Fermi National Accelerator Laboratory

FERMILAB-Conf-93/028

Charged Particle Optics without Detailed Field Maps

David C. Carey

Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois 60510

February 1993

Proceedings of the *Computational Accelerator Physics Conference 1993*, Pleasanton, California, February 22-26, 1993

Operated by Universities Research Association Inc. under Contract No. DE-AC02-76CHO3000 with the United States Department of Energy

CHARGED PARTICLE OPTICS WITHOUT DETAILED FIELD MAPS

David C. Carey
Fermi National Accelerator Laboratory*
P.O. Box 500, Batavia, Illinois 60510

ABSTRACT

For the initial design of a beam line or charged particle optical system, it is both useful and convenient to be able to describe the components in terms of a small number of parameters. These parameters are used in a calculation of a transfer map which represents the effect of the beam line on a particle trajectory. The transfer map is often expressed as some kind of series expansion. A calculation to first order requires the smallest number of descriptive parameters. Extension of the calculation to higher orders requires a greater number of parameters.

From our mathematical backgrounds we have come to have certain expectations as to the characteristics of a series expansion. These expectations may not always be commensurate with the physics of charged particle beam lines. The reconciliation of these expectations will be discussed.

The example used will be the program TRANSPORT and its extension to third order. The third-order expansion may represent the inherent limit of the series representation without numerical integration. We shall explain why we may have reached that limit.

INTRODUCTION

The effect of a beam line on the trajectory of a charged particle [1] is often described in terms of a transfer map. The transfer map can take the form of a multivariable Taylor series expansion [2] of TRANSPORT [3] or the symplectic maps of Alex Dragt and his collaborators [4]. In most of what follows, the exact nature of the map is not important. The details of the ensuing description will be described in terms of the multivariable Taylor series formalism, but most of the conclusions will be equally valid for other formalisms as well.

From our introductory courses in calculus that we took as undergraduates we came to have certain expectations about series expansions. The application of series expansions to charged particle optics gives us additional expectations. Finally, the desire that the mathematical formalism should produce useful results rounds out the set. A short list of the simplest of our expectations might be:

- the expansion converges rapidly
- The lower order coefficients should not depend on whether the higher orders are calculated
- The coefficents can be calculated analytically

The rapid convergence of the series expansion is a mathematically useful characteristic, since it means that the number of terms that must be calculated is relatively small. The use of the reference trajectory for the origin of the expansion facilitates rapid convergence. The helpful fact is that the origin of the series lies within the population of trajectories we wish to follow through the system. If, for example, the transformation of trajectories were to be expanded in absolute floor coordinates, the series would converge much less rapidly.

Rapid convergence of the series representation also describes a physical characteristic of the beam lines that physicists typically want to design. The elimination of nonlinearities is the most common use for correcting elements. A typical class of nonlinearities would be the chromatic aberrations. Sextupoles and octupoles can be used for correcting elements so that, as nearly as possible, all momenta can have the same focusing characteristics.

The invariance of the lower-order coefficients to whether the higher-order terms are calculated is a characteristic of expansions that we have come to take for granted. In elementary mathematics it is true of Taylor series and of expansions in orthogonal polynomials. It is not true of least squares fits. In the following discussions, we shall show that it is not necessarily an attainable ideal in charged particle optics.

That the coefficients can be calculated analytically is a fundamental prerequisite to the concept of transfer maps. If the coefficients have to be calculated numerically, then what we are doing is ray tracing and subsequently constructing transfer maps from the results of the ray tracing. Such a procedure may be useful in some instances. However, it is no longer the usual first step of describing a beam line in terms of a small number of parameters and then deriving a transfer map directly. It is more of a hybrid approach where the concept of the transfer map is used to characterize the effects of ray tracing.

For purposes of calculation, a charged particle optical system can be broken up into sections. These sections can be classified as being of two types:

1. translationally invariant fields
2. fringe fields

Below we shall show how other configurations are possible. We shall argue that inclusion of other types of field will violate our basic expectations for series expansions.

