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**All or Nothing:  
On the Small Fluctuations of Two-Dimensional  
String-Theoretic Black Holes**

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**ABSTRACT**

A comprehensive analysis of small fluctuations about two-dimensional string-theoretic and string-inspired black holes is presented. It is shown with specific examples that two-dimensional black holes behave in a radically different way from all known black holes in four dimensions. For both the  $SL(2, R)/U(1)$  black hole and the two-dimensional black hole coupled to a massive dilaton with constant field strength, it is shown that there are a *continuous infinity* of solutions to the linearized equations of motion, which are such that it is impossible to ascertain the classical linear response. It is further shown that the two-dimensional black hole coupled to a massive, linear dilaton admits *no small fluctuations at all*. We discuss possible implications of our results for the Callan-Giddings-Harvey-Ströminger black hole.

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## I. Overview

### I.1 Introduction

Seemingly insuperable problems may sometimes be forced to yield their secrets if one considers instead simpler problems which are appropriately chosen. Whether or not such a strategy will succeed in a specific case depends in the first instance on the proviso that the chosen, simpler problem be sufficiently closely related to the problem of authentic interest. The discovery by Stephen Hawking that, when considered in the semi-classical approximation, black holes may emit thermally-distributed radiation has led to a number of problems which have so far proved too difficult to solve. The supreme example is the problem posed by the possibility that black holes, in the event that they “evaporate” absolutely and completely, may irrevocably obliterate information *in principle*. If this is how Nature behaves then Quantum Mechanics would seem to be unable to provide an adequate account, for it is a *sine qua non* that the wave function must display a well-defined unitary time development [1]. It is clearly premature to abandon quantum mechanics before giving its principles a fair chance, and it is unlikely that this will have been achieved by relying solely on the semi-classical approximation. The discovery of black hole radiation was originally presented with the understanding that one was considering solutions to Einstein’s equations in four dimensions, and it is in this context that it has so far proved too difficult to go beyond the semi-classical approximation.

Recently a great deal of attention has been focussed on interesting related developments in string theory. Black hole solutions have been discovered to the still-unwritten equations of motion in string theory. One may say that these solutions are “unsatisfactory” in differing amounts and in various ways: They are available only at the level of the Born approximation in string theory; some of them are further approximate in that higher-order contributions to the sigma model on the sphere have been neglected in their derivation; some of them are actually solutions to a “string-inspired” theory and not to string theory; some of them are defined in a mythical two-dimensional universe. Nevertheless, and with specific regard to the last point, it is precisely because of the relative simplicity which may be found in two-dimensional solutions that it is hoped that their study may assist in the resolution of outstanding problems of four-dimensional black holes.

The deep puzzles of black hole physics which one would like to solve are in large part inherently quantum mechanical. In the case of four-dimensional black holes one is at present able to say much more about the classical mechanical behavior than about

the quantum mechanical behavior. It is hoped in particular that one may study the quantum mechanical aspects of two-dimensional black holes (for which purpose their origins in string theory are probably immaterial) and draw inferences therefrom which may be successfully applied to the quantum mechanics of four-dimensional black holes. As always, the correspondence principle must serve as our guide in moving between quantum mechanics and classical mechanics. In considering the possibly simpler quantum mechanics of two-dimensional black holes to aid in the understanding of four-dimensional black holes, it is natural to require that the correspondence limit of the substitute,  $2d$  configuration behave in a reasonably similar way to that of the  $4d$  configuration of authentic interest. In this paper we perform the first comprehensive analysis of the small fluctuations of specific two-dimensional string-theoretic and string-inspired black holes which have been the focus of recent research.<sup>3</sup> We demonstrate that these *two-dimensional black holes display classical behavior which differs radically from that of all known four-dimensional black holes*. We find this to be the case for the  $SL(2, R)/U(1)$  black hole [2] as well as for two-dimensional black holes coupled to a massive dilaton. In the cases of the Witten black hole and the black hole coupled to a massive dilaton with constant field strength we find that the linearized equations of motion admit a continuous infinity of solutions which are such that it is *in principle* impossible to ascertain the classical linear response, while we find that the black hole coupled to a massive linear dilaton admits *no small fluctuations at all*. We may say therefore that the physics of these two-dimensional black holes is an “all or nothing” proposition.

It is an element of geometry that the Einstein–Hilbert lagrange density in a two-dimensional theory of gravitation is a total divergence. It is furthermore the case that two-dimensional dilaton gravity is characterized by the absence of propagating degrees of freedom. We note that it is highly unlikely that the unusual behavior displayed by the various two-dimensional black holes studied in this paper is a consequence of this fact. We see very different types of linear response behavior for the various black hole examples we study, although they share the absence of propagating degrees of freedom. While the extremely unusual classical behavior found for specific two-dimensional black holes is not fully understood as to its origin, we may speculate on the possible implications of these results for other two-dimensional black holes. In particular, we discuss the so-called CGHS black hole [3], which is closely related to the Witten black hole, and which is being studied in an attempt to secure a better understanding of the physics of four-dimensional black holes. Based on the results of our analysis of different types of two-dimensional black holes, it would not

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<sup>3</sup>See Note 1 below which follows the conclusion section of this article.

be surprising to discover that the CGHS black hole too displays radically different classical linear response behavior from the known black holes in four dimensions. It is essential to repeat the calculation of the present paper for this configuration in order to determine if it is reasonable to expect that correct inferences applicable to four-dimensional black holes can be drawn from its study.

This report is organized as follows. The remainder of the Section I is devoted to a review of the recent work that has been done on two-dimensional black holes in Section I.2, after which we provide a précis of the group-theoretic derivation of the  $SL(2, R)/U(1)$  black hole in Section I.3. In Section II we perform the analysis of the small fluctuations of black holes: in Section II.1 we provide a general description of the technique for black holes in arbitrary dimensions which will be useful to those who are not familiar with this subject; in Section II.2 we specialize the analysis to the case of two dimensions by first providing an account of the general formulae relevant to two-dimensional theories of gravitation in Section II.2.a, after which we consider comprehensively in turn the  $SL(2, R)/U(1)$  black hole in Section II.2.b, the black hole coupled to a massive dilaton with constant field strength in Section II.2.c.i and the black hole coupled to a massive linear dilaton in Section II.2.c.ii. We present our conclusions in Section III, which is followed by a section of Notes detailing certain technical points, Tables of numerical results and an Appendix.

## I.2 Review of Related Work

In this section we present a brief survey of the recent research efforts devoted to two-dimensional black holes, and in the next section we review the group-theoretic derivation of the  $SL(2, R)/U(1)$  black hole, both of which will be useful to those who are not familiar with this subject. Experts may proceed directly to the analysis of the linearized equations of motion in Section II. For the particular case of two-dimensional black holes, a great deal of research has followed the observation by Witten [2] that the conformal field theory based on the non-compact coset model  $SL(2, R)/U(1)$ , which had been developed by Bars and Nemeschansky [4], Rocek, *et. al.* [5], and others, consists of a two-dimensional black hole coupled to the dilaton. More importantly, the asymptotic form of the metric is just the linear dilaton vacuum which is studied in the  $c = 1$  matrix model. Furthermore, the endpoint of the Hawking radiation process, *i.e.*, the  $M \rightarrow 0$  limit, where  $M$  is the mass of the black hole, also approaches the linear dilaton vacuum.

Using the algebraic structures inherent in the  $G/H$  construction of this model, a number of groups, including Dijkgraaf *et. al.* [6], Distler and Nelson [7], and Chaud-

huri and Lykken [8], have considered the spectrum of states and their correlation functions. In particular, Chaudhuri and Lykken [8] emphasize the  $W_\infty$ -like structure of the model's marginal operators, to which point we shall return in a later section.

Among other developments, Bars [9], and Ginsparg and Quevedo [10] have classified all  $G/H$  models which give rise to spacetimes with only a single time-like coordinate, in any number of dimensions. In addition to the obvious physical importance of having only a single time-like coordinate, it is argued by Bars [9] that models with more than one time-like coordinate are likely to be ill-behaved, since the Virasoro conditions (or equivalently, light-cone gauge) are generally sufficient to remove the negative norm states generated by only a single time-like coordinate. A Hamiltonian formalism is developed, in which the target space metric, antisymmetric tensor and dilaton are determined to *all* orders in  $\alpha'$ . Ginsparg and Quevedo [10] have stressed the connection between target space singularities and fixed points of the gauge transformation generated by  $H$ . Gibbons and Perry [11] have discussed the thermodynamics of the  $SL(2, R)/U(1)$  solution and related heterotic solutions.

Another model which has recently attracted great interest is the dilaton gravity model of Callan, Giddings, Harvey and Strominger (CGHS) [3]. The study of the model begins with the so-called “string-inspired” action, to which a set of minimally-coupled free scalar fields is added. In the initial model, they found that any scalar wave impinging on the linear dilaton vacuum creates a black hole. Calculating the Hawking radiation via its relation in two dimensions to the trace anomaly, one finds a divergent integrated flux. The resolution to this apparent dilemma lies in the neglect of backreaction on the metric. Therefore, CGHS modified their action to include the one-loop effects of the scalar fields.

While the initial hopes that the Hawking radiation could then be treated well within the semi-classical regime were later proven false [12], a number of groups continue to investigate the detailed behavior of the model. DeAlwis [13], as well as Bilal and Callan [14], have attempted to quantize the system by a Distler–David–Kawai approach (see also Hamada [15]). That is to say, they try to form a non-linear sigma model which solves the appropriate beta-function equations and reduces to the CGHS model in the semi-classical limit. As pointed out by Giddings and Strominger [16], such models generally do not have a well-defined ground state. They point out an ambiguity in the regularization of the path integral of the theory, with the result that essentially an infinite number of counterterms must be specified even though dilaton gravity is renormalizable. In other work, Hawking and Stewart [17] claim numerical evidence that the CGHS black hole will end in a “thunderbolt”, *i.e.*, a singularity which propagates out to infinity on a spacelike or null path. Russo, Susskind and

Thorlacius [18] discuss models in which a naked singularity forms, but claim that appropriate boundary conditions can be imposed which will prevent the loss of any quantum mechanical information.

Variations of these models have been treated recently by a number of authors. One variation with which we will be concerned here are models with a nonvanishing dilaton potential, considered recently by Gregory and Harvey [19], and by Horne and Horowitz [20] (the latter in four dimensions only). Others include charged and supersymmetric black holes [21].

In spite of all these efforts, many of the central questions concerning both black hole physics and nonperturbative string backgrounds remain essentially unanswered. The proper quantization of the CGHS model is needed in order to probe the problems of information loss and the endpoint of Hawking radiation, but even the full set of classical solutions of the model are not known. Starting from the string-theoretic  $SL(2, R)/U(1)$  model, one faces a similar problem, in that generally one is only able to perform calculations in the semi-classical limit.

### I.3 Review of Group-Theoretic Derivation of the $SL(2, R)/U(1)$ Black Hole

The first black hole we will consider is the  $SL(2, R)/U(1)$  model, discovered in various forms by Witten [2], Mandal *et. al.* [22], and Bars *et. al.* [4]. Here we briefly review its construction, generally following the notation of Witten [2].

