Fermi National Accelerator Laboratory

FERMILAB-CONF-93/343-T
November, 1993

Status of Lattice QCD[1]

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Significant progress has recently been achieved in the lattice gauge theory calculations required for extracting the fundamental parameters of the standard model from experiment. Recent lattice determinations of such quantities as the kaon B parameter, the mass of the b quark, and the strong coupling constant have produced results and uncertainties as good or better than the best conventional determinations. Many other calculations crucial to extracting the fundamental parameters of the standard model from experimental data are undergoing very active development. I review the status of such applications of lattice QCD to standard model phenomenology, and discuss the prospects for the near future.

INTRODUCTION

Our only existing experimental clues about the theory that lies beyond the standard model are the apparently arbitrary fundamental parameters of the standard model. The only experiments guaranteed to determine the origin of electroweak symmetry breaking and thus new clues into beyond the standard model physics were to have been performed at the SSC. The fundamental parameters of the standard model may prove to be our only window onto beyond the standard model physics for some time, unless we get lucky with a lower energy accelerator. The nonperturbative calculations which allow the extraction of these parameters from experiment will therefore take on increasing importance over the next few years.

The last few years have seen significant progress in some of these calculations with lattice QCD. Some of the simplest ones have now been completed with first attempts at quantitative estimation of all uncertainties. Lattice calculations of the mass of the b quark, m_b, and the kaon B parameter, B_K, are now believed by their authors to be more accurate than the best conventional phenomenological determinations of these quantities. Determinations of the strong coupling constant, α_s, are now of comparable quality to the best conventional determinations and will soon be significantly better. Many other calculations crucial to standard model phenomenology are undergoing rapid development.

[1]Talk presented at the Lepton-Photon Symposium, Cornell University, Aug. 10-15, 1993

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The foundations of the current advances were laid around 1980 with the development of explicit expressions for hadron masses and other hadronic quantities by Weingarten and by Parisi and their collaborators. (1) The recent developments have arisen in part because of dramatic developments in machine and algorithmic technology that will not be reviewed here. (2) The computers on which these calculations have been performed are around 10,000 times as powerful as the Vaxes on which the first hadron spectrum calculations were performed around 1980. Likewise, the speeds of algorithms for the inclusion of sea quark loops from first principles (algorithms for "unquenched" calculations) have increased by an even larger factor, one which is hard to measure because of the extreme slowness of the original algorithms.Numerous other methodological and technical improvements have also contributed to the reliability of the calculations.

These developments have occurred also because of an improved perspective regarding which lattice calculations are easiest to perform reliably, and are most useful to particle physics. Although the desire to understand the physics of nuclear energy levels was the initial spur to the study of strong interactions, the calculation of the energy levels of the uranium nucleus is not currently seen as the most promising test of QCD, or of standard model or beyond the standard model physics. Likewise, although the ability to calculate the proton mass from first principles seemed like a Holy Grail when lattice gauge theory was invented 20 years ago, the mass of the proton and the rest of the light hadron spectrum is not the only or even the most important application of lattice gauge theory. The pseudoscalar mesons π, K, D, and B, and the quarkonia (the ψ's and Υ's) are significantly simpler than the proton and other hadrons, as will be discussed. They will provide good tests of lattice methods and significant information about the standard model before calculations of the proton mass do.

QCD

QCD Phenomenology

The ψ and Υ Systems

The simplest hadrons to investigate on the lattice are the ψ and Υ systems. Quarkonia are smaller than most hadrons, resulting in smaller finite volume errors. Propagators for heavy quarks can be calculated much more rapidly than light quark propagators, and no extrapolation down to the physical quark mass is required as it is for light quarks. These facts have been particularly emphasized in Ref. (8).

Quarkonia have received relatively little attention from lattice theorists until recently. The reason may be that they have been well understood for a long time on the basis of nonrelativistic potential models (7), which become rigor-

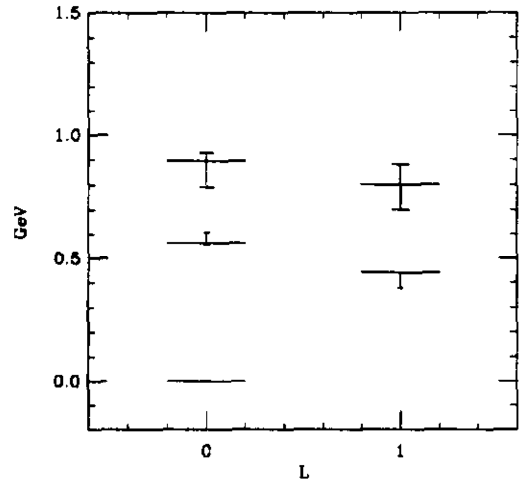

FIG. 1. The mass spectrum of the Υ system, $L = 0$ states (1S, 2S, and 3S) and $L = 1$ states (1P and 2P). Mass difference in GeV with the ground state is shown. Solid lines are experiment.

ous predictions of QCD in a well-defined limit, $m_Q \to \infty$. This fact should, to the contrary, place them among the most interesting and important of current lattice calculations because of the possibility of using nonrelativistic methods and reasoning to:

- guide physics expectations, and
- monitor the accuracy of approximations (finite lattice spacing a, finite volume V, quenched) and make corrections.

For example, the nonrelativistic picture tells us that the hyperfine splitting in quarkonia is a short distance quantity, sensitive to $\overline{\psi}\sigma \cdot B\psi$, the dimension five operator which is the leading operator which must be added to the action to correct for finite lattice spacing errors. This quantity is useful in testing and fine tuning the approximations used in lattice calculations. A spin-averaged quantity like the 1P–1S splitting should be insensitive to these leading finite lattice spacing errors. It is also insensitive to the precise value of the quark mass used, since it is almost the same for the ψ and Υ systems. It is a likely candidate to yield information about particle physics, as in the extraction of the strong coupling constant, to be discussed below.

