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SO(4) INVARIANT STATES IN QUANTUM COSMOLOGY

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ABSTRACT

The phenomenon of linearisation instability is identified in models of quantum cosmology that are perturbations of mini-superspace models. In particular, constraints that are second order in the perturbations must be imposed on wave functions calculated in such models. It is shown explicitly that in the case of a model which is a perturbation of the mini-superspace which has S^3 spatial sections these constraints imply that any wave functions calculated in this model must be $SO(4)$ invariant.

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1 Introduction

The phenomenon of linearisation instability in classical general relativity is well understood [1- 3]. It arises when approximations to solutions of the vacuum Einstein equations are sought by expanding the equations about a known solution which has compact Cauchy surfaces and non-trivial Killing vectors and solving the linearised equations for the perturbation. In this case, solving the linearised equations alone does not always yield a metric which is a good approximation to a solution of Einstein's equations i.e. a solution to the linearised equations may not be tangent to a curve of exact solutions.

The reason is that some of the constraints of general relativity are exactly zero to linearised order, in fact there is one such constraint for every Killing vector. Thus the first non-zero order is the second and there is one second order constraint for each Killing vector. Imposition of these second order constraints is what is needed to eliminate the spurious solutions. These complications can also be seen as a reflection of the structure of the space of solutions that are close to a solution with Killing vectors and compact Cauchy surfaces [4-6]. This space is not a manifold, since the diffeomorphism group does not act freely but has a fixed point which is precisely the background metric with isometries. Rather, it has a stratified stucture and the background geometry is a singular point in the space.

It has been pointed out that when one comes to quantise gravitational perturbations on backgrounds with compact Cauchy surfaces and Killing vectors, one must again take into consideration these second order constraints, now imposed as operators annihilating physical states. [7]. The consequences of this have been worked out in detail for the case of DeSitter space [8].

Although linearisation instability would not be expected to play a role in quantum cosmology in general since one integrates over all four-geometries, symmetries or not, it does turn out to be important in models of quantum cosmology in which departures from mini-superspace are considered small in some sense. In these cases, the mini-superspace has closed (compact without boundary) spatial sections and spatial Killing vectors and the same considerations as before must be made.

The purpose of the present paper is to demonstrate how linearisation instability arises in the quantum cosmology model of Halliwell and Hawking [9]. The paper proceeds as follows. In Section 2, we review classical linearisation instability. In Section 3 the model is described and we see explicitly how six of the linearised momentum constraints vanish identically. The expressions for the second order constraints are derived. Section 4 contains the calculation of the quantum second order constraints in a representation on wave functions that are functions of the scale factor and the mode coefficients of the harmonic expansion of the perturbation. It is shown that these six constraints obey the algebra of $SO(4)$. In section 5 it is shown In section 6 a scalar field is added the analysis repeated. Section 7 is a discussion.

2 Linearisation Instability

This brief discussion follows that of Moncrief [7]. Let M be a compact three-manifold without boundary. In the hamiltonian formulation of general relativity, the dynamical variables are (g, π), where $g = g_{ij}$ is a riemannian metric and $\pi = \pi^{ij}$ is its canonical momentum, a tensor density, on M. Due to diffeomorphism invariance, general relativity is a constrained theory. The constraint hypersurface in phase space is defined by $\Phi(g, \pi) = 0$, where Φ is the constraint map

$$\Phi(g, \pi) = \left(\mathcal{H}(g, \pi), \mathcal{H}^i(g, \pi)\right), \tag{2·1}$$

with

$$\mathcal{H}(g, \pi) \equiv (\mu_g)^{-1}\frac{1}{2}\left(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}\right)\pi^{ij}\pi^{kl} - \mu_g(R(g) - 2\Lambda), \tag{2·2}$$

$$\mathcal{H}^i(g, \pi) \equiv -2\pi^{ij}{}_{;j}. \tag{2·3}$$

Here, $\mu_g = (\det g)^{\frac{1}{2}}$, semi-colon denotes covariant derivative with respect to g, and units have been chosen in which $16\pi G = 1$.

Let (g_0, π_0) be a solution of the constraints. Suppose we are looking for a solution of Einstein's equations close to a background solution, ${}^4g_0$, for which (g_0, π_0) is the initial data. The new solution will have initial data $(g, \pi) = (g_0 + h, \pi_0 + \omega)$. One may expand out the constraints:

$$\Phi(g_0 + h, \pi_0 + \omega) = \Phi^{(1)}_{(g_0, \pi_0)}(h, \omega) + \Phi^{(2)}_{(g_0, \pi_0)}(h, \omega) + \dots \tag{2·4}$$

where $\Phi^{(1)}$ ($\Phi^{(2)}$ etc.) is linear (quadratic etc.) in the perturbation (h, ω). We will adopt similar notation from here on, so that a superscript (k) denotes a quantity that is kth order in (h, ω). The usual linear constraints are $\Phi_1(g_0, \pi_0) = 0$. However, if ${}^4g_0$ admits Killing vectors, imposing the linear constraints alone will not in general exclude nonintegrable perturbations.

