



# Fermi National Accelerator Laboratory

FERMI-PUB-92/383-T  
UMHEP-384

## EXACT PATH INTEGRALS BY EQUIVARIANT LOCALIZATION

Hans M. Dykstra

*Dept. of Physics and Astronomy  
University of Massachusetts  
Amherst, MA 01003*

Joseph D. Lykken and Eric J. Raiten

*Theory Group, MS106  
Fermi National Accelerator Laboratory  
P.O. Box 500, Batavia, IL 60510*

### Abstract

It is a common belief among field theorists that path integrals can be computed exactly only in a limited number of special cases, and that most of these cases are already known. However recent developments, which generalize the WKB method using equivariant cohomology, appear to contradict this folk wisdom. At the formal level, equivariant localization would seem to allow exact computation of phase space path integrals for an arbitrary partition function! To see how, and if, these methods really work in practice, we have applied them in explicit quantum mechanics examples. We show that the path integral for the 1-d hydrogen atom, which is not WKB exact, is localizable and computable using the more general formalism. We find however considerable ambiguities in this approach, which we can only partially resolve. In addition, we find a large class of quantum mechanics examples where the localization procedure breaks down completely.



## 1. Introduction

The major failing of the path integral method applied to quantum field theory or quantum mechanics is that most physically relevant path integrals cannot be computed. The known exceptions fall roughly into three categories. There are the Gaussian forms, which are the basis of perturbation theory in quantum field theory. There are also certain exactly solvable models, usually quantum mechanical integrable systems. In such cases the path integral is often computable due to the fact that the WKB approximation is exact. Thus the path integral localizes to a sum over classical trajectories. The third category is path integral representations of topological quantities such as the Euler character or the Witten index. Here one often finds that supersymmetry in the path integral implies an equivariant cohomology structure, which localizes the path integral onto zero modes or characteristic classes.

As emphasized in [1, 2, 3], the common denominator of most exactly computable path integrals is equivariant cohomology and localization. This raises the possibility that by studying these methods directly we may greatly expand our inventory of doable path integrals, as well as WKB-like approximation schemes. Though equivariant cohomology is usually associated with supersymmetric theories, it actually has very general application. The key ingredient required is some sort of differential form structure among the physical, auxiliary, or ghost variables. This is a rather weak requirement. In fact, as recently shown by Blau, Keski-Vakkuri, and Niemi[1], and by Niemi and Tirkkonen[4], equivariant cohomology can be exhibited in an *arbitrary* phase space path integral. Further, they showed formally that under seemingly weak conditions, this results in equivariant localization of these path integrals. They would thus be exactly computable!

Of course, we do not really expect arbitrary phase space path integrals to be exactly computable. Since equivariant localization proofs are very formal, there are ample opportunities for subtle difficulties to creep in. Also phase space path integrals are themselves rather disreputable. As pointed out in [1], there is a well known *false* method for computing arbitrary phase space path integrals via Hamilton-Jacobi theory and (invalid) canonical transformations of the path integral variables[5, 6].

In short, there is no substitute for simple, explicit examples, and that is the subject of this paper. In section 2. we briefly review equivariant cohomology and localization formulae for path integrals. In section 3. we make the formalism explicit by considering the phase space path integrals for the free particle and the harmonic oscillator. We use equivariant localization to write the partition function of a one-dimensional quantum mechanical system with an *arbitrary* static potential as an elementary contour integral. Along the way we see that this derivation breaks down for essentially all potentials which are bounded below.

In section 4. we consider the 1-D hydrogen atom, whose potential is not bounded below. Its path integral is not WKB exact, and is not amenable to Morse theory analysis since its classical trajectories all coalesce at two points in phase space. Nevertheless we show that it is exactly computable via equivariant localization, modulo an important ambiguity which we discuss.