The most extensive investigation of finite lattice spacing errors which has yet been performed in a phenomenological calculation has recently been completed by the NRQCD collaboration for the ψ and Υ systems. They use the formalism of Nonrelativistic QCD (8), a discretized version of the nonrelativistic expansion of the quark action. In previous work reported in Ref. (9), coefficients of operators for finite lattice spacing and nonrelativistic corrections were evaluated at tree level. Corrections were then determined by

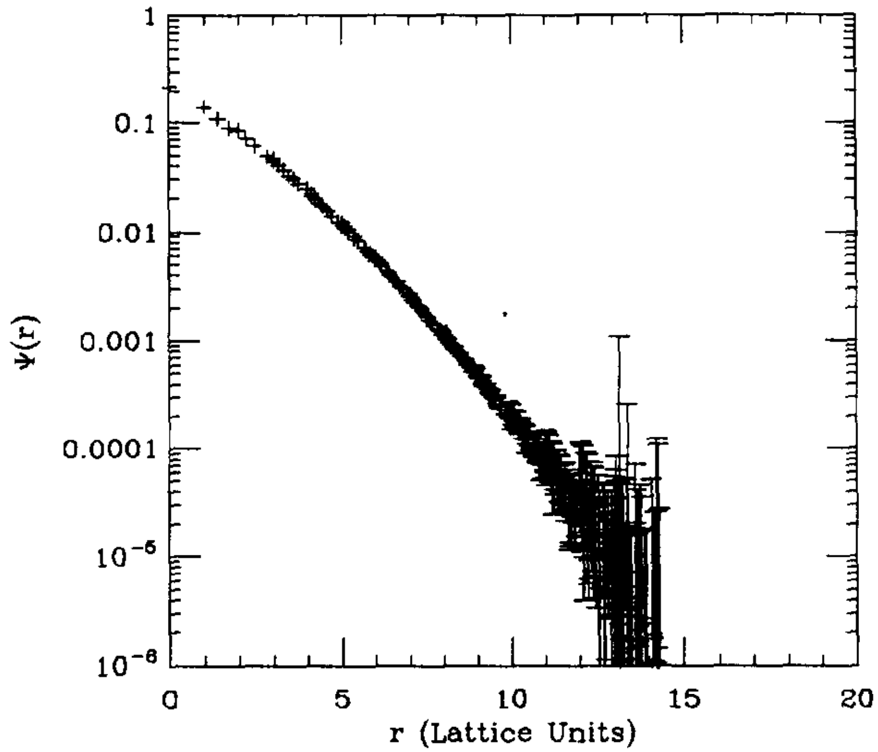

FIG. 2. The wave function $\Psi(r)$ of the Υ meson as a function of r in lattice units.

evaluating the operator expectation values in potential model wave functions. This year, these expected corrections were verified from first principles by including the required operators directly in the lattice calculations, with coefficients evaluated to one loop. (11) The resulting spectrum for the Υ is shown in Fig. 1. The mass of the Υ was used as an input (to fix the quark mass), and the overall energy scale (the lattice spacing in physical units) was chosen to obtaining the best fit to the remaining masses.

Fig. 2 shows the Coulomb gauge wave function, calculated on a 24^4 lattice with Wilson fermions. (10) Unlike a QED-like pure Coulomb potential which would produce a wave function with the form $\Psi(r) \propto \exp(-\alpha m r)$, QCD produces nonrelativistic bound states with wave functions that fall more slowly at short distances (smaller effective α at short distances) and faster at long distances (as expected from confinement). This wave function was calculated in the quenched approximation, which has slightly too much asymptotic freedom due to the absence of light quark loops. It would not be surprising to find that the same calculations repeated in the full theory showed slightly less concavity in the wave function, perhaps of order 20% less ($\approx \beta_0^{nf=3}/\beta_0^{nf=0} = 9/11$). The effects of finite boundary conditions can be seen half way across the lattice, at $r = 12$ in lattice units.

The Light Hadron Spectrum

Calculations involving light quarks are significantly more difficult than those with only heavy quarks. One cannot estimate as accurately in advance what order of correction operators must be added to the action to achieve a certain accuracy in finite lattice spacing errors, or what volume must be used to reduce finite volume errors to a negligible level. These things must be determined to a much greater degree by painstaking experimentation. Light quark propagators are much more costly computationally. Effects of light quark loops are likely to be more complicated than for quarkonia.

Current algorithms for calculating light quark algorithms fail when the light quark mass $m_l \equiv (m_u + m_d)/2$ is reduced toward its physical value. The extrapolation is likely to be reasonably straightforward for such quantities as m_π and f_π whose chiral behavior is well understood. The masses of particles like the ρ which become unstable in the small m, large V limit clearly require special care in extrapolating to $m_l \rightarrow 0$. (12) (The correlation functions from which M_ρ is determined become dominated by the two pion cut, rather than the ρ pole in the physical limit.) The proton mass is also known to have much larger nonlinear corrections in chiral perturbation theory than m_π and f_π do when m_l is raised above m_s. (13) Analogous effects in M_P occur in the quenched approximation with different coefficients. In addition, there are indications in quenched chiral perturbation theory calculations of pathologies as $m_l \rightarrow 0$, which so far have not been reconciled with numerical results. (14)

The most systematic attempt so far at calculating the hadron spectrum in the quenched approximation appeared this year from the GF11 collaboration. (15) The calculation was performed at several values each of lattice spacing, volume, and quark mass, with the results extrapolated to the physical values of each. The resulting spectrum is shown in Fig. 3. M_π and M_K have been used as inputs to set the quark masses. The error bars include statistics, finite volume, and $m_l \rightarrow 0$ extrapolations (assuming linear behavior in m_l). They do not include uncertainties due the finite lattice spacing extrapolation and to the quenched approximation. The dominant finite lattice spacing errors can be removed by adding a single correction operator to the action. A calculation with a tree level improved $O(a)$ corrected action has been performed at a single small lattice spacing, with results that appear to be consistent with these extrapolated results (though with larger statistical errors). (16)

Adding light quark loops to the calculation is much more expensive. The state of uncertainty analysis is therefore somewhat less advanced than for the quenched theory. Even more than for quenched calculations, algorithms begin to fail as the quark mass is reduced. The light quark mass can be reduced only to around 0.4 m_s; results must then be extrapolated to the physical limit.

