



SU(N) Gauge Theories with C -Periodic Boundary Conditions: II. Small Volume Dynamics

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ABSTRACT

The dynamics of SU(N) gauge theories, especially for $N = 3$, in a small C -periodic box are investigated. We identify the fields that minimize the energy—the torons—and determine which of these “classical” vacua are stable quantum mechanically. The stable torons break cubic symmetry, which has interesting consequences on the spectrum. At any of the stable torons there are also quartic modes. Since all C -periodic boundary conditions are gauge-equivalent, we choose a convenient version, for which the quartic modes are constant modes, and compute the effective Hamiltonian to one loop in perturbation theory.

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1 Introduction

For non-abelian gauge fields C -periodic boundary conditions were introduced in ref. [1]. With C -periodic boundary conditions fields at $\mathbf{x} + L\mathbf{e}_i$ are the charge conjugate of fields at \mathbf{x} . (Here \mathbf{e}_i is a unit vector in the i direction, and L denotes the length of the torus.) Gauge invariant operators with $C = +1$ are periodic and those with $C = -1$ are anti-periodic. An $SU(N)$ gauge field, however, is C -periodic up to a gauge transformation:

$$\mathbf{A}(\mathbf{x} + L\mathbf{e}_i) = \Omega_i(\mathbf{x})[\nabla + \mathbf{A}^*(\mathbf{x})]\Omega_i^{-1}(\mathbf{x}), \quad (1.1)$$

where $*$ denotes complex conjugation. (For details about our conventions, see [1].) Under local gauge transformations

$${}^g\mathbf{A}(\mathbf{x}) = g(\mathbf{x})[\nabla + \mathbf{A}(\mathbf{x})]g^{-1}(\mathbf{x}), \quad (1.2)$$

and

$${}^g\Omega_i(\mathbf{x}) = g(\mathbf{x} + L\mathbf{e}_i)\Omega_i(\mathbf{x})g^T(\mathbf{x}), \quad (1.3)$$

where $g(\mathbf{x})$ is an $SU(N)$ -valued function with arbitrary boundary condition. The superscript T denotes transpose, and for a unitary matrix $g^T = (g^{-1})^*$.

C -periodic boundary conditions are interesting because they share several topological properties of the infinite volume [1, 2]. For example, in Abelian gauge theories C -periodic boundary conditions permit a single charged particle in a finite volume, whereas periodic systems are always neutral [2], owing to Gauss' law. In pure $SU(N)$ gauge theories with periodic boundary conditions the Hamiltonian possesses \mathbf{Z}_N symmetries. With C -periodic boundary conditions there are no such symmetries for odd N , and they are merely \mathbf{Z}_2 for even N [1]. Furthermore, for odd N the topological (instanton) charge is an integer.

At non-zero temperature, spatial C -periodic boundary conditions explicitly break the (“temporal”) \mathbf{Z}_3 symmetry of the $SU(3)$ partition function. They act as a soft source, selecting a preferred value of the Polyakov loop [3]. This contrasts the behavior in a periodic box, where the average Polyakov loop vanishes because of tunneling between the three \mathbf{Z}_3 phases, even above the deconfinement temperature. Physically speaking, a single \mathbf{Z}_3 charge—the Polyakov loop represents a static quark—can exist in a C -periodic volume, while in a periodic volume the \mathbf{Z}_3 Gauss law forbids it. When dynamical quarks are included C -periodic boundary conditions break chiral symmetry in a similar fashion [3], which has interesting consequences for the low energy chiral Lagrangian describing the physics of the Goldstone pions. More generally, C -periodic boxes offer the possibility of approaching the infinite volume limit with different qualitative features than in traditional periodic boxes. In particular, a comparison of the two can be used to estimate the size of finite size effects, and to show that the physics ultimately becomes independent of the infrared cutoff.