To see this, let C be any function and $Y = Y^i$ be any vector field on M. Define the projection, $P_{(C,Y)}(\Phi)$, of Φ along (C, Y) by

$$\begin{aligned} P_{(C,Y)}(\Phi) &= \int_M d^3x \, \langle (C, Y), \Phi(g, \pi) \rangle \\ &= \int_M d^3x \left[C\mathcal{H}(g, \pi) + Y^i\mathcal{H}_i(g, \pi)\right], \end{aligned} \tag{2·5}$$

where $\mathcal{H}_i = g_{ij}\mathcal{H}^j$. For any (C,Y)

$$P_{(C,Y)}(\Phi(g,\pi)) = P^{(1)}_{(C,Y)}(\Phi(g,\pi)) + P^{(2)}_{(C,Y)}(\Phi(g,\pi)) + \ldots = 0. \tag{2·6}$$

where

$$P^{(k)}_{(C,Y)}(\Phi(g,\pi)) = \int_M \left\langle (C,Y), \Phi^{(k)}_{(g_0,\pi_0)}(h,\omega) \right\rangle, \; k = 1,2\ldots. \tag{2·7}$$

It can be shown that $P^{(1)}_{(C,Y)}(\Phi(g,\pi))$ vanishes if and only if C and Y are the normal and tangential projections on the initial surface of a Killing vector of ${}^4g_0$. In that case, the lowest non-trivial order for the constraint projected along the Killing direction is the second. Thus, in order to treat the constraints consistently, one must impose

$$\Phi_1(g_0,\pi_0) = 0, \tag{2·8}$$

the usual linear constraints and, in addition,

$$P^{(2)}_{(C,Y)}(\Phi(g_0 + h, \pi_0 + \omega)) = 0 \tag{2·9}$$

for each Killing vector (C,Y) of the background.

On quantisation of the perturbations on the background ${}^4g_0$, (2·8) and (2·9) can be implemented as operator constraints on physical states.

3 Perturbed Mini-Superspace

In general, one would not expect the phenomenon of linearisation instability to arise in quantum gravity since, roughly, one integrates over all four-geometries, with no restriction on symmetry properties. However, in quantum cosmology, motivated by the approximate homogeneity of the observed universe, models have been studied in which one imposes severe symmetries on the four geometries included in the path integral. Going beyond these "mini-superspace" models, attempts have been made to treat departures from homogeneity perturbatively. When the homogeneous "background" space is a three-sphere, linearisation instability emerges as expected. In this section we describe just such a model [9]. The three-metric, g_{ij} has the form

$$g_{ij} = a^2(t)\,(q_{ij} + h_{ij}) \tag{3·1}$$

where q_{ij} is the round metric on S^3, normalised so that $\int \sqrt{q}d^3x = 16\pi^2$ (note that $q_{ij} = 4\Omega_{ij}$ where Ω_{ij} is the metric induced by the embedding of S^3 in R^4). h_{ij} is a perturbation and to be considered as small.

There are six Killing vectors of the homogeneous background: the three left invariant plus the three right invariant vector fields on S^3, $\{e_A{}^i : A = 1,2,3\}$ and $\{\tilde{e}_A{}^i\}$, respectively. They satisfy

$$[e_A, e_B] = -\epsilon_{AB}{}^C e_C, \quad [\tilde{e}_A, \tilde{e}_B] = \epsilon_{AB}{}^C \tilde{e}_C, \quad \delta^{AB} e_A{}^i e_B{}^j = \delta^{AB} \tilde{e}_A{}^i e_B{}^j = q^{ij}. \tag{3·2}$$

They are, in fact, the lowest order vector harmonics on S^3 (Lifschitz harmonics $(S^{o,e}_{n=1})^i$ [10]). We introduce an alternative, "spherical", basis for the Killing vectors, $\{e_a{}^i : a = \pm 1, 0\}$ defined by

$$\begin{aligned} e_{\pm 1} &= \pm\frac{1}{\sqrt{2}}(e_1 \mp ie_2), \\ e_0 &= -ie_3, \end{aligned} \tag{3·3}$$

and the dual basis of one forms, $\{e^a{}_i\}$ such that $e^a{}_i e_b{}^i = \delta^a{}_b$. $\{\tilde{e}_a{}^i\}$ and $\{\tilde{e}^a{}_i\}$ are defined similarly.

