = (\dot{R}/\Phi)^2 = 2G_N M/R \simeq$

$c^{-2}(8\pi G_N \rho_{\text{in}}/3)R^2$ inside the void, the fractional change in the radius R over time scale Δt is roughly

$$f \equiv \frac{\Delta R}{R} = \frac{\dot{R}\Delta t}{R} = \sqrt{\frac{\rho_{\text{in}}(t)}{\rho_{\text{out}}(t)}} H_{\text{out}}(t) \Phi_{\text{in}}(t) \Delta t \simeq \Phi_{\text{in}}^{-1}(t_i) \sqrt{\frac{t_i}{t}} \frac{\Delta t}{H_{\text{out}}^{-1}(t_i)}, \quad (4.11)$$

where we have used the fact that $H_{\text{out}}^{-1}(t_i) = 2t_i$, $\sqrt{\rho_{\text{in}}(t)/\rho_{\text{out}}(t)} H_{\text{out}}(t) \simeq \Phi_{\text{in}}^{-2}(t_i) H_{\text{out}}(t_i)$ and $\Phi_{\text{in}}(t_i) = (\rho_{\text{out}}(t_i)/\rho_{\text{in}}(t_i))^{1/4}$. We require $f < 1$. Writing $t = \Delta t + t_i$, Eqn(4.11) becomes $\Delta t = f^2 \Phi_{\text{in}}^2(t_i) H_{\text{out}}^{-1}(t_i) [1 + \sqrt{1 + 1/(f\Phi_{\text{in}}(t_i))^2}]$. Since $\Phi_{\text{in}}(t_i) > 1$, we have $f\Phi_{\text{in}}(t_i) \gtrsim 1$. The condition for which the density inside of the void remains approximately constant during time Δt then, is $\Delta t/H_{\text{out}}^{-1}(t_i) \leq 2f^2 \Phi_{\text{in}}^2(t_i) \simeq \Phi_{\text{in}}^2(t_i)$. We now combine this with Eqn(4.6) to find the maximum allowed void size $\Delta R_A/H_{\text{out}}^{-1}(t_i)$ given $\Phi_{\text{in}}(t_i)$. We find $c^{-1}\Delta R_A/H_{\text{out}}^{-1}(t_i) \leq .5\Phi_{\text{in}}(t_i)[-1 + \sqrt{1 + 4(f\Phi_{\text{in}}(t_i))^2}] \simeq f\Phi_{\text{in}}^2(t_i)$. Eqn (4.6) is then valid when $c^{-1}\Delta R_A/H_{\text{out}}^{-1}(t_i) \leq f\Phi_{\text{in}}^2(t_i)$, and Eqn. (4.9) is valid when

$$c^{-1}R_{\text{wall}}(t_i) / H_{\text{out}}^{-1}(t_i) \lesssim \sqrt{\rho_{\text{out}}(t_i) / \rho_{\text{in}}(t_i)}. \quad (4.12)$$

Note that Eqn(4.10) is automatically satisfied.

The quick 1st-crossing time might seem completely counterintuitive. How can a photon travel a distance *much* larger than the Hubble radius in less than a Hubble time? The answer lies in describing how one measures the size of an object which is *not* a small perturbation in spacetime. If size is measured circumferentially, then the void is enormous because its circumferential size is $2\pi R_{\text{wall}}(t_i) \gg H_{\text{out}}^{-1}(t_i)$. (If spacetime were static, then it would take a photon time $\Delta t \simeq 2\pi R_{\text{wall}}(t_i)/c$ to encircle the void). However, if size is measured radially (the time taken for a photon to cross the object if spacetime were static), then the void is measured to be very small. In fact, an interesting comparison can be made to measuring the size of a black hole, an *overdense* region. If one measures its circumferential size, it is small (or at least finite), but its radial size is infinite.

The 1st-crossing time for a superhorizon-sized void was previously calculated incorrectly because t was assumed to be the proper time outside *and* inside the void; in our notation, it was implicitly assumed that $\Phi(t, r) = 1$. Comparing Eqns(4.1) and (4.7), we indeed see that the factor $\Phi_{\text{in}}(t_i) > 1$ is missing from Eqn(4.1). Note in addition that the factor of $\sqrt{t_i/t}$ in Eqn(4.7) does not come from spacetime expanding at the wall, but rather from the density outside the void redshifting, causing $\Phi_{\text{in}}(t)$ to decrease. In fact, spacetime is roughly non-expanding at the inner wall edge. If $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) > \Phi_{\text{in}}(t_i) \gg 1$, the actual position of photon A is approximately constant in time until the last moment: $R_A \simeq R_{\text{wall}}(t_i)$ until $t \simeq t_i + \Delta t_c$ (see Eqn(4.8)). Thus, spacetime at the inner edge of the void wall is neither expanding nor contracting.

We note the interesting fact that if a photon inside the void is in thermal equilibrium with average frequency ν , its frequency is T_{in} . If it moves outside the void, its frequency is blue-shifted to $\nu' = \nu\Phi_{\text{in}} = \nu(T_{\text{out}}/T_{\text{in}}) = T_{\text{out}}$, the average frequency of thermal photons outside the void. Thus, this photon is automatically in thermal equilibrium outside the void, so that an outside observer could not detect the void's initial presence unless non-thermal photons came out.^g This is essentially because the fluid is relativistic, or that ρ ,

^gBecause entropy is created at the shock (see VII), the photon with νT_{in} would actually have a slightly higher frequency than thermal outside the void.

p , n and ϵ depend only on the temperature (II-C). We can understand why the photon is blue-shifted by way of comparison to an (overdense) black hole. Suppose an observer falls into a black hole ticking off photons at a fixed frequency. As this observer crosses the event horizon, the frequency of the photon emitted last gets redshifted to infinity as observed by a stationary observer outside the black hole.¹² The opposite effect happens for a photon emitted from an underdense region. Because a photon leaving a void enters a region with a much larger gravitational potential, the frequency instead gets blue-shifted.

In conclusion, the 1st-crossing time for a superhorizon-sized relativistic void embedded in a FRW expanding universe is given by Eqn (4.9) (if Eqn(4.12) is satisfied), and depends sensitively on the quantity $R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i)$ $(\rho_{\text{in}}(t_i)/\rho_{\text{out}}(t_i))^{1/4}$. We emphasize the important point that if $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) < (\rho_{\text{out}}(t_i)/\rho_{\text{in}}(t_i))^{1/4} = T_{\text{out}}(t_i)/T_{\text{in}}(t_i)$, then the 1st-crossing time is less than the outside Hubble time: $\Delta t_c/H_{\text{out}}^{-1}(t_i) < 1$.

V. Pressureless Non-Viscous Voids

For the special case when the pressure and viscosity are zero, the equations of motion can be solved analytically. These give the Tolman-Bondi dust solutions,^{10,12} which we will review here briefly. It is important to study the pressureless case not only as a test problem, but also to see how removing fluid forces affects the evolution of a void. (Because $p = Q = 0$, the particles move only under gravitational forces: $\Phi^{-1}\dot{U} = -G_N M/R^2$ (see Eqn (2.22))).

Since $\Phi' = 0$ (Eqn (2.26)), we set $\Phi(t, r) \equiv 1$. The mass contained within r remains constant: $M(r) = \int_0^r 4\pi n R^2 dR/\Gamma$, since $\dot{M} = 0$ (see Eqn (2.9)). And from Eqn(2.30), $\Gamma(t, r) = \Gamma(t_i, r)$. For $R' \neq 0$, $\rho = n$ can be found from Eqn(2.19): $\rho(t, r) = \rho(t_i, r) R(t_i, r)' R(t_i, r)^2 / [R(t, r)' R(t, r)^2]$. The generalized pressureless Friedmann equation, Eq. (2.12), now becomes

$$\dot{R}^2 = c^2 [\Gamma(r)^2 - 1] + 2G_N M(r)/R. \quad (5.1)$$

The quantity $U^2/2 - G_N M/R = c^2(\Gamma^2 - 1)/2$ is conserved during evolution and can be interpreted as the generalized total energy. Eqn (5.1) can be integrated for a shell of radius r :

$$\begin{aligned} R &= \frac{c^{-2}G_N M}{\Gamma^2 - 1} (\cosh \eta - 1), & t &= \tau_0(r) + \frac{c^{-3}G_N M}{(\Gamma^2 - 1)^{3/2}} (\sinh \eta - \eta), & \text{for } \Gamma(r)^2 > 1 \\ R &= \frac{c^{-2}G_N M}{1 - \Gamma^2} (1 - \cos \eta), & t &= \tau_0(r) + \frac{c^{-3}G_N M}{(1 - \Gamma^2)^{3/2}} (\eta - \sin \eta), & \text{for } \Gamma(r)^2 < 1 \\ R &= (9G_N M/2)^{1/3} (t - \tau_0(r))^{2/3}, & & & \text{for } \Gamma(r) = 1. \end{aligned} \quad (5.2)$$

These are the Tolman-Bondi solutions.

In these models, shell-crossing can occur. This happens when two adjacent shells (labeled by r and $r + dr$) occupy the same position so that $R' = 0$ and $\rho \rightarrow \infty$. This may lead to a non-unique continuation of the solution,²⁷ and thus computations have to be stopped. This problem is believed to occur because the pressure has been artificially set to zero. It is generally thought that adding pressure would prevent this situation from occurring.^h As will be seen, when $\Gamma(r) > 1$ in the void wall region, shell-crossing does occur. The addition of enough artificial viscosity can prevent this from happening, however. Even though the viscosity given by Eqn (2.27) was designed to stabilize numerical shocks, it has been found to prevent shell-crossing.

^hThe author thanks T. Piran for this information.

Table 1: Convergence test for Tolman-Bondi model

$\Delta R(t_i)$	L_1	s	L_{1a}	s	L_{1b}	s
8	1.2×10^{-3}	...	8.33×10^{-4}	...	8.55×10^{-4}	...
4	4.13×10^{-4}	1.48	2.26×10^{-4}	1.89	2.16×10^{-4}	2.0
2	1.55×10^{-4}	1.32	6.09×10^{-5}	1.82	5.35×10^{-5}	1.97
1	6.56×10^{-5}	1.24	1.81×10^{-5}	1.75	1.41×10^{-5}	1.92

For the numerical simulations in this section, we set $G_N = 1$, $c = 1$, $t_i = 1$, $\rho(t_i, R) = 0$, $\epsilon(t_i, R_B) = 0$, $C = .3$, $\bar{f} = .005$ and $\gamma = 5/3$. Thus, the initial energy density and Hubble radius outside the void are $4\pi\rho_{\text{out}}(t_i) = 2/3$ and $H_{\text{out}}^{-1}(t_i) = 3/2$, respectively (see III-A).