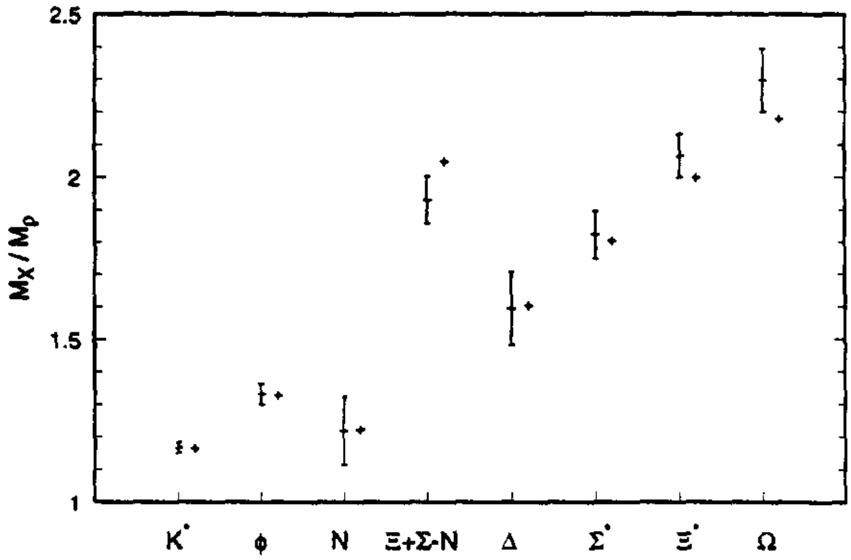

FIG. 3. The spectrum of the light hadrons in the quenched approximation, extrapolated to zero lattice spacing, infinite volume, and physical quark mass. Error bars do not include uncertainties due to the quenched approximation or to finite lattice spacing. + denotes experiment.

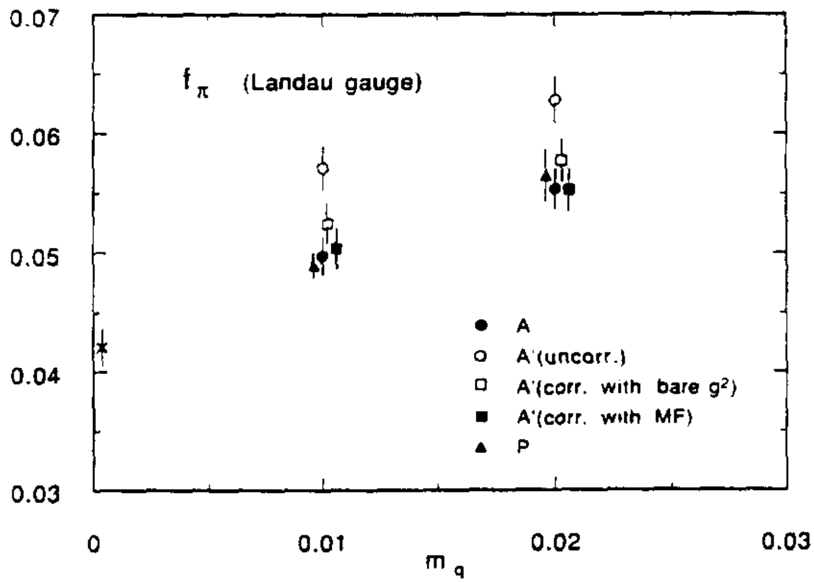

FIG. 4. The pion decay constant calculated in full QCD with two light flavors, as a function of the quark mass. Light hadron results must at present be extrapolated to the physical quark mass because algorithms fail when the mass becomes too small.

Fig. 4 shows an unquenched calculation of f_π (17), which may be the simplest light hadron quantity to obtain other than the pion mass. Chiral perturbation theory leads us to expect it to have a small and smooth extrapolation to the chiral limit, compared with M_ρ and M_P. The calculation used two light flavors of quarks in the sea, the lattice quark masses of 0.02 and 0.01 correspond roughly to m_s and $m_s/2$. The size of the extrapolation from $m_l \approx m_s/2$ ($M_\pi^2 \approx M_K^2$) is consistent with expectations based on the experimental value of f_K^2/f_π^2. Some sources of error have been carefully checked: consistency between values of f_π obtained from various operators has been tested, the effects of finite volume on M_π have been tested to be under 1% (finite volume effects on M_ρ and M_P are much larger). Yet to be checked are the effects of finite volume on f_π itself, the agreement of f_π as a function of m_l with chiral perturbation theory, and the effects of finite lattice spacing.

The algorithmic restriction to unphysically large light quark masses seems likely to be with us for a while. As the various sources of uncertainty in unquenched calculations gradually become better understood, one of the important questions for consumers of lattice calculations will become: which quantities have the smoothest and best understood extrapolations from $m_l \approx m_s/2$ down to the physical light mass limit?

The Fundamental Parameters of QCD

From the standpoint of standard model physics, one of the most crucial results of hadron spectrum calculations is the determination of the fundamental parameters of QCD: the strong coupling constant, α_s, and the quark masses. Such calculations have two elements. First, one calculates a measurable dimensionful quantity such as f_π or a level splitting in the ψ or Υ system, to set the lattice energy scale in physical units. This appears to be the least important source of uncertainty in such calculations. Second, one determines the physical coupling at short distances. This may be done either by a) relating the bare lattice parameter to a conventional definition ($\overline{MS}$, for example) via renormalization group improved, mean field improved perturbation theory, or better still, by b) ignoring the bare lattice parameters (which are often somewhat pathological compared with physically defined parameters) and extracting the physical parameters nonperturbatively from short distance lattice calculations.