Let us rename the projected constraints, $P_{(0,e_a)}(\Phi)$ as P_a and $P_{(0,\tilde{e}_a)}(\Phi)$ as $\tilde{P}_a$ and expand them out in the perturbation. First consider P_a.

$$\begin{aligned} P_a &= -2\int d^3x \pi^{ij}{}_{;j} e_a{}^k g_{ik} \\ &= -2a^2 \int d^3x \pi^{ij}{}_{;j} e_a{}^k (q_{ik} + h_{ik}). \end{aligned} \tag{3·4}$$

The zeroth order constraint is zero since that relates to the background which is homogeneous: $(\pi^{ij}{}_{;j})^{(0)} = 0$. The first order constraint is

$$\begin{aligned} \left(\pi^{ij}{}_{;j}\right)^{(1)} =& \pi^{(1)ij}{}_{|j} + \pi^{(0)kj}\Gamma^{(1)i}_{kj} \\ =& \pi^{(1)ij}{}_{|j} + \pi^{(0)kj} h^i{}_{k|j} - \frac{1}{2}\pi^{(0)ij} h^k{}_{k|j} \end{aligned} \tag{3·5}$$

where vertical bar denotes covariant derivative with respect to q_{ij}, all tensor indices are (now and henceforth) raised and lowered with q and Γ^i_{jk} is the Christoffel symbol of the metric g_{ij}.

So

$$\begin{aligned} P^{(1)}_a &= -2a^2 \int d^3x (\pi^{ij}{}_{;j})^{(1)} e_a{}^k q_{ik} \\ &= -2a^2 \int d^3x \left[\pi^{(1)ij}{}_{|j} + \pi^{(0)kj} h^i{}_{k|j} - \frac{1}{2}\pi^{(0)ij} h^k{}_{k|j}\right] e_a{}^l q_{il}. \end{aligned} \tag{3·6}$$

Using ${e_a}^{(i|j)} = 0$ and $\pi^{(0)ij} \propto q^{ij}$, it can be shown that $P_a^{(1)} = 0$ as expected and similarly $\tilde{P}_a^{(1)} = 0$.

Now let us consider the second order,

$$P_a^{(2)} = -2a^2 \int d^3x \left[({\pi^{ij}}_{;j})^{(2)} {e_a}^k q_{ik} + ({\pi^{ij}}_{;j})^{(1)} {e_a}^k h_{ik} \right]. \tag{3·7}$$

We have

$$({\pi^{ij}}_{;j})^{(2)} = {\pi^{(2)ij}}_{|j} + \pi^{(1)kj} \Gamma^{(1)i}_{kj} + + \pi^{(0)kj} \Gamma^{(2)i}_{kj}, \tag{3·8}$$

whence

$$P_a^{(2)} = -2a^2 \int d^3x \left[-\frac{1}{2} h_{kj|i} \pi^{(1)kj} {e_a}^i - h_{ik} \pi^{(1)jk} {{e_a}^i}_{|j} \right]. \tag{3·9}$$

4 The Algebra of the Second Order Constraints

In this section we will expand h_{ij} and π^{ij} in spin-2 hyperspherical spinor harmonics on S^3 (more details of which can be found in [11,12]), and calculate the second order constraints.

$$\begin{aligned} h_{ij}(\gamma, t) =& {e^a}_i(\gamma) {e^b}_j(\gamma) \begin{pmatrix} 1 & 1 & m \\ a & b & 2 \end{pmatrix} \sum_{LJ} h^{NM}_{LJ}(t) Y^{2LJ}_{mNM}(\gamma) \\ &+ q_{ij}(\gamma) \sum_J x^{NM}_J(t) {\mathcal{D}^J}_{NM}(\gamma) \sqrt{\frac{2J+1}{16\pi^2}}, \end{aligned} \tag{4·1}$$

$$\begin{aligned} \pi^{ij}(\gamma, t) =& \frac{1}{48\pi^2} q^{ij}(\gamma) \sqrt{q} \Big[(2a)^{-1} \pi_a - a^{-2} \sum_{LJ} h^{NM}_{LJ}(t) {\pi_h}^{LJ}_{NM}(t) \\ &- a^{-2} \sum_J x^{NM}_J(t) {\pi_x}^J_{NM}(t) \Big] \\ &+ \frac{1}{3} a^{-2} q^{ij} \sqrt{q} \sum_J {\pi_x}^{NM}_J(t) {\mathcal{D}^J}_{NM}(\gamma) \sqrt{\frac{2J+1}{16\pi^2}} \\ &+ 5a^{-2} \sqrt{q} {e_a}^i(\gamma) {e_b}^j(\gamma) \begin{pmatrix} a & b & m \\ 1 & 1 & 2 \end{pmatrix} \sum_{LJ} {\pi_h}^{NM}_{LJ}(t) Y^{2LJ}_{mNM}(\gamma) \end{aligned} \tag{4·2}$$