Our notation will be such that the two transverse directions are x and y, with y being vertical. The longitudinal direction is z if rectilinear and s if curvilinear. The complete six-component vector of trajectory coordinates is then:

$$(x, x', y, y', \ell, \delta)$$

The prime indicates differentiation with respect to s. The longitudinal coordinate is neither z nor s because here it represents the difference between the individual

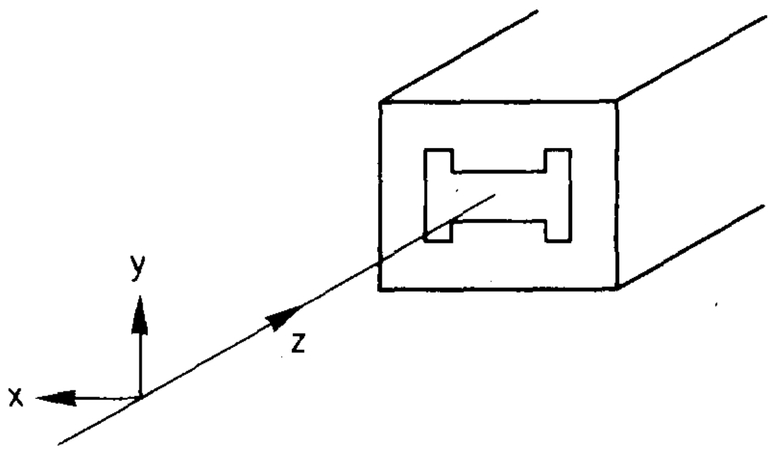

Figure 1: The trajectory coordinate system at the entrance to a magnet.

trajectory and the reference trajectory. The symbol ℓ represents the longitudinal separation, and δ the fractional momentum difference. The complete vector is sometimes denoted by the vector symbol $\mathbf{X}$, and the individual components by x_i. The coordinate system at the entrance to a magnet is shown in figure 1.

With these essentials established, we begin our analysis of the different types of field region.

TRANSLATIONALLY INVARIANT FIELDS

Standard Beamline Components

The interior fields of the standard multipole components are translationally invariant along the reference trajectory. These components include the combined function bending magnet, the quadrupole, the sextupole, the octupole, and the solenoid. For the translationally invariant fields we can add another expectation to our list.

- The transfer matrix should segment longitudinally

This statement means that we can arbitrarily choose a point on the reference trajectory interior to the magnet. The portions of the magnet before and after this point constitute two shorter magnets. Transfer maps can be calculated separately for these shorter magnets. When combined the two transfer maps should then be identical to the transfer map for the entire magnet calculated as a whole. This statement will be true only to the order to which all the maps are calculated. If, for example, the transfer maps are all calculated to second order, the result of multiplying the two transfer maps will produce terms of third and fourth order.

These terms will not be found in the transfer map for the entire magnet considered as a whole, unless this transfer map is also calculated to third or fourth order.

The differential equations of motion of the particle trajectory coordinates take the form:

$$x_i'' + k^2 x_i = f_i \tag{1}$$

where f is known as the driving term. The term f contains all the nonlinearities in the trajectory coordinates. It can be expanded to third order as:

$$f_i = \sum_j D_{ij}x_j + \sum_{jk} E_{ijk}x_j x_k + \sum_{jk\ell} F_{ijk\ell}x_j x_k x_\ell \tag{2}$$

These equations are solved by iteration. A solution is first found to the homogeneous equation, where f_i is set to zero. Each iteration then raises by one the order of the solution. The solution found is substituted into the right side of equation (2) and solved again. At a particular stage, the solution takes the form:

$$x_i = \text{homogeneous solution} + \int G(t,\tau)f_i(\tau)d\tau \tag{3}$$

The letter G denotes the Green's function, which is made up out of the solutions of the homogeneous equation.