We first examine the situation in which each mass shell's velocity initially compensates for the gravitational attraction inwards: $\Gamma(t_i, R) = 1$. Then $U(t_i, R) = \sqrt{2G_N M/R}$. Figure 1 shows the energy density versus $R R(t_i, R_B)/R(t, R_B) (\equiv R R_{j_B}(t_i)/R_{j_B}(t))$ for a superhorizon-sized compensated void with $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) = 333$, $\Delta R_{\text{wall}}(t_i) = 15$, $\alpha = .001$, $\Delta R(t_i) = 2.5$ and $k^2 = 0$. We show the analytic and numerical results at times $t = 1, 10, 100$, and 300 where $R_{j_B}(t) = 1002, 4651, 2.16 \times 10^4$, and 4.49×10^4 respectively. The triangles and squares are the numerical and Tolman-Bondi solutions, respectively, although they are difficult to distinguish because the numerical results agree so well with the analytic results. By $t \sim 300$, the density everywhere is approximately constant; there is hardly a trace of the void's initial presence. Identical results are obtained for subhorizon-sized voids. In addition, an initially uncompensated void evolves similarly.

In Table 1, we show the results of a convergence test for $q = \rho$ applied to the same initial conditions as in Figure 1, but for variable $\Delta R(t_i)$ (see III-C). (We only do this test for ρ , because the accumulated error in M and R are much smaller). We set up analytic conditions initially and integrate until $t = 1.5$. Because the inner grid point or two ends up being the numerical culprit for non-second order convergence, we also calculate $L_{1a} \equiv \frac{1}{N-2} \sum_{i=3}^N e_i$ and $L_{1b} \equiv \frac{1}{N-10} \sum_{i=10}^N e_i$, where e_i is the relative error. (This is because the code consistently underestimated ρ at the innermost few grid points). To the right of each global error estimate, the convergence rate is shown ($L_i \propto \Delta R(t_i)^s$). We see that the convergence rate for L_1 is less than second-order, whereas that for L_1 is nearly second order.

We have just seen that if $U = U_{\text{GRAV}}$ ($\Gamma(t_i, R) = 1$), the void disappears. What happens when $U > U_{\text{GRAV}}$ ($\Gamma(t_i, R) > 1$) in the wall region? Recall that this corresponds to a net outward peculiar velocity (II-D). Figure 2 shows the result for a compensated void with $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) = 333$, $\Delta R_{\text{wall}}(t_i) = 15$, $\alpha = .001$, $\Delta R(t_i) = 2.5$, and $k^2 = 0$. We choose the initial velocity to be given by Eqn(3.5). Therefore, $\Gamma(t_i, R) = 1$ everywhere except in the wall region, where $\Gamma(t_i, R) > 1$. Again, the triangles and squares represent the numerical and analytic solutions, respectively. The initial time (t_i) and $t = 1.02, 1.048$ are shown. Again, the difference between the numerical and analytical results are small except at $t = 1.048$, where shell crossing occurs in both solutions, a good check on the code. The comoving radius of the shell with the highest

density, $r_{\text{shell}}(t)$, remains approximately constant in time. Because each shell in the wall has constant total energy $c^2(\Gamma^2 - 1)/2$, the wall expands outward. (Since $U > 0$, R increases. But since the total energy is constant and $M(r)$ is constant, U must increase). Identical results are obtained for subhorizon voids.

Figures 3a and 3b show the density as a function of $R R(t_i, R_B)/R(t, R_B)$ for a subhorizon-sized, shallow, uncompensated and compensated void, respectively, at $t = 1, 50, 100, 400$ and 1000 . Note that shell-crossing would have occurred at $t = 116$ and 2.1 for the uncompensated and compensated voids, respectively. Here, $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) = .0067$, $\Delta R_{\text{wall}}(t_i) = .001$, $\Delta R(t_i) = .0001$, $\alpha = .5$, and the velocity $U(t_i, R)$ is given by Eqn (3.5). In addition, for Figure 3a, $k^2 = 4$ and $R_{j_B}(t) = .035, .49, .79, 2.13$ and 4.26 , while for Figure 3b, $k^2 = 8$ and $R_{j_B}(t) = .035, .48, .76, 1.9$ and 3.5 . As can be seen, the initial perturbation grows with time and eventually forms a thin, dense shell. Again, the comoving coordinate for the shell with the highest density is approximately constant in time after the shell has formed. As the shell travels outward, it pushes mass in front of it, producing a shock. This situation is similar to that of a fast car colliding with slower ones; although the faster car is not allowed to move through the slower cars, momentum is still transferred to them. Note that the initially compensated perturbation forms a thick shell more quickly than the uncompensated perturbation, although it then proceeds to grow more slowly.

As long as the voids formed remain subhorizon-sized, they will eventually grow according to a known similarity solution.²⁸ An initially compensated (uncompensated) perturbation in an expanding FRW $p = 0$ universe will eventually form a dense, thin shell that expands outward as $R_{\text{shell}} \equiv R(t, r_{\text{shell}}) \propto t^{4/5}$ ($t^{8/9}$). In Figure 3c, we show the position of the void wall versus time for the results of Figures 3a and 3b. The triangles and squares connected by lines are the numerical solutions for the compensated and uncompensated cases, respectively, and the dashed and dotted lines are the self-similar solutions for the compensated and uncompensated cases, respectively. As can be seen, the initially compensated perturbation approaches the similarity solution more quickly than the initially uncompensated perturbation, but for $t \gtrsim 800$, both solutions are self-similar.

VI. FRW Homogeneous Cosmologies

We now test our code against the exact FRW homogeneous and isotropic solution for relativistic fluids with $p = \rho/3$. As discussed in II-A,B and III-A, the FRW solution is $R(t, r) = ra(t) = R(t_i, r)(t/t_i)^\xi$ where $\xi = 1/2$. In addition, $4\pi G_N \rho(t) = 3c^2/(8t^2)$ and $U = R/(2t)$. For the numerical results shown in this section, we set $G_N = 1$, $c = 1$, $\gamma = 4/3$, $C = .3$, $t_i = 1$, $\epsilon(t_i, R_B) = 10^6$ and $c^{-1}R_B(t_i)/H^{-1}(t_i) = 250$ so that $4\pi\rho(t_i, R_B) = 3/8$.

Figure 4 shows the relative error in ρ (III-C) for a simulation with $\Delta R(t_i) = .5$ and $k^2 = 0$. The analytical solution was set up initially, and the code was run until $t = 1.1$. The solid, dotted and dashed lines are for $\bar{f} = .01$, $\bar{f} = .005$ and $\bar{f} = .0025$. The relative error at the outer boundary is seen to be very sensitive to \bar{f} ; it is .05%, .15% and 2% for $\bar{f} = .0025, .005$ and $.01$ respectively. Note also the underprediction of ρ at the innermost few grid points.

Table 2 shows the accumulated error at time $t = 1.1$ for simulations with $k^2 = 4$.ⁱ Again, $L_{1b} \equiv \frac{1}{N-10} \sum_{i=10}^N e_i$, where e_i is the relative error for $q = \rho$. In the first 7 columns, we show the results for $N = 25, 50, 100, 200, 400, 800$ and 1600 (i.e. $\Delta R(t_i) = N/250$)

ⁱWhen evolving voids, viscosity is absolutely necessary. It is therefore important to see how it affects the solution where the fluid is approximately homogeneous and isotropic. It turns out that it is virtually unaffected by $Q \neq 0$, as it should be.

Table 2: Ultrarelativistic Homogeneous convergence test

	Number of grid points (N)							\bar{f}	L_1 $\Delta R \rightarrow 0$	s
	25	50	100	200	400	800	1600			
L_1	.016	.010	.0073	.0058	.0050	.0046		.01	.004	...
L_{1b}	.0071	.0056	.0049	.0045	.0044	.0043				
L_1	.012	.0067	.0038	.0023	.0016	.0012		.005	.00085	2.00
L_{1b}	.0015	.0012	.0010	.00091	.00088	.00087				
L_1	.011	.0059	.0031	.0017	.00096	.00060	.00042	.00025	.00025	1.77
L_{1b}	.00057	.00041	.00032	.00027	.00026	.00025	.00025			

and for $\bar{f} = .01, .005,$ and $.0025$. For a given value of \bar{f} , as N increases, L_1 and L_{1b} approach the same constant value even though L_1 starts out much larger. After this constant value has been reached, L_1 and L_{1b} remain unchanged when $\Delta R(t_i)$ is further decreased, even though the relative error near the origin improves. This is because beyond a certain point, all the accumulated error comes from the outer boundary, which is immune to changes in $\Delta R_{\text{wall}}(t_i)$ (see III-B).

Knowing that a given value of \bar{f} limits convergence of the code, we can calculate the convergence as a function of the asymptotic value of L_1 . We assume that $L_1(\Delta R(t_i) \rightarrow 0) \propto \bar{f}^s$. Column 9 gives the estimated value for $L_1(\Delta R(t_i) \rightarrow 0)$, and column 10 estimates the value of s . We see that s is nearly 2, which means that convergence in \bar{f} is nearly second order given a small enough value for $\Delta R(t_i)$.

VII. Numerical Evolution of Voids

A. Nonrelativistic Fluids

In this subsection we examine the evolution of voids composed of nonrelativistic particles in zero gravity ($G_N = 0$). If T is the fluid temperature and μ is a fluid particle's mass, $T/\mu \ll 1$. Since $H^{-1} \propto G_N^{-1}$, these voids are subhorizon-sized. For the simulations in this subsection, we set $G_N = 0$, $c = 10^{10}$, $t_i = 1$, $C = .3$, $\gamma = 5/3$, $C = .3$, $4\pi\rho(t_i, R_B) = 2/3$ and $\epsilon(t_i, R_B) = 1$.

We start with the shock tube problem, a standard test of 1-D slab codes.²⁹ In addition, it provides insight into the dynamics of collapsing voids. In a shock tube, the fluid is initially at rest and is separated into two regions with different pressures and densities. The pressure discontinuity produces a shock wave which propagates into the low pressure region and a rarefaction wave which propagates into the high pressure region. An analytic similarity solution exists for a perfect fluid with slab geometry. It does not exist for the spherically symmetric geometry however.³⁰ Far from the origin however, the spherically symmetric solution approaches the slab solution for small times and distances.¹⁴ We will thus set up a spherically symmetric shock tube by evolving an uncompensated void far from the origin, and compare the results to the exact slab similarity solution (briefly reviewed in Appendix D).

In Figure 5 we show the results for the shock tube problem with $U(t_i, R) = 0$ (or $\Gamma(t_i, R) = 1$), $R_{\text{wall}}(t_i) = 20$, $\Delta R_{\text{wall}}(t_i) = .01$, $\Delta R(t_i) = .01$, $\alpha = .01$ and $k^2 = 5$. (We do not take the pressure gradient to be discontinuous initially, because the solution in the original wall area is not as accurate then.^j) We plot the pressure, number density, velocity and specific energy as a function of the position R at t_i and at $t = 1.15$. The triangles connected by lines is the numerical solution, and the dashed lines is the slab similarity solution. At $t = 1.15$, a strong shock wave (located at $R \simeq 19.68$) moves inward and a rarefaction wave (between $19.85 \lesssim R \lesssim 20.14$) moves outward. Note that the shock is spread out over $k^2 \simeq 4-5$ grid points. The numerical and analytical solutions are seen to agree well in this limit, because the spherical geometrical effects are small ($(R_{\text{shock}}(t) - R_{\text{wall}}(t_i))/R_{\text{wall}}(t_i) \sim .4/20 = .02$). The distortion of the velocity distribution is due to the small geometrical effect; the shock gets slightly stronger directly behind the shock due to the smaller effective volume $4\pi R^2 \Delta R$ those mass shells occupy relative to shells further back. An important point to emphasize is that although initially the fluid is everywhere stationary ($U(t_i, R) = 0$), it acquires a net momentum to the left in the wall region.