If the calculations are done in the quenched approximation, one must also estimate the associated corrections and uncertainties. The expected effects in the Υ meson are illustrated in Fig. 5 Omission of quark loops from the theory results in too large a β function. If parameters are set by adjusting middle distance physics such as the 1P-1S splitting to be right, the coupling constant at short distances will be slightly too small, as illustrated in the left hand figure. The size of the effect may be estimated with the aid of potential models in advance of including sea quarks from first principles. Very

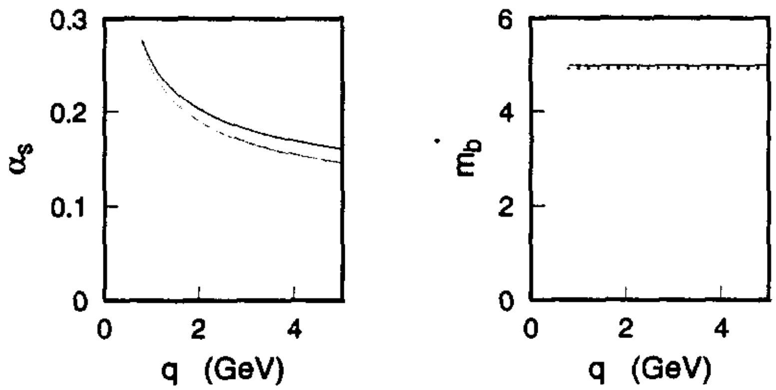

FIG. 5. Expected running of the strong coupling constant (left) and the b quark mass (right) between the energy scale of Υ physics (around 600-1100 MeV) and the lattice spacing scale π/a at which the coupling is extracted (around 3-6 GeV). Dotted lines are the quenched approximation, solid lines are full QCD.

roughly, the expected size of the effect is of order $\beta_0^{n_f=3}/\beta_0^{n_f=0} = 9/11$. The "running mass" of the b quark, on the other hand, does not run for $q^2 < m_b^2$. The effective mass determined at the lattice spacing scale is the same mass governing the dynamics at the much lower scale of Υ physics, whether or not the effects of light quark loops are included. This fact makes m_b the most reliably known of the parameters of QCD. The light quark masses are more difficult, since we have no way of reliably estimating the effects of quark loops without including them from first principles.

The Strong Coupling Constant

To obtain the the lattice scale in physical units for use in a lattice determination of the strong coupling constant, spin-averaged level splittings in the ψ and Υ systems are particularly useful quantities, for reasons already discussed. Uncertainties arising from usual lattice errors such as finite V, finite a, and statistics are quite small and seem to be under very good control. The dominant uncertainties in α_s determinations are arise from perturbation theory, and, for the present, the quenched approximation.

There are very large perturbative one loop corrections in the relation between the bare lattice coupling constant and physically defined coupling constants. These; however, are no more significant than the large corrections in the relation between the MS and $\overline{MS}$ couplings. (18) The most sensible expansion parameter need not be the one which is simplest in terms of the regulator: it must be determined from physical quantities. The bare coupling constants may be rather pathological expansion parameters. The origin of the large corrections can also be understood: they arise from tadpoles due to higher dimension operators in the Wilson lattice action. If such effects are taken into account with mean field theory estimates, a large number of perturbative series for pure gauge theory, Wilson fermions, and NRQCD become very convergent. Mean field theory improvement may also be used be used to estimate higher order uncalculated terms in lattice perturbation theory.

In Ref. (19), a mean field improved perturbation theory was used to extract the renormalized coupling from the bare lattice coupling. Subsequent nonperturbative extractions of α_s from a variety of lattice quantities yielded results which were systematically a few per cent higher than the coupling obtained through the mean field improved relation with the bare coupling. (18,20) (See also (21).) This has resulted in a small but significant increase in the present values of α_s over those reported in Ref. (19).

The largest source of uncertainty in current determinations of α_s, and the one under poorest control is the use of potential models and perturbation theory to estimate the effects of light quark loops on the results. It is clear that the quenched theory, with too strong a β function, ought to have too weak an α_s at short distances. For the Υ system, potential models and ordinary perturbation theory yield similar estimates for the increase in the running for the quenched theory between the scale of Υ physics and the cutoff scale. (Potential models suggest 600–1100 MeV for typical gluon momentum transfers in the Υ. (11)) The agreement of the much more sensitive correction for the ψ system provides some check on the consistency of the analysis.

The first checks of these estimates from first principles have now appeared. (22). With large errors, the results so far are consistent with the quenched analysis.

The preliminary results for the latest analyses of α_s determined in the quenched approximation (10,11) are consistent with

$$\alpha_s(M_Z) = 0.110 \pm 0.004. \tag{1}$$

The Heavy Quark Masses

Since the effective mass of fermions does not run at energy scales below the pole mass, determinations of m_c and m_b do not suffer from the largest effect due to the quenched approximation in the determinations of α_s and m_l: the running of the effective coupling or mass between the physics scale and the short distance scale at which the coupling is determined. The NRQCD collaboration has determined m_b in two independent ways, with compatible results. In method 1, one calculates the binding energy, the difference between the bare quark masses and the physical mass of the Υ. This lattice result is then subtracted from the experimental mass of the Υ to obtain $2m_b$. In method 2, one determines the coefficient $1/2m$ required in the quark's kinetic energy to obtain the correct energy–momentum relation for the Υ. The largest uncertainty in each case arises from perturbation theory, estimated at 1–2%

for method 1, and 2–3% for method 2. Uncertainties arising from finite lattice spacing, finite volume, and statistics are estimated to be 1% or less. The result for the pole mass is

$$m_b = 4.7 \pm 0.1 \text{ GeV}. \tag{2}$$

Using (23)

$$m_{\overline{MS}}(\mu = m) = m_{\text{pole}}/(1 + 4/3\frac{\alpha_s(m)}{\pi} + 12.4(\frac{\alpha_s(m)}{\pi})^2 + \ldots) \tag{3}$$

and $\alpha_{\overline{MS}}(4.7 \text{ GeV}) \approx 0.18$ one obtains

$$m_{\overline{MS}}(4.7 \text{ GeV}) = 4.2 \pm 0.1 \text{ GeV} \tag{4}$$

for the b quark.