The integrals can be evaluated analytically if k^2 and the coefficients D, E, F, and their higher-order equivalents are constants. Then, for any order, the terms inside the integral become sums and products of terms of lower order. These terms are, in turn, made up of solutions R_{ij} to the first-order equation and are nothing more than sums and products of trigonometric and hyperbolic functions. These last named functions can be evaluated analytically to give the higher-order transfer matrix elements T_{ijk} and $U_{ijk\ell}$.

The Acceleration Element

The accelerator element is not entirely translationally invariant. The field is translationally invariant, but since the element accelerates, the reference momentum is not constant. As a result, the wavelengths of the trigonometric and hyperbolic functions representing both the transverse and longitudinal motion change continuously as the element is traversed. For many years, the only representation of an acceleration element in TRANSPORT was for a massless particle. This made some kind of sense since TRANSPORT was originated at SLAC, which is an electron laboratory. The massless matrix element possessed the characteristic of longitudinal segmentation, in spite of its simple analytical form.

When this same formulation was applied to the massive particle, it failed miserably. The expressions did not agree at all with the results of numerical integration. The formula also did not come close to satisfying the segmentation test.

Fortunately, the mathematics of particle motion with continuously varying wavelength is already well developed because of quantum mechanics. The WKB method, applied to the acceleration cavity for massive particles, produces results of reasonably high accuracy [5]. Since the technique is perturbative, the expressions are not exact. However, they agree with the results of numerical integration to approximately the accuracy with which the transfer matrices are printed in TRANSPORT.

Magnet Mispowering and Violations of Midplane Symmetry

A bending magnet which is mispowered and/or has skew multipoles [6] in its field can still be translationally invariant. However, the path along which it is translationally invariant may no longer be the path that the reference particle would take in passing through the magnet. It is the path the reference particle would take if the magnet were correctly powered and the skew multipoles were absent.

It is now important to retain the field expansion about the path along which the translational invariance occurs. The integration will also be done along this path. Only then can the integrals described in the last section continue to be tractable analytically. Otherwise the field expansion will have to be paramaterized as a function of distance along the new trajectory. This parameterization will be an approximation, possibly to something which already has a known analytic form. The integrals will contain terms from this parameterization and possibly have to be evaluated numerically.

We will have lost one of the big advantages of the transfer map formalism. For this reason, even the skew dipole must be included in the skew field expansion. A combined function bending magnet with a skew dipole component cannot be rotated to eliminate that skew dipole component.

The coefficients D, E, and F, on the right side of equation (3) will have contributions from the skew components. The equations of motion will become considerably more complicated. Since sums and products of terms are used in making up higher-order terms, the number of terms in the expressions will considerably more than double. Since some of the terms are of first order (the skew dipole and quadrupole terms), iteration will continue to alter terms of all orders. It is clear that we need some criterion to terminate the iteration.

One criterion is that the matrix elements calculated be only linear in the skew fields. This is a reasonable approximation if the skew fields are small. By small we mean that the effect of quadratic or higher-order terms in the skew components is negligible for whatever purpose the beam line is being used for.

Smallness is indeed typically a characteristic of skew fields. Skew fields are present either through error of fabrication or for steering or correction purposes. These reasons all require small fields in well-designed beams. If the inherently skew fields need to be large, the beam designer should reconsider the beam configuration.

Finally, we should emphasize the difference between a magnet with skew multipoles and a rotated midplane-symmetric magnet. A magnet can often be rotated

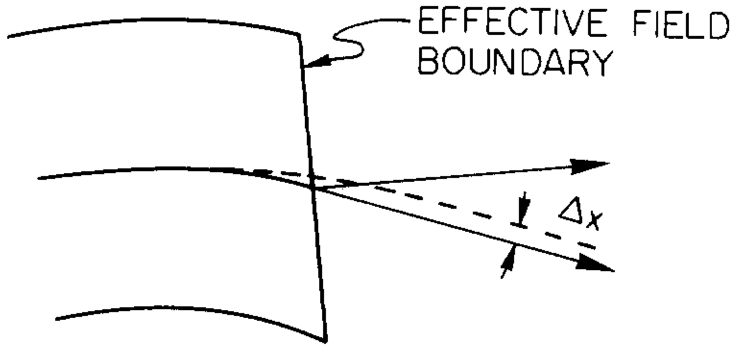

Figure 2: The parallel displacement of the reference trajectory in the fringe field of a dipole magnet

by a large angle. One possible purpose for rotating a set of quadrupoles might be to exchange horizontal and vertical phase space. A reason for rotating a dipole might be to provide vertical bending.