## 2. Equivariant Cohomology and Localization

Equivariant cohomology is a simple and powerful generalization of ordinary de Rham cohomology and the calculus of differential forms (for details, see [7, 8]). To describe differential forms with respect to some  $d$ -dimensional manifold  $M$ , it is convenient to introduce a contravariant vector of anticommuting Grassmann variables  $c^a$ . Then differential forms can be represented as covariant tensor functions on  $M$  contracted with the  $c^a$ 's:

$$\begin{aligned}
 0 - \text{forms} : & \quad \phi(x), \\
 1 - \text{forms} : & \quad \phi_a(x)c^a, \\
 2 - \text{forms} : & \quad \phi_{ab}(x)c^a c^b, \\
 & \quad \text{etc..}
 \end{aligned} \tag{2.1}$$

The exterior derivative operator  $d$  can be written

$$d = c^a \frac{\partial}{\partial x^a}. \tag{2.2}$$

It takes  $n$ -forms to  $n+1$ -forms and is nilpotent:  $d^2 = 0$ . De Rham cohomology is the classification of closed forms (forms which are annihilated by  $d$ ) modulo exact forms (which are  $d$  of something).

To define equivariant cohomology, we first introduce a new operation,  $\iota_\chi$ , which is interior multiplication with respect to an (arbitrary) vector  $\chi^a(x)$ . Explicitly:

$$\iota_\chi = \int dc_a \chi^a \tag{2.3}$$

Clearly  $\iota_\chi$  takes an  $n$ -form to an  $n-1$  form:

$$\begin{aligned}
 \iota_\chi \phi(x) &= 0, \\
 \iota_\chi \phi_a(x)c^a &= \phi_a \chi^a, \\
 \iota_\chi \phi_{ab}(x)c^a c^b &= \phi_{ab}(\chi^a c^b - \chi^b c^a).
 \end{aligned} \tag{2.4}$$

Furthermore  $\iota_\chi$  is nilpotent.

We can now define an equivariant exterior derivative  $d_\chi$  by

$$d_\chi = d + \iota_\chi. \tag{2.5}$$

Note that  $d_\chi$  takes an  $n$ -form to both an  $n+1$ -form and an  $n-1$  form. Now,  $d_\chi$  is not generally nilpotent:

$$d_\chi^2 = (d\iota_\chi + \iota_\chi d) = \mathcal{L}_\chi, \tag{2.6}$$

where  $\mathcal{L}_\chi$  is the Lie derivative with respect to the vector  $\chi^a(x)$ . Thus  $d_\chi$  is nilpotent only on the subspace of differential forms which are annihilated by the Lie derivative  $\mathcal{L}_\chi$ .

Without going into more rigorous mathematical detail, we are already in a position to sketch the general derivation of equivariant localization formulae. Suppose we want to compute an integral over the manifold  $M$  of some general form  $\alpha$ , where by a general form we mean a linear combination of different  $n$ -forms (including a top-form):

$$\int_M \alpha \quad (2.7)$$

In order to apply equivariant localization, we must be able to find a vector  $\chi^a$  such that, with respect to  $\chi^a$ ,  $\alpha$  is equivariantly closed:

$$d_\chi \alpha = 0. \quad (2.8)$$

In addition, we must be able to find some other differential form  $\beta$  which is in the subspace annihilated by  $\mathcal{L}_\chi$ :  $\mathcal{L}_\chi \beta = 0$ , and is not equivariantly exact. When this can be done, one constructs the following modified integral:

$$\int_M \alpha e^{-\lambda d_\chi \beta}, \quad (2.9)$$

where  $\lambda$  is a numerical parameter. It is then straightforward to show that, formally, this modified integral is independent of the value of  $\lambda$ :

$$\begin{aligned} \frac{d}{d\lambda} \int_M \alpha e^{-\lambda d_\chi \beta} &= -\lambda \int_M (d_\chi \beta) \alpha e^{-\lambda d_\chi \beta} \\ &= \lambda \int_M \beta d_\chi (\alpha e^{-\lambda d_\chi \beta}) \\ &= 0. \end{aligned} \quad (2.10)$$

In the above we used  $d_\chi \alpha = 0$ ,  $d_\chi d_\chi \beta = 0$ , and the fact that the integral of an equivariantly exact form vanishes.

