

Renormalization of Massive Lattice Fermions

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The renormalization of a general action for massive lattice fermions is discussed. The analysis applies for all $m_0 a$. Preliminary results for the self energy at one loop in perturbation theory are presented.

1. INTRODUCTION

This paper is a progress report of efforts to calculate the renormalization of fermion masses and bilinear currents in one-loop perturbation theory [1]. When these calculations are finished, they will permit a determination of heavy-quark masses, they will give one-loop predictions for the tuning of improvement parameters in the action, and they will give a one-loop guide for extrapolating matrix elements to the continuum.

After specifying a general action in sect. 2, sect. 3 sketches an all-orders derivation for mass and wavefunction renormalization in terms of the fermion self energy. We have the self energy to one-loop for the simplest action, and present results with and without tadpole improvement. Sect. 4 discusses current renormalization.

2. THE ACTION

Consider the action $S = S_0 + S_B + S_E + \dots$,

$$S_0 = m_0 \int \bar{\psi}(x)\psi(x) - \frac{1}{2}ar\zeta \int \bar{\psi}(x)\Delta^{(3)}\psi(x) + \zeta \int \bar{\psi}(x)\gamma \cdot \mathbf{D}\psi(x) \quad (1)$$

$$+ \frac{1}{2} \int \bar{\psi}(x)[(1 + \gamma_0)D_0^- - (1 - \gamma_0)D_0^+]\psi(x),$$

$$S_B = -\frac{1}{2}ac_B\zeta \int \bar{\psi}(x)\boldsymbol{\Sigma} \cdot \mathbf{B}(x)\psi(x), \quad (2)$$

$$S_E = -\frac{1}{2}ac_B\zeta \int \bar{\psi}(x)\boldsymbol{\alpha} \cdot \mathbf{E}(x)\psi(x). \quad (3)$$

Special cases are the Wilson action [2], which sets $r = \zeta = 1$, $c_B = c_E = 0$, and the Sheikholeslami-

Wohler action [3], which sets $r = \zeta = 1$, $c_B = c_E$. To remove lattice artifacts in general the parameters $m_0 a$, r , ζ , c_B and c_E must all be adjusted [4]. In a non-relativistic setting, however, it is enough to adjust $m_0 a$, c_B , and c_E [4].

3. THE SELF ENERGY

The self energy $\Sigma(p)$ is related to the momentum-space propagator by

$$S^{-1}(p) = S_0^{-1}(p) - \Sigma(p), \quad (4)$$

where $S_0(p)$ is the free propagator. In perturbation theory $\Sigma(p)$ is the sum of all one-particle irreducible graphs. The p_0 -Fourier transform $C(t, p) = (2\pi)^{-1} \int dp_0 e^{ip_0 t} S(p)$ obeys

$$C(t, p) = Z_2(p)e^{-E_p|t|}Q + \dots, \quad (5)$$

where E_p is the energy of a one fermion state with momentum p , and Q is a Dirac matrix satisfying $(Q\gamma_0)^2 = Q\gamma_0$. The \dots denote multi-particle states, which are irrelevant here.

The self energy has the decomposition

$$\Sigma(p) = i \sum_{\mu} \gamma_{\mu} \sin p_{\mu} a A_{\mu}(p) + C(p) \quad (6)$$

in Dirac matrices. For a Euclidean invariant cutoff C and $A_{\mu} = A \forall \mu$ are functions of p^2 only. With the lattice cutoff, however, they are constrained only by (hyper)cubic symmetry. For emphasis it is convenient to write, say, $C(p_0, p)$.

To obtain an expressions for E_p and Z_2 one carries out the p_0 integration with the residue theorem. For arbitrary p the energy E_p is the



solution of the implicit equation

$$1 + m_0 a + \frac{1}{2} r \zeta \mathcal{P}^2 a^2 - C = \cosh E a + (1 - A_0) \sqrt{1 - \mathcal{P}^2} \sinh E a, \quad (7)$$

for E . The abbreviation

$$\mathcal{P}^2 = \frac{\sum_j (\zeta - A_j)^2 \sin^2 p_j a}{(1 - A_0)^2 \sinh^2 E a}. \quad (8)$$