The field of such magnets, if analyzed in the external coordinate system, might be said to have large skew components. However, such magnets do have a magnet midplane about which the field is symmetric. About this midplane the field can then be expanded entirely in terms of symmetric multipoles.

A magnet where skew components are required for the field representation has no single plane of symmetry. In a multipole expansion about a reference curve, each multipole will have its own plane of symmetry. However, these planes for different multipoles will not necessarily coincide and there will be no necessary single magnetic midplane.

FRINGE FIELDS

The Reference Trajectory

In what is now a classic paper [7], Harald Enge derives the effect of an extended fringe field on the reference trajectory and on the first-order transfer matrices. His derivation is for a pure dipole field, where the interior of the magnet has no quadrupole or higher-order multipole terms. An equivalent sharply-cut-off field may be defined where the field integral is the same as for the extended fringe field case. The effect of the extended fringe field is to cause a parallel displacement of the reference trajectory compared to the case for a sharply cut off field. This parallel displacement is illustrated in figure 2.

The displacement is parallel because the angle by which the trajectory is deflected is proportional to the integral of the magnetic field traversed. Since the magnetic field is independent of the horizontal transverse coordinate the field integral does not depend on whether the field ends suddenly or is extended. The displacement of the reference trajectory is given by:

$$\Delta\xi = \sec^2\beta\frac{g^2}{\rho}I_1 \tag{4}$$

where β is the pole face rotation angle, g is the magnetic gap, and ρ is the radius of curvature of the reference trajectory in the interior field. The integral I_1 is sometimes called a "form factor". It is given by

$$I_1 = \int_{-z_1}^{\infty}\int_{-z_1}^{z}\frac{B_y^o - B_y}{g^2 B^o}dz'dz \tag{5}$$

Matsuda et al. consider the case of a transformation through the fringe field of a combined-function bending magnet [8]. For a combined function bending magnet, the field strength is dependent on the horizontal transverse coordinate. The transverse displacement due to the extended fringe field causes the reference trajectory to get into a region where the field strength differs from the value on the reference trajectory for the sharply- cut-off field. The angle of deflection is then different, and the reference trajectory is not parallel to what it had been in the sharply-cut-off case.

In other words, there is no simple field integral which can be used to determine the deflection of the reference trajectory. Matsuda et al define what they call the "ideal field boundary for normal incidence." It is the boundary of a sharply-cut-off field that does give the same angular deflection as occurs in the reference trajectory with the extended fringe field. I believe that they have done as well as is possible in a difficult situation. However, the inevitable conclusion is that even the determination of the physical reference trajectory may require detailed ray tracing.

Form Factors

Getting back to Enge's work, he also derives the first-order transformation through an extended fringe field for a purely dipole interior field. In this case, one additional form factor is needed. It is given by:

$$I_2 = \int_{-z_1}^{\infty}\frac{B_y(B_o - B_y)}{gB_o^2}dz \tag{6}$$

It is used in the modification of the vertical focusing strength for an extended fringe field. In the case of a sharply-cut-off field, the horizontal and vertical focusing strengths of the magnet entrance or exit are equal and opposite. In the case of an extended fringe field region they are not.

As the order of the calculation is increased, the number of form factors necessarily also increases. For a given order, the number of form factors also increases with the detail to which the calculation is done. The level of detail roughly corresponds to the number of times the equations of motion are iterated. We shall discuss this at greater length below.