This paper concentrates on $N = 3$ and studies the dynamics in a small C -periodic box using perturbation theory. For the periodic box this approach was pioneered by Lüscher [4]. To all orders of perturbation theory, he found that the spectrum possesses an N^d -fold degeneracy among all sectors of 't Hooft electric flux [5]. The degeneracy is lifted non-perturbatively by tunneling between different perturbatively degenerate states [6]. For $SU(3)$ gauge theory with C -periodic boundary conditions there are no \mathbf{Z}_3

sectors at all. Therefore, one might hope that the spectrum in a C -periodic box does not get rearranged by tunneling, and that the approach to the infinite volume limit might be smoother than in the periodic case. Unfortunately, it turns out that states with different quantum numbers of the cubic rotation group are degenerate to all orders of perturbation theory. Once again, non-perturbative tunneling removes the degeneracy.

We follow Lüscher’s line of attack [4]. First, one identifies the configurations that minimize the potential energy—the so-called torons. Second, one integrates out all modes other than the torons, yielding an effective potential \mathcal{V}_{eff} , the minima of which are the starting point of perturbation theory. As with periodic boundary conditions, it turns out that there are other modes (other than the torons) that are quartic at the minima of \mathcal{V}_{eff} . They cannot be integrated out perturbatively, and the small-volume dynamics is described by an effective Hamiltonian H_{eff} for the quartic modes and the torons. Note that the two effective descriptions apply in different regions of configuration space, as illustrated in fig. 1.

The paper is organized as follows: Sect. 2 explores the structure of the C -periodic toron valley, which consists of real, constant, abelian fields. The appendices contain the detailed proof that the torons are gauge-equivalent to real, constant, abelian fields. Sect. 3 presents the calculation of the toron effective potential \mathcal{V}_{eff} . It has four absolute minima and one local minimum. Any of the four absolute minima is an appropriate starting point for perturbation theory, but they all break cubic invariance. The consequences of this symmetry breaking on the spectrum in small volumes is sketched in sect. 4. Sect. 5 explains how to evaluate H_{eff} by evaluating the (imaginary) time evolution amplitude and presents the results of the one-loop approximation to H_{eff} . Finally,

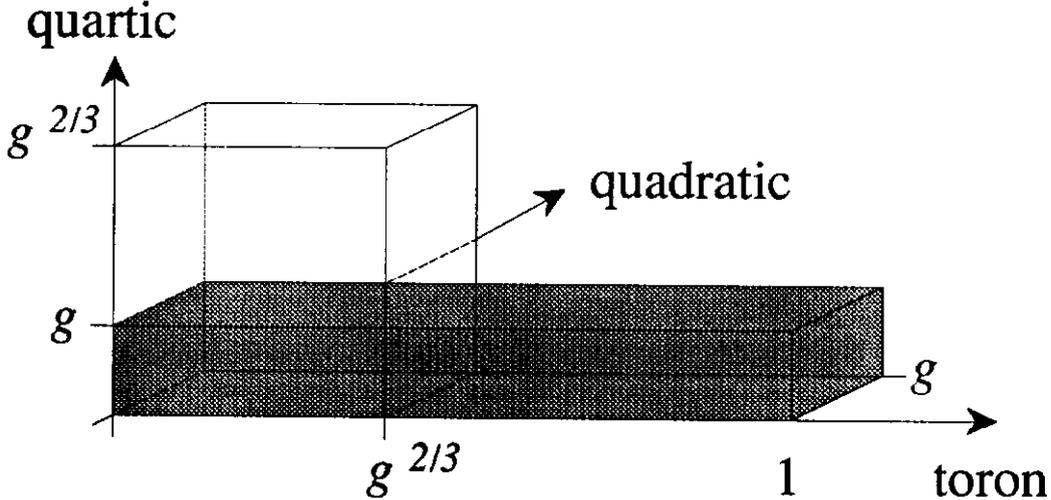


Figure 1: Regions of validity of \mathcal{V}_{eff} vs. H_{eff} . The modes corresponding to the direction labeled “quadratic” are integrated out in both cases. For the expansions to be valid, their fluctuations must be $O(g_0)$. \mathcal{V}_{eff} is valid in the gray box, when the “quartic” fluctuations are $O(g_0)$ but the torons are $O(1)$. H_{eff} is valid in the clear box, when the quartic and toron fluctuations are $O(g_0^{2/3})$.

sect. 6 offers some concluding remarks.