The mass of the c quark will soon be reported based on these same methods. It will be relatively less accurate then m_b because the errors arising from an additive mass renormalization are larger relative to m_c then to m_b, and because the required perturbation theory is less accurate at ψ energy scales (under 700 MeV, according to potential models) then at Υ energy scales (under 600-1100 MeV).

The t quark is expected to decay before it forms QCD bound states. Lattice methods are unlikely to contribute to determining its mass.

The Light Quark Masses

The light quark masses are the most difficult of the fundamental parameters of QCD to determine. There is certainly significant running of the quark masses between the lattice spacing scale and the scale of light hadron physics. However, there is no hope of estimating the effects of light quark loops on this running at large distances: unquenched calculations are required from the start. However, m_π, the obvious choice to determine m_l, is by far the easiest of the light hadron masses to determine, so this calculation is likely to be among the first to performed reliably in unquenched calculations. The current status of determinations of m_l in the $\overline{MS}$ scheme is summarized in Fig. 6 (24). Calculations have been performed with Wilson fermions and staggered (KS, or Kogut-Susskind) fermions, in the quenched approximation ($N_f = 0$) and with two flavors of light quarks ($N_f = 2$). The calculations employing staggered fermions are nicely independent of the lattice spacing, while those using Wilson fermions show significant variation, seeming to approach the staggered results as the lattice spacing is reduced. (This may be related to the fact that the lattice spacing errors in the quark propagators start at $O(a)$ for Wilson fermions and $O(a^2)$ for staggered.) This is a pity, because the perturbation theory for the relation between the bare staggered fermion mass and the $\overline{MS}$ mass is much worse behaved than the analogous perturbation

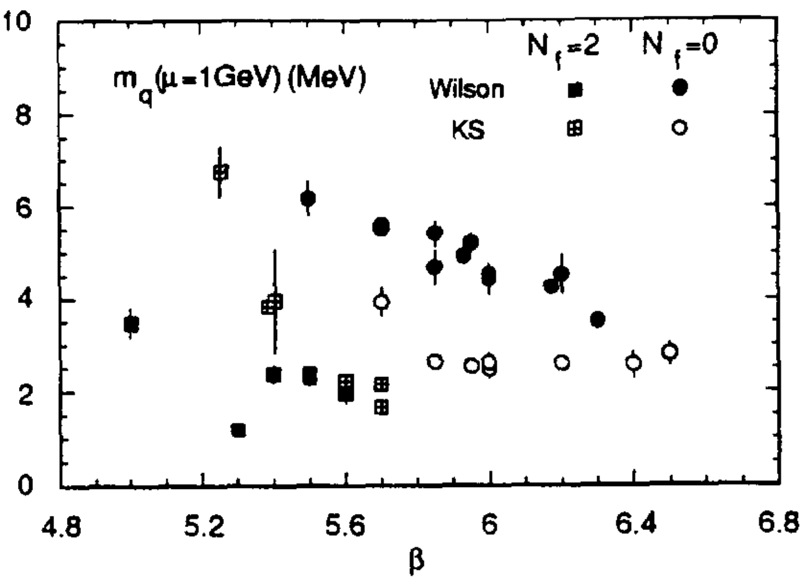

FIG. 6. Lattice determinations of the light quark mass in MeV, as a function of $\beta \equiv 6/g^2$. Smaller lattice spacings (i. e., more correct results) are to the right.

theory Wilson fermions or NRQCD. There is roughly a 40% effect in this relation which has not been understood in terms of either renormalization group logarithms or mean field theory tadpoles.

Current results from staggered fermions ($m_l \sim 2$ MeV) are at the low end of the conventional range, but it is not known yet how reliable these are.

There are thus several ways in which the determination of the light quark masses is more difficult than the determination of α_s and the heavy quark masses: unquenched calculations are required from the start, the perturbation theory is less well understood, and nonperturbative techniques for extracting the short distance quark mass are less well developed. On the other hand, m_l is known from existing phenomenology to within only a factor of three (as opposed to 5–20% for the other quantities) so the payoff will ultimately be bigger.

WEAK MATRIX ELEMENTS

As long as it is the case that the only clues available to us about the theory lying behind the standard model are the apparently arbitrary "fundamental" parameters of the model, one of the most important applications of lattice gauge theory will be the calculations of the hadronic weak matrix elements that allow the extraction of the elements of the Cabibbo-Kobayashi-Moskawa matrix elements from hadron decay data. The hadronic matrix elements for extracting V_{ud} and V_{us} can be estimated with sufficient accuracy by employing

$SU(2)$ and $SU(3)$ flavor symmetry, respectively, so that lattice calculations are unlikely to be of much assistance until they are much more accurate. For the remaining CKM matrix elements, lattice calculations will eventually play a crucial role. For the elements connecting b and c quarks for lighter quarks, exclusive semileptonic meson decays are the most feasible lattice calculations. Cabibbo suppressed semileptonic decays of the t quark are not likely to be observed any time soon. The indirect effects of the t quark in neutral meson mixing amplitudes are the likeliest sources of information on CKM matrix elements involving the t quark. We will discuss some of these amplitudes in order of increasing difficulty.

B_K

The simplest and best understood of these weak matrix elements is the kaon "B parameter",

$$B_K = \frac{\langle \bar{K}^0 | \mathcal{O}_{\Delta S=2} | K^0 \rangle}{\frac{8}{3} m_K^2 f_K^2}, \tag{5}$$

which is required to relate CP violation in kaons to the parameters of the CKM matrix. There is a variety of reasons for this.