We begin with the ungauged  $SL(2, R)$  Wess–Zumino–Witten (WZW) action

$$S_{WZW} = \frac{k}{8\pi} \int_{\Sigma} d^2z \sqrt{h} h^{ij} \text{Tr}(g^{-1} \partial_i g g^{-1} \partial_j g) + ik\Gamma, \quad (1)$$

where  $\Sigma$  is a Riemann surface with metric  $h$ ,  $g$  is an  $SL(2, R)$ -valued field on  $\Sigma$ , and  $k$  is real and positive.  $\Gamma$  is the Wess–Zumino term, which is usually represented as

$$\Gamma = \frac{1}{12\pi} \int_B d^3y \text{Tr}(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg), \quad (2)$$

where  $B$  is a three-dimensional manifold with boundary equal to  $\Sigma$ . In this expression,  $g$  has been extended from a field on  $\Sigma$  to a field on  $B$ , but  $\Gamma$  is independent of this choice.

The Euclidean version of the black hole is now obtained by gauging the  $U(1)$  subgroup the infinitesimal action of which is given by

$$\delta g = \epsilon \{ \mathcal{G}g + g\mathcal{G} \}, \quad (3)$$

where  $\mathcal{G}$  is the constant  $SL(2, R)$  element

$$\mathcal{G} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4)$$

To gauge this symmetry, we introduce a gauge field  $A$  with the transformation law

$$\delta A_i = -\partial_i \epsilon. \quad (5)$$

In local complex coordinates  $z, \bar{z}$ , the gauge invariant action now takes the form

$$S = S_{WZW} + \frac{k}{2\pi} \int_{\Sigma} d^2 z \{ A_{\bar{z}} \text{Tr}(\mathcal{G} g^{-1} \partial_z g) + A_z \text{Tr}(\mathcal{G} \partial_{\bar{z}} g g^{-1}) + A_z A_{\bar{z}} (-2 + \text{Tr}(\mathcal{G} g \mathcal{G} g^{-1})) \}. \quad (6)$$

One now fixes the gauge by setting

$$g = \cosh r + \sinh r \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}. \quad (7)$$

The gauge field  $A$  appears quadratically and without derivatives. Integrating it out and dropping the Wess–Zumino term (as it is a total derivative) one finds the effective action

$$I_0 = \frac{k}{4\pi} \int d^2 x \sqrt{h} h^{ij} (\partial_i r \partial_j r + \tanh^2 r \partial_i \theta \partial_j \theta). \quad (8)$$

This has the form of a nonlinear sigma model with target space metric

$$ds^2 = \frac{k}{2} ((dr)^2 + \tanh^2 r (d\theta)^2). \quad (9)$$

It is well-known [23] that upon integrating out the gauge field one finds that the integration measure yields a finite correction to the action:

$$I = I_0 - \frac{1}{8\pi} \int d^2 x \sqrt{h} \Phi(r, \tau) R, \quad (10)$$

where  $R$  is the world sheet curvature and  $\Phi$  is the target space dilaton. In the present case, one finds

$$\Phi = 2 \ln \cosh r + \eta, \quad (11)$$

where  $\eta$  is a constant related to the black hole mass. This form of the dilaton can also be seen from the target space action which we will consider in the next section.

The Lorentzian signature form of the black hole, which we shall use in the next section, can be obtained most simply by the analytic continuation  $\theta \rightarrow it$ , or by gauging a different  $U(1)$  subgroup, in which the matrix  $\mathcal{G}$  above is replaced by

$$\mathcal{G} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (12)$$

As noted by Witten [2], if one computes the central charge from this action, it differs from the  $SL(2, R)/U(1)$  value of  $2 + \frac{6}{k-2}$  by an amount of order  $\frac{1}{k^2}$ , implying that there are further corrections from the integration over the gauge field. These corrections would presumably appear in terms higher order in the sigma model coupling  $\alpha'$ . We will return to this point in a later section.

Nearly all of the 2-d black holes in the recent literature are related in some way to the  $SL(2, R)/U(1)$  black hole. For example, the analysis of the CGHS model [3] begins with the  $M = 0$  limit of the  $SL(2, R)/U(1)$  black hole (which corresponds to  $\eta \rightarrow -\infty$ ). A set of minimally-coupled scalars is added to the action, and one finds that *any* incoming scalar wave creates a black hole. Of course, for a given incoming scalar distribution, it is not known whether the resulting background solution corresponds to a conformal field theory.

The massive dilaton models recently considered by Gregory and Harvey [19] are also related to the  $SL(2, R)/U(1)$  model, in that they are solutions of the same target space action, but with the addition of an explicit potential for the dilaton (though they do not contain the scalars of the CGHS model). By taking the mass to zero, one can recover the  $SL(2, R)/U(1)$  model. Of course, the mass terms imply that these models definitely do not correspond to a conformal field theory. While such terms do not appear in string perturbation theory, it is widely speculated that they are related to supersymmetry breaking. Furthermore, a mass must be generated since the dilaton is related to the string coupling constant. Experimental tests of conventional Brans-Dicke models also put tight constraints on very light scalars, though there are recent models in which such limits are evaded if the metric is chosen to couple differently to a “dark matter” dilaton than to ordinary “visible” matter [24].

## II. Analysis of Linearized Equations

### II.1 Small-Fluctuation Analyses of Black Holes

In the following sections we shall explicitly analyze the perturbations of two-dimensional string-theoretic black holes. Here we shall first survey the general procedure used in the analysis of the perturbations of black holes in any number of dimensions [25]. We suppose that one has found a black hole solution to the coupled field equations of an interacting system consisting of gravitation and, in general, additional “matter” fields of various possible types, including different spins. The different species of “matter” will be denoted by labels  $\psi^{(1)}$  through  $\psi^{(n)}$ , where possible tensor indices have been

suppressed. The field configuration which defines the black hole solution, which we will refer to as the *background*, will be denoted by the collection of  $g_{\mu\nu}^B$  and  $\psi_B^{(i)}$ . The coupled field equations to which the background provides a solution are then given by

$$R_{\mu\nu} = T_{\mu\nu}^{(1)} + \cdots + T_{\mu\nu}^{(n)} , \quad (13)$$

$$\begin{aligned} \hat{\mathcal{H}}^{(1)}(\psi^{(1)}) &= \mathcal{I}^{(1)} , \\ &\vdots \\ \hat{\mathcal{H}}^{(n)}(\psi^{(n)}) &= \mathcal{I}^{(n)} , \end{aligned} \quad (14)$$

where the  $T_{\mu\nu}^{(i)}$  are the various stress tensors associated with the different “matter” fields, the  $\hat{\mathcal{H}}^{(i)}$  are in general coupled, nonlinear, tensor-valued, second-order partial differential operators which may depend on the different fields and the  $\mathcal{I}^{(i)}$  are possible source terms. From these coupled nonlinear equations one now computes the associated first-order variations, which yields<sup>4</sup>

$$\delta R_{\mu\nu} = \delta T_{\mu\nu}^{(1)} + \cdots + \delta T_{\mu\nu}^{(n)} , \quad (15)$$

$$\begin{aligned} 0 &= \delta \left[ \hat{\mathcal{H}}^{(1)}(\psi^{(1)}) \right] - \delta \mathcal{I}^{(1)} , \\ &\vdots \\ 0 &= \delta \left[ \hat{\mathcal{H}}^{(n)}(\psi^{(n)}) \right] - \delta \mathcal{I}^{(n)} . \end{aligned} \quad (16)$$

One next substitutes the background field values  $g_{\mu\nu}^B$  and  $\psi_B^{(i)}$  into these equations and then works out the reduction of the system which results upon identifying any integrability conditions and imposing any kinematical constraints. This leads to the following system of *linear*, coupled partial-differential equations

$$\begin{aligned} \hat{\Theta}^{(1)} \delta f_1 &= \Xi^{(1)} , \\ &\vdots \\ \hat{\Theta}^{(m)} \delta f_m &= \Xi^{(m)} , \end{aligned} \quad (17)$$

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<sup>4</sup>It is important to note that there is a proper order in which to perform these computations: it is only *after* calculating the abstract variations that one may substitute the background field values into eqs. (15) through (16). If this order is not respected one will in general miss those terms which vanish in the background but do not fluctuate to zero.

where the  $\delta f_i$  are the distinct perturbations of the background fields and the  $\hat{\Theta}^{(i)}$  are linear partial-differential operators which depend on the background but are independent of the various perturbations. In these equations, for a given  $\delta f_i$  the corresponding  $\Xi^{(i)}$  is a function of as many as  $m-1$  of the *remaining* perturbations and their derivatives and in general one has  $m \leq n$ . In the subsequent treatment of these equations one usually assumes that all field perturbations have a time-dependence  $\propto e^{i\omega t}$ , where  $\omega$  is a non-dispersive frequency, and a temporal Fourier analysis is performed.

One next looks for an appropriate separation of variables in order to transform eqs. (17) into a set of coupled ordinary differential equations. The chosen separation must be consistent with the boundary conditions imposed on the field perturbations. It is then usually convenient to introduce integrating factors which serve to eliminate all first derivative terms, after which one attempts to decouple the resulting system of ordinary differential equations in two steps. One first searches for a transformation of the dependent variables which will allow the system to be expressed in the form:

$$\begin{pmatrix} \hat{D}^2 & & \\ & \ddots & \\ & & \hat{D}^2 \end{pmatrix} \begin{pmatrix} \delta f_1 \\ \vdots \\ \delta f_m \end{pmatrix} = \begin{pmatrix} P_{11} & \cdots & P_{1m} \\ \vdots & \ddots & \vdots \\ P_{m1} & \cdots & P_{mm} \end{pmatrix} \begin{pmatrix} \delta f_1 \\ \vdots \\ \delta f_m \end{pmatrix} = \begin{pmatrix} \mathcal{P}_1 \\ \vdots \\ \mathcal{P}_m \end{pmatrix}, \quad (18)$$

where  $\hat{D}^2 = d^2 + \omega^2$  (here  $d$  is the spatial derivative), the  $P_{ij}$  are scalar functions and the  $\mathcal{P}_i = \mathcal{P}_i(\delta f_1, \dots, \delta f_m)$  are therefore in general linear functions of all the distinct perturbations, but not of their derivatives. We say that the system in this form has been only *differentially decoupled*. In the second step we diagonalize the matrix  $(P_{ij})$ , after which the completely decoupled system of equations may be expressed in the form

$$\hat{D}^2 \delta p_i = v_i \delta p_i, \quad (19)$$

where the physical perturbation functions  $\delta p_i$  are linear combinations of the  $\delta f_i$  appropriate to the diagonalization of  $(P_{ij})$ , and the scalar functions  $v_i$  are the *perturbation potentials* which surround the black hole as a consequence of the presence of the small fluctuations. Thus the original system has been reduced to a set of completely decoupled Schrödinger-like radial equations. As a result, once one has worked out the explicit expressions for the  $v_i$  it is possible to study the properties of any possible bound states, to calculate the various scattering coefficients associated with different incident perturbations and in general to determine completely the linear response of

the black hole to diverse types of incoming waves of small to moderate intensity. It must be emphasized, however, that there is no guarantee that it will be possible in all cases to secure a suitable transformation of the dependent variables which will allow the system of equations to be decoupled. Indeed, in the general case this is an extremely challenging mathematical problem, and as we shall see, it is in precisely this regard that two-dimensional black holes display unexpected properties.