2 Vacuum valley for C -periodic boundary conditions

In $SU(3)$ all choices of the transition function Ω_i in eq. (1.1) are physically equivalent [1], so, except in sect. 5, we shall work with $\Omega_i = 1$. Then eq. (1.1) becomes

$$\begin{aligned} A^A(\mathbf{x} + L\mathbf{e}_i) &= A^A(\mathbf{x}), \quad A \in \{2, 5, 7\}, \\ A^a(\mathbf{x} + L\mathbf{e}_i) &= -A^a(\mathbf{x}), \quad a \in \{1, 3, 4, 6, 8\}, \end{aligned} \quad (2.1)$$

in Lie-algebra components. More generally, the periodic components take values in an $\mathfrak{so}(N)$ subalgebra of $\mathfrak{su}(N)$, generated by the N real generators. The Hamiltonian can be written as $H = {}^+H + {}^-H$, where

$$\begin{aligned} {}^+H &= \int d^d\mathbf{x} \left[\frac{1}{2}g_0^2(E_i^A)^2 + \frac{1}{4}g_0^{-2}(F_{ij}^A)^2 \right], \\ {}^-H &= \int d^d\mathbf{x} \left[\frac{1}{2}g_0^2(E_i^a)^2 + \frac{1}{4}g_0^{-2}(F_{ij}^a)^2 \right]. \end{aligned} \quad (2.2)$$

(We work in d space dimensions, but we are most interested in $d = 3$.) The appendices show in detail that the minima of the potential energy are gauge-equivalent to real, constant, abelian fields. Roughly speaking, the anti-periodic modes must vanish because otherwise $\nabla_i A_j^a \neq 0$ and, hence, $F_{ij}^a \neq 0$. This leaves the real $\mathfrak{so}(N)$ subalgebra with Hamiltonian ${}^+H$, whose potential $\int d^d\mathbf{x} (F_{ij}^A)^2 / (4g_0^2)$ is minimized by constant, abelian fields.

In $SU(2)$ and $SU(3)$ the manifold of distinct torons resembles the orbifold that appears for $SU(2)$ with periodic boundary conditions. The real, constant, abelian field is

$${}^a\mathbf{A} = \frac{C}{L}T^a, \quad (2.3)$$

where $T^a = i\sigma^a/2$ in $SU(2)$ and $T^a = i\lambda^a/2$ in $SU(3)$. At the outset, it appears that $C \in \mathbb{R}^d$, but there are discrete gauge symmetries identifying certain C [6]. The transformation

$$\exp(-4\pi T^2 \mathbf{x}_i / L) : C_i \mapsto C_i + 4\pi \quad (2.4)$$

reduces the manifold to a torus. Writing $l = \sum_{i=1}^d e_i$, the transformation

$$\exp(-3\pi T^2 \mathbf{x} \cdot l / L) \exp(\pi T^3) \exp(-\pi T^2 \mathbf{x} \cdot l / L) : C \mapsto -C + 2\pi l \quad (2.5)$$

reduces the manifold to an orbifold. For $SU(2)$ the orbifold is $[0, 4\pi]^d / \mathbf{Z}_2$, as with periodic boundary conditions. This is no great surprise, since for $SU(2)$ C -periodic boundary conditions are gauge equivalent to periodic ones [1].

For $SU(3)$ there is yet another discrete symmetry:

$$\exp(4\pi T^5 \mathbf{x} \cdot l / L) : C \mapsto -C \quad (2.6)$$

reducing the orbifold to $[0, 4\pi]^d / (\mathbf{Z}_2 \times \mathbf{Z}_2)$. The singular points of this orbifold are (for $d = 3$) in the set $\Pi = \{(\pi, \pi, \pi), (\pi, \pi, -\pi), (\pi, -\pi, \pi), (\pi, -\pi, -\pi)\}$. Based on previous experience, we expect these points to play an important role.

3 Effective potential for C

To work out the toron effective potential, we start by parametrizing the SU(3) gauge field as follows:

$$LA(\mathbf{x}) = g^{-1}(\mathbf{x}) \left(L\nabla + CT^2 + g_0 \sum_{\nu} q^A(\nu) T^A \exp[i2\pi\nu \cdot \mathbf{x}/L] \right. \\ \left. + g_0 \sum_{\nu} q^a(\nu) T^a \exp[i\pi(2\nu + l) \cdot \mathbf{x}/L] \right) g(\mathbf{x}) \quad (3.1)$$

where $q^2(\mathbf{0}) = 0$, $q^A(-\nu) = q^A(\nu)^*$, $q^a(-\nu - l) = q^a(\nu)^*$. Upper-case color indices run over $A \in \{2, 5, 7\}$, whereas lower-case color indices run over $a \in \{1, 3, 4, 6, 8\}$.