- The amplitude involves only pseudo scalars, which have the best statistics and finite volume errors of the light hadrons.
- Calculations for kaons may be performed with $m_l = m_s/2$ and extrapolated in $m_s - m_d$ to the physical values. Chiral perturbation theory shows this to be a more benign extrapolation then the usual $m_d \to 0$ limit. (It is a higher order effect in chiral perturbation theory.)
- B_K is a ratio of two very similar amplitudes, in which many errors (such as those arising from perturbation theory and the quenched approximation are likely to cancel.

As is typical in calculations with light pseudoscalar mesons, the cleanest results are produced with staggered fermions, which preserve an exact chiral symmetry at the expense of "doubling" of the light flavors. (25)

Many of the assumptions in the calculation have been checked in the last year. The one loop perturbative corrections have been checked. (26) The independence of the result on the use of gauge variant vs. gauge invariant operators has been tested. (27) The hope that the effects of the quenched approximation are small has been tested explicitly. (27,28)

The most important new result is an improved understanding of the source of the rather large finite lattice spacing dependence, which has previously dominated the uncertainty. Such lattice spacing dependence in weak amplitudes can arise from powers of a due to discretization errors, and from powers

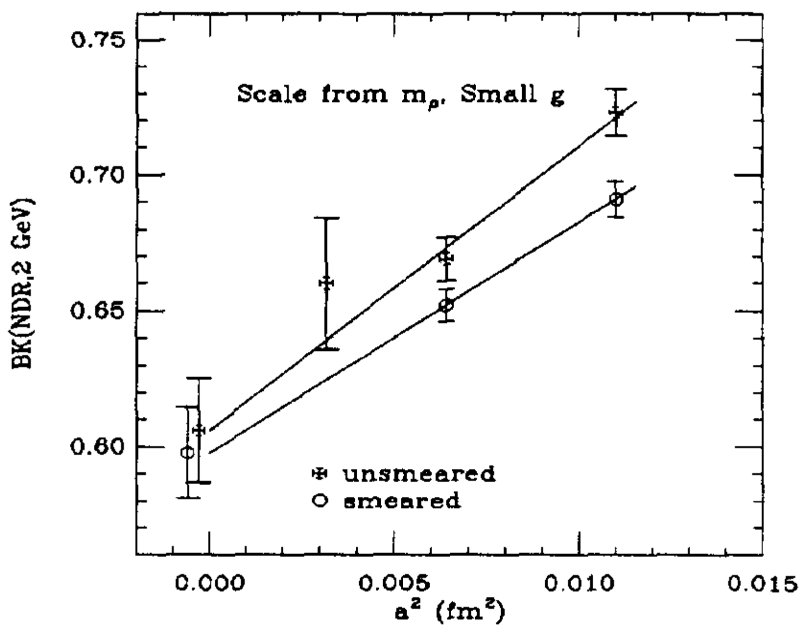

FIG. 7. B_K in the naive dimensional reduction scheme as a function of the lattice spacing squared. Results for two different types of four-quark operators have the same $a^2 \to 0$ limit.

of $1/\ln a$ due to perturbation theory. The perturbative corrections are small, mostly canceling between the numerator and denominator in Eqn. 5. They are unlikely to contribute much to the finite a effects. (This is in contrast with some previous examples of finite a dependence such as in the string tension, where a large dependence of σ/Λ_{lat}^2 on a is now understood to have arisen predominantly from the use of bare lattice perturbation theory rather than renormalized perturbation theory.) This leaves the question of the power in a of the effects of discretization. The Staggered Collaboration has very recently completed an examination of all of the dimension 7 operators capable of producing $O(a)$ errors in B_K for staggered fermions. (29) They have found that among the many such operators, none has the right flavor and lattice symmetries to contribute to the amplitude for B_K. They therefore extrapolate their small lattice spacing data in $a^2 \to 0$ (Fig. 7) to obtain their final answer.

The current result in the naive dimensional reduction scheme, with estimates of the known sources of uncertainty, is

$$B_K(NDR, 2\text{GeV}) = 0.616 \pm 0.020\ (stat)$$

$$
\begin{aligned}
&\pm 0.014\ (g^2) \\
&\pm 0.009\ (scale) \\
&\pm 0.004\ (operator) \\
&\pm 0.002\ (correction) \\
= {}& 0.616 \pm 0.020 \pm 0.017.
\end{aligned}
$$

For the renormalization group invariant B parameter, they obtain

$$
\begin{aligned}
\widehat{B}_K &\equiv B_K(NDR, 2\text{GeV})\ \alpha_s(2\text{GeV})^{-6/25} \\
&= 0.825 \pm 0.027 \pm .023.
\end{aligned}
$$

For comparison, the $1/N$ expansion predicts $\widehat{B}_K = 0.7 \pm 0.1$. (30) A further check which has yet to be done is to test explicitly whether the extrapolation to the physical value of $m_s - m_d$ is as small when quark loops are included as it has been shown to be in the quenched approximation.

Heavy Meson Decay Constants

The special simplicities arising in B_K from the fact that it is a ratio of two similar amplitudes are not shared by typical weak decay amplitudes such as heavy meson decay constants. In particular, there is no reason to expect perturbative corrections and the effects of the quenched approximation to be particularly small.

f_{D_s} is the only heavy meson decay constant which can be directly compared with experiment. It will therefore play an important role in validating methods for calculating decay constants as theory and experiment become more precise. Fig. 8 shows the lattice results for this quantity, using "improved" (in terms of finite lattice spacing errors) light quark methods. They are $f_{D_s} = 230 \pm 35$ MeV (31) and 218^{+50}_{-8} MeV (32). They may be compared with the experimental numbers 232 ± 69 MeV from WA75 (33) and 344 ± 76 MeV from CLEO (34). The analysis going into the lattice uncertainties is not as detailed as that behind B_K. The numbers have remained reasonably stable, however, as the calculational methods have improved over the last few years.