In Figure 6a we show the long-term results for an initial configuration with $R_{\text{wall}}(t_i) = 1$ but otherwise identical to Figure 5. The pressure, number density, velocity and specific energy are shown at the initial time $t_i = 1$ with dashed lines, and long after the collision at $t = 2.5$ with triangles and connecting lines. Although it is not shown for clarity, the fluid configuration before the shock rebounds at the origin is similar to Figure 5: a shock heads toward the origin and a rarefaction wave moves away from the origin. When the (spherical) shock crashes into the origin, the volume effect proves harsh as the fluid collides with itself in a vanishingly small volume. This causes the pressure at the origin to become very large in order to repel the fluid (not pictured here). Note that the fluid at the shock as well as far behind it must be repelled, since it is all moving toward the origin.^k When the dust settles, we find that a weak shock has rebounded back. This outward-moving shock can be seen at $R(t=2.5) \simeq 1.3$. Due to the volume effect however, this shock will become weaker as it moves outward further. Also seen at $t = 2.5$ is the original (outgoing) rarefaction wave located between $1.3 \lesssim R \lesssim 2.4$. Note that at $t = 2.5$, the fluid is (and will approximately remain) at rest near the origin because the velocity is zero and the pressure is constant. However, a large distortion has been left behind in the fluid in the form of low density and high kinetic energy. This consists entirely of the fluid originally in the void. The low- n , high- ϵ values for the first 5-6 grid points, however, is artificial. This is a consequence of using VonNeumann-type artificial viscosity called “wall-heating”, and is caused by the collision between 2 shocks.³¹

In Figure 6b we show the initial and long-term pressure, number density, velocity and specific energy as a function of the radius for a compensated, nonrelativistic void. The

^jThis is primarily because we calculate n via $\dot{n} \propto (R^2 dR)^{-1}$ rather than $\dot{n} \propto (r^2 dr)^{-1}$

^kIt is this reversal that requires a very robust difference scheme. All of the “obvious” difference schemes failed after the inbound shocks reached the origin. Setting $f = r^2$ in Eqn (2.20) and using those difference schemes listed in Appendix B are the very necessary requirements.

Table 3: Nonrelativistic collapse times

Void type	$\Delta R_{\text{wall}}(t_i) = .02$	$\Delta R_{\text{wall}}(t_i) = .04$	$\Delta R_{\text{wall}}(t_i) = .08$	$\Delta R_{\text{wall}}(t_i) = .12$
	Collapse Time Δt_c Percent Change: $(\Delta t_c - \Delta t_c(.02)_{\text{un}})/\Delta t_c(.02)_{\text{un}}$			
uncompensated	.38 41 8%	.45 18%	.50 32%
compensated	.15 -61%	.20 -47%	.27 -29%	.32 -16%

initial distribution is identical to Figure 6a except for the energy density, and $\Delta R_{\text{wall}}(t_i) = .02$. Again, the fluid configuration at $t_i = 1$ is shown as dashed lines and that at $t = 2.5$ is shown as triangles connected by lines. An inbound shock is again formed from the initial pressure gradient at the inner edge of the void wall. Unlike the uncompensated case however, a weak outgoing shock wave is formed instead of a rarefaction wave. At $t = 2.5$, it is located at $R = 3.2$. Like the uncompensated void, the pressure at the origin becomes very large after the shock collides with itself there, and a weak shock rebounds. At $t = 2.5$, this rebounded shock is located at $R \simeq 2.3$, which is farther out than that for the uncompensated void ($R \simeq 1.3$). This is because the inbound shock produced from the compensated case is much stronger than for the uncompensated case. And again there is a large distortion left near the origin containing all the fluid initially in the void, although it is somewhat different spatially.

We now compare the collapse times for uncompensated and compensated voids of varying wall thicknesses. (The collapse time is defined to be the time taken for the shock to reach the origin). We set $R_{\text{wall}}(t_i) = 1$, $\alpha = .01$, $k^2 = 3$ and $\Delta R(t_i) = .01$. Table 3 shows the results for $\Delta R_{\text{wall}}(t_i) = .02, .04, .08$ and $.12$, where we calculate the percent change by comparing the collapse time with the uncompensated $\Delta R_{\text{wall}}(t_i) = .02$ collapse time of $\Delta t = .38$. As $\Delta R_{\text{wall}}(t_i)$ increases, the collapse time increases. In addition, the collapse time for compensated voids is substantially smaller than for uncompensated voids.

In conclusion, nonrelativistic voids with zero gravity collapse in the form of a shock, the strength of which depends on the details of the void wall. Some time after collapsing, the fluid is virtually at rest everywhere, with n and ϵ inhomogeneous near the origin.

B. Special Relativistic Fluids

In this subsection we consider the evolution of special relativistic voids with $T/\mu \gg 1$, where T is the temperature and μ is the mass of a fluid particle. Thus, $\epsilon/c^2 \gg 1$, and we set $G_N = 0$, $c = 1$, $t_i = 1$, $C = .3$, $\gamma = 4/3$, $\epsilon(t_i, R_B) = 10^3$, $U(t_i, R) = 0$ (or $\Gamma(t_i, R) = 1$) and $4\pi\rho(t_i, R_B) = 3/8$ in this subsection.

In Figure 7a, we show M , Γ , $4\pi\rho$ and Φ as a function of R for a relativistic shock tube problem. Here $R_{\text{wall}}(t_i) = 1$, $k^2 = 3$, $\Delta R_{\text{wall}}(t_i) = .02$, $\Delta R(t_i) = .01$ and $\alpha = 10^{-4}$. As in the nonrelativistic case, an inbound shock is formed. (This is observed most easily in the plot of $4\pi\rho$ versus R). However, an outgoing ‘‘rarefaction wave’’ is not observed, as it is in

the nonrelativistic case (see Figure 5). If a wave were to propagate outward, it could go at most the distance a photon would travel. But from Eqn(4.3), we know that a photon starting from the outer wall edge only moves the distance $\Delta R_B = .04$ (.065) in time $\Delta t = .04$ (.065). Since this is of order the grid point thickness, if an outbound wave were present, it would not be observed at this time anyway. On the other hand, an inbound photon starting from the inner wall edge would travel the distance $\Delta R_A = .4$ (.65) (from Eqn(4.4)) in time $\Delta t = .04$ (.065), since $\rho_{\text{out}}(t_i)/\rho_{\text{in}}(t_i) \simeq 10^4$. Reexamining Figure 7a, we now notice an important result; the inbound shock's position is approximately equal to photon A's location—the shock moves inward at roughly the speed of light. This is actually not so surprising, because the speed of sound for a perfect fluid with $p = \rho/3$ is $.57c$ (III-C).

Consider next voids compensated in energy density. Because of relativistic-particle diffusion, we expect the inbound shock to again travel at approximately the speed of light. Since the value of $\Phi_{\text{in}}(t_i)$ does not depend on the functional form of ρ in the void wall (Eqn (2.14)), the shock should move the same distance per time as for the uncompensated void. We ran a compensated void simulation with the same initial conditions as from Figure 7a (except for ρ in the void wall), and found that at $t = 1.04$ and 1.065 , the compensated shock is ahead by only $\Delta R = .05$. However, this can be accounted for by the slightly different values of ρ initially at the inner edge of the wall. Thus, special relativistic compensated and uncompensated voids collapse at approximately the same speed, unlike nonrelativistic voids (see Table 3).

In Figure 7b, we show the numerical results for an uncompensated void with $R_{\text{wall}}(t_i) = 1$, $k^2 = 3$, $\alpha = 10^{-4}$ and $\Delta R(t_i) = .01$, and for $\Delta R_{\text{wall}}(t_i) = .02, .04, .06$ and $.1$. It is clear from this figure that in all four cases the shock reaches the origin at $\Delta t_c \sim .08$; Δt_c is approximately independent of $\Delta R_{\text{wall}}(t_i)$. (Compare this with Table 3). Using Eqn (4.4), we estimate the time for light to reach the origin at $\Delta t_c \simeq R_{\text{wall}}(t_i)/\Phi \simeq .09$, in agreement with the simulations.^l (During this time, photon B would only move outward the distance $\Delta R_B = \Delta t_c \simeq .09$). The value of $\Delta R_{\text{wall}}(t_i)$ however, does influence shock formation-time and strength. As $\Delta R_{\text{wall}}(t_i)$ increases, the shock formation time increases and the shock strength decreases.

We now examine the what happens to the void after the shock collides at the origin. In Figure 7c, we show the numerical results for an uncompensated void with $R_{\text{wall}}(t_i) = 1$, $\Delta R_{\text{wall}}(t_i) = .02$, $\alpha = 10^{-4}$, $k^2 = 8$ and $\Delta R(t_i) = .01$.^m We show p , n , U and ϵ as a function of R at the initial time t_i and at times 1.04 and 3.0 , where the second and third times are before and after collision at the origin, respectively. At $t = 3$, $p' \simeq 0$ and $U' \simeq 0$ near the origin; the fluid there is roughly at rest. Two weak outgoing waves are

^lBecause Φ_{in} actually increases slightly due to the dissipation of energy at the shock, the actual travel time for the shock to reach the origin, Δt_c , is decreased slightly.

^mAfter colliding, a very weak shock rebounds. The generalized functional form for the artificial viscosity, however, was derived in the strong shock limit. Thus, a large value of k^2 was needed to maintain numerical stability after the collision. A new functional form for Q will have to be used in future simulations to stabilize the weak outgoing shock.³⁴

observed: the shock (at $R \sim 1$) and a “rarefaction wave” (between $1.1 \lesssim R \lesssim 2.1$). In addition, in contrast with nonrelativistic voids, $n' \sim 0$ and $\epsilon' \sim 0$ near the origin.ⁿ This result is expected; since $n \propto p^{3/4} \propto T^3$ and $\epsilon \propto p^{1/4} \propto T$ (from the discussion following Eqns (2.28)) for a non-viscous fluid, if $p' \simeq 0$, then it follows that $n' \simeq 0$ and $\epsilon' \simeq 0$. This is a consequence of the fact that ρ , p , n and ϵ depend only on the temperature for $\epsilon/c^2 \gg 1$ and $Q = 0$. We therefore find that after the collapse, there is only a small trace of the void’s initial presence!

We conclude that a special relativistic void collapses in the form of a shock which travels at approximately the speed of light into the void. Thus, the collapse time is of order the 1st-crossing time. Some time after the collapse, the fluid becomes approximately homogeneous and isotropic everywhere.

C. General Relativistic Fluids

In this subsection we study the evolution of general relativistic voids for $T/\mu \gg 1$, where T is the temperature and μ is the mass of a fluid particle. We set $G_N = 1$, $c = 1$, $t_i = 1$, $C = .3$ and $\gamma = 4/3$. Therefore, the outside Hubble radius and density (that outside the void) are $cH_{\text{out}}^{-1}(t_i) = 2$ and $4\pi G_N \rho(t_i, R_B) = 3/8$.