Thus, provided the limits  $\lambda \rightarrow 0$  and  $\lambda \rightarrow \infty$  exist, we obtain the following localization formula:

$$\int_M \alpha = \lim_{\lambda \rightarrow \infty} \int_M \alpha e^{-\lambda d_\chi \beta}. \quad (2.11)$$

The original integral over  $M$  has been localized to a subspace  $M_\chi$ , which is the support for the nontrivial equivariant cohomology ( $d_\chi \beta = 0$ ).

When applied to ordinary (finite dimensional) integrals over compact manifolds, the above arguments can be made rigorous, and the right hand side of (2.11) (for suitable choice of  $\beta$ ) reduces to the WKB formula. This is the Duistermaat-Heckman theorem.

At the formal level, this result was extended in [1] to apply to general phase space path integrals. Consider a general bosonic quantum mechanical phase space path integral:

$$\dot{Z}(T) = \int [d(\text{Liouville})] e^{iS} = \int [d\phi^a] \sqrt{\det \|\omega_{ab}\|} \exp i \int_0^T dt (\theta_a \dot{\phi}^a - H(\phi)), \quad (2.12)$$

where  $\phi^a(t)$  are the coordinates of phase space (satisfying periodic boundary conditions),  $\theta_a(\phi)$  are their conjugates,  $H$  is the hamiltonian, and  $\omega_{ab}$  is an antisymmetric tensor giving the symplectic 2-form whose matrix inverse defines Poisson brackets:

$$\omega_{ab} = \partial_a \theta_b - \partial_b \theta_a . \quad (2.13)$$

Introducing real Grassmann auxiliary coordinates  $c^a(t)$ , which also satisfy periodic boundary conditions, we can write:

$$Z(T) = \int [d\phi^a][dc^a] \exp i \int_0^T dt \left( \theta_a \dot{\phi}^a - H(\phi) + \frac{1}{2} c^a \omega_{ab} c^b \right) . \quad (2.14)$$

We can then identify loop space differential forms as tensor functionals of the  $\phi^a(t)$ 's contracted with the  $c^a(t)$ 's. There are two vectors of obvious significance for defining equivariant operations. One is the hamiltonian vector field, which is defined by the relation:

$$\partial_a H(\phi) = \omega_{ab} \chi_H^b . \quad (2.15)$$

The other is  $\dot{\phi}^a$ , since  $\dot{\phi}^a \partial_a$  generates time translations on our differential forms.

In [1], equivariant cohomology is defined with respect to the vector

$$\chi_S^a(t) = \dot{\phi}^a(t) - \chi_H^a(\phi(t)) , \quad (2.16)$$

since, as one can easily verify,  $d_S S = 0$ . Furthermore, the zeroes of  $\chi_S^a(t)$  correspond to the classical trajectories. Thus, if we shift the action by  $d_S$  of a 1-form  $\beta$

$$S \rightarrow S + \lambda d_S \beta , \quad (2.17)$$

and choose  $\beta$  proportional to  $\chi_S^a$ , we can expect to obtain a WKBJ localization of  $Z(T)$ . Since both  $\chi_S^a$  and  $c^a$  are contravariant, we cannot construct such a  $\beta$  without also introducing a metric on loop space,  $G_{ab}(t_1, t_2)$ . Then we may define  $\beta$  as

$$\beta = \int dt_1 dt_2 G_{ab}(t_1, t_2) \chi_S^a(t_1) c^b(t_2) . \quad (2.18)$$

Recalling that the localization argument requires  $\mathcal{L}_S \beta = 0$ , one finds that this reduces to the condition

$$\mathcal{L}_S G_{ab}(t_1, t_2) = 0 . \quad (2.19)$$

In other words,  $\chi_S^a$  must be a Killing vector of the loop space metric. It is easy to see that  $\mathcal{L}_{\dot{\phi}^a}$  automatically annihilates  $G_{ab}(t_1, t_2)$  provided it takes the form

$$G_{ab}(t_1, t_2) = g_{ab}(\phi(t)) \delta(t_1 - t_2) . \quad (2.20)$$

Then (2.19) reduces to the constraint

$$\mathcal{L}_H g_{ab}(\phi) = 0 . \quad (2.21)$$

In other words,  $\chi_H^a$  must be a Killing vector of the finite dimensional metric  $g_{ab}$ .