The B meson decay constant f_B is of even greater phenomenological interest because of its role in describing $B^0\overline{B^0}$ mixing, and much work has been invested in it recently. Initial lattice calculations in the static limit produced results which were very high (over 300 MeV), compared with expectations from quark model estimates. (35) Subsequent work revealed several sources of mostly negative corrections, and a final consensus has not yet emerged, even in the quenched approximation and in the static limit. Current estimates range from 185 to 370 MeV. (31,32,36–38)

An example of a correction which is still in the process of being sorted out is shown in Fig. 9. (38) More dependence on the lattice spacing is observed than

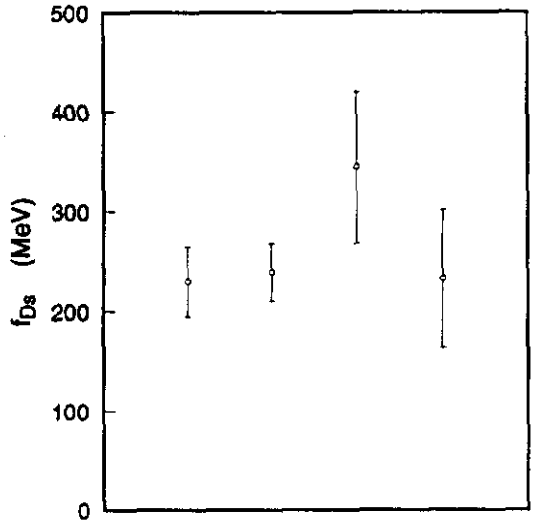

FIG. 8. The decay constant of the D_s meson, from lattice calculations (left two points), compared with experimental results from CLEO (third point) and WA75 (last point).

was apparent originally. (36,38) Part of this a dependence arises from higher orders of perturbation theory which fall like logarithms of a as $a \to 0$, and can be at least partially ameliorated with the use of improved perturbation theory. On the other hand, part of it may also come from discretization errors which fall as powers of a. Until the functional form of the a dependence is understood, an $a \to 0$ extrapolation cannot be made with confidence.

Ratios of decay constants can be calculated more accurately. Current results for f_{B_s}/f_B ($= f_{D_s}/f_D(1+O(m_s/m_c)$ (39)) in the quenched approximation lie in the range 1.11–1.22. (31,32,36,37) The effects of adding quark loops to these calculations may be estimated from the one loop chiral perturbation theory calculations of these quantities, which give $f_{B_s}/f_B = f_{D_s}/f_D \approx 1.1$. (40) Ultimately, on would hope to calculate the deviations of these ratios from unity as accurately as the decay constants themselves: that is, $f_{B_s}/f_B - 1$, $f_{D_s}/f_D - 1$, and eventually $f_B/f_D - 1$ to perhaps 20%.

The hadronic amplitude for $B^0\overline{B^0}$ mixing may be written in terms of f_B and a B parameter, following the notation of the K system. In the standard model, the parameter measuring the experimentally observed $B^0\overline{B^0}$ mixing is given by

$$x_d = (\text{known factors})\,|V_{td}^* V_{tb}|^2 f_B^2 B_B. \quad (6)$$

Pilot studies of B_B and B_{B_s} have been performed which yielded results close

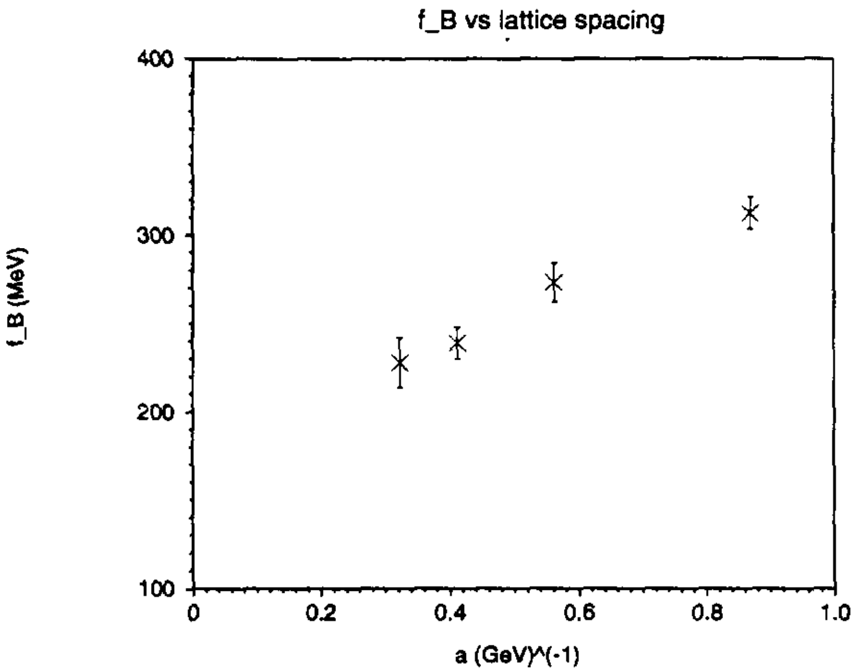

FIG. 9. The decay constant of the B meson, f_B, in the static approximation, as a function of the lattice spacing a.

to the "vacuum saturation" value of one. (41)

Semileptonic Decays

Semileptonic decay amplitudes share all of the difficulties of decay constants. In addition,

- large momentum in the decay meson leads to worse finite lattice spacing errors and worse statistics,
- finite lattice volume leads to a coarse decay momentum discretization, and
- calculations for many decay momenta are required, each of which is as difficult as a decay constant calculation.