Here the self-energy functions $A_\mu(p_0, \mathbf{p})$ and $C(p_0, \mathbf{p})$ are evaluated at $p_0 = iE$. The solution of eq. (7), $E = E_{\mathbf{p}}$, defines the (lattice-distorted) mass shell of the fermion. The residue is

$$Z_2^{-1}(\mathbf{p}) = (1 - A_0) \cosh E_{\mathbf{p}} a - \dot{C} \sqrt{1 - \mathcal{P}^2} + (\sqrt{1 - \mathcal{P}^2} + \dot{A}_0) \sinh E_{\mathbf{p}} a + \frac{\sum_j \dot{A}_j (\zeta - A_j) \sin^2 p_j a}{(1 - A_0) \sinh E_{\mathbf{p}} a}. \quad (9)$$

The notation $\dot{f} = (ia)^{-1} (df/dp_0)$. In eq. (9) the self-energy functions $A_\mu(p_0, \mathbf{p})$ and $C(p_0, \mathbf{p})$ are evaluated on shell, i.e. $p_0 = iE_{\mathbf{p}}$.

The \mathbf{p} dependence of $Z_2(\mathbf{p})$ is an artifact of the lattice cutoff. An acceptable definition of the wavefunction renormalization constant is

$$Z_2^{-1} = e^{M_1 a} - A_0 \cosh M_1 a + \dot{A}_0 \sinh M_1 a - \dot{C} \quad (10)$$

at $\mathbf{p} = \mathbf{0}$, where $M_1 \equiv E_0$ is the (all-orders) rest mass of the fermion.

Eq. (7) at $\mathbf{p} = \mathbf{0}$ determines the rest mass via

$$e^{M_1 a} = 1 + m_0 a + A_0 \sinh M_1 a - C \quad (11)$$

and the dynamic mass $M_2 = (d^2 E_{\mathbf{p}} / dp_1^2)_{\mathbf{p}=\mathbf{0}}^{-1}$ via

$$\frac{e^{M_1 a}}{M_2 a} = r \zeta + \frac{(\zeta - A_1)^2}{(1 - A_0) \sinh M_1 a} + \frac{d^2}{dp_1^2} \left[A_0 (iE_{\mathbf{p}}, \mathbf{p}) \sinh E_{\mathbf{p}} a - C(iE_{\mathbf{p}}, \mathbf{p}) \right]. \quad (12)$$

A total p_1 -derivative includes an explicit part and an implicit part through the $E_{\mathbf{p}}$ dependence. In eqs. (11) and (12) the self-energy functions and derivatives are evaluated at $p_0 = iM_1$ and $\mathbf{p} = \mathbf{0}$.

For a massless fermion, $M_1 = M_2 = 0$. The bare mass that induces $M_1 = 0$ obeys

$$m_{0c} a = C(0, \mathbf{0}; m_{0c} a). \quad (13)$$

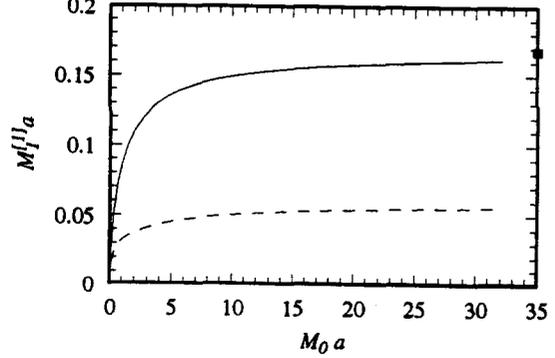


Figure 1. Plot comparing $M_1^{[1]} a$ vs. $M_0 a$ (solid curve) and $\tilde{M}_1^{[1]} a$ vs. $\tilde{M}_0 a$ (dashed curve). The static point is indicated by the box [5].

The third argument of C denotes the parametric dependence. It is useful to take care of this term once and for all, and write

$$e^{M_1 a} = 1 + M_0 a + A_0 \sinh M_1 a - \tilde{C} \quad (14)$$

where $M_0 a \equiv m_0 a - m_{0c} a = (2\kappa)^{-1} - (2\kappa_c)^{-1}$, and $\tilde{C}(iM_1 a, \mathbf{0}; m_0 a) = C(iM_1 a, \mathbf{0}; m_0 a) - m_{0c} a$.

We turn now to one-loop results for $r = \zeta = 1$, $c_B = c_E = 0$, with and without tadpole improvement. In perturbation theory the rest mass has an expansion

$$M_1 a = \log(1 + \tilde{M}_0 a) + \sum_{l=1}^{\infty} g_0^{2l} \tilde{M}_1^{[l]} a. \quad (15)$$

In the tadpole improved version $\tilde{M}_0 a = M_0 a / u_0$, where u_0 is a suitable (gauge invariant) average link. In applications both u_0 and κ_c would be taken from Monte Carlo calculations. Below we choose $u_0 = (8\kappa_c)^{-1}$. Figure 1 shows the one-loop correction to the rest mass M_1 . As expected, $\tilde{M}_1^{[1]}$ is significantly smaller than $M_1^{[1]}$.