As shown in [1], the formal procedure just outlined results in a WKB localization of the path integral (2.12). Thus in the case where these formal steps actually carry through, the path integral is WKB exact. As already mentioned, there are a number of known examples where this occurs. This is well-trodden territory, and we have not attempted to explore it further.

Instead, we turn to the recent paper by Niemi and Tirkkonen[4], in which they apply a *different* equivariant localization scheme to the general phase space integral (2.12). The difference lies in the choice of the 1-form  $\beta$ : instead of (2.18) they choose

$$\beta = \frac{1}{2}g_{ab}\dot{\phi}^a c^b, \quad (2.22)$$

where we suppressed the  $t$  integrations which accompany index contractions. The equivariant localization procedure is otherwise the same as described above. One then obtains a localization of  $Z(T)$  not onto classical trajectories, but rather onto the time independent modes of  $\phi^a(t)$  and  $c^a(t)$ . Their result (correcting a minor typo) is

$$Z(T) = \int d\phi_0^a dc_0^a e^{-iTH(\phi_0) + i\frac{1}{2}Tc_0^a \omega_{ab} c_0^b} \sqrt{\text{Det} \left[ \frac{\frac{1}{2}(\Omega_{ab} + R_{ab})}{\sinh \left[ \frac{T}{2}(\Omega_{ab} + R_{ab}) \right]} \right]}. \quad (2.23)$$

In this expression, the determinant is an ordinary matrix determinant, and the integrals are finite dimensional integrals over the time-independent real variables  $\phi_0^a$  and real Grassmann variables  $c_0^a$ . The tensors  $\Omega_{ab}$  and  $R_{ab}$  can be expressed as geometric quantities in terms of the metric  $g_{ab}$  and the hamiltonian vector  $\chi_H^a$  which is a Killing vector of  $g_{ab}$ :

$$\begin{aligned} \Omega_{ab} &= 2\chi_{a;b}^H, \\ R_{ab} &= R_{abcd}c_0^c c_0^d. \end{aligned} \quad (2.24)$$

Now (2.23) is a remarkably simple result. If this localization proved to be valid for some large class of physically relevant partition functions, it would be an exciting and important achievement. Less optimistically, we have here at the very least an alternative to WKB methods which is equally nonperturbative. Further, both WKB and the Niemi-Tirkkonen formula are merely examples of much more general equivariant localization techniques.

### 3. Application to 1-D Quantum Mechanics with Static Potentials

The simplest possible application of (2.23) is to one dimensional quantum mechanics with static potentials. In this case there are only two phase space coordinates,  $x(t)$  and  $p(t)$ , and the Hamiltonian is of the form

$$H = \frac{1}{2}p^2 + V(x). \quad (3.1)$$

For the moment, let us confine ourselves to potentials which are bounded below. Then it is convenient to define ‘‘harmonic’’ and polar coordinates as follows:

$$\begin{aligned} p &= r \sin \theta, \\ V(x) - V_{\min} &= \frac{1}{2}\omega^2 y^2 = r \cos \theta, \\ H &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 y^2 + V_{\min} = \frac{1}{2}r^2 + V_{\min}. \end{aligned} \quad (3.2)$$

Note that we only change variables in the time independent coordinates of the localized integral, never in the original phase space path integral.

The advantage of working in polar coordinates is that the hamiltonian vector field has only one nonvanishing component:

$$\chi^\theta = \chi = -\omega \frac{dy}{dx}, \quad (3.3)$$

and it is straightforward to solve the constraint  $\mathcal{L}_H g_{ab} = 0$ , which is equivalent to:

$$\begin{aligned} \chi \partial_\theta g_{\theta\theta} + 2g_{\theta\theta} \partial_\theta \chi &= 0, \\ \partial_\theta(\chi g_{r\theta}) + g_{\theta\theta} \partial_r \chi &= 0, \\ \chi \partial_\theta g_{rr} + 2g_{r\theta} \partial_r \chi &= 0. \end{aligned} \quad (3.4)$$