## II.2 Analysis of the Linearized Equations of Motion in Two Dimensions

### II.2.a General Formulae for Two-Dimensional Theories of Gravitation

In considering the small fluctuations of two-dimensional black holes we first note that the most sufficiently general form for the perturbed metric associated with a given initial configuration can be represented by a diagonal matrix. This is always possible to arrange through a transformation of the coordinates, as a result of which we note that we will not encounter the analogues of the “axial” perturbations which arise in the study of black hole perturbations in more than two dimensions. The first-order perturbations of two-dimensional black holes are entirely “polar”, and thus the metric tensor corresponding to the squared line element

$$ds^2 = -e^{2f_0} dt^2 + e^{2f_1} dr^2 , \quad (20)$$

will experience perturbations in the form

$$\begin{pmatrix} -e^{2f_0} & 0 \\ 0 & e^{2f_1} \end{pmatrix} \rightarrow \begin{pmatrix} -e^{2f_0+2\delta f_0} & 0 \\ 0 & e^{2f_1+2\delta f_1} \end{pmatrix} . \quad (21)$$

We note in passing that, having taken the metric tensor to be diagonal, and thus taken the perturbed metric tensor to be diagonal as well, the choice of gauge in the perturbed system has been partially fixed. We shall consider the residual gauge freedom in the perturbed system presently. We see that with the reasonable assumption described in the previous section that all field perturbations carry a time-dependence  $\propto e^{i\omega t}$ ,<sup>5</sup> the various equations for the different small fluctuations are *automatically* separated in the coordinates. For any two-dimensional black hole, then, the small fluctuations are determined by a system of coupled ordinary differential equations.

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<sup>5</sup>It is the case that the background spacetimes considered in this paper are all characterized by a Killing vector. The norm of this vector in the perturbed metric is *indefinite*, as a result of which perturbations about the background are in general time-dependent.

We will consider the physics determined by the two-dimensional action:<sup>6</sup>

$$S = (2\pi)^{-1} \int d^2x \sqrt{-g} e^{-2\Phi} \left[ R + 4(\nabla\Phi)^2 + 4\Lambda^2 - e^{-2\Phi} V(\Phi) \right], \quad (22)$$

where  $\Phi$  is the dilaton field,  $V$  is a generic ‘‘potential’’ for the dilaton, and  $\Lambda$  is the cosmological constant. Extremization of the action with respect to the gravitational and dilaton fields, respectively, leads to the following equations of motion:

$$2e^{-2\Phi} \left\{ \nabla_\mu \nabla_\nu \Phi + g_{\mu\nu} \left[ (\nabla\Phi)^2 - \nabla^2\Phi - \Lambda^2 + \frac{1}{4} e^{-2\Phi} V(\Phi) \right] \right\} = 0, \quad (23)$$

$$e^{-2\Phi} \left\{ R + 4\Lambda^2 + 4\nabla^2\Phi - 4(\nabla\Phi)^2 + e^{-2\Phi} \left[ \frac{1}{2} \frac{\partial V}{\partial \Phi} - 2V(\Phi) \right] \right\} = 0. \quad (24)$$

Upon contracting both sides of eq.(23) with the metric tensor, substituting the result into eq.(24), resolving the resulting equation into components again and thereafter employing the convenient substitution <sup>7</sup>  $\Phi \rightarrow -\Phi/2$  one may rewrite these equations as:

$$0 = \nabla^2\Phi + (\nabla\Phi)^2 - 4\Lambda^2 + e^\Phi \tilde{V}, \quad (25)$$

and

$$R_{\mu\nu} = \nabla_\mu \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} e^\Phi \left( \frac{1}{2} \tilde{V}' - \tilde{V} \right), \quad (26)$$

where  $\tilde{V} \equiv V|_{\Phi \rightarrow -\Phi/2}$  and a prime denotes differentiation with respect to the dilaton field.

The variations of  $\nabla^2\Phi$ ,  $(\nabla\Phi)^2$  and  $\nabla_\mu \nabla_\nu \Phi$  are given by

$$\begin{aligned} \delta(\nabla^2\Phi) &= \Phi_{,\nu} \delta g^{\mu\nu}_{,\mu} + \left[ (f_0 + f_1)_{,\mu} \Phi_{,\nu} + \Phi_{\mu\nu} \right] \delta g^{\mu\nu} + g^{\mu\nu} \Phi_{,\nu} (\delta f_0 + \delta f_1)_{,\mu} \\ &\quad + g^{\mu\nu} \delta \Phi_{,\mu,\nu} + \left[ (f_0 + f_1)_{,\mu} g^{\mu\nu} + g^{\mu\nu}_{,\mu} \right] \delta \Phi_{,\nu}, \end{aligned} \quad (27)$$

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<sup>6</sup>We employ the sigma-model metric throughout the following analysis.

<sup>7</sup>This substitution is in accord with the convention for the relation between the string coupling and the exponential of the dilaton field employed by Witten in [2]. It differs from the convention chosen in [19,21], which are devoted to the analysis of two-dimensional black holes coupled to a massive dilaton. For these cases, when viewed as solutions to a two-dimensional theory of gravitation, the choice is intrinsically unimportant (in particular since there is no electromagnetic field present and hence no duality transformation relating possible electric and magnetic solutions), and may be in any event irrelevant when viewed in the context of string theory since the massive dilaton black holes considered in [19] may have little, and possibly nothing whatsoever, to do with string theory.

$$\delta [(\nabla\Phi)^2] = \Phi_{,\mu}\Phi_{,\nu}\delta g^{\mu\nu} + g^{\mu\nu}\Phi_{,\nu}\delta\Phi_{,\mu} + g^{\mu\nu}\Phi_{,\mu}\delta\Phi_{,\nu} , \quad (28)$$

$$\delta(\nabla_\mu\nabla_\nu\Phi) = \delta\Phi_{,\mu,\nu} - \Phi_{,\lambda}\delta\Gamma_{\mu\nu}^\lambda - \Gamma_{\mu\nu}^\lambda\delta\Phi_{,\lambda} , \quad (29)$$

and thus we derive from eqs. (25) and (26) the following linearized perturbation equations:

$$\begin{aligned} 0 = & g^{\mu\nu}\delta\Phi_{,\mu,\nu} + \left[ g^{\mu\nu}_{,\nu} + g^{\mu\nu}(f_0 + f_1)_{,\nu} + 2g^{\mu\nu}\Phi_{,\nu} \right] \delta\Phi_{,\mu} + e^\Phi \left( \delta\tilde{V} + \tilde{V}\delta\Phi \right) \\ & + \Phi_{,\nu}\delta g^{\mu\nu}_{,\mu} + \left[ (f_0 + f_1)_{,\mu}\Phi_{,\nu} + 2\Phi_{,\mu}\Phi_{,\nu} \right] \delta g^{\mu\nu} + g^{\mu\nu}\Phi_{,\nu}(\delta f_0 + \delta f_1)_{,\mu} , \end{aligned} \quad (30)$$

and

$$\begin{aligned} \delta R_{\mu\nu} = & \delta\Phi_{,\mu,\nu} - \Phi_{,\lambda}\delta\Gamma_{\mu\nu}^\lambda - \Gamma_{\mu\nu}^\lambda\delta\Phi_{,\lambda} \\ & - \frac{1}{2}e^\Phi \left[ \left( \frac{1}{2}\tilde{V}' - \tilde{V} \right) \delta g_{\mu\nu} + g_{\mu\nu} \left( \frac{1}{2}\delta\tilde{V}' - \delta\tilde{V} \right) + g_{\mu\nu} \left( \frac{1}{2}\tilde{V}' - \tilde{V} \right) \delta\Phi \right] . \end{aligned} \quad (31)$$

We now note that for the general metric given by eq.(20) computation reveals that the components of the Ricci tensor are given by:

$$R_{00} = e^{2f_0-2f_1} (f_{0,r,r} + f_{0,r}^2 - f_{0,r}f_{1,r}) - (f_{1,0,0} + f_{1,0}^2 - f_{0,0}f_{1,0}) , \quad (32)$$

$$R_{01} = R_{10} = 0 , \quad (33)$$

and

$$R_{11} = -e^{-2f_0+2f_1} R_{00} . \quad (34)$$

One thus finds that the perturbations in the Ricci tensor are given by:

$$\begin{aligned} \delta R_{00} = & e^{2f_0-2f_1} \left[ \delta f_{0,r,r} + (2f_{0,r} - f_{1,r})\delta f_{0,r} - f_{0,r}\delta f_{1,r} \right. \\ & \left. + 2(\delta f_0 - \delta f_1)(f_{0,r,r} + f_{0,r}^2 - f_{0,r}f_{1,r}) \right] - \delta f_{1,0,0} \\ & - (2f_{1,0} - f_{0,0})\delta f_{1,0} + f_{1,0}\delta f_{0,0} , \end{aligned} \quad (35)$$

$$\delta R_{01} = \delta R_{10} = 0 , \quad (36)$$

and

$$\begin{aligned} \delta R_{11} = & - \left[ \delta f_{0,r,r} + (2f_{0,r} - f_{1,r}) \delta f_{0,r} - f_{0,r} \delta f_{1,r} \right] \\ & + e^{-2f_0+2f_1} \left[ \delta f_{1,0,0} + (2f_{1,0} - f_{0,0}) \delta f_{1,0} - f_{1,0} \delta f_{0,0} - 2(\delta f_0 - \delta f_1) (f_{1,0,0} + f_{1,0}^2 - f_{0,0} f_{1,0}) \right] . \end{aligned} \quad (37)$$

### II.2.b The $SL(2, R)/U(1)$ Black Hole

The  $SL(2, R)/U(1)$  black hole with Lorentzian signature is the solution to eqs. (25) and (26) for  $\tilde{V} = \tilde{V}' = 0$  given by the metric tensor of eq.(20) characterized by the Wick rotation of the metric components given in eq.(9) above:

$$g_{00} = -e^{2f_0} = -\frac{k}{2} \tanh^2 r , \quad (38)$$

$$g_{11} = e^{2f_1} = \frac{k}{2} , \quad (39)$$

where  $k = 2\Lambda^{-2}$  is the level of the underlying Wess–Zumino action, along with a dilaton field given by

$$\Phi = \log \cosh^2 r + \eta , \quad (40)$$

where  $\eta$  is a constant which is related to the mass  $M$  of the black hole through the equation  $e^\eta = (k/2)^{1/2} M$ .