For the dynamic mass it is better to define a renormalization factor via $M_2 = m_2 Z_M$, where

$$m_2 a = \frac{M_0 a (1 + M_0 a) (2 + M_0 a)}{2\zeta^2 (1 + M_0 a) + r\zeta M_0 a (2 + M_0 a)} \quad (16)$$

is the tree-level expression for the dynamic mass, except that the linear mass divergence is absorbed, order by order, into $M_0 a$. The tadpole improvement is $M_2 = \tilde{m}_2 \tilde{Z}_M$, where \tilde{m}_2 is

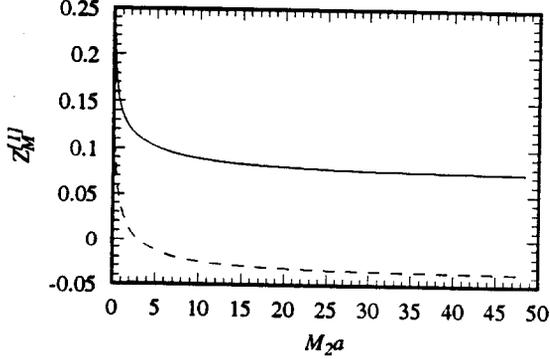


Figure 2. Plot comparing $Z_M^{[1]}$ (solid curve) and $\tilde{Z}_M^{[1]}$ vs. $M_2 a$ (dashed curve).

given by the right-hand side of eq. (16), but with $M_0 a \mapsto \tilde{M}_0 a$. The factors Z_M have series

$$\tilde{Z}_M^{(\sim)} = 1 + \sum_{l=1}^{\infty} g_0^{2l} \tilde{Z}_M^{[l]}, \quad (17)$$

Figure 2 shows the one-loop renormalization of the dynamic mass. Again, $|\tilde{Z}_M^{[1]}|$ is significantly smaller than $|Z_M^{[1]}|$.

To define the perturbative coefficients for the wavefunction renormalization constant, factor out $e^{M_1 a}$. The tadpole improved constant is $\tilde{Z}_2 = u_0 Z_2$. The perturbative series are

$$e^{M_1 a} \tilde{Z}_2^{(\sim)} = 1 + \sum_{l=1}^{\infty} g_0^{2l} \tilde{Z}_2^{[l]}. \quad (18)$$

The one-loop coefficient has an infrared divergence, which can be regulated with a gluon mass λ . Figure 3 plots the IR-finite $\tilde{z}_2^{(\sim)[1]} = \tilde{Z}_2^{[1]} + \log(\lambda^2 a^2)/(6\pi^2)$. Once again, $\tilde{z}_2^{[1]}$ is significantly smaller than $z_2^{[1]}$.

4. VERTEX CORRECTIONS

A full vertex function takes the form

$$V(p|q) = S(p)\Gamma(p|q)S(q). \quad (19)$$

In perturbation theory Γ is given by the sum of all truncated three-point diagrams. To put the external lines on shell, one Fourier transforms

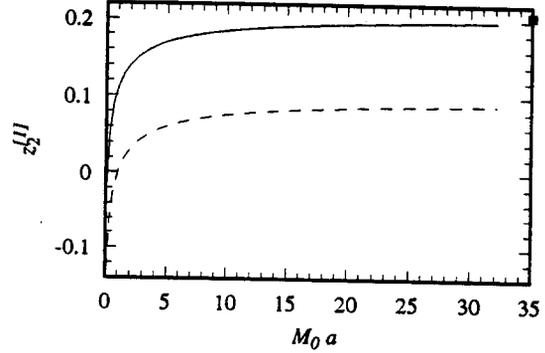


Figure 3. Plot comparing $z_2^{[1]}$ vs. $M_0 a$ (solid curve) and $\tilde{z}_2^{[1]}$ vs. $\tilde{M}_0 a$ (dashed curve). The static point is indicated by the box [5].

in p_0 and q_0 . Poles arise precisely as in the self-energy derivation, so the on-shell truncated vertex function is $\Gamma(iE_p, p|iE_q, q)$. Consequently, when normalization conditions introduce a factor $1 + m_0 a$ at tree level, the all-orders generalization is $e^{M_1 a}$, just as with Z_2^{-1} .

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