The general solution is:

$$\begin{aligned} g_{\theta\theta} &= \frac{f(r)}{\chi^2}, \\ g_{r\theta} &= \frac{f(r)}{\chi^2} \int d\theta \partial_r \left( \frac{1}{\chi} \right) + \frac{f_2(r)}{\chi}, \\ g_{rr} &= \frac{\chi^2}{f(r)} (g_{r\theta})^2 + f_3(r). \end{aligned} \quad (3.5)$$

Note that the metric is not completely determined by the constraint; the general solution (3.5) involves three arbitrary functions of  $r$ :  $f$ ,  $f_2$ , and  $f_3$ . This is not surprising, since we expect that the localized integral (2.23) is at least partially metric independent.

A much worse feature of the general solution (3.5) is that  $g_{r\theta}(r, \theta)$  is not, in general, a single-valued function. To see this, observe that  $g_{r\theta}(\theta=0) = g_{r\theta}(\theta=2\pi)$  only obtains if

$$\partial_r \int_0^{2\pi} \frac{1}{\chi} = 0, \quad (3.6)$$

which is equivalent to

$$\int_0^{2\pi} d\theta \frac{dx}{dy} = \text{constant}. \quad (3.7)$$

The harmonic oscillator gives the only solution of (3.7). Thus we conclude that, for essentially all static potentials which are bounded below, the Niemi-Tirkkonen equivariant localization procedure fails, due to the nonexistence of a single-valued metric satisfying the Lie derivative constraint.

What of the harmonic oscillator, the sole survivor of this calamity? In this case we have

$$\begin{aligned} \chi &= -\omega, \\ g_{\theta\theta} &= \frac{f(r)}{\omega^2}, \quad g_{r\theta} = -\frac{f_2(r)}{\omega}, \quad g_{rr} = \frac{f_2^2}{f} + f_3(r). \end{aligned} \quad (3.8)$$

The antisymmetric tensor  $\Omega_{ab}$  and the curvature tensor  $R_{abcd}$  each have only one independent component:

$$\Omega_{\theta r} = -\frac{f'(r)}{\omega}, \quad R_{\theta r \theta r} = \frac{-c'(r)}{2\omega c(r)} \Omega_{\theta r}, \quad (3.9)$$

where we have introduced the (as yet) arbitrary function  $c(r)$ :

$$c(r) = \frac{\Omega_{\theta r}}{2\sqrt{g}} = -\frac{f'}{2\sqrt{ff_3}}. \quad (3.10)$$

It is now straightforward to plug into the localization formula (2.23). Performing the Grassmann integrals and the  $\theta$  integral leads to the following simple result:

$$Z(T) = \frac{1}{\omega} \int_0^\infty dr \frac{d}{dr} \left( \frac{c}{\sin cT} e^{-i\frac{1}{2}r^2T} \right). \quad (3.11)$$

So, in the special case of the harmonic oscillator, the localization formula reduces to a total derivative. The final result is

$$Z(T) = \frac{1}{\omega} \frac{c(0)}{\sin[c(0)T]}. \quad (3.12)$$

It is surprising that, even in such a simple case as the harmonic oscillator, the final result of the localization procedure is ambiguous, i.e. it depends on the value at  $r=0$  of the arbitrary function  $c(r)$ . The correct result for the path integral is only obtained if we impose the additional ad hoc boundary condition

$$\lim_{r \rightarrow 0} c(r) = \frac{1}{2}\omega. \quad (3.13)$$

As we will see in the next section, such ambiguities appear to be a general feature of equivariant localization of phase space path integrals. Thus the localization formulae, even when they are otherwise valid, are incomplete. This could have been anticipated even in the formal derivation. An obvious requirement[9] which must be incorporated into the localization procedure is that the 1-form  $\beta$  should be homotopically equivalent to 0 under the ‘‘supersymmetry’’ transformation generated by  $d_S$ . Thus one expects generally that additional inputs are required when choosing  $g_{ab}$ , in order to ensure that we are in the trivial homotopy class.