In order to study the small fluctuations around this background configuration we specialize eqs. (30) and (31) to the case of  $\tilde{V} = \tilde{V}' = \delta\tilde{V} = \delta\tilde{V}' = 0$ , which yields:

$$\begin{aligned} 0 = & g^{\mu\nu} \delta \Phi_{,\mu,\nu} + \left[ g^{\mu\nu}_{,\nu} + g^{\mu\nu} (f_0 + f_1)_{,\nu} + 2g^{\mu\nu} \Phi_{,\nu} \right] \delta \Phi_{,\mu} \\ & + \Phi_{,\nu} \delta g^{\mu\nu}_{,\mu} + \left[ (f_0 + f_1)_{,\mu} \Phi_{,\nu} + 2\Phi_{,\mu} \Phi_{,\nu} \right] \delta g^{\mu\nu} + g^{\mu\nu} \Phi_{,\nu} (\delta f_0 + \delta f_1)_{,\mu} , \end{aligned} \quad (41)$$

and

$$\delta R_{\mu\nu} = \delta\Phi_{,\mu,\nu} - \Phi_{,\lambda}\delta\Gamma_{\mu\nu}^{\lambda} - \Gamma_{\mu\nu}^{\lambda}\delta\Phi_{,\lambda} . \quad (42)$$

The *basic perturbation equations* for the Witten black hole are then obtained by substituting eqs. (38), (39) and (40) into eqs. (41) and (42), which yields

$$\delta f_0'' + 2\coth r \delta f_0' - \operatorname{sech} r \operatorname{csch} r \delta f_1' + \omega^2 \coth^2 r \delta f_1 + \operatorname{sech} r \operatorname{csch} r \delta\Phi' + \omega^2 \coth^2 r \delta\Phi = 0 \quad (43)$$

$$\delta f_0'' + 2\operatorname{sech} r \operatorname{csch} r \delta f_0' - \tanh r (\operatorname{csch}^2 r + 2) \delta f_1' + \omega^2 \coth^2 r \delta f_1 + \delta\Phi'' = 0 \quad (44)$$

$$\delta\Phi'' + \tanh r (\operatorname{csch}^2 r + 4) \delta\Phi' + \omega^2 \coth^2 r \delta\Phi - 2\tanh r \delta f_1' - 8\delta f_1 + 2\tanh r \delta f_0' = 0 \quad (45)$$

$$0 = i\omega (\delta\Phi' - 2\tanh r \delta f_1 - \operatorname{sech} r \operatorname{csch} r \delta\Phi) , \quad (46)$$

where in these equations a prime denotes differentiation with respect to  $r$ . As expected for any two-dimensional black hole, as described above, we find that these form a system of coupled, linear, ordinary differential equations. As noted in the Introduction, this model of two-dimensional dilaton gravity does not incorporate propagating degrees of freedom. This obviously does not imply that the linearized equations of motion may not be reduced to differentially-decoupled form. In order to proceed we would like to attempt to follow the prescription outlined in the previous section. Thus we must search for a suitable transformation which will put the system into differentially decoupled form, after which the final reduction to a completely decoupled set of equations would proceed without difficulty. In this connection we note that eq.(46) fixes the relation between  $\delta f_1$  and  $\delta\Phi$ , as a consequence of which we would naively expect a final reduction to two decoupled second-order equations for the physical perturbation functions. However, we also see that the distinct field perturbations appear on an unequal footing in these equations: neither  $\delta f_0$  nor  $\delta f_1''$  appear in these equations and, as we shall discover, this fact portends unusual consequences.

We will begin the attempt to differentially decouple the system given in eqs. (43) through (46) by noting that, in virtue of the so-called Curci-Paffuti equations [26]

$$\frac{1}{2}\nabla_{\nu}\beta^{(\Phi)} = \nabla^{\mu}\beta_{\mu\nu}^{(g)} - 2\beta_{\mu\nu}^{(g)}\nabla^{\mu}\Phi , \quad (47)$$

where  $\beta^{(\Phi)}$  and  $\beta_{\mu\nu}^{(g)}$  are the beta-functions<sup>8</sup> for the dilaton and gravitational fields, respectively, we are guaranteed that any single one of the equations of motion of

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<sup>8</sup>  $\beta_{\mu\nu}^{(g)}$  and  $\beta^{(\Phi)}$  are beta-functions of the non-linear sigma model with target space metric given by eqs. (38) and (39), and generate the equations of motion for the  $SL(2, R)/U(1)$  black hole.

the background fields is automatically satisfied if the beta-functions corresponding to the remaining equations vanish. This in turn allows us to proceed to attempt to decouple the system of perturbation equations by considering first eqs. (43) through (45), without imposing eq.(46). To that end we can first eliminate  $\delta\Phi''$  between eqs. (44) and (45) to obtain:

$$\begin{aligned} \delta f_0'' + 2\delta f_0'(\operatorname{sech} r \operatorname{csch} r - \tanh r) - \delta\Phi'(\operatorname{sech} r \operatorname{csch} r + 4\tanh r) \\ - \omega^2\delta\Phi\coth^2 r - \operatorname{sech} r \operatorname{csch} r \delta f_1' + \delta f_1(\omega^2\coth^2 r + 8) = 0 . \end{aligned} \quad (48)$$

We now write eqs. (43) and (48) as a simultaneous system:

$$\mathcal{M}X = Y , \quad (49)$$

where

$$\mathcal{M} = \begin{pmatrix} \omega^2\coth^2 r & -\operatorname{sech} r \operatorname{csch} r \\ \omega^2\coth^2 r + 8 & -\operatorname{sech} r \operatorname{csch} r \end{pmatrix} , \quad (50)$$

$$X = \begin{pmatrix} \delta f_1 \\ \delta f_1' \end{pmatrix} , \quad (51)$$

$$Y = \begin{pmatrix} -\delta f_0'' - 2\coth r \delta f_0' - \operatorname{sech} r \operatorname{csch} r \delta\Phi' - \omega^2\coth^2 r \delta\Phi \\ -\delta f_0'' - 2\delta f_0'(\operatorname{sech} r \operatorname{csch} r - \tanh r) + \delta\Phi'(\operatorname{sech} r \operatorname{csch} r + 4\tanh r) + \omega^2\coth^2 r \delta\Phi \end{pmatrix} . \quad (52)$$

Using

$$\mathcal{M}^{-1} = \frac{1}{8} \begin{pmatrix} -1 & 1 \\ -\sinh r \cosh r (\omega^2\coth^2 r + 8) & \omega^2\coth r \cosh^2 r \end{pmatrix} , \quad (53)$$

we find:

$$4\delta f_1 = 2\tanh r \delta f_0' + (\operatorname{sech} r \operatorname{csch} r + 2\tanh r) \delta\Phi' + \omega^2\coth^2 r \delta\Phi , \quad (54)$$

and

$$\begin{aligned} 4 \operatorname{sech} r \operatorname{csch} r \delta f_1' &= \\ &= 4\delta f_0'' + 2(\omega^2 + 4) \coth r \delta f_0' \\ &\quad + [\omega^2\coth^2 r (\operatorname{sech} r \operatorname{csch} r + 2\tanh r) + 4\operatorname{sech} r \operatorname{csch} r] \delta\Phi' \\ &\quad + \omega^2\coth^2 r (\omega^2\coth^2 r + 4) \delta\Phi . \end{aligned} \quad (55)$$

Differentiating eq.(54) we get:

$$4\delta f_1' = 2 \tanh r \delta f_0'' + 2\text{sech}^2 r \delta f_0' + (\text{sech } r \text{ csch } r + 2\tanh r) \delta \Phi'' - (\text{sech}^2 r \text{csch}^2 r - \omega^2 \coth^2 r) \delta \Phi' - 2\omega^2 \coth r \text{csch}^2 r \delta \Phi, \quad (56)$$

and combining eqs. (55) and (56) we obtain:

$$2 \tanh r (\sinh^2 r + \cosh^2 r) \delta f_0'' + 2 \cosh^2 r (\omega^2 + 4 - \text{sech}^4 r) \delta f_0' - (\text{sech } r \text{ csch } r + 2\tanh r) \delta \Phi'' + (4 + \text{sech}^2 r \text{csch}^2 r + 2\omega^2 \cosh^2 r) \delta \Phi' + \omega^2 \coth r \left[ (\omega \cosh r \coth r)^2 + 2 (2 \cosh^2 r + \text{csch}^2 r) \right] \delta \Phi = 0. \quad (57)$$

Returning now to the basic perturbation equations (eqs. (43) through (46)), we substitute the values for  $\delta f_1$  and  $\delta f_1'$  dictated by eq.(54) into eq.(45), which, after some manipulation, yields

$$\text{sech}^2 r \delta \Phi'' - \left[ \omega^2 \coth r + \text{sech}^3 r \text{csch } r (\sinh^2 r + \cosh^2 r) \right] \delta \Phi' - 2\omega^2 \delta \Phi - 2 \tanh^2 r \delta f_0'' - 2 \tanh r (2 + \text{sech}^2 r) \delta f_0' = 0. \quad (58)$$

Upon eliminating  $\delta f_0''$  between eqs. (57) and (58) we find:

$$-\coth^2 r \delta \Phi' + \coth^3 r (\omega^2 \cosh^2 r + 2) \delta \Phi + 2 \cosh^2 r \delta f_0' = 0. \quad (59)$$

Given this reduction, it is evident that the original system of equations is not amenable to further reduction to differentially–decoupled form through the use of a transformation of the dependent variables. The most general simultaneous *linear* transformation<sup>9</sup> of all of the dependent variables when substituted into the set of equations given by (43) through (45) fails to differentially decouple the system, which at first appears to be surprising.<sup>10</sup> That this is the case, however, becomes apparent when one notices that eq.(46) is in fact equivalent to eqs. (54) and (59). Thus, one may check that the

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<sup>9</sup>A nonlinear or more complicated non–local transformation would be inconsistent with the restriction to small fluctuations we have imposed throughout the analysis.

<sup>10</sup>The algebraic manipulations required in the analysis of the general case are very involved. A machine symbolic manipulation program, such as *Mathematica*, proves extremely useful in sorting out the many pieces.

original system of perturbation equations given in eqs.(43) through (46) is actually *entirely a consequence* of the following two first–order differential equations:

$$0 = 2\delta f_0' - \operatorname{csch}^2 r \delta\Phi' + \cosh r \operatorname{csch}^3 r (\omega^2 \cosh^2 r + 2) \delta\Phi , \quad (60)$$

$$0 = 2\delta f_1 - \coth r \delta\Phi' + \operatorname{csch}^2 r \delta\Phi . \quad (61)$$

We therefore find that, for a given frequency  $\omega$ , the spatial evolution of the small fluctuations of the Witten solution is completely determined by eqs. (60) and (61). We note that these are two equations in three unknowns, and that only  $\delta\Phi$  and its derivative appear in both eqs. (60) and (61). The consequence of this is that one finds that a consistent first–order<sup>11</sup> perturbation solution may be found for *any* choice of functional form for  $\delta\Phi$ , which is a completely unprecedented result. This result is altogether different from the corresponding results one finds for any of the solved small–fluctuation problems involving the known black hole solutions in the general theory of relativity, where to date one has always encountered eigenvalue equations (*cf* eq.(19)) for *some* perturbation potential, and where in virtue of the decoupling constraints it is *not* possible to find solutions for arbitrary perturbations.<sup>12</sup> In the usual case, the classical linear response of the black hole may be determined once the perturbation potentials are known. The linear response is defined by the scattering coefficients, which, for an assumed asymptotic behavior, are uniquely predicted by the radial eigenvalue equations. In the present case, in contrast, the equations admit a continuous infinity of solutions. Within this set of solutions are entirely distinct functions with identical asymptotic behavior, and thus the equations do not uniquely determine scattering coefficients. Therefore, although eqs. (60) and (61) may be unambiguously solved, it is nevertheless impossible to unambiguously ascertain the linear response of the black hole. We may thus say that the equations which normally determine the linear response of the black hole are in this case *physically unproductive*.

<sup>13</sup>

We may gain perspective on the surprising behavior of the  $SL(2, R)/U(1)$  black hole which we have discovered by viewing our results in the context of the underlying conformal field theory. In particular we shall consider the dimension (1, 1) operators

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<sup>11</sup>Technical details involving boundary conditions on the fields and residual gauge invariance, respectively, are discussed in Notes 2 and 3 below, following the conclusion section.

<sup>12</sup>One also finds results similar to those which obtain in general relativity in the case of the known “string–inspired” solutions in four dimensions, *i.e.*, for dilaton gravity in four dimensions [27].