where $\begin{pmatrix} 1 & 1 & m \\ a & b & 2 \end{pmatrix}$ is a three-j symbol (with the spin-2 index m raised, see eq.(4.6)) and $Y^{2LJ}_{mNM}(\gamma)$ is a spin-2 hyperspherical spinor harmonic on S^3, the point of S^3 being written as an element, γ, of $SU(2)$.

$$Y^{2\,L\,J}_{m\,N\,M}(\gamma) = \sqrt{\frac{(2J+1)(2L+1)}{16\pi^2}}\mathcal{D}^{L}{}_{N}{}^{N'}(\gamma)\begin{pmatrix} L & J & 2 \\ N' & M & m \end{pmatrix} \tag{4·3}$$

where $\mathcal{D}^{L}{}_{N}{}^{N'}(\gamma)$ is a spin-L representation matrix of $SU(2)$. From (4·3) we see there is a condition on L and J namely $|L-J| \leq 2$. The harmonics with $L = J$ correspond to the "scalar" traceless tensor harmonics of Lifschitz, P^n_{ij}, those with $|L-J| = 1$ to the "vector" traceless tensor harmonics, S^n_{ij} and those with $|L-J| = 2$ to the tranverse traceless harmonics G^n_{ij} with $n = L+J+1$ in each case.

The harmonics are normalised so that

$$\int d^3x\sqrt{q}\;\; Y^{2\,L\,J}_{m\,N\,M}Y^{m\,L'\,J'}_{2\,N'\,M'} = \delta_{LL'}\delta_{JJ'}C^{L}_{NN'}C^{J}_{MM'} \tag{4·4}$$

where

$$C^{L}_{NN'} = C^{LNN'} = (-1)^{L-N}\delta_{N,-N'} \tag{4·5}$$

is the spin-L metric with which all spin-L indices are lowered and raised according to

$$U_N = C_{NN'}U^{N'} \quad \text{and} \quad V^N = V_{N'}C^{N'N}. \tag{4·6}$$

Repeated indices, one upstairs and one downstairs, are summed over.

The expansion coefficients of π^{ij} are found using:

$$\begin{aligned} \pi_{x}{}^{L}_{N\,M} &\equiv \frac{\partial \mathcal{L}}{\partial \dot{x}^{N\,M}_{L}} \\ &= \int d^3x \frac{\delta \mathcal{L}}{\delta \dot{g}_{ij}(x)}\frac{\partial \dot{g}_{ij}(x)}{\partial \dot{x}^{N\,M}_{L}} \\ &= \int d^3x \pi^{ij}(x)\frac{\partial \dot{g}_{ij}(x)}{\partial \dot{x}^{N\,M}_{L}}, \end{aligned} \tag{4·7}$$

$$\begin{aligned} \pi_{h}{}^{L\,J}_{N\,M} &\equiv \frac{\partial \mathcal{L}}{\partial \dot{h}^{N\,M}_{L\,J}} \\ &= \int d^3x \pi^{ij}(x)\frac{\partial \dot{g}_{ij}(x)}{\partial \dot{h}^{N\,M}_{L\,J}}, \end{aligned} \tag{4·8}$$

$$\begin{aligned} \pi_{a} &\equiv \frac{\partial \mathcal{L}}{\partial \dot{a}} \\ &= \int d^3x \pi^{ij}(x)\frac{\partial \dot{g}_{ij}(x)}{\partial \dot{a}}. \end{aligned} \tag{4·9}$$

Note that the harmonics $\mathcal{D}$ and Y are complex and reality conditions on the expansion coefficients are needed to ensure that h_{ij} and π^{ij} are real. The conditions are that the complex conjugate of any coefficient with both indices upstairs equals that coefficient with its indices lowered e.g. $(h^{N\,M}_{L\,J})^* = h^{L\,J}_{N\,M}$.