Let us define background-field covariant derivatives

$$+\mathcal{D}^{AB} = i2\pi\nu\delta^{AB} + C f^{A2B} \quad (3.2)$$

acting on $q^A(\nu)$ and

$$-\mathcal{D}^{ab} = i\pi(2\nu + l)\delta^{ab} + C f^{a2b} \quad (3.3)$$

acting on $q^a(\nu)$. To $O(g_0^2)$ the kinetic energy of C and interactions can be neglected and the Hamiltonian is

$$+H = \frac{1}{2L} \sum_{\nu} \left[\mathbf{p}^A(-\nu) \cdot \mathbf{p}^A(\nu) + q_j^A(-\nu) + \Omega_{jk}^{AB} q_k^B(\nu) \right] \quad (3.4)$$

with $\mathbf{p}^A(\nu)$ the canonical momentum conjugate to $q^A(\nu)$ and

$$+\Omega_{jk}^{AB} = (- +\mathcal{D}^2 \delta_{jk} + +\mathcal{D}_j + \mathcal{D}_k)^{AB}. \quad (3.5)$$

The expressions for $-H$ and $-\Omega_{jk}^{ab}$ are obtained by replacing $+\mathcal{D}^{AB}$ by $-\mathcal{D}^{ab}$, q^A by q^a , etc. To $O(g_0^2)$ the Hamiltonians $\pm H$ describe the $\nu \neq \mathbf{0}$ modes as a collection of harmonic oscillators whose squared frequencies are the eigenvalues of $\pm\Omega$. Adding up the zero-point energy of these oscillators gives the effective potential for C :

$$L\mathcal{V}_{\text{eff}}(C) = \frac{1}{2} \text{Tr}[+\Omega^{1/2}(C)] + \frac{1}{2} \text{Tr}[-\Omega^{1/2}(C)]. \quad (3.6)$$

It is natural to work in the background-field Coulomb gauge

$$+\mathcal{D}^{AB} \cdot \mathbf{q}^B(\nu) = 0, \quad -\mathcal{D}^{ab} \cdot \mathbf{q}^b(\nu) = 0. \quad (3.7)$$

Since C parameterizes an abelian field, the background-field covariant derivatives commute, so in Coulomb gauge $\pm\Omega = -\pm\mathcal{D}^2$. The eigenvalues are

$$+\Omega(C) : \begin{cases} (2\pi\nu)^2, & \nu \neq 0, \quad A = 2 \text{ sector} \\ (2\pi\nu \pm \frac{1}{2}C)^2, & A \in \{5, 7\} \text{ sector} \end{cases} \quad (3.8)$$

and

$$-\Omega(C) : \begin{cases} (2\pi\nu + \pi l \pm C)^2, & a \in \{1, 3\} \text{ sector} \\ (2\pi\nu + \pi l \pm \frac{1}{2}C)^2, & a \in \{4, 6\} \text{ sector} \\ (2\pi\nu + \pi l)^2, & a = 8 \text{ sector} \end{cases} \quad (3.9)$$

all of which have multiplicity $d - 1$. Performing the traces yields

$$\mathcal{V}_{\text{eff}}(\mathbf{C}) = \hat{\mathcal{V}}(\frac{1}{2}\mathbf{C}) + \hat{\mathcal{V}}(\mathbf{C} - \pi\mathbf{l}) + \hat{\mathcal{V}}(\frac{1}{2}\mathbf{C} - \pi\mathbf{l}) \quad (3.10)$$

where [4, 6]

$$\hat{\mathcal{V}}(\mathbf{C}) = -\frac{d-1}{L\pi^2} \sum_{\nu \neq \mathbf{0}} \frac{1}{\nu^4} \cos(\mathbf{C} \cdot \nu). \quad (3.11)$$