Before moving on to more interesting problems, let us consider the phase space path integral for the free particle. In this case  $H = \frac{1}{2}p^2$ , and  $\chi^z = \chi = p$ . The localization procedure again has ambiguities, which can be eliminated by fixing the metric for the free particle to be:

$$g_{ab} = \begin{pmatrix} g_{xx} & g_{xp} \\ g_{px} & g_{pp} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.14)$$

The localization formula can be reduced to

$$Z(T) = \int dp dx \frac{1}{p} \frac{d}{dp} \left( \frac{c(p)}{\sin cT} e^{-i\frac{1}{2}p^2 T} \right). \quad (3.15)$$

With the metric (3.14) we have  $c(p)$  vanishing identically, and the above produces the correct result.

#### 4. The 1-D Hydrogen Atom

The 1-d hydrogen atom is the static potential problem with hamiltonian[10]

$$H = \frac{1}{2}p^2 - \frac{e^2}{|x|}. \quad (4.1)$$

While the partition function can be computed exactly by solving the Schrodinger equation, the path integral for the partition function cannot be directly performed by standard techniques. In particular it is not WKBJ exact. The classical bound state orbits coalesce at  $x=0$ ,  $p \rightarrow \pm\infty$ , making this system highly unsuitable for localization onto classical trajectories. We wish to examine whether another form of equivariant localization, such as (2.23), may apply to systems of this type.

In considering bound states it will again be useful to change variables (of the time independent modes only!). We define hyperbolic coordinates

$$\begin{aligned} p &= |r| \sinh \alpha, \\ x &= \frac{2e^2}{r|r| \cosh^2 \alpha}, \end{aligned} \quad (4.2)$$

so that  $H = -\frac{1}{2}r^2$ . The original phase space maps into  $-\infty \leq r, \alpha \leq \infty$ . The hamiltonian vector has only one nonvanishing component:

$$\chi^\alpha = \chi = -\frac{1}{4e^2} r^3 \cosh^3 \alpha. \quad (4.3)$$

The Lie derivative constraint  $\mathcal{L}_H g_{ab} = 0$  has exactly the same form as (3.4), with  $(r, \theta)$  replaced by  $(r, \alpha)$ . Thus the general solution for  $g_{ab}$  has precisely the form (3.5). However in the present case, since  $\alpha$  is a hyperbolic coordinate, we do not encounter a single-valuedness problem in defining  $g_{r\alpha}$ . Explicitly:

$$g_{r\alpha} = \frac{12e^2 f(r)}{r^4 \chi} \left[ \frac{\sinh \alpha}{2 \cosh^2 \alpha} + \frac{1}{2} \tan^{-1}(\sinh \alpha) \right] + \frac{f_2(r)}{\chi}, \quad (4.4)$$

which is perfectly well-defined.

A trivial calculation gives

$$\Omega_{\alpha r} = \frac{f'(r)}{\chi}. \quad (4.5)$$

The curvature can then be computed rather easily by exploiting the relation

$$\Omega_{\alpha r; r} = 2R_{\alpha r r} \chi. \quad (4.6)$$

One obtains

$$R_{\alpha r r} = \frac{1}{2\chi^2} \left( f'' - \frac{(f')^2}{2f} - \frac{f' f_3'}{2f_3} \right). \quad (4.7)$$

With these results, one can easily show that the localization formula (2.23) can be reduced to the following simple expression:

$$Z(T) = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} d\alpha \frac{1}{\chi} \frac{d}{dr} \left( \frac{c}{\sin cT} e^{i\frac{1}{2}r^2 T} \right), \quad (4.8)$$

where the function  $c(r)$  is defined by

$$c(r) = \frac{f'}{2\sqrt{f f_3}}. \quad (4.9)$$

In fact (4.9) is a general result; it follows from (2.23) for any static potential which is unbounded below.