Substituting the expansions into (3·9) and using

$$e^a{}_{i|j} e_b{}^i e_c{}^j = \frac{1}{2}\sqrt{6}\begin{pmatrix} a & 1 & 1 \\ 1 & b & c \end{pmatrix}, \tag{4·10}$$

$$\mathcal{D}^J{}_N{}^{M|i} e^a{}_i = (-1)^{2J}\sqrt{J(J+1)(2J+1)}\mathcal{D}^J{}_N{}^{N'}\begin{pmatrix} J & M & a \\ N' & J & 1 \end{pmatrix} \tag{4·11}$$

and the angular momentum recoupling formula

$$\begin{pmatrix} j_1 & l_2 & \mu_3 \\ m_1 & \mu_2 & l_3 \end{pmatrix}\begin{pmatrix} \mu_1 & j_2 & l_3 \\ l_1 & m_2 & \mu_3 \end{pmatrix}\begin{pmatrix} l_1 & \mu_2 & j_3 \\ \mu_1 & l_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \tag{4·12}$$

(see e.g. Edmonds [13]) we obtain

$$\begin{aligned} P^{(2)}_a = &\sum_{LJ}\begin{pmatrix} 1 & M & J \\ a & J & M' \end{pmatrix}\sqrt{J(J+1)(2J+1)}\, h^{N\,M'}_{L\,J}\pi^{(1)}_h{}^{L\,J}_{N\,M} \\ &+\sum_{J}\begin{pmatrix} 1 & M & J \\ a & J & M' \end{pmatrix}\sqrt{J(J+1)(2J+1)}\, x^{N\,M'}_J \pi^{(1)}_x{}^{J}_{N\,M}. \end{aligned} \tag{4·13}$$

One way to deal with the constraints on quantisation is to take wave functions to be functions of the coefficients, $\Psi \equiv \Psi(a, h^{N\,M}_{L\,J}, x^{N\,M}_J)$ and to enforce the constraints as conditions on physical states, representing π_* by $-i\frac{\partial}{\partial *}$. Here we make the approximation of representing

$$\begin{aligned} \pi^{(1)}_x{}^{J}_{N\,M} &\to -i\frac{\partial}{\partial x^{N\,M}_J} \\ \pi^{(1)}_h{}^{L\,J}_{N\,M} &\to -i\frac{\partial}{\partial h^{N\,M}_{L\,J}}. \end{aligned} \tag{4·14}$$

We have

$$\begin{pmatrix} 1 & M & J \\ a & J & M' \end{pmatrix}\sqrt{J(J+1)(2J+1)} = -i\left(j^J_a\right)_{M'}{}^{M} \tag{4·15}$$

where $\left(j^J_a\right)$ are the matrix generators of the spin-J representation of $SU(2)$. Thus, imposing the second order constraint, (4·13), gives us finally

$$-\left[\sum (j^J_a)_{M'}{}^{M}\delta_{N'}{}^{N} h^{N'\,M'}_{L\,J}\frac{\partial}{\partial h^{N\,M}_{L\,J}} + \sum (j^J_a)_{M'}{}^{M}\delta_{N'}{}^{N} x^{N'\,M'}_J\frac{\partial}{\partial x^{N\,M}_J}\right]\Psi = 0. \tag{4·16}$$

Similarly we can show that

$$\hat{P}_a^{(2)} = -\sum_{LJ} \begin{pmatrix} 1 & N & L \\ a & L & N' \end{pmatrix} \sqrt{L(L+1)(2L+1)}\, h_{L\ J}^{N'\,M} \pi_h^{(1)}{}_{L\ J}^{\ \ NM}$$
$$- \sum_{J} \begin{pmatrix} 1 & N & J \\ a & J & N' \end{pmatrix} \sqrt{J(J+1)(2J+1)}\, x_J^{N'\,M} \pi_x^{(1)}{}_{J}^{\ NM} \tag{4·17}$$

and thus the full set of second order quantum constraints consists of (4·16) together with

$$\left[\sum (j_a^L)_{N'}{}^N \delta_{M'}{}^M h_{L\ J}^{N'\,M'} \frac{\partial}{\partial h_{L\ J}^{N\,M}} + \sum (j_a^J)_{N'}{}^N \delta_{M'}{}^M x_J^{N'\,M'} \frac{\partial}{\partial x_J^{N\,M}}\right] \Psi = 0. \tag{4·18}$$

(4·16) and (4·18) imply that Ψ is $SO(4)$ invariant since it is easy to see that

$$\left[\hat{P}_A^{(2)}, \hat{P}_B^{(2)}\right] \Psi = -\epsilon_{AB}{}^C \hat{P}_C^{(2)} \Psi \tag{4·19}$$

$$\left[\hat{\bar{P}}_A^{(2)}, \hat{\bar{P}}_B^{(2)}\right] \Psi = \epsilon_{AB}{}^C \hat{\bar{P}}_C^{(2)} \Psi \tag{4·20}$$

$$\left[\hat{P}_A^{(2)}, \hat{\bar{P}}_B^{(2)}\right] \Psi = 0 \tag{4·21}$$

where hats denote quantum operators. Thus the constraints generate the algebra of $SO(4) \cong SU(2) \times SU(2)/Z_2$.