<sup>13</sup>Note that this is not the same as the condition of a vanishing perturbation potential,  $v_i = 0$ , for which one would have a reduced system of the form  $\delta p_i'' + \omega^2 \delta p_i = 0$ .

of the conformal field theory: the so-called marginal operators. These operators have the property that when one or more of them is incorporated into the definition of the sigma model the value of the central charge is preserved. There is a *subset* of these operators which are further distinguished by the property that the conformal dimensions of all operators in this subset are preserved as well in the modified model. This special class of marginal operators are known as exactly marginal operators. Their properties have been elaborated in [8], where it was shown how to explicitly compute them to first-order in an expansion in  $1/k$ .<sup>14</sup> The spacetime effect on the background fields of specific exactly marginal operators (evaluated to first-order in  $1/k$ ) was derived in [8]. We shall now consider the compatibility of this action with the conditions embodied within the basic perturbation equations of the  $SL(2, R)/U(1)$  black hole. The particular exactly marginal operators investigated in [8] were the operators  $L_0^1 \bar{L}_0^1$  and  $L_0^2 \bar{L}_0^2$ .<sup>15</sup> It was shown in [8] that the addition of the operator  $L_0^2 \bar{L}_0^2$  to the action of the non-linear sigma model generates the deformed lagrangian  $L$  given by ( $\alpha$  is an arbitrary parameter)

$$L = \partial_z r \partial_{\bar{z}} r \left[ 1 - 2\alpha (\operatorname{csch}^2 r + \operatorname{sech}^2 r) \right] + \partial_z \theta \partial_{\bar{z}} \theta \left[ \sinh^2 r + 2\alpha - \frac{(\sinh^2 r + 2\alpha)^2}{\cosh^2 r + 2\alpha} \right]. \quad (62)$$

We would like to determine whether or not this deformation, produced by an exactly marginal operator, is encompassed within the continuous infinity of allowed deformations we have discovered in our analysis of the small fluctuations of the black hole.<sup>16</sup> In comparing the deformation produced by the operator  $L_0^2 \bar{L}_0^2$  with our analysis of the small fluctuations it is important to note that the calculation in [8] leading to eq.(62) was performed with the neglect of terms in the sigma model which were of higher than bilinear order in derivatives. With the proviso that the metric is asymptotically-flat it is legitimate to neglect these terms in the limit  $r \rightarrow \infty$ . Thus, we may read off from eq.(62) the appropriate fields to substitute into the perturbation equations given in eqs. (60) and (61), taking care to work in the large- $r$  limit. It is straightforward to verify that eqs. (60) and (61) are indeed satisfied in this limit, and we thus find that a *particular* example of a first-order fluctuation which is consistent with the linear

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<sup>14</sup>In the limit that  $k \rightarrow \infty$  one obtains purely *classical* conformal field theory.

<sup>15</sup>Here  $L_n^s$  is defined as  $L_n^s = V_n^s + \tilde{V}_n^s$ , where  $V_n^s$  and  $\tilde{V}_n^s$  are the  $n$ 'th Fourier components of two of the generators of the super- $W_\infty$  algebra, and  $s$  is the  $W_\infty$  "spin" of the algebra [8].

<sup>16</sup>Note that it is appropriate to ask this question since the deformation in eq.(62) has been computed to lowest-order in an expansion in  $1/k$ , which is to say that it represents a classical conformal field theoretic effect. As such, it is consistent to compare it with our analysis of the small fluctuations since it has also been (implicitly) performed at lowest order in  $1/k$ .

constraints given by the basic perturbation equations is provided by the operator  $L_0^2 \bar{L}_0^2$ . However, the analogous calculation applied to the operator  $L_0^1 \bar{L}_0^1$  reveals that the deformation it generates corresponds to the excitation of a *non-linear* departure from the background [28]. Specifically, the tachyon field, which is implicitly present in the background with zero field strength in the Witten solution, appears as a second-order perturbation. However, we have restricted our analysis to *small* fluctuations understood to be of first-order, and it is thus inappropriate to compare the effect of this operator with our results. The complete set of all exactly marginal operators is believed to constitute a *countably* infinite set, since the quantum numbers which distinguish them are discretely valued. Clearly these cannot encompass all of the allowed deformations we have discovered, since, as we have demonstrated above, the linearized perturbation equations of the  $SL(2, R)/U(1)$  black hole allow a *continuous* infinity of solutions. Although we have demonstrated that the operator  $L_0^2 \bar{L}_0^2$  at large  $r$  generates a particular one of the continuous infinity of deformations we have discovered, the fact that the operator  $L_0^1 \bar{L}_0^1$  does *not* generate a small fluctuation suggests that only some (and perhaps none) of the remaining exactly marginal operators excite small fluctuations.<sup>17</sup> Thus the mere existence of an infinite set of exactly marginal operators does not imply that there are an infinity of allowed small fluctuations, and even if it did, this would have accounted for only a countable infinity. We have thus discovered a new, continuously infinite class of motions the fundamental origin of which awaits explanation.

One may therefore enquire as to precisely where the new continuous infinity of allowed small fluctuations we have discovered fits in the description of the physics of the  $SL(2, R)/U(1)$  black hole. The (subset of the) countably infinite set of exactly marginal operators consistent with the basic perturbation equations is evidently insufficient to describe all of the allowed motions of the black hole. This black hole is actually a particular two-dimensional solution to the equations of motion of string theory. More precisely, the Witten black hole is an *approximate* solution to the string equations of motion: It is a solution at the level of the Born approximation in string theory since the sigma model has been formulated on a sphere and thus all higher-loop (and, more generally, non-perturbative) string corrections have been ignored; It is evidently an approximate solution to the sigma model as well, as reflected in the presence of  $O(1/k^2)$  corrections to the value of the central charge pointed out by Witten in [2]. It is natural to speculate that what is missing from the picture lies in

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<sup>17</sup>Actually, the fact that we have shown that the exactly marginal operator  $L_0^2 \bar{L}_0^2$  excites an allowed small fluctuation serves to verify the *consistency*, to first-order in  $1/k$ , of the two  $1/k$  expansions, used to derive the black hole and to explicitly compute  $L_0^2 \bar{L}_0^2$ , respectively.

the corrections that have been neglected in the higher string-loop contributions, or in the higher-order  $1/k$  contributions on the sphere, or perhaps some combination of both contributions. The underlying  $W_\infty$  structure of this model appears to be related to the existence of an infinite number of exactly marginal operators. However, as we have stressed, it is not obvious that these operators generate a (countably infinite) set of small fluctuations, and we expect this to remain true even if one were to consider the effect of the exactly marginal operators computed to *all* orders in  $1/k$ . As we have also stressed, however, even if the exactly marginal operators computed to all orders in  $1/k$  *did* excite a countably infinite set of modes, this would not account for the continuous infinity of perturbations we have found. Ideally, one would like to compare the effects of exactly marginal operators, calculated to all orders in  $1/k$ , to small fluctuations as determined by the *exact* beta-functions. The latter, unfortunately, are not known at present, although the solution to the equations they correspond to (with the same leading order behavior as the Witten black hole) has been calculated [6,9]. It should in any event be worthwhile to extend our results by examining the next-to-leading-order corrections in  $1/k$ .

### II.2.c Massive Dilaton Black Holes

We will now study the small fluctuations of two-dimensional black hole configurations in which the dilaton is massive, and we thus return to eqs. (25) and (26). In order to proceed it is necessary to select a particular form for the potential energy density  $V(\Phi)$ . Here one has a great deal of latitude since, apart from a special case such as the Witten solution (*i.e.*, choosing  $\tilde{V} = \tilde{V}' = 0$ ) which furnishes a solution at the level of the Born approximation to the equations of motion of string theory, the models defined by the action of eq.(22) are no more than “string-inspired” models. Thus, the fact that it is not today known how (or better, if) string-theoretic principles determine the form of the dilaton potential is to a certain extent unimportant. We shall here follow the choice made in recent studies of these configurations [19] in which the potential is chosen by *fiat* to be of the form:

$$V(\Phi) = m^2\Phi^2, \tag{63}$$

where  $m$  is the mass of the dilaton.<sup>18</sup> This is certainly the simplest non-trivial choice

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<sup>18</sup>It must be remembered *throughout* the following analysis that, as stated above in the text and in footnote #7, in our calculations we make the substitution  $\Phi \rightarrow -\Phi/2$  in the equations of motion. This should be borne in mind when comparing certain expressions below with corresponding expressions given in [19]. In particular with the choice of  $V$  given in eq.(63), one has  $\tilde{V} = \frac{1}{4}m^2\Phi^2$ .

one may make for the potential, and it is conceivable that such a choice may prove to be useful. Upon substituting eq.(63) into eqs. (25) and (26) one then obtains:

$$0 = \nabla^2 \Phi + (\nabla \Phi)^2 - 4\Lambda^2 + \frac{1}{4}m^2 e^\Phi \Phi^2 , \quad (64)$$

$$R_{\mu\nu} = \nabla_\mu \nabla_\nu \Phi + \frac{1}{2}m^2 e^\Phi g_{\mu\nu} \left( \frac{1}{4}\Phi^2 + \frac{1}{2}\Phi \right) . \quad (65)$$

In recent studies a putative massive dilaton black hole configuration was studied by employing the ansatz of eq.(20) for the metric tensor, with the metric functions taking the values

$$g_{00} = -e^{2f_0} = -A^2, \quad g_{11} = e^{2f_1} = A^{-2} , \quad (66)$$

where  $A = A(r)$  is to be determined by solving the field equations. It was shown that there exist two possible black hole solutions: one for which the dilaton field strength is given by a constant:  $\Phi = p_0$ , say, and another for which the dilaton field is proportional to  $r$ :  $\Phi = p_1 r$ .

### II.2.c.i Constant Dilaton Solution

In the case of a constant dilaton field,  $\Phi = p_0$ , one may prove that the constant scalar curvature is given by<sup>19</sup>

$$R = -(A^2)'' = 4\Lambda^2 (1 + 2p_0^{-1}) , \quad (67)$$

as a result of which we find the metric solution  $A^2 = ar^2 + br + c$ , where

$$a = -2\Lambda^2 (1 + 2p_0^{-1}) , \quad (68)$$

and  $b$  and  $c$  are integration constants. We will now consider the basic perturbation equations (eqs. (30) through (31)) for the constant dilaton solution. We observe first that the (01)-component of the linearized Einstein equations is given by

$$0 = i\omega (\delta\Phi' - A^{-1}A'\delta\Phi) , \quad (69)$$

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<sup>19</sup>See the comment in footnote #18.

which immediately yields the integral  $\delta\Phi = \kappa A$  with  $\kappa$  a constant. We must now ensure that our small-fluctuation approximation is valid, which is the case if  $|\delta\Phi/\Phi| \ll 1$ . For the constant dilaton solution this means that we must have

$$|\delta\Phi(r)/\Phi| = \left| \frac{\kappa}{p_0} \sqrt{ar^2 + br + c} \right| \ll 1 . \quad (70)$$

We will now prove that this inequality dictates that we must take  $\kappa = 0$  for the value of the integration constant. The smallness constraint must be satisfied everywhere in order to justify the neglect of terms of higher than first-order in our analysis, and in particular in the limit  $r \rightarrow \infty$ . From eq.(70) we see that we must have

$$\lim_{r \rightarrow \infty} |\delta\Phi/\Phi| = \lim_{r \rightarrow \infty} \left| \frac{\kappa a^{1/2}}{p_0} r \right| \ll 1 , \quad (71)$$

which implies that  $\kappa = 0$ , or that  $a=0$ , or both.<sup>20</sup> However, if  $a = 0$  we have

$$\lim_{r \rightarrow \infty} |\delta\Phi/\Phi| = \lim_{r \rightarrow \infty} \left| \frac{\kappa b^{1/2}}{p_0} r^{1/2} \right| \ll 1 , \quad (72)$$

which implies that  $\kappa = 0$ , or that the integration constant  $b = 0$ , or both. However, if  $a=b=0$ , one has  $A^2 = c$ , in which case for arbitrary non-vanishing<sup>21</sup>  $c$  the metric tensor is constant and non-singular (*cf* eq.(66)), and the configuration is no longer a black hole at all.<sup>22</sup> Therefore we must require that  $\kappa = 0$ , as a result of which we have found that

$$\delta\Phi = 0 , \quad (73)$$

and thus all first-order fluctuations in the dilaton field have exactly vanishing amplitude. As a result of this we observe that the linearized dilaton equation (*cf* eq.(30)) vanishes identically. Furthermore, making use of eq.(31) we find that the (00)- and (11)-components of the Einstein equations simplify dramatically, and we obtain

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<sup>20</sup>From eq.(68) we see that  $a$  can vanish for special values of  $p_0$  or  $\Lambda$ .