In Figure 8, we show the pressure, number density, velocity and specific energy for a general relativistic void at the initial time t_i and for $t = 1.04$ and $t = 1.065$. The initial conditions are identical to those of Figure 7a, except that $k^2 = 4$ and $U(t_i, R) = U_{\text{GRAV}} = \sqrt{2G_N M/R}$ (or $\Gamma(t_i, R) = 1$). Because the void is subhorizon-sized ($c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) = 1/2$), its evolution looks virtually the same as that for the special relativistic void shown in Figure 7a; at the void wall, the gravitational force is $M/R^2 + 4\pi pR/c^2 \simeq 8\pi\rho R/3 \simeq 10^{-1}$, which is much less than the fluid force $\Gamma^2 p'/(4pR') \simeq (4\Delta R(t_i))^{-1} \simeq 12$ (see Eqn. (2.8)).

In Figure 9a, we show the pressure as a function of R $R(t_i, R_B)/R(t, R_B)$ ($\equiv R(t, r) R_{j_B}(t_i)/R_{j_B}(t)$) for a general relativistic superhorizon-sized void at t_i , $t = 2.0$ and $t = 8.0$ with $\Gamma(t_i, R) = 1$, $R_{\text{wall}}(t_i) = 50$, $\Delta R_{\text{wall}}(t_i) = 1.$, $k^2 = 4$, $\alpha = 10^{-4}$, $\epsilon(t_i, R_B) = 10^3$ and $\Delta R(t_i) = .5$. (This void is $50/2 = 25$ times the outside Hubble radius). In addition, $R_{j_B}(t) = 100.2$, 139.1 , and 271.8 . Even though $G_N \neq 0$ here, a strong inward shock still forms. This is because particles are diffusing into the void, having been accelerated away from the high-pressure wall. (Note that at the wall, the acceleration due to gravity is $G_N M/R^2 + 4\pi pR/c^2 \simeq 10$, whereas that due to the fluid force is only $\Gamma^2 p'/(4pR') \simeq .25$. This small relative amount however, is enough to form the shock). Note that because of expansion, the pressure outside (and to a lesser extent inside) the void redshifts. By $t = 8$, the pressure outside the void has redshifted from $p \simeq 10^{-1}$ to 2×10^{-3} , the expected amount since $p \propto 1/t^2$ so that $p_{\text{out}}(8) \simeq 10^{-1}/64 \simeq 2 \times 10^{-3}$. At the same time, the pressure on the inside has only redshifted from $p \simeq 10^{-5}$ to 5×10^{-6} , because the inside Hubble time is larger than that outside: $H_{\text{in}}^{-1}(t_i)/H_{\text{out}}^{-1}(t_i) = \sqrt{\rho_{\text{out}}(t_i)/\rho_{\text{in}}(t_i)} = 100$. From Eqn(4.9), the 1st-crossing time is $\Delta t_c \simeq 50/10(1 + 25/(2 \times 10)) = 11.25$, in rough

ⁿFor the innermost 7-8 grid points, ϵ and n are too large and too small, respectively. This is again due to “wall-heating”.

agreement with the numerical collapse time of $\Delta t_c \simeq 8$. Thus, the collapse time is found to be approximately equal to the 1st-crossing time—the the shock moves inward at roughly the speed of light. This is not surprising however, because the speed of sound is $.57c$ (III-C).

Figures 9b and 9c show the pressure as a function of $R R(t_i, R_B)/R(t, R_B)$ for a general relativistic superhorizon-sized void with $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) = 25$, $\Gamma(t_i, R) = 1$, $k^2 = 4$, $\Delta R_{\text{wall}}(t_i) = 1$, $\epsilon(t_i, R_B) = 10^6$ and $\Delta R(t_i) = .5$. In addition, $\alpha = 10^{-6}$ and $R_{j_B}(t) = 100.2, 125.1, \text{ and } 148.7$ for Figure 9b, and $\alpha = 10^{-10}$ and $R_{j_B}(t) = 100.2, 102.6, \text{ and } 104.4$ for Figure 9c. The numerical collapse times in Figures 9b and 9c are $\Delta t_c = 1.3$ and $.09$, respectively, which are both smaller than the outside Hubble time. (Note that $\Delta t_c = .09$ is $1/20^{\text{th}}$ the outside Hubble time). Since $\Phi(t_i)$ equals 31.6 and 316, respectively, the 1st-crossing times from Eqn(4.9) are $\Delta t_c \simeq 50/31.6(1 + 25/31.6) \simeq 2.2$ and $50/316(1 + 25/316) = .17$, respectively. The 1st-crossing times are larger than the numerical collapse times because $\Phi_{\text{in}}(t)$ increases during the collapse due to the dissipation of energy at the shock. Thus, the gain in entropy at the shock only makes the collapse time shorter. For example, in Figure 9c, $\Phi_{\text{in}}(t)$ increases from its original value of $\Phi_{\text{in}} = 316$ to $\Phi_{\text{in}} \sim 450 - 500$ during the collapse. Using $\Phi_{\text{in}} = 500$, we would predict $\Delta t_c \sim .1$, which is roughly correct.

We note however, that in Figure 9c (and to a lesser extent in Figure 9b), only part of the void has been filled in at the collapse time $t_c \equiv t_i + \Delta t_c$. Thus, thermalization and homogenization has not been achieved by t_c . We note from Figures 9a,b and c that as the initial relative energy density inside the void decreases, the fraction of the void filled in at the collapse time decreases. However, since the energy density inside the somewhat filled void is still much less than $\rho_{\text{out}}(t_c)$, $\Phi(t_c)$ inside the “void” is still much greater than one. And because $\Gamma(t_c)$ is also larger than one inside the “void”, the distance light can travel is still greater inside than outside the void. Thus, although the void has not yet homogenized, this may only take an additional small amount of time.

Figures 10a and 10b show the pressure versus $R R(t_i, R_B)/R(t, R_B)$ for a superhorizon-sized uncompensated and compensated void, respectively. Here, $\Gamma(t_i, R) = 1$, $R_{\text{wall}}(t_i) = 500$, $\Delta R_{\text{wall}}(t_i) = 10.$, $k^2 = 4$, $\epsilon(t_i, R_B) = 10^6$, $\Delta R(t_i) = 5.$, and $\alpha = 10^{-10}$. (These voids are 250 times the outside Hubble radius). The pressure is shown at $t_i = 1$ and $t = 1.7$ for each void, and at $t = 2.1$ and $t = 2.0$ for the uncompensated and compensated voids, respectively. In addition, in Figure 10a, $R_{j_B}(t) = 1002, 1288, \text{ and } 1424$, while in Figure 10b, $R_{j_B}(t) = 1002, 1307, \text{ and } 1418$. The important point to note is that the shock reaches the origin at $t \simeq 2.1$ for both voids. This is due to the diffusion of particles into the void, and does not depend on the compensatedness of the void because $\Phi_{\text{in}}(t_i)$ depends only on $\rho_{\text{in}}(t_i)/\rho_{\text{out}}(t_i)$. (This is similar to that for special relativistic relativistic voids (VII-B)). This roughly agrees with the predicted collapse time $\Delta t_c = 500/316(1 + 250/(2 \times 316)) \simeq 2.2$ from Eqn(4.9). Note also that at $t \simeq 2$, the difference between Figures 10a and 10b is small. For the compensated void, there is no perceptible outbound shock, and the original density “bump” in the void wall is stretched and damped out. In fact, the fluid initially in the compensated void’s wall is moving inward at $t \simeq 2.1$.

Up to this point, we have shown the evolution of general relativistic uncompensated and compensated voids for the initial velocity profile $U = U_{\text{GRAV}} = \sqrt{2G_N M/R}$ (or $\Gamma(t_i, R) = 1$), where the initial velocity per shell just balances gravity. If the wall initially has an outward peculiar velocity (as expected from first-order inflation, for example), then $U(t_i, R_{\text{wall}}(t_i)) > U_{\text{GRAV}}(t_i, R_{\text{wall}}(t_i))$ (or $\Gamma(t_i, R_{\text{wall}}(t_i)) > 1$), as discussed in II-D. In Figure 11, we show $4\pi p$, M , Γ and Φ versus $R R(t_i, R_B)/R(t, R_B)$ for a compensated superhorizon-sized void with $U/c = R\sqrt{8\pi G_N \rho}/3$ (Eqn(3.5)), $R_{\text{wall}}(t_i) = 500$, $\Delta R_{\text{wall}}(t_i) = 10.$, $k^2 = 4$, $\epsilon(t_i, R_B) = 10^6$, $\Delta R(t_i) = 2.5$, $\bar{f} = .005$ and $\alpha = 10^{-10}$. The voids are superhorizon-sized: $R_{\text{wall}}(t_i)/H(t_i, R_B)^{-1} = 250$. In addition, $R_{j_B}(t) = 601.2, 783.9, \text{ and } 911.8$, and we show the configurations at t_i and at times $t = 1.7$ and $t = 2.3$. The void wall is initially moving outward with a very large peculiar velocity, since $\Gamma(t_i, R_{\text{wall}}(t_i)) \simeq 900$.

We find the very interesting result that even though the wall has a large outward peculiar velocity, the inner part of the void still collapses. This is because the fluid near the base of the void wall gets accelerated into the void right away, pulling adjacent fluid with it. It is true however, that at $t \sim 2$ the density profile looks different than that from Figure 10b. This is because the fluid in the void wall takes more time to lose its outward velocity. Since the numerical collapse time is $\Delta t_c = 1.3$ from Figure 11, we find that the extra time taken for the void to collapse is approximately $1.3 - 1.1 \simeq .2$, which is still less than the outside Hubble time. (This follows qualitatively from our discussion in Section II-D, where it was argued that Γ would decrease very quickly at the inner edge of the wall, since $\dot{\Gamma} \propto -\Gamma^2 p'$). These new results show that the collapse time of a superhorizon-sized void with a large outward peculiar wall velocity can be of order the 1st-crossing time. Since the minimum thermalization and homogenization time is the 1st-crossing time, the time for thermalization and homogenization of this void may be short.