Another way to see that (4·16) and (4·18) mean that Ψ is $SO(4)$ invariant is to note that under a rotation $\gamma \to \xi\gamma\eta^{-1}$, with $\xi,\ \eta \in SU(2)$, the coefficients x_J^{NM} and $h_{L\ J}^{N\,M}$ transform as

$$h_{L\ J}^{N\,M} \to h'^{N\,M}_{L\ J} = h_{L\ J}^{N'\,M'} \mathcal{D}^L{}_{N'}{}^N(\xi^{-1}) \mathcal{D}^J{}_{M'}{}^M(\eta^{-1})$$
$$x_J^{N\,M} \to x'^{N\,M}_J = x_J^{N'\,M'} \mathcal{D}^J{}_{N'}{}^N(\xi^{-1}) \mathcal{D}^J{}_{M'}{}^M(\eta^{-1}) \tag{4·22}$$

If ξ and η are infinitesimal we have

$$\delta h_{L\ J}^{N\,M} = -i\left(\xi^A (j_A^L)_{N'}{}^N \delta_{M'}{}^M + \eta^A (j_A^J)_{M'}{}^M \delta_{N'}{}^N\right) h_{L\ J}^{N'\,M'} \tag{4·23}$$

$$\delta x_J^{N\,M} = -i\left(\xi^A (j_A^J)_{N'}{}^N \delta_{M'}{}^M + \eta^A (j_A^J)_{M'}{}^M \delta_{N'}{}^N\right) x_J^{N'\,M'} \tag{4·24}$$

where $\{\xi^A\}$ and $\{\eta^A\}$ are two sets of three real parameters. $\Psi(x,h)$ is invariant under all rotations iff

$$\left[\sum \delta h_{L\ J}^{N\,M} \frac{\partial}{\partial h_{L\ J}^{N\,M}} + \sum \delta x_J^{N\,M} \frac{\partial}{\partial x_J^{N\,M}}\right] \Psi = 0 \quad \forall\, \xi^A, \eta^A \tag{4·25}$$

which conditions are exactly (4·16) and (4·18).

We note that $\Psi \equiv \Psi(h^2, x^2)$, where $h^2 = h_{L\,J}^{N\,M} h_{N\,M}^{L\,J}$ and $x^2 = x_J^{N\,M} x_{N\,M}^{J}$, is $SO(4)$ invariant. More generally, a wave function is invariant if all the "left" indices (i.e. indices that transform under ξ) are contracted together with metrics and/or three-j symbols and similarly for all the "right" indices (that transform under η).

5 The Physical Degrees of Freedom

We are used to identifying the transverse traceless modes of the perturbation of the gravitational field as the physical degrees of freedom. In this section we will see that the second order constraints can be reduced to a form that reflects this.

We can write the constraint (4·13) as a sum of "scalar", "vector" and "tensor" (transverse traceless) parts

$$P_a^{(2)} = {}^sP_a + {}^vP_a + {}^tP_a \tag{5·1}$$

where

$${}^sP_a = \sum_J \sqrt{J(J+1)(2J+1)} \begin{pmatrix} 1 & M & J \\ a & J & M' \end{pmatrix} \left(x_J^{N\,M'} \pi_x^{(1)}{}_{N\,M}^{J} + h_{J\,J}^{N\,M'} \pi_h^{(1)}{}_{N\,M}^{J\,J} \right) \tag{5·2}$$

$${}^vP_a = \sum_J \sqrt{J(J+1)(2J+1)} \begin{pmatrix} 1 & M & J \\ a & J & M' \end{pmatrix} \left(h_{J-1\,J}^{N\,\,M'} \pi_h^{(1)}{}_{N}^{J-1\,J} + h_{J+1\,J}^{N\,\,M'} \pi_h^{(1)}{}_{N\,\,M}^{J+1\,J} \right) \tag{5·3}$$

$${}^tP_a = \sum_J \sqrt{J(J+1)(2J+1)} \begin{pmatrix} 1 & M & J \\ a & J & M' \end{pmatrix} \left(h_{J-2\,J}^{N\,\,M'} \pi_h^{(1)}{}_{N}^{J-2\,J} + h_{J+2\,J}^{N\,\,M'} \pi_h^{(1)}{}_{N\,\,M}^{J+2\,J} \right). \tag{5·4}$$