<sup>21</sup>In the special case  $a = b = c = 0$  the metric function  $A^2$  vanishes identically in which case the metric tensor is ill-defined globally. In any event, we note in passing that for this case one obtains eq.(73) automatically.

<sup>22</sup>Note that this argument is distinct from the observation of the fact that when  $a = 0$  the curvature vanishes (*cf* eq.(67)). The constant dilaton configuration is a black hole in virtue of the fact that there is an event horizon, and *not* because there is a curvature singularity, as indeed there is not.

$$\begin{aligned}
& A^4 \left[ \delta f_0'' + 3A^{-1}A'\delta f_0' - A^{-1}A'\delta f_1' + 2(\delta f_0 - \delta f_1) \left( A^{-1}A'' + A^{-2}A'^2 \right) \right] + \omega^2 \delta f_1 \\
& = -\frac{1}{2}A^2 m^2 e^{p_0} \left( p_0 + \frac{1}{2}p_0^2 \right) \delta f_0 , \tag{74}
\end{aligned}$$

for the (00)–equation, and

$$-\delta f_0'' - 3A^{-1}A'\delta f_0' + A^{-1}A'\delta f_1' - \omega^2 A^{-4} \delta f_1 = \frac{1}{2}A^{-2} m^2 e^{p_0} \left( p_0 + \frac{1}{2}p_0^2 \right) \delta f_1 , \tag{75}$$

for the (11)–equation. Upon multiplying the (11)–equation by  $A^4$  and adding the result to the (00)–equation one obtains

$$(\delta f_0 - \delta f_1) \left[ A^3 A'' + A^2 A'^2 + \frac{1}{4}A^2 m^2 e^{p_0} \left( p_0 + \frac{1}{2}p_0^2 \right) \right] = 0 . \tag{76}$$

The equations of motion of the background fields may now be used to obtain the relation  $16\Lambda^2 = m^2 p_0^2 e^{p_0}$ . Upon substituting this expression into eq.(76), along with the value of  $A^2$  with  $a$  given by eq.(68), one finds that the quantity in the square brackets vanishes identically, and thus the two gravitational equations form a redundant system and there is only one independent equation. The consequence of this is that the black hole coupled to a massive dilaton with constant field strength behaves in a manner similar to that of the  $SL(2, R)/U(1)$  black hole [2]: a solution for one of the gravitational perturbations may be found for *any* choice of the other one. As before, we must ensure that the solutions are sufficiently small to be considered as first–order perturbations. Since we have shown that eqs. (74) and (75) are equivalent, we may check that the fluctuations are acceptable by considering either one of them. To that end we note that the (11)–equation may be written as

$$0 = A^4 \delta f_0'' + 3A^3 A' \delta f_0' - A^3 A' \delta f_1' + \omega^2 \delta f_1 + \varpi A^2 \delta f_1 , \tag{77}$$

where  $\varpi = \frac{1}{2}m^2 e^{p_0} (p_0 + \frac{1}{2}p_0^2)$  is a constant. Eq.(77) can be rewritten as

$$0 = A (A^3 \delta f_0')' - \frac{1}{4} (A^4)' \delta f_1' + (\omega^2 + \varpi A^2) \delta f_1 , \tag{78}$$

which may be integrated to yield

$$A^3 \delta f_0' = \psi + \int_{r_h}^r dr \left[ \frac{1}{3} (A^3)' \delta f_1' - (\omega^2 + \varpi A^2) A^{-1} \delta f_1 \right] , \tag{79}$$

where  $r_h$  is the position of the event horizon and  $\psi$  is the constant of integration. Now, since  $A^2 = ar^2 + br + c$ , near the horizon one has

$$A \sim (r - r_h)^{1/2}, \quad A^3 \sim (r - r_h)^{3/2}, \quad (80)$$

*etc.* Noting that  $\delta f_0' = A^{-1}A'$ , we will find it convenient to ensure that  $|\delta f_0/f_0| \ll 1$  by proving the sufficient condition that  $|\delta f_0'/f_0'| \ll 1$ . Substituting eq.(80) into eq.(79) we find

$$(r - r_h)^{-3/2} \left\{ \psi + (r - r_h)^{1/2} \delta f_0 - \int_{r_h}^r dr \left[ \omega^2 (r - r_h)^{-1/2} + \varpi (r - r_h)^{1/2} \right] \delta f_1 \right\} \ll (r - r_h)^{-1}. \quad (81)$$

This condition requires that we take  $\psi = 0$  for the value of the integration constant, as a result of which the constraint will be satisfied as long as  $\delta f_1$  is regular in the limit  $r \rightarrow r_h$ . One can similarly check that a continuous distribution of small gravitational fluctuations can be found in the limit  $r \rightarrow \infty$ . Thus, as is the case for the Witten black hole, there exist a continuous infinity of small-fluctuation solutions to the linearized equations of motion for the massive dilaton black hole with constant field strength, and it is therefore *in principle* impossible to unambiguously determine the classical linear response of the black hole. Of course, unlike the  $SL(2, R)/U(1)$  black hole, here the fluctuation in the dilaton field is constrained to vanish, but there remains an uncountably infinite ambiguity in the gravitational perturbations. Although this black hole is characterized by a massive dilaton and is therefore not described in terms of a conformal field theory, whereas the  $SL(2, R)/U(1)$  black hole is so described, the two different two-dimensional configurations display similar behavior: the classical linear response is indeterminate, an unusual situation which differs radically from the behavior of all known black holes in four dimensions.

### II.2.c.ii Linear Dilaton Solution

In the case of a linear dilaton solution with  $\Phi = p_1 r$  the equations of motion for the background fields have been solved by Gregory and Harvey [19], who find the following expression for the metric function  $A$ :<sup>23</sup>

$$A^2 = 1 - 2M e^{\pm p_1 r} - \frac{m^2}{16p_1^2} e^{\mp p_1 r} (2p_1^2 r^2 \pm 2p_1 r + 1), \quad (82)$$

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<sup>23</sup>See the comment in footnote #18.

with  $M$  (not to be confused with the dilaton mass  $m$ ) arbitrary. With the background metric specified by eq.(66) we may consider the basic perturbation equations for the black hole. We obtain

$$0 = i\omega (\delta\Phi' - A^{-1}A'\delta\Phi - p_1\delta f_1) , \quad (83)$$

from the (01)–component of the linearized Einstein equation, and

$$\begin{aligned} 0 = & A^2\delta\Phi'' + \left[ (A^2)' + 2p_1A^2 \right] \delta\Phi' + \left[ \omega A^{-2} + \frac{1}{4}p_1m^2re^{p_1r} (2 + p_1r) \right] \delta\Phi \\ & + p_1A^2\delta f_0' - 2p_1 \left[ (A^2)' + p_1A^2 \right] \delta f_1 - p_1A^2\delta f_1' , \end{aligned} \quad (84)$$

for the linearized dilaton equation. For the (00)–component of the linearized Einstein equations we obtain

$$\begin{aligned} & A^4 \left[ \delta f_0'' + 3A^{-1}A'\delta f_0' - A^{-1}A'\delta f_1' + 2(\delta f_0 - \delta f_1) \left( A^{-1}A'' + A^{-2}A'^2 \right) \right] + \omega^2\delta f_1 \\ & = -\omega^2\delta\Phi - \frac{1}{2}A^2(A^2)'\delta\Phi' + p_1A^2 \left[ (A^2)'\delta f_1 - (A^2)'\delta f_0 - A^2\delta f_0' \right] \\ & \quad - \frac{1}{2}e^\Phi \left[ A^2m^2 \left( \Phi + \frac{1}{2}\Phi^2 \right) \delta f_0 + A^2m^2 \left( \frac{1}{4}\Phi^2 + \Phi + \frac{1}{2} \right) \delta\Phi \right] , \end{aligned} \quad (85)$$

and we find

$$\begin{aligned} & -\delta f_0'' - 3A^{-1}A'\delta f_0' + A^{-1}A'\delta f_1' - \omega^2A^{-4}\delta f_1 \\ & = \delta\Phi'' - p_1\delta f_1' + A^{-1}A'\delta\Phi' \end{aligned} \quad (86)$$

for the (11)–component of the linearized Einstein equations. By appropriately combining these equations and making use of the expression for  $A(r)$  given in eq.(82), we may rewrite the system as<sup>24</sup>

$$0 = \alpha(r)\delta\Phi' + \beta(r)\delta\Phi , \quad (87)$$

$$0 = \gamma(r)\delta f_0 + \epsilon(r)\delta\Phi' + \rho(r)\delta\Phi , \quad (88)$$

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<sup>24</sup>We have taken  $M = 0$ , to ensure an asymptotically–flat metric, as well as the upper choice of sign in eq.(82).

$$0 = -p_1 \delta f_0' + \tau(r) \delta \Phi' + \sigma(r) \delta \Phi . \quad (89)$$

In these equations the scalar functions  $\alpha(r)$  and  $\beta(r)$  are given by<sup>25</sup>

$$\alpha(r) \equiv -8e^{3p_1 r} m^2 r (2 + p_1 r) , \quad (90)$$

$$\begin{aligned} \beta(r) \equiv & I^{-1} (7m^4 - 832e^{2p_1 r} m^2 p_1^2 + 40m^4 p_1 r + 48e^{3p_1 r} m^4 p_1 r + 1792e^{2p_1 r} m^2 p_1^3 r + 192m^4 p_1^2 r^2 \\ & + 216e^{3p_1 r} m^4 p_1^2 r^2 - 512e^{2p_1 r} m^2 p_1^4 r^2 - 576m^4 p_1^3 r^3 - 288e^{3p_1 r} m^4 p_1^3 r^3 + 192m^4 p_1^4 r^4 \\ & - 192e^{3p_1 r} m^4 p_1^4 r^4) , \end{aligned} \quad (91)$$

where

$$I \equiv m^2 - 64e^{2p_1 r} p_1^2 + 4m^2 p_1 r + 8m^2 p_1^2 r^2 . \quad (92)$$

The analysis of this coupled system of differential equations proceeds as follows. One first differentiates eq.(88), which may be used to eliminate all terms proportional to both  $\delta f_0$  and  $\delta f_0'$  across eqs. (88) and (89), and hence from the complete system since no such terms appear in eq.(87). Then eq.(87) and its derivative may be used to eliminate all terms proportional to  $\delta \Phi'$  and  $\delta \Phi''$  from the system as well. The result of these successive operations is a single equation of the form<sup>26</sup>

$$\chi(r) \delta \Phi(r) = 0 . \quad (93)$$

The next step in the analysis entails a numerical examination of the function  $\chi(r)$ , which demonstrates that in general one has  $\chi(r) \neq 0$ , as may be seen in Table 1 where representative values of  $\chi(r)$  are displayed. This result suggests that  $\delta \Phi = 0$ . One may then also note that eq.(87) can be directly integrated to yield

$$\delta \Phi = \text{const.} \exp \left( - \int^r dr \beta / \alpha \right) . \quad (94)$$

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<sup>25</sup>The expressions for  $\gamma$ ,  $\epsilon$ ,  $\rho$ ,  $\tau$  and  $\sigma$  are huge and will not be displayed here. A *Mathematica* routine which generates these functions will be provided via electronic mail upon request.