In Figure 12, we show the void radius versus cosmic time for a void with initial size $R_{\text{wall}}(t_i) = c10^{23} H_{\text{out}}^{-1}(t_i)$. If the temperature outside the void initially is $T_{\text{out}}(t_i) = 10^{14} \text{ GeV}$, then the initial time is $t_i = 10^{-33}$ seconds.² And because recombination occurs at $t_{\text{rec}} \simeq 10^{12}$ seconds, $t_{\text{rec}}/t_i = 10^{45}$, which is near where the dashed lines intersect at the top of Figure 12. Therefore, the plot consists the radiation dominated epoch in the early universe, where the scale factor outside the void is $a(t) = a(t_i)\sqrt{t/t_i}$. Because $R(t_i, R_B)/R(t, R_B) = \sqrt{t_i/t}$, any comoving point (defined as $R \propto \sqrt{t/t_i}$, not $r = \text{const}$) is a vertical line in this plot. The dashed line labeled r_{CM} shows the void size if it were comoving with spacetime, as previously suggested (see IV). The Hubble radius outside the void is $cH_{\text{out}}^{-1} \equiv R_{\text{HOR}} = 2ct$, and is shown as the dashed line labeled by r_{HOR} . As can be seen, r_{HOR} and r_{CM} intersect at $t/t_i \simeq 10^{46}$ (or 10^{-11} seconds), shortly after recombination. (This also follows from Eqn(4.2), since $\Delta t_c \simeq H_{\text{out}}^{-1}(t_i)(c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i))^2 \simeq 10^{46}$). If a superhorizon-sized void were to conformally expand with spacetime, then this void would just barely be around to distort the microwave background at recombination.⁸

The dash-dot lines show the position of the 1st-crossing photons (i.e. the inner void wall) from Eqn(4.8). We show $\Phi_{\text{in}}(t_i) = 5 \times 10^{11}$, 10^{15} and 10^{20} (since we require $\Phi_{\text{in}}(t_i) \gtrsim \sqrt{10^{23}}$ (from Eqn(4.12))), with the predicted 1st-crossing times of $t_c/t_i \simeq \Delta t_c/t_i = 4 \times 10^{22}$, 10^{16} and 10^6 respectively. If $\Delta t_c/H_{\text{out}}^{-1}(t_i) > 1$, the radius of the inner edge of the wall is constant ($R \simeq \text{const}$) until time $t \simeq t_c \simeq t_i(\Phi_{\text{in}}^{-1}(t_i)R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i))^2$, at which point R rapidly decreases to zero (see Eqn(4.8)). If the reheat temperature is $T_{\text{RH}} = T_{\text{out}}(t_i) = 10^{14}\text{GeV}$, then the void with $\Phi_{\text{in}}(t_i) = T_{\text{out}}(t_i)/T_{\text{in}}(t_i) = 10^{20}$ may thermalize by $t \simeq 10^{-27}$ seconds (or at $T \simeq T_{\text{RH}}\sqrt{t_i/t} \simeq 10^{11}\text{GeV}$), far before recombination or nucleosynthesis. (Note that this value of $\Phi_{\text{in}}(t_i)$ corresponds to an initial void temperature of $T_{\text{in}}(t_i) = T_{\text{RH}}\Phi_{\text{in}}(t_i)^{-1} = 10^{-6}\text{GeV} = 1\text{keV}$).

In conclusion, we have seen that a compensated or uncompensated superhorizon-sized void collapses as fluid in the wall diffuses into the void at the speed of light. Thus, the collapse time is found to be approximately equal to the 1st-crossing time calculated in Section IV, which is *much* smaller than previously thought.

VIII. Discussion

In this paper, the evolution of a superhorizon-sized void embedded in a Friedmann-Robertson-Walker (FRW) universe was studied by numerically evolving the spherically symmetric general relativistic equations in the Lagrangian gauge and synchronous coordinates. (A superhorizon-sized void is a void larger than the Hubble radius outside the void: $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) > 1$, where t_i is the initial cosmic time, $cH_{\text{out}}^{-1}(t_i)$ is the Hubble radius outside the void and $R_{\text{wall}}(t_i)$ is the radius of the void). The particles are assumed to be in local thermal equilibrium so that they can be described as a fluid, and the perfect fluid equation of state is chosen. The inside of the void is initially chosen to be homogeneous and non-expanding. We are particularly interested in the evolution of voids with $T/\mu \gg 1$, where T is the local temperature and μ is the mass of a fluid particle. This corresponds to a radiation-dominated period in the early universe. We find that for $p = \rho/3$, general relativistic voids collapse via a shock propagating inward from the void wall. The shock is formed from the steep pressure gradient at the inner edge of the wall. It moves at approximately the speed of light; this is not surprising, since the speed of sound is $.57c$. Thus the void collapse time (the time taken for the shock to reach the origin) roughly equals the photon 1st-crossing time (the time taken for a photon initially at the inner wall edge to reach the origin). At the time this shock reaches the origin, much of the fluid in the original wall area is moving toward the origin behind the shock. The energy density of this fluid (i.e. the fluid behind the shock) is less than that outside the void. At the collapse time then, only part of the void has been filled in with fluid. In particular, as the initial energy density inside the void decreases, the fraction of the void filled in at the collapse time decreases.

Because the shock moves inward at roughly the speed of light, we can calculate the approximate collapse time. For $p = \rho/3$, the 1st-crossing time is

$$\Delta t_c = c^{-1}R_{\text{wall}}(t_i) \left(\frac{\rho_{\text{in}}(t_i)}{\rho_{\text{out}}(t_i)} \right)^{1/4} \left[1 + \frac{c^{-1}R_{\text{wall}}(t_i)}{2H_{\text{out}}^{-1}(t_i)} \left(\frac{\rho_{\text{in}}(t_i)}{\rho_{\text{out}}(t_i)} \right)^{1/4} \right] \quad (8.1)$$

for $R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) \lesssim \sqrt{\rho_{\text{out}}(t_i)/\rho_{\text{in}}(t_i)}$, where $\rho_{\text{in}}(t_i)$ and $\rho_{\text{out}}(t_i)$ are the initial fluid energy densities inside and outside the void, respectively, and $\rho = T^4$. If $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) < (\rho_{\text{out}}(t_i)/\rho_{\text{in}}(t_i))^{1/4}$, then $\Delta t_c/H_{\text{out}}^{-1}(t_i) \lesssim 1$; the 1st-crossing time is less than or comparable to the initial Hubble time outside the void. In fact, as the density inside the void approaches zero, the collapse time goes to zero!

It may seem contradictory that the 1st-crossing time for a *superhorizon-sized* relativistic void can be less than the outside Hubble time (i.e. or that light travels comparatively farther inside a void than outside). There are several points to make about this. First, if we choose Eulerian instead of Lagrangian synchronous coordinates to evolve special relativistic voids ($G_N = 0$), then the 1st-crossing time is not fast. This is because spacetime is sliced differently in time inside the void. Since a similar 1st-crossing time is obtained for general relativistic voids, it might be argued that the quick collapse time is a figment of the Lagrangian gauge used. However because the FRW metric has a coordinate singularity at the Hubble radius when transformed to Eulerian synchronous coordinates, superhorizon-sized voids cannot be evolved in these coordinates. Any prior intuition from special relativistic voids in these coordinates therefore, must be carefully applied.

Second, calling a void “superhorizon-sized” is deceptive. Although the 1st-crossing time for a superhorizon-sized *perturbation* in the FRW universe is greater than the outside Hubble time, this does not imply that the same is true for a superhorizon-sized void. This is because the void is not a small perturbation in general (i.e. $(\rho_{\text{out}} - \rho_{\text{in}})/\rho_{\text{out}} \simeq 1$). If the void’s size is defined to be the spacelike circumferential radius relative to the *outside* Hubble radius, then the void is superhorizon-sized if $R_{\text{wall}} > cH_{\text{out}}^{-1}$. However, if the void’s size is defined to be its radius relative to the Hubble radius *inside* the void, then its size is smaller since $c^{-1}R_{\text{wall}}(t_i)/H_{\text{in}}^{-1}(t_i) = c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i)\sqrt{\rho_{\text{in}}(t_i)/\rho_{\text{out}}(t_i)}$; it is *subhorizon-sized* if $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) < \sqrt{\rho_{\text{out}}(t_i)/\rho_{\text{in}}(t_i)}$! (We call this latter size the “radial size”, because it is the time taken for a photon to cross a static void multiplied by c). This is because the Hubble radius is much larger inside than outside the void. Since the void is an underdense region, we expect the opposite relative size problem to occur for overdense regions where the gravitational potential is large, not small. As an example, a black hole’s circumferential size is small (or at least finite), while its “radial size” is infinite.

The original motivation for studying this problem was to determine the evolution of voids formed from first-order (e.g. extended) inflation. These voids are compensated in energy density, and the walls initially have large outward peculiar velocities.³⁴ Previous authors estimated the minimum homogenization time by determining the 1st-crossing time. They found it to be^{6,8,9}

$$\Delta t_c \simeq H_{\text{out}}^{-1}(t_i) \left(\frac{c^{-1}R_{\text{wall}}(t_i)}{H_{\text{out}}^{-1}(t_i)} \right)^2. \quad (8.2)$$

This estimate was based on the assumption that a superhorizon-sized void would conformally expand with spacetime. The present work throws considerable doubt on this

assumption and its implications (i.e. a long thermalization time for superhorizon-sized voids) for several reasons. First, the 1st-crossing time is much shorter than this estimate. For $\Delta t_c/H_{\text{out}}^{-1}(t_i) > 1$, this estimate is off by the factor $\sqrt{\rho_{\text{in}}(t_i)/\rho_{\text{out}}(t_i)}$ (see Eqn(8.1)). Since the minimum thermalization time is the 1st-crossing time, this implies that thermalization and homogenization may happen much quicker than suggested. This would have profound effects upon our understanding of the evolution of superhorizon-sized voids in the early universe. Second, the inner edge of the void collapses (not expands) in roughly the 1st-crossing time, even (apparently) for voids with large outward peculiar velocities. This collapse occurs because the wall decelerates from the positive wall pressure ($\dot{U} \propto -p' < 0$). If the wall pressure were negative instead (e.g. the wall pressure of a vacuum bubble), the wall would accelerate outward. Thus, fluid from at least part of the wall rushes into the void, partially “filling it” at the 1st-crossing time. Because the light travel distance is still large there, the void may still fill up in a relatively short amount of time. Third, the void wall can cease expanding by many mechanisms. Because fluid rushes into the void, large velocity gradients are formed in the original wall area which “pull” on the wall and can slow it down. In addition, when the peculiar wall velocity is much larger than the gravitational velocity, the deceleration of the wall is proportional to the velocity squared which can be enormous. Finally, an outgoing wave will be damped and slowed down in any case, by virtue of the volume effect caused by the spherical geometry. Thus even if the wall initially expands outward conformally (or faster), it may stop doing so rather quickly, as the simulations appear to suggest. In any case, even if the outer part of the wall *can* carry out some mass for a long period of time, the interior region will continue to fill in and homogenize. Since overdensities will most likely form near the origin after collapse, perturbations, small waves and/or overdensities will most likely be around at recombination. These perturbations could have interesting amplitudes, and therefore could alter the standard Harrison-Zeldovich density spectrum. In any case, because the qualitative void evolution picture previously suggested is at least partially incorrect, it is reasonable to expect the estimated thermalization time to be erroneous; the “big-bubble problem” might not be a problem after all.

It is important to point out that in order not to distort the microwave background, the thermalization time only needs to decrease somewhat from the estimate given by Eqn(8.2). The largest superhorizon-sized void from minimalist inflation is roughly $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) \simeq 10^{27}$.⁸ If the initial time after reheat is $t_i = 10^{-33}$ seconds ($T_{\text{RH}} = 10^{14}\text{GeV}$), then at recombination ($t_{\text{rec}} = 10^{12}$ seconds), this void would not have thermalized according to Eqn(8.2). Using these initial conditions with Eqn(8.2), it is easy to see that voids for which $c^{-1}R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) \gtrsim 10^{23}$ were traditionally troublesome. However, since 10^{23} is such an enormous number, a very slight change in the functional form for the thermalization time, Δt_H , can cause thermalization to occur before recombination. For instance, if we suppose that the thermalization time can be written as $\Delta t_H \simeq c^{-1}H_{\text{out}}^{-1}(t_i)(R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i))^p$, then an acceptable range for p is $p \lesssim 1.6$, which is not dramatically different from 2. The point is that the “big-bubble problem” can only be resolved by *accurate* knowledge of the thermalization time, because estimates off by

a somewhat small amount can lead to erroneous conclusions.