The form factors, of course, depend on the detailed shape of the fringe field profile. They can all be derived from a simple field model to give order-of-magnitude estimates on the effect of the extension of the fringe field. However, as the order and accuracy of the calculation are increased the exact values of the form factors become more important. Eventually we reach the point where the parameterization contains as much information as a detailed field map. The combined procedure of evaluating the form factors and transforming trajectories then becomes tantamount to numerical ray tracing.

Consequences of Maxwell's Equations

We have seen that a model of the extended fringe field is required in order to be physically realistic. We shall now investigate the requirements on the fringe field imposed by the mathematics of the situation. We shall, in this section, discuss only the case of the pure dipole field, where the interior field in the magnet is uniform.

From Maxwell's equations we can relate the derivatives of the magnetic field components:

$$\frac{\partial B_y}{\partial z} = \frac{\partial B_z}{\partial y} \tag{7}$$

The first term, $\frac{\partial \mathbf{B_y}}{\partial \mathbf{z}}$ is evaluated on the magnetic midplane. The field in the vertical direction B_y is the main bend field. It is the component which, when plotted against z, gives the drop off in the field as the magnet is exited.

The horizontal component of the magnetic field has the effect of vertical focusing. The field component B_z seems at first to be a longitudinal component. However, in this case, the z direction is taken perpendicular to the magnet pole face. If the magnet pole face is rotated, then B_z will have a component transverse to the beam. This component will be proportional to $\tan\beta$, where β is the angle by which the magnet pole face is rotated.

By symmetry the component B_z is zero on the magnetic midplane. The field value itself is given by

$$B_z = \Delta y \frac{\partial B_z}{\partial y} = \Delta y \frac{\partial B_y}{\partial z} \tag{8}$$

The deflection per unit length is proportional to the field strength. The total deflection is calculated by integrating longitudinally through the fringe field region. The derivative with respect to z integrates out and the deflection is simply proportional to the difference in field strength B_y in the interior and exterior. In the first iteration, at least, it does not matter if the field is sharply cut off.

To calculate the secord-order fringe-field transmission characteristics, we need to determine the consequences of Maxwell's equation on the second derivatives of the field components:

$$\frac{\partial^2 B_y}{\partial z^2} = -\frac{\partial^2 B_y}{\partial y^2} \tag{9}$$

The longitudinal rate of fall off of the vertical dipole field is now related to the horizontal transverse rate of change of the same component. This component affects the horizontal focusing of the beam. The change in vertical strength due to the second derivative is:

$$\Delta B_y = \frac{1}{2}(\Delta y)^2 \frac{\partial^2 B_y}{\partial y^2} = \frac{1}{2}(\Delta y)^2 \frac{\partial^2 B_y}{\partial z^2} \tag{10}$$

Integrating again longitudinally through the fringe field region, we get that the angular deflection due to this term is proportional to the difference in the first derivative $\frac{\partial B_y}{\partial z}$ inside and outside the magnet. Since the field is asymptotically constant both in the interior and exterior, the first derivative is zero in both regions. Hence, in second order a sharply-cut-off field can be used to calculate transfer matrix elements.

In third order, we must consider the field third derivatives. Maxwell's equations give us:

$$\frac{\partial^3 B_y}{\partial z^3} = -\frac{\partial^3 B_z}{\partial y^3} \tag{11}$$

Here, as in first order, the influence is on the vertical focusing when the pole face is rotated. As with the two lower orders, we can integrate the field third derivative through the fringe field region to get the difference of two second derivatives. The two second derivatives are zero, and the effect on the focusing would seem to vanish.

However, in this case, the order of the partial derivative is so high that its effect does not entirely integrate out. In the course of traversing the fringe field region, the direction of the reference trajectory changes. This in turn is reflected by a change in angle between the field component B_z and the velocity vector. The combination of these terms gives a contribution which is proportional to the form factor

$$\frac{1}{B_o^2} \int_{-z_1}^{z_2} \left(\frac{dB_y}{dz} \right)^2 dz \tag{12}$$

This form factor is not finite in the sharply-cut-off case. In order that the third-order matrix elements be finite, the function describing the vertical field component B_y must have a finite derivative almost everywhere. A linear fall off of the field strength in the fringe field region will produce finite third-order matrix elements.