<sup>26</sup>The explicit form of the function  $\chi(r)$  is extremely complicated and will not be given here. A *Mathematica* routine which generates this function will be provided via electronic mail upon request. The interested reader is warned that the output file is exceedingly large, consuming approximately 100 kilobytes of computer memory.

Given that generically  $\chi(r) \neq 0$ , and that this is true in particular for values of  $r$  for which  $\beta/\alpha$  is finite (as may easily be checked), the above equation can be consistent with the remaining equations (*i.e.*, eqs. (88) and (89), or, what is the same thing, with eq.(93)) only if the integration constant vanishes identically. Inspection of eq.(83) reveals that one must take  $\delta f_1 = 0$  for consistency. Finally, a numerical analysis of the function  $\gamma(r)$  demonstrates that in general one has  $\gamma(r) \neq 0$ , as may be seen in Table 2, where representative values of  $\gamma(r)$  are displayed, in virtue of which one must take  $\delta f_0 = 0$  for consistency (*cf* eq.(88)). This analysis demonstrates quite generally that the only consistent simultaneous solution of the coupled system of perturbation equations is the trivial solution in which *all* of the small fluctuations are constrained to vanish. Thus, we have found another unexpected result: the linear dilaton species of two–dimensional massive dilaton black hole does not admit *any* small fluctuations around the background configuration, in complete contrast once again to the corresponding results which have been obtained for the black holes of four–dimensional general relativity. The result indicates that the black hole coupled to a massive, linear dilaton represents an isolated point in the space of field configurations of two–dimensional dilaton gravity.

### III. Conclusions

We have found that the Witten black hole behaves in a radically different way from all other known black hole solutions, whether in the conventional general theory of relativity or in four–dimensional dilaton gravity. For those solutions one may perform an analysis (as outlined in Section II.1 above) of the linear response of the black hole to incoming waves which leads to decoupled eigenvalue equations for the physical fluctuations characterized by specific perturbation potentials. For these various black holes one finds different perturbation potentials corresponding to different varieties of uniquely determined scattering behavior, and indicative of whether or not a bound state can form. In the case of the Witten solution, however, the equations for the small fluctuations cannot be brought into completely decoupled form. In contrast to the situation which obtains for all previously studied black holes, there exist a continuous infinity of acceptable (*i.e.*, sufficiently small to be considered of first–order) solutions to the linearized equations of motion about the background. We have further shown that as a consequence of this it is impossible to unambiguously determine the classical linear response of the black hole, since the reduced perturbation equations do not uniquely determine the scattering coefficients for specified asymptotic behavior.

In studying a two–dimensional conformal field theory it is interesting to study the exactly marginal (1, 1) operators. In the case of the conformal field theory underlying

the  $SL(2, R)/U(1)$  black hole some of the exactly marginal operators may generate deformations of the action of the underlying sigma model which correspond to small fluctuations of the background fields of the black hole. We have explicitly confirmed this for the particular case of the exactly marginal operator  $L_0^2 \bar{L}_0^2$  by verifying that the small fluctuations it produces do indeed satisfy the linearized equations of motion. However, the exactly marginal operators constitute only a countably infinite set, and in any event, as we have discussed, only *some* of them will excite physically-acceptable small fluctuations. Thus it is necessary to look elsewhere in order to account for the complete, uncountably infinite set of small motions which our equations allow the black hole to perform. It is very surprising to encounter such an intrinsic ambiguity in the classical analysis of the linear response. However, we may recall that the  $SL(2, R)/U(1)$  black hole is a solution at the level of the Born approximation to the equations of motion of a string propagating in two dimensions. The black hole configuration is approximate as well in that higher-order corrections in an expansion in powers of  $1/k$  are neglected in obtaining the solution. Although this approximate character is well-known, the hope has been expressed by many authors that the black hole solution is nevertheless “very useful for getting a qualitative picture of the physics.” We suspect that the behavior we have uncovered, which is highly unusual, is sufficiently different from the behavior of all known four-dimensional black holes that it may be misleading to utilize this black hole model at all as a point of reference in studying the properties of physically-realistic black holes in four dimensions. It is natural to wonder whether a proper classical linear response can be restored by considering instead a black hole solution which is *exact*. Of course, the word exact has a double meaning here. One’s chief desire would be to have in hand a black hole solution which is truly exact in the sense of string theory, in which all string-loop corrections have been accounted for. Such a solution is not available at the moment, and may not be known for a long, long time. On the other hand, when considered solely as a black hole *qua* a solution to a two-dimensional theory of gravitation, one might hope that a proper linear response would be obtained by analyzing instead the corresponding two-dimensional black hole solution in which higher-order  $1/k$  corrections on the sphere have been included. Dijkgraaf, *et. al.* [6] and Bars and Sfetsos [9] have claimed to have derived such a solution, and work is in progress in extending the analysis of this paper to that black hole.

As discussed in Section I.2, there are a number of related black hole constructions which have been discovered recently. In particular, we examined two specific examples of related black hole solutions which have been found. These are both two-dimensional black holes coupled to a massive dilaton. In the somewhat special case

in which the background dilaton is characterized by a *constant* field strength, we find behavior reminiscent of the  $SL(2, R)/U(1)$  black hole, in that a continuous infinity of small fluctuations is admitted by the linearized equations of motion, and it is again impossible in principle to ascertain the classical linear response of the black hole. That this black hole behaves in a manner similar to the  $SL(2, R)/U(1)$  black hole is surprising in that one might have thought that the essential source of this unusual behavior in the case of the Witten solution might lie in its origin as a conformal field theory. However, since the constant dilaton black hole is in particular coupled to a massive dilaton, and is thus not derived from a conformal field theory, that explanation is open to question. We also analyzed the linear response of the two-dimensional black hole solution coupled to a massive, *linear* dilaton. This is an important example to consider since the linear dilaton vacuum is roughly analogous to four-dimensional Minkowski space. Here we found entirely different, but again unexpected, behavior as compared to the linear response of known four-dimensional black holes. In striking contrast to the other examples we studied, the black hole with massive linear dilaton is intrinsically constrained so that no small fluctuations are allowed at all. Thus this black hole configuration is an isolated point in the space of field configurations of the theory of two-dimensional dilaton gravity, and as such represents an unusual occurrence in a generally covariant theory.

These surprising results do not appear to be a consequence of the fact that the underlying dilaton gravity theories do not incorporate propagating degrees of freedom. All of the black holes we have studied share this property, yet they display two vastly different linear response behaviors. In this connection we note that attention has recently turned to the study of the CGHS black hole [3]. This black hole is being closely studied in an attempt to resolve questions of four-dimensional black hole physics, such as: What is the nature of the final result of the Hawking radiation process? Do black holes destroy information? If they do, does this signal that the very tenets of the quantum theory itself must be modified? Thus, the obvious candidate two-dimensional black hole which must, and which remains to be, analyzed using the methods of this paper is the CGHS solution. As we have discussed, the fundamental cause of the infinite classical fluctuation ambiguity found for the  $SL(2, R)/U(1)$  black hole is not yet clear. Nevertheless, the CGHS model has its origin in a non-linear sigma model which is closely related to that which underlies the Witten black hole. This suggests that the CGHS model may well also display classical linear response behavior which is radically different from that of all known four-dimensional black holes. Recall that the problems for which the CGHS model is being studied are inherently quantum mechanical in origin, and that the correspondence principle dic-

tates that one must properly recover classical mechanics from quantum mechanics in the appropriate limit. However, the two-dimensional black holes we have analyzed in detail in this paper do not display linear response behavior which is in any way characteristic of their four-dimensional counterparts. If the classical mechanical behavior of the CGHS black hole is indeed shown to be radically different from that of four-dimensional black holes, then its use as a toy model from which to draw inferences applicable to the outstanding problems of four-dimensional black holes must be treated with caution.

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## Notes

### Note 1

As the present paper was being completed the authors received a preprint (reference [29] by Diamandis, *et. al.*, in which related issues involving black holes with time-dependent tachyons are treated.

### Note 2

In order to ensure consistency with our assumption throughout that all fluctuations are small (and hence that only linear terms need be retained) one must, of course, choose  $\delta\Phi$  such that  $|\delta\Phi/\Phi| \ll 1$ . With the help of eq.(40) we see that this requires that

$$|\delta\Phi/\Phi| = \left| \frac{\delta\Phi}{\ln \cosh^2 r + \frac{1}{2} \ln(k/2) + \ln M} \right| \ll 1 . \quad (\text{N2.1})$$

Having chosen  $\delta\Phi$ , as stated in the text, one may always find a solution for the fluctuations in the metric by substituting it into eqs. (60) and (61). We must restrict our attention, however, to solutions for the metric perturbations which satisfy the constraints  $|\delta f_0/f_0| \ll 1$  and  $|\delta f_1/f_1| \ll 1$ . It is easy to see that there are a continuous infinity of simultaneous solutions which satisfy these constraints. For instance, by solving eq.(61) for  $\delta f_1$  and using eq.(39) we find that

$$|\delta f_1/f_1| = \left| \coth r \frac{\delta\Phi'}{\ln(k/2)} - \text{csch}^2 r \frac{\delta\Phi}{\ln(k/2)} \right| . \quad (\text{N2.2})$$

We are interested only in the behavior of fields at points outside of the event horizon, which is located at  $r = 0$ . Thus it is clear that we must consider the amplitudes of the

field perturbations at the two extreme locations:  $r = 0$  and  $r \rightarrow \infty$ , since it is only at these positions that it is possible for the necessary “smallness” constraints to be violated. We find that we must choose those perturbations in the dilaton field such that  $\delta\Phi$  vanishes at infinity and goes to zero faster than  $r$  at the horizon. Similarly, we may solve eq.(60) for  $\delta f_0'$  and then integrate both sides of the resulting equation. After an integration by parts, and making use as well of eq.(38), we obtain

$$|\delta f_0/f_0| = \left| \frac{\operatorname{csch}^2 r \delta\Phi}{\ln[(k/2) \tanh r]} - \frac{\omega^2 \int_0^r dr \coth^3 r \delta\Phi}{\ln[(k/2) \tanh r]} \right|. \quad (\text{N2.3})$$

We see that in order to satisfy the condition  $|\delta f_0/f_0| \ll 1$  it is necessary again to require that  $\delta\Phi$  vanish at infinity and go to zero faster than  $r$  at the event horizon. These simple conditions can obviously be satisfied for a *continuous* distribution of choices of values of the fluctuation  $\delta\Phi$ . Having established that the magnitudes of the ratios of the fluctuations to the background fields are finite at the horizon and at infinity we are done, since it follows from the above equations that they are finite at all intermediate values of  $r$ . To see this, and in particular to see that the ratios are both finite and small, recall that we are performing a classical analysis, which is to say that we are actually working in the limit  $k \rightarrow \infty$ . Thus, we are assured that the smallness constraints are satisfied for all of the fluctuations, in view of which we have confirmed that there are an uncountable infinity of physically acceptable solutions to the basic perturbation equations of the  $SL(2, R)/U(1)$  black hole.