Of the many questions which remain unanswered, the most important is to determine what happens to a superhorizon-sized void after it collapses. If it thermalizes and homogenizes, at what time does this happen? As mentioned previously, at the time of 1st-crossing, the energy density within the former void region is greater than $\rho_{\text{in}}(t_i)$, but is still less than $\rho_{\text{out}}(t)$. In addition, the bulk motion of much of the original wall fluid is inward so that spacetime is collapsing and not expanding there; thermalization and homogenization has not occurred. However, since the “void” is still relatively underdense at this time, the light travel distance will be comparatively large, thereby allowing for the possibility of relatively fast thermalization and homogenization. We have found that special relativistic voids eventually nearly homogenize after creating an overdense region at the origin. In this case, all but the fluid near the origin becomes roughly homogeneous fairly quickly. The fluid near the origin however, is overdense for a relatively long time. If general relativistic voids behave similarly, then after collapsing, an *overdense* region would form at the origin. This would take a relatively long amount of time to diffuse away, and would eventually result in a nearly homogeneous and isotropic universe. In any case, since this overdense region would form very quickly, it is unlikely that empty voids would be around any time near recombination, as previously suggested.^{32,8,9} Instead, overdensities, perturbations and small waves with strange fluid velocities may be. Future work will study the consequent evolution of a general relativistic void after collapse,³³ as well as the evolution of a superhorizon-sized void from first-order (e.g. extended) inflation.³⁴

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Appendix A: Transformation of the FRW metric to Eulerian Coordinates

As discussed in Section II A., the metric in Eq. (2.1) can be subjected to transformations of the type $t = f_1(t', r')$ and $r = f_2(t', r')$ without altering its spherical symmetry. If we set $r^2 = k(t, r)$, then we can rewrite Eq. (2.1) as

$$ds^2 = -(N^2 - S^2)dt^2 + 2Sdt dr + h^2 dr^2 + r^2 d\Omega^2, \quad (\text{A.1})$$

where we have introduced the lapse and shift functions as $N(t, r)$ and $S(t, r)$, respectively. This is the Eulerian gauge with synchronous coordinates if we choose $S = 0$. As we shall see, when the FRW spatially flat metric is transformed in this manner, the resulting metric has a coordinate singularity at the horizon. Thus, the study of superhorizon-sized voids is not possible in these coordinates.

Consider the FRW $k = 0$ metric given by Eqn (2.3). Letting $\tilde{r} = ra(t)$, we can rewrite

this as

$$ds^2 = - \left(1 - \tilde{r}^2 \left(\frac{\dot{a}}{a} \right)^2 \right) dt^2 - 2\tilde{r} \frac{\dot{a}}{a} d\tilde{r} dt + d\tilde{r}^2 + \tilde{r}^2 d\Omega^2, \quad (\text{A.2})$$

where we set $c = 1$ everywhere in this appendix. This is the Eulerian asynchronous form of the FRW metric. We notice that this takes the form of Eq. (A.1) with $N = 1$, $S = \tilde{r}\dot{a}/a$ and $h = 1$. Since $a(t) \propto t^\xi$, where $\xi = 1/2$ and $2/3$ in radiation ($p = \rho/3$) and matter ($p = 0$) dominated universes respectively, we can eliminate the $d\tilde{r}dt$ term by defining

$$t = \xi \tilde{r} / g(\tilde{t}, \tilde{r}) \quad (\text{A.3})$$

and ensuring that the condition

$$\frac{\partial g}{\partial \tilde{r}} = g \frac{\xi + (1 - \xi)g^2}{\xi \tilde{r}(1 - g^2)} \quad (\text{A.4})$$

hold. Defining $\kappa \equiv (1 - \xi)/\xi$ and $\alpha \equiv 1/[2(1 - \xi)]$, this last expression can be integrated to obtain

$$h(\tilde{t})\tilde{r} = g/(1 + \kappa g^2)^\alpha, \quad (\text{A.5})$$

where $h(\tilde{t})$ is an arbitrary (integration) function of \tilde{t} only. The metric then becomes

$$ds^2 = - \left(\frac{h\tilde{r}}{g} \right)^4 \frac{(1 + \kappa g^2)^{2(\alpha+1)}}{1 - g^2} d\tilde{t}^2 + \frac{1}{1 - g^2} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2. \quad (\text{A.6})$$

This metric has a coordinate singularity at $g = 1$, which from Eqn(A.3) occurs at $r = t/a(t) \xi^{-1}$. But this is the comoving radius of the Hubble radius (r_{HOR}), since $H^{-1} = t/\xi = r_{\text{HOR}} a(t)$. Therefore, we have shown that for the FRW models expressed in Eulerian, synchronous coordinates, a coordinate singularity occurs at the Hubble radius.

As an interesting example we take $\xi = 1/2$, which corresponds to radiation-domination ($p = \rho/3$). In addition, we choose $h \equiv \xi/\tilde{t}$. Using Eqs. (A.5) and (A.3) together with $\tilde{r} = r a(t) = r a(t_i)(t/t_i)^{1/2}$, we find that $\alpha = \kappa = 1$, $\tilde{t} = t + (a(t_i)r)^2/(4t_i)$ and $g = (1 - \sqrt{1 - 4(\tilde{r}/\tilde{t})^2})/(\tilde{r}/\tilde{t})$. In addition, the metric simplifies to $ds^2 = -(1 - g^2)^{-1} d\tilde{t}^2 + (1 - g^2)^{-1} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2$. Note that even though the energy density is spatially constant on a hypersurface $t = \text{constant}$, on a hypersurface of constant \tilde{t} , $\rho \propto t^{-2} = 4g^2\tilde{r}^{-2} = 4\tilde{t}^2(1 - \sqrt{1 - 4(\tilde{r}/\tilde{t})^2})^2/\tilde{r}^4$ which is *not* spatially constant.

Appendix B: Differencing of the Equations

In this appendix, we list the expressions as they are differenced in the code. We use the shorthand notation $\Delta G \equiv G_{i+1}^j - G_i^j$ or $\Delta G \equiv G_i^j - G_{i-1}^j$ for forward or backward differencing, respectively. The equations involving spatial derivatives are

$$\begin{aligned} \dot{U} &= -\Phi \left(\frac{G_N M}{R^2} + \frac{4\pi G_N (p + Q) R}{c^2} \right) - 3 \frac{\Gamma \Phi R^2}{w \Delta(r^3)} 4\pi \Delta(p + Q) \\ \dot{n} &= -\frac{n \Phi \Delta(R^2 U)}{R^2 \Delta R} \\ \dot{c} &= -\frac{4\pi \Phi (p + Q) \Delta(R^2 U)}{\Gamma r^2 \Delta r}, \end{aligned} \quad (\text{B.1})$$

We note that there are many other schemes which could potentially be used. (For example, one could write $\Delta r^3/3$ instead of $r^2\Delta r$ in the expression for $\dot{\epsilon}$). Although these variations can produce and propagate the shock and rarefaction waves, they can not handle the bounce of the shock at the origin. The only set of difference equations which we have found to do this are given above. For the Tolman-Bondi pressureless dust models (Section V), it is found that more accurate solutions are obtained near the origin when differencing the \dot{n} equation as follows: $\dot{n} = -(n^2\Phi\Delta(R^2U))/(\Gamma r^2\Delta r)$. The Tolman-Bondi figures shown in this paper, however, use the difference equations as written in Eqns(B.1).

Appendix C: General Relativistic Jump Conditions

In this appendix, we derive the jump conditions for general relativistic shocks. This closely follows the derivation by May and White (1965)¹⁴. These conditions will then be used to derive a more general artificial viscosity expression. In addition, the shock jump conditions for ultra special relativistic shocks are derived.

Let the variables a and b represent labels for comoving observers ahead and behind the shock, respectively. If each observer measures the invariant interval separating the same two events on the world surface of a shock, using Eqn(2.2), we obtain

$$[ds^2] = [-c^2\Phi^2 dt^2 + \Lambda^2 dr^2 + R^2 d\Omega^2] = 0, \quad (\text{C.1})$$

where $[G] \equiv G_a - G_b$. Because the shocks are radial, we can choose the two events to have $dr = dt = 0$ but $d\Omega \neq 0$. Therefore, R is continuous across the shock: $[R] = 0$. Now consider two events with $dr \neq 0$ and $dt \neq 0$. Then from Eqn(C.1),

$$[c^2\Phi^2 - M_s^2/n^2] = 0, \quad (\text{C.2})$$

where r_s is the position of the shock in comoving coordinates, dr_s/dt is the shock “speed”, and $M_s \equiv f(dr_s/dt)/(4\pi R^2)$ from Eqns(2.19) and (2.5). This is the first of the jump condition equations.

It can be shown that the jump conditions for the Schwarzschild metric (but *not* for the comoving metric) are^{35,14}

$$[T_\nu^\mu \partial g / \partial x^\mu] = 0, \quad (\text{C.3})$$

where g is the equation for the world surface of the shock. Therefore we must first find the jump conditions in the Schwarzschild frame and then transform back to the comoving frame. The Schwarzschild metric is $ds^2 = -c^2 A^2 dT^2 + B^2 dR^2 + R^2 d\Omega^2$, where R is now the “Eulerian” coordinate radius. Since R is a coordinate, $[R] = 0$, and using the same argument as above for the two observers,

$$[c^2 A^2 - B^2 S^2] = 0, \quad (\text{C.4})$$

where S is the shock “speed” in this frame: $cS \equiv dR_s/dT$. For the Schwarzschild metric, the solution for the metrics functions are well known: $B^2 = 1 - 2MG/(Rc^2) = A^{-2}$ where $M(R, T) = 4\pi c^{-2} \int_0^R \rho R^2 dR$. As long as ρ is not infinite in the shock, the mass is continuous across the shock, $[M] = 0$. Then $[B] = [A] = 0$. And using Eqn (C.4), we

find that $[S] = 0$. We conclude that in the Schwarzschild frame, the metric components and the shock “speed” are continuous across the shock: $[A] = [B] = [R] = [S] = 0$.