As the order of the calculation is further increased, additional continuity conditions will be required for the higher-order transfer matrix elements to be finite. Ultimately the field will need to possess finite derivatives of all orders. The fact that the field model must become more refined as the order of the calculation is increased means that the lower orders will be affected also. Thus we are in seeming violation of one of our basic principles, namely that the lower order coefficients should not depend on whether the higher orders are calculated. The only solution

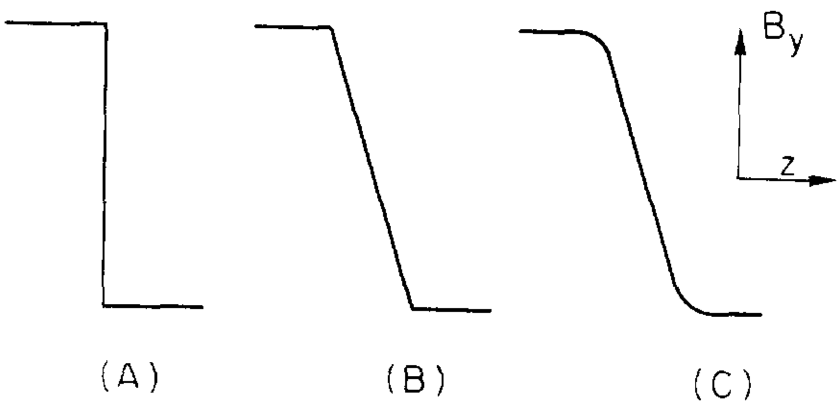

Figure 3: Three possible fringe field profiles: (A) A sharply-cut-off field where the field strength is discontinuous. (B) A linear fall off where the field strength is continuous. (C) A curved profile where the derivative of the field strength is continuous.

is to use at the outset a field representation which satisfies Maxwell's equations.

Relations Among Orders

There is still another reason why the calculation of a given order may affect lower orders. The equations of motion are solved by iteration. In the interior field of a magnet, at each iteration the lowest order affected is increased by one. Such a mathematical convenience does not hold for the fringe field. Here even the reference trajectory is determined by an iterative procedure.

In order to identify singularities and to better understand the iterative procedure, Sagalovsky [9] has developed a formalism based on an expansion in the quantity ϵ. The quantity ϵ is defined as the ratio of the separation g of magnetic poles to the interior radius of curvature ρ of the reference trajectory. The transverse coordinates are then all normalized to ρ while the longitudinal coordinate is normalized to g. The normalized coordinates are denoted by a bar over the letter, such as $\bar{z}$ for $\frac{z}{g}$. The effect of the iteration procedure is then such that the lowest order terms in ϵ that are changed increase by one with each iteration. The differential equations to be solved for the second-order transfer matrix elements are:

$$\begin{aligned} \frac{d}{d\bar{z}} T_{1ij} &= \epsilon T_{2ij} \qquad (13) \\ \frac{d}{d\bar{z}} T_{2ij} &= -3\epsilon \Gamma_1^2 \Gamma_2 h T_{2ij} - f_{2ij} \end{aligned}$$

$$
\begin{aligned}
\frac{d}{d\bar{z}}T_{3ij} &= \epsilon T_{4ij} \\
\frac{d}{d\bar{z}}T_{4ij} &= -\epsilon\Gamma_1^2\Gamma_2\frac{d}{d\bar{z}}(hT_{3ij}) + f_{4ij}
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_1 &= (1+\Delta^2)^{\frac{1}{2}} \\
\Gamma_2 &= \frac{\Delta}{(1+\Delta^2)^{\frac{1}{2}}} \\
\Delta &= -\frac{h}{\rho}(1+\Delta^2)^{\frac{3}{2}} \\
\Delta(z) &= \epsilon z\tan\beta - \epsilon^2\sec^3\beta\int_{z_1}^{z}\int_{z_1}^{z'} h(z'')dz''dz'
\end{aligned}
\tag{14}
$$

Here h is the ratio of vertical magnetic field $B_y(z)$ at a given point in the fringe field to its value B_y^0 in the interior field. The terms denoted by the letter f are termed "driving terms" and are tabulated by Sagalovsky.