One can calculate the linearised momentum constraints and they are

$$\frac{\pi_a^{(0)}}{48\pi^2}\left[-\frac{1}{2}\frac{(n^2-1)}{4} x_J^{N\,M} + f_J h_{J\,J}^{N\,M}\right] + \frac{(n^2-1)}{6a}\pi_x^{(1)}{}_{J}^{N\,M} + f_J \frac{10}{a}\pi_h^{(1)}{}_{J\,J}^{N\,M} = 0 \tag{5·5}$$

$$\frac{\pi_a^{(0)}}{48\pi^2} h_{J\pm1\,J}^{N\,\,\,\,\,\,M} + \frac{10}{a}\pi_h^{(1)}{}_{J\pm1\,J}^{N\,\,\,\,\,\,M} = 0 \tag{5·6}$$

where $f_J = \frac{1}{2}(-1)^{2J+1}\sqrt{(n^2-4)(n^2-1)/30}$ and $n = 2J+1$.

We also have the zeroth order and linear hamiltonian constraints

$$\frac{1}{8a}\frac{1}{(48\pi^2)^2}\pi_a^{(0)} a \pi_a^{(0)} + \frac{1}{2}a^2 - \frac{2}{3}\Lambda a^4 = 0 \tag{5·7}$$

$$\left[3a^3\Lambda - \frac{1}{2}a\left(n^2+\frac{1}{2}\right) - \frac{1}{(48\pi^2)^2}\frac{3}{16a^2}\pi_a^{(0)}a\pi_a^{(0)}\right]x_J^{NM} + af_J h_J^{NM}{}_J - \frac{1}{48\pi^2}\frac{\pi_a^{(0)}}{2a^2}\pi_x^{(1)}{}_J^{NM} = 0. \tag{5·8}$$

If (5·5)-(5·8)hold then it can be shown that

$$\pi_a \, {}^vP_a = \pi_a \, {}^sP_a = 0. \tag{5·9}$$

This uses the fact that $\begin{pmatrix} 1 & J & J \\ a & M & M' \end{pmatrix} c_{NN'}$ is antisymmetric under interchange of (M, N) with (M', N').

On quantisation, the constraints become operators that annihilate the wave function Ψ. If a factor ordering is chosen as in (5·5)-(5·8) (i.e. just put hats on everything as it stands) then

$$\hat{\pi}_a \, {}^v\hat{P}_a\Psi = \hat{\pi}_a \, {}^s\hat{P}_a\Psi = 0 \tag{5·10}$$

so

$$\hat{\pi}_a \, \hat{P}_a^{(2)}\Psi = \hat{\pi}_a \, {}^t\hat{P}_a\Psi. \tag{5·11}$$

Since $\hat{\pi}_a\Psi \neq 0$ and $\hat{\pi}_a$ commutes with $\hat{P}^{(2)}$ and ${}^t\hat{P}$ this implies that

$$\hat{P}_a^{(2)}\Psi = {}^t\hat{P}_a\Psi. \tag{5·12}$$

Thus the second order constraints on the wave function may be reduced to the condition that its dependence on the transverse traceless modes be $SO(4)$ invariant.

6 A Scalar Field

So far we've dealt only with vacuum cosmologies. The treatment of classical linearisation instability was originally confined to the vacuum case. Results in the non-vacuum case vary according to the problem being considered. Kastor and Traschen [14] investigate the case where the perturbations in the energy-momentum of the matter are prescribed at some initial time, either directly or by specifying how the constituent fields vary. This leads, in the case where the background spacetime has "Integral Constraint Vectors", to constraints on the possible metric variations allowed. Arms [15] investigates the linearisation stability of the Einstein-Maxwell equations without specifying the matter perturbations. She finds that linearisation instability will occur if the (spatially compact) background space-time has Killing vectors which generate diffeomorphisms under which the $U(1)$ connection is invariant. A similar calculation is done for Einstein-Yang-Mills [16].

With this in mind, suppose we want to add a massive minimally coupled scalar field, Φ, to the model, where Φ has a background homogeneous part and an inhomogeneous perturbation. Now, the spatial Killing vectors generate rotations which leave the background scalar field invariant. Thus we expect linearisation instability to occur. Indeed, the matter part of the momentum constraint is given by

$$\mathcal{H}_m^i = g^{ij}\frac{\partial\Phi}{\partial x^j}\pi_\Phi. \tag{6·1}$$

It is easy to see that smearing this with a Killing vector and calculating the first order part will give identically zero since $\mathcal{L}_{e_a}\Phi^{(0)} = \mathcal{L}_{e_a}\pi_\Phi^{(0)} = 0$.