We now derive the jump conditions in the Schwarzschild frame. If a shock is located at position $R_s(T)$ at time T , the equation for the world surface of a shock is $g = R_s(T) - R = 0$. In addition, the perfect fluid stress-energy tensor in the Schwarzschild frame^o is

$$T_{\mu}^{\nu} = c^{-2}(\rho + p)g'_{\mu\lambda}u'^{\mu}u'^{\lambda} + pg'_{\mu\lambda}g'^{\nu\lambda} = nwg'_{\mu\lambda}u'^{\mu}u'^{\lambda} + pg'_{\mu\lambda}g'^{\nu\lambda}, \quad (\text{C.5})$$

and the conservation of mass equation is $[nu'^{\nu}\partial g/\partial x^{\nu}] = 0$.^{35,14} Using Eqns (C.3), the junction conditions become

$$[T_T^T S - T_T^R] = c^2 [(-A^2(u'^T)^2 nw + p)S + nwA^2 u'^T u'^R] = 0 \quad (\text{C.6})$$

$$[T_R^T S - T_R^R] = [SnwB^2 u'^T u'^R - (nwB^2(u'^R)^2 + p)] = 0 \quad (\text{C.7})$$

$$[nSu'^T - nu'^R] = 0. \quad (\text{C.8})$$

We would like to express these equations in terms of the comoving metric functions. First, we can rewrite the shock velocity in the comoving frame as $cS = (R_t + R_r \dot{r}_s)/(T_t + T_r \dot{r}_s)$. In addition, we can relate the metric functions from the comoving frame to those in the Schwarzschild frame by $g'^{\mu\nu} = (\partial x'^{\mu}/\partial x^{\sigma})(\partial x'^{\nu}/\partial x^{\lambda})g^{\sigma\lambda}$. We then obtain $(cA)^{-2} = (c\Phi)^{-2}T_t^2 - \Lambda^{-2}T_r^2$, $B^{-2} = -(c\Phi)^{-2}R_t^2 + \Lambda^{-2}R_r^2$ and $(c\Phi)^{-2}T_t R_t = \Lambda^{-2}T_r R_r$. In addition, the “mass” $M(r, t)$ contained within comoving radius r is continuous across the shock, since $[M] = [nR^2(1 + \epsilon/c^2)]dR \xrightarrow{dR \rightarrow 0} 0$. Using Eqn (2.12), we find that $[\Gamma^2 - U^2/c^2] = 0$.

We can use these relations to calculate the 4-velocity as measured in the Schwarzschild frame. Since the 4-velocity in the comoving frame is $u^{\mu} = (-c\Phi^{-1}, 0, 0, 0)$, and the velocity transforms as $u'^{\lambda} = (\partial x'^{\lambda}/\partial x^{\sigma})u^{\sigma}$, $u'^{\lambda} = -\Phi^{-1}(c\dot{T}, \dot{R}, 0, 0)$, where $\dot{T} = \partial T/\partial t$ and $\dot{R} = \partial R/\partial t$. Since $U \equiv \dot{R}/\Phi$, and using the fact that the 4-velocity is normalized to $u'^{\lambda}u'_{\lambda} = -c^2 = c^2 A^2(u'^T)^2 - B^2(u'^R)^2$, we can rewrite the 4-velocity as $u'^{\lambda} = -(A^{-1}\sqrt{B^2 U^2 + c^2}, U, 0, 0)$. In the special relativistic limit ($G_N = 0$), $A = 1$ and $B = 1$ and this becomes

$$u'^{\lambda} = -((U^2 + c^2)^{1/2}, U, 0, 0) = -(c\Gamma, \Gamma v, 0, 0), \quad (\text{C.9})$$

where we have defined $v \equiv U/\Gamma$ so that $\Gamma = (\sqrt{1 - (v/c)^2})^{-1}$. In the special relativistic limit, then, v is the fluid particle’s radial velocity, Γ is usual gamma-factor (i.e. energy per particle mass), and U is the particle momentum per particle mass.

We can now rewrite Eqn (C.6)-(C.8) and $[S] = 0$ in terms of the comoving metric functions using the fact that $T_r/T_t = c^{-2}U\Lambda/(\Gamma\Phi)$ and $[\dot{r}_s] = 0$. We obtain

$$[UM_s/(nc^2) + \Phi\Gamma] = 0 \quad (\text{C.10})$$

$$[c^2 M_s \Gamma(1 + \epsilon/c^2) - pU\Phi] = 0 \quad (\text{C.11})$$

$$[M_s U(1 + \epsilon/c^2) - p\Gamma\Phi] = 0 \quad (\text{C.12})$$

$$[M_s \Gamma/n + \Phi U] = 0. \quad (\text{C.13})$$

^oWe denote quantities in the Schwarzschild frame by primes.

Eqn (C.2) and Eqns (C.10)-(C.13) make up the required shock conditions. One of them however, is redundant. It can be shown that in the nonrelativistic limit, the above conditions reduce to the Lagrangian shock jump conditions given in Ref 21.

The case considered by May and White (1967) is the penetration of a shock into a non-relativistic medium. We set $G_N = 0$ and $\epsilon_a = p_a = U_a = 0$ and take $\epsilon = p/[(\gamma - 1)n]$. Using Eqns (C.11) and (C.12) with the fact that $U_b^2 = \Gamma_b^2 - 1$, the energy behind the shock is found to be $\epsilon_b = \Gamma_b - 1$. Using Eqns (C.12) and (C.13), we eliminate Φ_b and find that $\eta \equiv n_b/n_a = (1 + \Gamma_b\gamma)/(\gamma - 1)$. From Eqn (C.12), we find that

$$M = (\gamma - 1)n_b\epsilon_b\Phi_b/U_b, \quad (\text{C.14})$$

and plugging this into Eqn (C.2), we determine that $\Phi_b = 1/(1 + \gamma(\Gamma_b - 1))$. Finally, we plug this into Eqn (C.14) and find $M_s = cn_a \sqrt{(\Gamma_b - 1)/(\Gamma_b + 1)} (1 + \Gamma_b\gamma)/[1 + \gamma(\Gamma_b - 1)]$. (It turns out that Eqn (C.10) gives no new information). Thus, all quantities behind the shock can be expressed in terms of Γ_b , the ‘‘strength’’ of the shock.

How does this affect the artificial viscosity needed to smooth the shock? Von-Neumann’s viscosity²² is $Q = k^2n(U')^2dr^2$ for $U' < 0$ and 0 otherwise, with the shock being spread over roughly k^2 grid points. Over the k^2 points, $Q \sim n(U_b - U_a)^2 = nU_b^2 = c^2n(\Gamma_b^2 - 1) = c^2n(\Gamma_b - 1)(\Gamma_b + 1)$. Using the fact that $\epsilon_b/c^2 = \Gamma_b - 1$, $Q \sim n\epsilon_b\Gamma_b \sim p\Gamma$ which is greater than the pressure p by the factor of $\Gamma \geq 1$. Since Q should be of order the pressure in the shock region, we divide by Γ : $Q = k^2n(U')^2dr^2/\Gamma$ for $U' < 0$ and $Q = 0$ otherwise. In the nonrelativistic limit, $\Gamma = 1$ and Von-Neumann’s artificial viscosity is obtained.

In this paper, we are more interested in the case where the fluid on both sides of the shock is ultrarelativistic. We first set $G_N = 0$ and $U_a = 0$. Then Eqn(C.2) and Eqns(C.10)-(C.13) can be manipulated to yield

$$U_b = \sqrt{\Gamma_b^2 - 1} \quad (\text{C.15})$$

$$M = cn_a\eta\sqrt{\Gamma_b^2 - 1} \Phi_a / (\Gamma_b\eta - 1) \quad (\text{C.16})$$

$$\eta = \Gamma_b(1 + \gamma\epsilon_b/c^2) - (1 + \epsilon_a/c^2) / [\epsilon_b/c^2(\gamma - 1)] \quad (\text{C.17})$$

$$\epsilon_b/c^2 = -1 + \Gamma_b(1 + \epsilon_a/c^2) + (\gamma - 1)(\Gamma_b - \eta^{-1})\epsilon_a/c^2 \quad (\text{C.18})$$

$$\Phi_b = |\eta - \Gamma_b| \Phi_a / (\Gamma_b\eta - 1). \quad (\text{C.19})$$

(It can be shown that these equations reduce correctly in the $\epsilon_a \rightarrow 0$ limit). Now, if we set $\gamma = 4/3$ and take the ultrarelativistic limit, $\epsilon_b/c^2 > \epsilon_a/c^2 \gg 1$, Eqn (C.17) becomes $\eta = 4\Gamma_b\{1 - 3\epsilon_a/(4\Gamma_b\epsilon_b)\} \simeq 4\Gamma_b$. Then the equations can be approximately solved to give

$$\epsilon_b \simeq 4\epsilon_a\Gamma_b(1 - (4\Gamma_b)^{-2}) / 3, \quad (\text{C.20})$$

which to lowest order is $\epsilon_b \simeq 4\epsilon_a\Gamma_b/3$. Therefore, ϵ_b depends not only on Γ_b but also on ϵ_a . We generalize the artificial viscosity to

$$\begin{aligned} Q &= k^2n \left(1 + \epsilon / (\Gamma c^2)\right) (U')^2dr^2 / \Gamma \quad \text{for } U' < 0 \\ Q &= 0 \quad \text{otherwise.} \end{aligned} \quad (\text{C.21})$$

For a strong shock then, $Q \sim n\epsilon U^2/(\Gamma^2 c^2) \sim n\epsilon \sim p$, as desired. And when $c^2 \rightarrow \infty$, this becomes VonNeumann's original expression. Note that because this was derived in the limit of strong shocks, it does not work as well for weaker special or general relativistic shocks.³³

We now write down the exact solution of Eqns (C.15)-(C.19) for conditions behind the shock in terms of Γ_b only and quantities in front of the shock. Because $\rho = n\epsilon$ in this limit, we can rewrite Eqns (C.17) and (C.18) as $\eta = 4\rho_b\Gamma_b/(3\rho_a + \rho_b)$ and $\eta = (3\rho_b + \rho_a)/(4\rho_a\Gamma_b)$ respectively. Setting them equal, we can solve for ρ_b . Then, all others quantities are determined. They are

$$U_b = \sqrt{\Gamma_b^2 - 1} \quad (C.22)$$

$$\frac{\rho_b}{\rho_a} = \frac{16\Gamma_b^2 - 10}{6} \left[1 + \sqrt{1 - 36(16\Gamma_b^2 - 10)^{-2}} \right] \quad (C.23)$$

$$M = \pm \frac{n_a \Phi_a (3\rho_b/\rho_a + 1)}{4\Gamma_b \sqrt{3\rho_b/\rho_a}} \quad (C.24)$$

$$\eta \equiv \frac{n_b}{n_a} = \frac{4\Gamma_b}{3(\rho_b/\rho_a)^{-1} + 1} \quad (C.25)$$

$$\frac{\epsilon_b}{\epsilon_a} = \frac{3 + \rho_b/\rho_a}{4\Gamma_b} \quad (C.26)$$

$$\Phi_b = \frac{\epsilon_a}{\epsilon_b} \Phi_a = \frac{3 + (\rho_b/\rho_a)^{-1}}{4\Gamma_b} \Phi_a. \quad (C.27)$$

We note from Eqn (C.27) that $[\Phi w] = 0$.