We want the solution to be at least of order zero in the parameter ϵ. The zero-order terms are those which do not depend on the spatial extent of the fringing field. The equations are solved by iteration. The driving terms f_{ijk} contain products of powers of the first-order transfer matrix elements R_{ij}. The second-order driving terms f_{ijk} contain an expression which is of order ϵ^{-1}. In order to solve the second-order equations to order ϵ^0, we need to include terms of order ϵ in the solution for the first order matrix elements R_{ij}.

The equations to be iterated to obtain the third-order matrix elements are similar to those used in second order. The driving terms are different and have also been tabulated by Sagalovsky.

$$
\begin{aligned}
\frac{d}{d\bar{z}}U_{1ij} &= \epsilon U_{2ij} \\
\frac{d}{d\bar{z}}U_{2ij} &= -3\epsilon\Gamma_1^2\Gamma_2 hU_{2ij} + g_{2ij} \\
\frac{d}{d\bar{z}}U_{3ij} &= \epsilon U_{4ij} \\
\frac{d}{d\bar{z}}U_{4ij} &= -\epsilon\Gamma_1^2\Gamma_2\frac{d}{d\bar{z}}(hU_{3ij}) + g_{4ij}
\end{aligned}
\tag{15}
$$

These driving terms now contain a term of order ϵ^{-2}. The driving terms contain both the first- and second-order matrix elements. Again we wish to obtain a solution which is of at least zeroeth order in ϵ. Now we are required to include terms of order ϵ in the second-order matrix elements T_{ijk} and of order ϵ^2 in the first-order matrix elements R_{ij}.

We see that in general the extension of the calculation to each new order in the transverse trajectory coordinates requires an extension to a higher order in

ϵ for all the lower orders in the trajectory coordinates. We find again that the lower-order coefficients will depend on whether the higher orders are calculated. In this case, the preceding statement is true even if the fringe field representation is perfectly accurate from the outset of the calculation.

Multipole Equivalents

The final blow to our conventional notions about how mathematics can be used to solve physical problems comes from the consideration of multipole equivalents. A bending magnet fringe field can be configured so that optically it acts as a multipole element. If the bending magnet itself is combined function, so that it has multipole components in its interior field, then multipoles can also arise from the combined effect of the interior and fringing fields. The multipole components can be evaluated for a sharply-cut-off field, even when other parts of the transfer matrix would diverge.

An equivalent quadrupole component is produced by a flat but rotated pole face. The transformation does not depend on the characteristics of the interior field of the magnet. The calculation is performed in a coordinate system where two of the coordinates are in the face of the magnet and the third is perpendicular to it. There is no curvature to consider.

In second order, an equivalent sextupole component can be produced in either of two ways. The first and more straightforward is simply a curvature of the pole face. The effect of this curvature is independent of any nonuniformities of the interior field of the magnet. It also does not enter into the expressions for the first-order transformation. Its effect may be calculated by using a curvilinear coordinate system where one coordinate is along the curved pole face of the magnet.

The second sextupole equivalent arises from the combined effect of the transverse gradient of the interior field and the rotation angle of the pole face. The gradient of the interior field is measured in the curvilinear interior coordinate system, where one of the coordinates is along the reference trajectory. Thus we have to reconcile a curvilinear coordinate system with a pole face which is not perpendicular to the longitudinal coordinate at its intersection with the reference trajectory. Here we have the beginning of a bit of complication.