Although lattice calculations are first principles calculations, it is perhaps fair they are treated in competition with sum rules and quark models at the present stage of the game. (42) Getting the level of detail and the accuracy of the uncertainty analysis for these processes to match that already obtained for B_K will be a long process, even if all goes relatively well. However, there

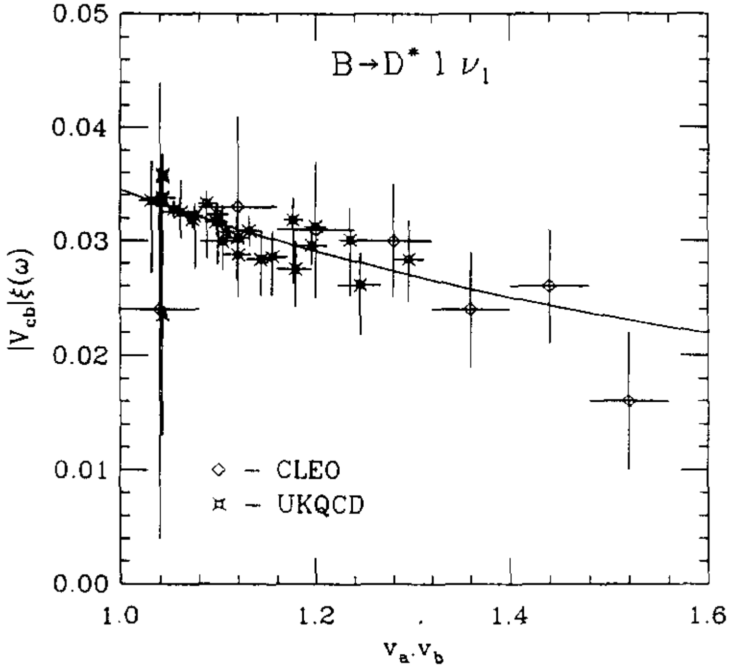

FIG. 10. The Isgur-Wise function calculated on the lattice compared with B decay data from CLEO.

is no obstacle presently known to eventually getting semileptonic decays into comparably good shape (other than the requirement of doing a lot of work).

Most lattice work on this area in the last two years has focused on the calculation of the Isgur-Wise function $\xi(v \cdot v')$ on the lattice. Two approaches have been investigated:

1. Direct formulation of an action for quarks in the $m \to \infty$ limit for finite velocity. (43) (This is analogous to the static approximation for $v = 0$.)

2. Use of light quark methods to calculate D meson elastic scattering (44,45), using

$$\langle D_{v'}|\bar{c}\gamma_\mu c|d_v\rangle = \xi(v \cdot v')(p + p')_\mu + O(\Lambda/m_c). \tag{7}$$

Fig. 10 shows the Isgur-Wise function calculated in the second approach in Ref. (45), compared with B decay data from CLEO. The lattice error bars do not explicitly include uncertainties arising from finite lattice spacing, finite volume, the quenched approximation, or $1/m_c$ effects. $\xi(0) = 0$ has been used to normalize the lattice results. The lattice calculations are not yet accurate enough to determine the curvature of the function, so what is really being calculated is the slope. Using the Stech-Neubert-Rieckert parameterization of the function, the lattice groups quote for the shape parameter $\rho^2 = 1.41(0.19)(0.19)$ (44), and $\rho^2 = 1.6^{+7}_{-6}$ (45), which are compatible results from sum rules and from fitting the shape of the data directly. The values of $|V_{cb}|$ obtained by the lattice groups are therefore compatible with those obtained from other analyses. Normalizing to a B lifetime of 1.50 ps, they obtain $|V_{cb}|\sqrt{\tau_B/1.50\text{ps}} = 0.044$ in Ref. (44) and $0.043(2)(^{6}_{5})$ in Ref. (45). In the errors quoted in Ref. (45), the first error is experimental, the second is part of the theoretical uncertainty.

CONCLUSIONS

There are now several lattice calculations (B_K, m_b, α_s) for which at least a first attempt has been made to examine all of the largest sources of uncertainty quantitatively. These uncertainty estimates are not yet on a par with the analysis of $g-2$ for the electron (although eventually they should be), but they are quite competitive with the analysis of theoretical uncertainties in short distance perturbative QCD processes.

The calculations which are currently best understood are in one way or another special cases, simpler than the generic lattice calculation. However, they and many others of the most interesting phenomenological calculations (decay constants, B parameters, many semileptonic decays) share certain other simplicities, which put them into a class which is likely to be doable over the next few years, assuming only programmatic rather than revolutionary improvements in methods. They involve hadronically stable mesons, either pseudoscalars or heavy quark-antiquark. They involve processes with a single hadron existing at a time.

Baryons and unstable mesons are likely to prove a bit more demanding, though still well within the range of current methods. More demanding still will be processes involving more than one hadron. (Conceptual problems involving final state interactions in imaginary time, as is used in lattice calculations, have yet to be worked out in practical applications.) The most phenomenologically important of these are the hadronic kaon decay amplitudes necessary for the analysis of CP violation in the K system. Farther away still are such things as a full nonperturbative calculation of high energy PP scattering, which are certainly not immediate prospects. Setting our sights even higher, one would like eventually to have lattice methods that worked for chiral gauge theories, so that nonperturbative beyond the standard model physics could be investigated in a reliable and straightforward way. No proposed method for such theories has so far been proven to work. It is not yet known whether this is a result of simple technical difficulties which are unusually complicated, or whether it is an indication of something deep about these theories which has not yet been sufficiently appreciated.

There are many goals ahead of us, not all of which are yet within either our grasp or our reach. On the other hand, most (though not all) of the calculations which are most crucial in extracting the fundamental parameters of the standard model from experiment are in the simplest class of lattice calculations: they involve single, stable mesons. The simplest of these have now been completed with uncertainty estimates. There is a good hope that these uncertainty estimates can be made very solid, and that many more simple but important calculations will join them over the next few years.

ACKNOWLEDGMENTS

I would like to thank many of the authors of work discussed here for correspondence. I would also like to thank C. Bernard, S. Gottlieb, A. S. Kronfeld, G. P. Lepage, G. Martinelli, S. Sharpe, D. Toussaint, and A. Ukawa for helpful discussions.

Fermilab is operated by Universities Research Association under contract with the U. S. Department of Energy.

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