Appendix D: Nonrelativistic Shock Tube Problem

In this section, we sketch the derivation of the slab shock tube solution for non-relativistic fluids.²⁹ We start with a fluid in which the pressure p and specific volume $V \equiv 1/n$ (where n is the number density) to the left and right of x_0 are p_1, V_1 and p_2, V_2 respectively—thus, p and V are discontinuous across $x = x_0$. We assume here that our coordinates are oriented so that $p_1 < p_2$ and $V_1 > V_2$. In addition, the fluid is initially at rest: $v_1 = v_2 = 0$. Because there is no scale to the problem, the solution is a function of x/t only, where x is the Eulerian coordinate. At time t later, there are 4 regions. Region 1 and 2 are at rest with $p = p_1, V = V_1$ and $v = v_1$, and $p = p_2, V = V_2$ and $v = v_2$ respectively. Separating region 1 and 3 is a shock wave, with position x_d . Region 3 contains the fluid behind the shock, and region 3' is wedged between regions 3 and 2 and contains the rarefaction wave. The boundary between region 3 and 3' is called a contact discontinuity and is at position x_c . The velocity and pressure are constant across this boundary: $p_{3'} = p_3$ and $v_{3'} = v_3$. The number density, however, is not. Finally the boundary between region 3' and region 2 is located at x_a . We will calculate p, V and v for regions 3 and 3' at time t , assuming that $\epsilon = pV/(\gamma - 1)$. We will solve for the following unknowns: $p_3, v_3, n_3, n_{3'}$ and $p, n,$ and v in the rarefaction wave.

Across a shock front, $v_1 - v_3 = \sqrt{(p_3 - p_1)(V_1 - V_3)}$ and $\epsilon_1 - \epsilon_3 + \frac{1}{2}(V_1 - V_3)(p_1 + p_3) = 0$.

Using $\epsilon = pV/(\gamma - 1)$, they become

$$V_3 = V_1[(\gamma + 1)p_1 + (\gamma - 1)p_3]/[(\gamma - 1)p_1 + (\gamma + 1)p_3] \quad (\text{D.1})$$

$$v_3 = -(p_3 - p_1)\sqrt{2V_1/[(\gamma - 1)p_1 + (\gamma + 1)p_3]} \quad (\text{D.2})$$

The similarity solution for a rarefaction wave is ²⁰

$$p = p_2[1 - (\gamma - 1)|v|/(2c_2)]^{2/(\gamma-1)} \quad (\text{D.3})$$

$$V = V_2(p_2/p)^{1/\gamma} \quad (\text{D.4})$$

$$v = 2(c - c_2)/(\gamma - 1) \quad (\text{D.5})$$

$$|v| = 2(c_2 - x/t)/(\gamma + 1), \quad (\text{D.6})$$

where the local speed of sound is given by $c_i = \sqrt{\gamma p_i V_i}$. We can equate the speed of the fluid v_3 to the fluid velocity in the rarefaction wave between regions 3 and 3'. We find

$$v_{3'} = 2c_2[(p_2/p_3)^{(1-\gamma)/(2\gamma)} - 1]/(\gamma - 1) \quad (\text{D.7})$$

Combining Eqns (D.2) and (D.7), we find that

$$p_3 = p_1 + \frac{1}{\gamma - 1} \sqrt{\frac{2\gamma p_2 n_1}{n_2}} \sqrt{(\gamma - 1)p_1 + (\gamma + 1)p_3} [1 - (p_3/p_2)^{(\gamma-1)(2\gamma)}]. \quad (\text{D.8})$$

This equation can now be solved numerically for p_3 . We can then determine v_3 , n_3 , and $n_{3'}$ using Eqns (D.2), (D.1) and (D.5). In addition, we find p , n and v from Eqns (D.3), (D.4) and (D.5). in the rarefaction wave.

The final step to the solution is determining the location of the boundaries separating these regions. Since the rarefaction wave is moving at the sound speed in region 2, $x_a = tc_2$. Since the velocity at b is $v_b = v_{3'}$ from Eqn (D.7), we then use Eqn (D.6) to find its position: $x_b = tc_2[1 - (\gamma + 1)/(\gamma - 1) [1 - (p_2/p_3)^{(1-\gamma)(2\gamma)}]]$. The contact discontinuity moves with the fluid and its position is thus $x_c = v_3 t$. Finally, the shock position can be determined by transforming to a frame in which the shock is constant and using mass conservation. We then transform back to find the the velocity of the shock to be $v_s = (n_3/(n_3 - n_1))v_3$ so that $x_d = v_s t$.

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25. Note that in order for the fluid to begin and remain in local thermal equilibrium, we must require the interaction length of the mediating particle to be smaller than the grid spacing at any time t . If λ is the proper interaction length, then $\lambda < \Lambda_i^j \Delta r = \Delta R_i^j / \Gamma_i^j$. If a gauge boson is the mediating particle (e.g. a photon), then $\lambda \simeq (\alpha^2 T)^{-1} \simeq g_*^{1/4} / (\alpha^2 \rho^{1/4})$, where $\sqrt{4\pi\alpha}$ is the gauge coupling, $\rho = \pi^2 g_* / 30$ and g_* is the number of effectively massless degrees of freedom.²
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FIGURE CAPTIONS

Fig. 1: The evolution of a compensated, pressureless void with radius $R_{\text{wall}}(t_i) = 333cH_{\text{out}}^{-1}(t_i)$ and for velocity $U(t_i, R) = \sqrt{2G_N M/R}$. The energy density as a function of

$RR(t_i, R_B)/R(t, R_B) \simeq R(t_i/t)^{2/3}$ is shown at the initial time $t_i = 1$ and at times $t = 10, 100$ and 300 .

Fig. 2: Same as for Figure 1 but with velocity $U(t_i, R) = c^{-1}R\sqrt{8\pi G_N\rho/3}$. A dense, thin, outward-moving shell is formed. Shell-crossing occurs at $t = 1.048$.

Fig. 3a: The evolution of a pressureless, uncompensated void with radius $R_{\text{wall}}(t_i) = .0067cH_{\text{out}}^{-1}(t_i)$ and velocity $U(t_i, R) = c^{-1}R\sqrt{8\pi G_N\rho/3}$. Artificial viscosity is used to prevent shell-crossing. The energy density as a function of $RR(t_i, R_B)/R(t, R_B) \simeq R(t_i/t)^{2/3}$ is shown at the initial time $t_i = 1$ and at times $t = 50, 100, 400$ and 1000 .

Fig. 3b: Same as Figure 3a but for a compensated void.

Fig. 3c: The radius of the shell versus time for the results pictured in Figures 3a and 3b. The predicted self-similar solutions are overlaid as a dotted and dashed line for an uncompensated ($R_{\text{shell}} \propto t^{8/9}$) and compensated ($R_{\text{shell}} \propto t^{4/5}$) void respectively.

Fig. 4: The relative error in the energy density versus $RR(t_i, R_B)/R(t, R_B) \simeq R\sqrt{t_i}/t$ for an initially homogeneous and isotropic FRW universe with $p = \rho/3$ at time $\Delta t/t_i = .1$. The results are for $\bar{f} = .01, .005$, and $.0025$ shown with solid, dotted and dashed lines, respectively.

Fig. 5: The nonrelativistic shock tube for $R_{\text{wall}}(t_i) = 20, \Delta R_{\text{wall}}(t_i) = .01$ and $G_N = 0$. The dashed lines represent the exact slab similarity solution.

Fig. 6a: The nonrelativistic evolution of an uncompensated void with $G_N = 0$. The dashed line is the initial time $t_i = 1$, and the triangles with connecting lines is at time $t = 2.5$ after the inbound shock has bounced and the fluid has begun to settle down.

Fig. 6b: Same as Figure 6a but for a compensated void.

Fig. 7a: Collapse of a special relativistic, uncompensated void for $R_{\text{wall}}(t_i) = 1, \Delta R_{\text{wall}}(t_i) = .02, T/\mu \gg 1, G_N = 0, U(t_i, R) = 0$ and $\alpha = 10^{-4}$. Shown is $M, \Gamma, 4\pi\rho$ and Φ versus R for the initial time $t_i = 1$ and for times $t = 1.04$ and 1.065 (before the shock reaches the origin).

Fig. 7b: We plot $4\pi p$ versus R for special relativistic voids with initial varying wall thicknesses at the initial time (unlabeled) and at later times $t = 1.04$ and $1.07, 1.08$ or 1.065 . The parameters are the same as in Figure 7a but with $\Delta R_{\text{wall}}(t_i) = .02, .04, .06$ and $.1$.

Fig. 7c: Collapse, thermalization and homogenization of a special relativistic void. The parameters are the same as in Figure 7a. Shown is the initial time t_i and times $t = 1.04$ (void collapsing) and $t = 3.0$ (after collapse).

Fig. 8: Collapse of a general relativistic, uncompensated void for $R_{\text{wall}}(t_i) = 1$, $\Delta R_{\text{wall}}(t_i) = .02$, $T/\mu \gg 1$, $U(t_i, R) = \sqrt{2G_N M/R}$ and $\alpha = 10^{-4}$ at the initial time $t_i = 1$ and at times $t = 1.04$ and 1.065 .

Figs. 9a,b,c: Collapse of general relativistic, uncompensated voids for $R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) = 25$, $\Delta R_{\text{wall}}(t_i) = 1$, $T/\mu \gg 1$, $U(t_i, R) = \sqrt{2G_N M/R}$. Figures 9a, 9b and 9c show the pressure versus $RR(t_i, R_B)/R(t, R_B) \simeq R\sqrt{t_i/t}$ for $\alpha = 10^{-4}$, $\alpha = 10^{-6}$ and $\alpha = 10^{-10}$, respectively.

Figs. 10a,b: Collapse of general relativistic, uncompensated and compensated void in 10a and b, respectively. $R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) = 250$, $\Delta R_{\text{wall}}(t_i) = 10$, $T/\mu \gg 1$, $U(t_i, R) = \sqrt{2G_N M/R}$ and $\alpha = 10^{-10}$ for both. Shown is the pressure versus $RR(t_i, R_B)/R(t, R_B) \simeq R\sqrt{t_i/t}$.

Fig. 11: Collapse of a general relativistic, compensated void for $R_{\text{wall}}(t_i)/H_{\text{out}}^{-1}(t_i) = 250$, $\Delta R_{\text{wall}}(t_i) = 10.$, $T/\mu \gg 1$, $U(t_i, R) = c^{-1}R\sqrt{8\pi G_N \rho/3}$ and $\alpha = 10^{-10}$. Shown is the pressure versus $RR(t_i, R_B)/R(t, R_B) \simeq R\sqrt{t_i/t}$.

Fig. 12: 1st-crossing times versus $R\sqrt{t_i/t}$ for general relativistic voids with initial radius $R_{\text{wall}}(t_i) = c10^{23}H_{\text{out}}^{-1}(t_i)$ and outside temperature $T_{\text{out}} = T_{\text{RH}}$, where T_{RH} is the reheat temperature. The dash-dot lines are for $\Phi_{\text{in}}(t_i) = T_{\text{out}}(t_i)/T_{\text{in}}(t_i) = 5 \times 10^{11}$, 10^{15} and 10^{20} and the temperature at which each void potentially thermalizes is $T = 500$, 10^6 and 10^{11} GeV respectively. We also plot (long dashes) the evolution of a conformally stretched perturbation with the same initial size, r_{CM} , and the Hubble radius (short dashes) outside the void, r_{HOR} . If $T_{\text{RH}} = 10^{14}$ GeV, then recombination occurs at $t/t_i = 10^{45}$ (or at temperature $T_{\text{rec}} \simeq 3 \times 10^{-9}$ GeV), where r_{CM} and r_{HOR} intersect.