### Note 3

In this note we discuss the residual gauge freedom implicit in our construction, and its effect on the infinite set of solutions to the linearized equations of motion. We have chosen (*cf* eq.(21)) the following ansatz for the perturbed metric

$$\begin{pmatrix} -e^{2f_0} & 0 \\ 0 & e^{2f_1} \end{pmatrix} \rightarrow \begin{pmatrix} -e^{2f_0+2\delta f_0} & 0 \\ 0 & e^{2f_1+2\delta f_1} \end{pmatrix}. \quad (\text{N3.1})$$

In setting the off-diagonal components to zero one has only partially fixed the gauge. Clearly, the ansatz in eq.(N3.1) is unaffected by coordinate transformations of the form

$$t \rightarrow \tilde{t} = \tilde{t}(t), \quad r \rightarrow \tilde{r} = \tilde{r}(r), \quad (\text{N3.2})$$

where  $\tilde{t}$  and  $\tilde{r}$  are arbitrary functions of  $t$  and  $r$ , respectively. However, the background should remain unchanged under this transformation, and therefore we must have

$$\tilde{t} = t + g(t), \quad \tilde{r} = r + h(r), \quad (\text{N3.3})$$

where  $g(t)$  and  $h(r)$  are of the same order of smallness as the  $\delta f_i$ . Upon utilizing the standard transformation laws for the metric tensor and for the dilaton, one finds

$$\delta f_0 \rightarrow \delta f_0 + g'(t) + f_{0,t}g(t) + f_{0,r}h(r), \quad \delta f_1 \rightarrow \delta f_1 + h'(r) + f_{1,t}g(t) + f_{1,r}h(r), \quad (\text{N3.4})$$

$$\delta \Phi \rightarrow \delta \Phi + g(t)\Phi_{,t} + h(r)\Phi_{,r}, \quad (\text{N3.5})$$

where a prime denotes differentiation with respect to the argument. Recall that we require that all perturbations have a time-dependence given by  $e^{i\omega t}$ . Also, note that all of the backgrounds which we have considered have the property  $f_{i,t} = \Phi_{,t} = 0$ . From the first component of eq.(N3.4), we see that consistency requires that  $h(r) = 0$ , and that we must also have  $g(t) \sim e^{i\omega t}$ , which merely results in an additive constant in  $\delta f_0$ . Similar analyses of the second component of eq.(N3.4) and of eq.(N3.5) yield no additional constraints. Since eqs. (60) and (75) do not contain any terms proportional to  $\delta f_0$  without derivatives, we see that the integration constant implicit in eqs. (60) and (75) is actually a gauge artifact. Thus, after taking account all residual gauge freedom, one is left with an uncountably infinite number of distinct solutions to the linearized equations of motion for both the Witten black hole and the black hole coupled to a constant, massive dilaton.

We remark briefly on the overall choice of gauge in studying the small fluctuations of two-dimensional black holes. One may enquire as to the consequences of choosing conformal gauge in our analysis, as well as in possible generalizations of our analysis to other configurations such as the CGHS black hole. In this case, following the procedure of reference [3], one would write the metric tensor as  $g_{\mu\nu} = e^{2\rho}\eta_{\mu\nu}$  with  $\rho$  a scalar function. In effecting the variation, one must be careful to allow  $g_{00}$  and  $g_{11}$  to vary *independently*. Thus one must take  $g_{00} \rightarrow e^{2\rho+2\delta f_0}\eta_{00}$  and  $g_{11} \rightarrow e^{2\rho+2\delta f_1}\eta_{11}$ . At this point, it may naively appear to be the case that the residual gauge freedom is fixed upon choosing  $\delta f_0 = \delta f_1$ , thereby restoring conformal gauge. In fact, for general  $\delta f_0$  and  $\delta f_1$ , and in particular when both are proportional to  $e^{i\omega t}$ , this cannot be done. It follows from eq.(N3.4) that one has

$$\delta f_0 - \delta f_1 \rightarrow \delta f_0 - \delta f_1 + g'(t) - h'(r) \quad (\text{N3.6})$$

where the fact that  $f_0 = f_1$  in conformal gauge has been used. It is assumed that  $\delta f_0 - \delta f_1 \neq 0$  initially. It is clearly not necessary that  $\delta f_0 - \delta f_1$  be equal to the sum of

a function of  $r$  alone and a function of  $t$  alone. It follows that, in general, functions  $g(t)$  and  $h(r)$  cannot be found which are consistent with the restoration of conformal gauge. This is in particular obvious if, as we require, both  $\delta f_0$  and  $\delta f_1$  vary with time as  $e^{i\omega t}$ .

We finally note that, throughout our analysis, we have made use of the coordinate system  $(x^0 \ x^1) = (t \ r)$  rather than light cone coordinates. This choice is consistent with our interest in what occurs outside of the event horizon, as opposed to what occurs throughout the maximally-extended spacetime.

## Appendix

The Christoffel symbols and variations of same which are relevant to the analysis of the perturbation equations are for convenience recorded below.

In the general case, one finds:

$$\begin{aligned}\Gamma_{00}^0 &= f_{0,0} & \Gamma_{01}^0 &= f_{0,1} & \Gamma_{11}^0 &= e^{2f_1-2f_0} f_{1,0} \\ \Gamma_{00}^1 &= e^{2f_0-2f_1} f_{0,1} & \Gamma_{01}^1 &= f_{1,0} & \Gamma_{11}^1 &= f_{1,1}\end{aligned}\quad (\text{A1})$$

and

$$\begin{aligned}\delta\Gamma_{00}^0 &= \delta f_{0,0} & \delta\Gamma_{01}^0 &= \delta f_{0,1} & \Gamma_{11}^0 &= e^{2f_1-2f_0} \left[ \delta f_{1,0} + 2(\delta f_1 - \delta f_0) f_{1,0} \right] \\ \delta\Gamma_{00}^1 &= e^{2f_0-2f_1} \left[ \delta f_{0,1} + 2(\delta f_0 - \delta f_1) f_{0,1} \right] & \delta\Gamma_{01}^1 &= \delta f_{1,0} & \delta\Gamma_{11}^1 &= \delta f_{1,1}\end{aligned}\quad (\text{A2})$$

### The $SL(2, R)/U(1)$ Black Hole

With the help of eqs. (38) and (39) one may derive the following:

$$\begin{aligned}\Gamma_{00}^0 &= 0 & \Gamma_{01}^0 &= \text{sech } r \text{ csch } r & \Gamma_{11}^0 &= 0 \\ \Gamma_{00}^1 &= \tanh r \text{ sech}^2 r & \Gamma_{01}^1 &= 0 & \Gamma_{11}^1 &= 0,\end{aligned}\quad (\text{A3})$$

$$\begin{aligned}\delta\Gamma_{00}^0 &= i\omega\delta f_0 & \delta\Gamma_{01}^0 &= \delta f_{0,r} & \delta\Gamma_{11}^0 &= i\omega \text{coth}^2 r \delta f_1 \\ \delta\Gamma_{00}^1 &= -2 \tanh r \text{ sech}^2 r \delta f_1 + 2 \tanh r \text{ sech}^2 r \delta f_0 + \tanh^2 r \delta f_{0,r} \\ \delta\Gamma_{01}^1 &= i\omega\delta f_1 & \delta\Gamma_{11}^1 &= \delta f_{1,r}.\end{aligned}\quad (\text{A4})$$

### Massive Dilaton Black Hole

With the help of eq.(66) one may derive the following:

$$\begin{aligned}\Gamma_{00}^0 &= 0 & \Gamma_{01}^0 &= A^{-1}A' & \Gamma_{11}^0 &= 0 \\ \Gamma_{00}^1 &= \frac{1}{2}A^2 (A^2)' & \Gamma_{01}^1 &= 0 & \Gamma_{11}^1 &= \frac{1}{2}A^2 (A^{-2})',\end{aligned}\quad (\text{A5})$$

$$\begin{aligned}\delta\Gamma_{00}^0 &= i\omega\delta f_0 & \delta\Gamma_{01}^0 &= \delta f_{0,r} & \delta\Gamma_{10}^1 &= i\omega\delta f_1 \\ \delta\Gamma_{11}^0 &= i\omega A^{-4}\delta f_1 & \delta\Gamma_{00}^1 &= -A^2 \left[ (A^2)' \delta f_1 - (A^2)' \delta f_0 - A^2 \delta f_{0,r} \right] & \delta\Gamma_{11}^1 &= \delta f_{1,r}.\end{aligned}\quad (\text{A6})$$

**Table 1:** The Function  $\chi(r)$ 

| $r$  | $m = 1$         | $m = 2$         | $m = 3$         | $m = 4$         |
|------|-----------------|-----------------|-----------------|-----------------|
| 1.0  | -26.9           | -1438.5         | -18312.6        | $-1.35.10^5$    |
| 2.0  | $-6.99.10^3$    | $-4.42.10^5$    | $-5.32.10^6$    | $-3.27.10^7$    |
| 3.0  | $-8.69.10^5$    | $-5.56.10^7$    | $-6.43.10^8$    | $-3.70.10^9$    |
| 4.0  | $-7.06.10^7$    | $-4.52.10^9$    | $-5.16.10^{10}$ | $-2.92.10^{11}$ |
| 5.0  | $-4.38.10^9$    | $-2.81.10^{11}$ | $-3.20.10^{12}$ | $-1.80.10^{13}$ |
| 6.0  | $-2.27.10^{11}$ | $-1.45.10^{13}$ | $-1.65.10^{14}$ | $-9.30.10^{14}$ |
| 7.0  | $-1.03.10^{13}$ | $-6.60.10^{14}$ | $-7.51.10^{15}$ | $-4.22.10^{16}$ |
| 8.0  | $-4.24.10^{14}$ | $-2.71.10^{16}$ | $-3.09.10^{17}$ | $-1.74.10^{18}$ |
| 9.0  | $-1.61.10^{16}$ | $-1.03.10^{18}$ | $-1.18.10^{19}$ | $-6.61.10^{19}$ |
| 10.0 | $-5.77.10^{17}$ | $-3.69.10^{19}$ | $-4.21.10^{20}$ | $-2.36.10^{21}$ |

The above table displays representative values of the function  $\chi(r)$  which arises in the analysis of the small fluctuations of the two-dimensional black hole coupled to a massive, linear dilaton, as discussed above and below eq.(93) in the text.

**Table 2:** The Function  $\gamma(r)$ 

| $r$  | $m = 1$     | $m = 2$     | $m = 3$     | $m = 4$     |
|------|-------------|-------------|-------------|-------------|
| 1.0  | 1.98        | 7.26        | 13.8        | 18.29       |
| 2.0  | 14.62       | 56.41       | 119.13      | 192.36      |
| 3.0  | 75.08       | 297.35      | 657.87      | 1141.77     |
| 4.0  | 327.34      | 1306.38     | 2928.15     | 5177.72     |
| 5.0  | 1298.41     | 5191.21     | $1.17.10^4$ | $2.07.10^4$ |
| 6.0  | 4841.00     | $1.94.10^4$ | $4.36.10^4$ | $7.74.10^4$ |
| 7.0  | $1.73.10^4$ | $6.91.10^4$ | $1.55.10^5$ | $2.76.10^5$ |
| 8.0  | $5.96.10^4$ | $2.38.10^5$ | $5.37.10^5$ | $9.54.10^5$ |
| 9.0  | $2.00.10^5$ | $8.02.10^5$ | $1.80.10^6$ | $3.21.10^6$ |
| 10.0 | $6.61.10^5$ | $2.64.10^6$ | $5.95.10^6$ | $1.06.10^7$ |

The above table displays representative values of the function  $\gamma(r)$  (*cf* eq.(88)) which arises in the analysis of the small fluctuations of the two-dimensional black hole coupled to a massive, linear dilaton, as discussed above and below eq.(93) in the text.

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