In third order, we have three ways of producing an octupole equivalent. The first, of course, is to give a cubic dependence to the pole face profile. Once again, the effect of this cubic dependence is independent of any inhomogeneities of the interior field of the bending magnet. It also does not affect any lower order. The system where one coordinate runs along the pole face now becomes a bit complicated, but is still analytically representable by a series expansion in rectilinear coordinates.

The second and third methods arise from combinations of pole face characteristics with those of the interior field. The second contribution of an octupole effect is due to the combined effect of the pole-face curvature and the linear dependence of the interior field. The third is due to the combined effect of the pole face rotation with the second transverse derivative of the interior field.

We shall first discuss the third method of producing an octupole equivalent since it is similar to one of the second order case. That second order case is the combined effect of the pole face rotation and the linear transverse variation of the interior field. The interior curvilinear system plus the pole face rotation again gives us a hint of impending complication.

The second contribution, mentioned above, to the effective octupole involves the reconciliation of two curvilinear coordinate systems. The effect of the pole face curvature falls off asymptotically in the interior or the magnet. However, there is no reason to think that the quadratic dependence of the interior field falls off asymptotically exterior to the magnet any more rapidly than does the dipole component. There is therefore no geometrical simplification for the field dependence. In third order the construction of a transfer map through an extended fringe field region inevitably points us in the direction of ray tracing through a detailed field map.

REFERENCES

[1] David C. Carey, *The Optics of Charged Particle Beams*, Harwood Academic Publishers, New York, 1987.

[2] Karl L. Brown, *A First- and Second-Order Matrix Theory for the Design of Beam Transport Systems and Charged Particle Spectrometers*, SLAC Report No. 75, Rev. 4, 1982.

[3] Karl L. Brown, Frank Rothacker, David C. Carey, and Christoph Iselin, *TRANSPORT, A Computer Program for Designing Charged Particle Beam Transport Systems*, SLAC Report No. 91, Rev. 2, 1977.

[4] Alex J. Dragt, "Lectures on Nonlinear Orbit Dynamics," Physics of High Energy Particle Accelerators, AIP Conference Proceedings No. 87, p. 147-313; Alex J. Dragt, Filippo Neri, Govindan Rangarajan, David R. Douglas, Liam M. Healy, and Robert D. Ryne, "Lie Algebraic Treatment of Linear and Nonlinear Beam Dynamics," Annual Review of Nuclear and Particle Science 38, 455-496 (1988).

[5] David C. Carey, "The WKB Approximation and the Travelling-Wave Acceleration Cavity," Record of the 1991 IEEE Particle Accelerator Conference, 1618.

[6] David C. Carey, "Two Dogmas of Charged Particle Optics," Proceedings of the Second International Conference on Charged Particle Optics, 417; David C. Carey, "Violation of Midplane Symmetry in Bending Magnets," IEEE Transactions on Nuclear Science, NS-32, 2255 (1985).

[7] Harald Enge,"Deflecting Systems," Focusing of Charged Particles, ed, Albert Septier, Academic Press, New York, 1967.

[8] H. Matsuda and H. Wollnik, "The Influence of an Inhomogeneous Magnetic Fringing Field on the Trajecories of Charged Particles in a Third Order Approximation," Nuclear Instruments and Methods 77, 40 (1970); H. Matsuda and H. Wollnik, "Third Order Transfer Matrices of the Fringing Field of an Inhomogeneous Magnet," Nuclear Instruments and Methods 77, 283 (1970); T. Sakurai, T. Matsuo and H. Matsuda, "The Vertical Component of an Ion Trajectory in a Homogeneous Magnet to a Third-Order Approximation," International Journal of Mass Spectroscopy and Ion Processes 91, 51 (1989).

[9] Leonid Sagalovsky, *Third-Order Charged Particle Beam Optics*, University of Illinois Doctoral Thesis, 1989.

*Operated by Universities Research Association, Inc., under contract with the United States Department of Energy.