As for the harmonic oscillator, we have obtained a remarkably simple result, which is unfortunately ambiguous due to its dependence on the (as yet) arbitrary function  $c(r)$ . Actually the ambiguity appears to be much worse, since the harmonic oscillator result only depended on  $c(0)$ . We can improve the situation by performing some additional manipulations on (4.9). We perform the  $\alpha$  integration, and expand  $\sin cT$  as a power series. Then, assuming that we can interchange integrations and summations, we have a simple expression for the integrated partition function:

$$\int_0^{\infty} dT Z(T) = -12\pi e^2 \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dr \frac{1}{r^4} \frac{1}{(2n-1) - \left(\frac{r^2}{2c(r)}\right)}. \quad (4.10)$$

We then make the mild assumption that the only singularities of the integrand in (4.10), assumed to be an analytic function of complex  $r$ , are poles on the real axis. We can then convert (4.10) into a contour integral, adding the prescription that the contour includes all poles except the one at  $r = 0$  (if we include all the poles, the integral simply vanishes). However, since the contour can be closed above or below, this is the same as including *only* the pole at  $r = 0$ . We therefore obtain

$$\int_0^{\infty} dT Z(T) = 4\pi^2 e^2 i \sum_{n=1}^{\infty} \lim_{r \rightarrow 0} \frac{d^3}{dr^3} \left( \frac{1}{(2n-1) - \left(\frac{r^2}{2c(r)}\right)} \right). \quad (4.11)$$

Thus we see that the result depends only on the behavior of  $c(r)$  as  $r$  goes to zero. To resolve this remaining ambiguity, we need an additional input. To see what this is, let us consider the main difference between the WKB localization of path integrals and the localization procedure which led to (2.23) and (4.9). In the former case

the original path integral is reduced to a sum of small fluctuation integrals around the classical trajectories. In the latter case, the original path integral is reduced to a sum of small fluctuation integrals around individual points in the phase space parametrized by the time independent modes  $(x, p)$ . From this fact we conclude that the contribution to (4.9) from the region  $x \rightarrow \infty$ ,  $p \sim 0$ , should be approximately equal to the corresponding contribution in (3.39) for the free particle, provided that  $T$  is not near  $\infty$ .

We have applied this matching condition by computing the general metric for the 1-d hydrogen atom in  $(x, p)$  coordinates. We then matched to the free particle metric (3.14) to leading order in large  $x$ , and to leading and next-to-leading order in small  $p^2 x$ . This is sufficient to give the leading order behavior of  $c(r)$  for small  $r$ :

$$c(r) = \frac{r^3}{2e^2} (1 + c_1 r + c_2 r^2 + \dots) . \quad (4.12)$$

Actually, the evaluation of (4.11) requires knowledge of the corrections  $c_1$  and  $c_2$ . While these can be computed in principle via perturbation theory, we have not done so. Regardless, the leading order behavior is enough to see that (4.11) reproduces the correct form for the bound state spectrum of the 1-d hydrogen atom:

$$E_n \propto \frac{e^4}{n^2} . \quad (4.13)$$

We consider this a nontrivial success of equivariant localization techniques applied to physically relevant path integrals. While we are still operating at a rather primitive level compared to standard WKB, we believe that the potential for obtaining new results via these more general methods of equivariant localization is much greater.

1. M. Blau, E. Keski-Vakkuri, and A. Niemi, *Phys. Lett.* **246B** (1990) 92.
2. A. Morozov, A. Niemi, and K. Palo, *Phys. Lett.* **271B** (1991) 365.
3. A. Hietamäki, A. Morozov, A. Niemi, and K. Palo, *Phys. Lett.* **263B** (1991) 417.
4. A. Niemi and O. Tirkkonen, *Phys. Lett.* **293B** (1992) 339.
5. L. Schulman, "Techniques and Applications of Path Integration", John Wiley & Sons (New York) 1981.
6. R. Dashen, B. Hasslacher, and A. Neveu, *Phys. Rev.* **D12** (1975) 2443.
7. J-M Bismut, *Comm. Math. Phys.* **103** (1986) 127.
8. N. Berline, E. Getzler, and M. Vergne, "Heat Kernels and Dirac Operators", Springer-Verlag (Berlin) 1992.
9. A. Niemi and P. Pasanen, *Phys. Lett.* **253B** (1991) 349.
10. R. Loudon, *Am. J. Phys.* **27** (1959) 649.