We expand Φ and π_Φ in scalar hyperspherical harmonics, which are the $SU(2)$ representation matrices,

$$\begin{aligned}\Phi(\gamma,t) &= \phi(t) + \sum f_J^{NM}(t)\mathcal{D}^J{}_{NM}(\gamma)\sqrt{\frac{J(J+1)}{16\pi^2}}\\ \pi_\Phi(\gamma,t) &= \frac{1}{16\pi^2}\sqrt{q}\pi_\phi(t) + \sqrt{q}\sum \pi_f{}_J^{NM}(\gamma)\mathcal{D}^J{}_{NM}(\gamma)\sqrt{\frac{J(J+1)}{16\pi^2}}\end{aligned} \tag{6·2}$$

where $\phi(t)$ and π_ϕ are "background quantities" and the rest is the perturbation. Then, we see that $\mathcal{H}_m^{(0)i} = 0$ and

$$(P_m)_a^{(1)} = a^2\int d^3x q_{ij}\mathcal{H}_m^{(1)i}e_a{}^j = 0. \tag{6·3}$$

The second order of the matter part of the constraint is

$$(P_m)_a^{(2)} = a^2\int d^3x\left(q_{ij}\mathcal{H}_m^{(2)i} + h_{ij}\mathcal{H}_m^{(1)i}\right)e_a{}^j \tag{6·4}$$

which can be calculated to be

$$(P_m)_a^{(2)} = \sum_{JNM}\begin{pmatrix}1 & M & J\\ a & J & M'\end{pmatrix}\sqrt{J(J+1)(2J+1)}\, f_J^{NM'}\pi_f^{(1)}{}_{NM}^J \tag{6·5}$$

and similarly

$$(\tilde{P}_m)_a^{(2)} = -\sum_{JNM}\begin{pmatrix}1 & N & J\\ a & J & N'\end{pmatrix}\sqrt{J(J+1)(2J+1)}f_J^{N'M}\pi_f^{(1)}{}_{NM}^J. \tag{6·6}$$

Thus, the $SO(4)$ invariance extends to the matter dependence of the wave function.

7 Discussion

We have seen how linearisation instability arises in a model of quantum cosmology in which departures from homogeneity are treated perturbatively. It gives rise to second order constraints on the wave function which imply that the wave function is $SO(4)$ invariant. This is as it should be of course since a field configuration on the three-sphere and a rotated configuration are the *same* as far as quantum cosmology is concerned. Similar considerations would arise in any model of perturbations around a mini-superspace with closed spatial sections.

We saw how linearisation instability manifested itself in a non-vacuum model in which the background matter field was invariant under the transformation generated by the Killing fields. It might be possible to prove a general result along these lines.

In Section 5, we used the zeroth order hamiltonian constraint to show that the vector and scalar parts of the second order constraints were redundant once the lower order constraints were imposed. In ref.[9] it is not the zeroth order hamiltonian constraint that is imposed on the wavefunction but the homogeneous projection of the hamiltonian constraint. This is (5·7) plus a part which is quadratic in the perturbation. Note that while it is not clear how this is justified in the perturbative approach, using this homogeneous hamiltonian constraint or the zeroth order hamiltonian constraint does not affect our result since the difference will be a higher order than that to which we are working.

Finally, this calculation shows how neatly the hyperspherical spinor harmonics exploit the group structure of S^3. One could use them to calculate explicitly the action of the DeSitter group on wave functions of gravitational perturbations on a DeSitter background. Six of the ten second order constraints are those calculated in section 4. The remaining four correspond to the boost Killing vectors, B^μ_α. In the coordinate system in which the metric is $ds^2 = -dt^2 + \frac{1}{4}\cosh^2 t\, q_{ij}dx^i dx^j$,

$$B^\mu_\alpha = (Q_\alpha, 4a^{-1}\dot{a}Q^{|i}_\alpha) \tag{7·1}$$

where Q_α, $\alpha = 1, \ldots 4$, are the four lowest inhomogeneous scalar harmonics on S^3 i.e. $\mathcal{D}^{\frac{1}{2}}{}_m{}^n$ ($Q_{n=2}$ Lifschitz), $a = \cosh t$, and the index i is raised using q^{ij}. Thus, the relevant constraint arises from the second order term in

$$\int d^3x \left(Q_\alpha \mathcal{H} + \tanh t\, g_{ij} Q^{|j}_\alpha \mathcal{H}^i\right). \tag{7·2}$$

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