Near its minimum at $\mathbf{C} = \mathbf{0}$, $\hat{\mathcal{V}}(\mathbf{C}) \approx \hat{\mathcal{V}}(\mathbf{0}) + 2|\mathbf{C}|/L$, cf. ref. [6]. Hence, the effective potential $\mathcal{V}_{\text{eff}}(\mathbf{C})$ has conical minima at $\mathbf{C} = \mathbf{0}$, from the first term in eq. (3.10), and at the 2^{d-1} points in Π , from the second term. (The third term implies a minimum at $2\pi\mathbf{l}$ which is equivalent to $\mathbf{0}$ by eq. (2.5).) To decide which point is the absolute minimum one must perform the sums explicitly. For $d = 3$

$$\begin{aligned} L\mathcal{V}_{\text{eff}}(\mathbf{0}) &= -1.78447\dots, \\ L\mathcal{V}_{\text{eff}}(\mathbf{C} \in \Pi) &= -3.25229\dots \end{aligned} \quad (3.12)$$

Thus, there are four degenerate, absolute minima at $\mathbf{C} \in \Pi$, as well as a local minimum at $\mathbf{C} = \mathbf{0}$.

Any of the four minima in Π is an appropriate starting point for perturbation theory. It may seem peculiar that $\mathbf{C} = \mathbf{0}$ is *not* appropriate, but it has precedent [7]. In sect. 5 we shall set up perturbation theory, expanding around $\mathbf{C} = (\pi, \pi, \pi)$. The next section discusses, in general terms, how the cubic symmetry breaking influences the spectrum.

4 Degeneracies to all orders in perturbation theory

Let us review the structure of the effective potential for $SU(N)$ gauge theories with periodic boundary conditions. The symmetry group of the Hamiltonian is $G = \mathbf{Z}_N^d \cdot O(d, \mathbf{Z})$, the semi-direct product of the central conjugations and the rotation group of the d -cube. The effective potential for the torons has N^d minima, each with stability subgroup $\mathcal{G} = O(d, \mathbf{Z})$. If one calculates the spectrum perturbatively, by expanding about a given minimum, one finds that the spectrum is labeled by the irreducible representations of \mathcal{G} . But there is an N^d -fold degeneracy, coming from states localized at the other minima. That is why the 't Hooft electric flux sectors are degenerate with the glueballs, to all orders in perturbation theory. Of course, non-perturbative tunneling between the N^d minima removes the degeneracies.

There are analogous phenomena for $N = 3$ with C -periodic boundary conditions. Now the symmetry group of the Hamiltonian is $G = O(d, \mathbf{Z})$, because there are no central conjugations. The effective potential for the torons has 2^{d-1} minima; below we identify the stability group \mathcal{G} . Again, the irreducible representations of \mathcal{G} label the perturbative spectrum, but each multiplet is 2^{d-1} -fold degenerate. Combining 2^{d-1} degenerate \mathcal{G} -multiplets into a (reducible) representation of G then reveals that the Hamiltonian has degeneracies, to all orders in perturbation theory. And again, tunneling removes them.

To work out the details one needs to describe the states enough to understand their symmetry properties. The quantum mechanical system at hand contains the degrees of

Table 1: Values of \mathcal{L}_{ij}^\pm for all $\mathbf{C} \in \Pi$ for $d = 3$. Note that, at these points, $\mathcal{L}_{ij}^- = -\mathcal{L}_{ij}^+$ always.

	(π, π, π)	$(\pi, \pi, -\pi)$	$(\pi, -\pi, \pi)$	$(\pi, -\pi, -\pi)$
\mathcal{L}_{12}^+	-	-	+	+
\mathcal{L}_{13}^+	-	+	-	+
\mathcal{L}_{23}^+	-	+	+	-
\mathcal{L}_{12}^-	+	+	-	-
\mathcal{L}_{13}^-	+	-	+	-
\mathcal{L}_{23}^-	+	-	-	+

freedom \mathbf{C} governed by the potential $\mathcal{V}_{\text{eff}}(\mathbf{C})$. A “perturbative eigenstate” centered at minimum $\mathbf{C}^\pi \in \Pi$ obeys

$$\langle \mathbf{C} \rangle = \mathbf{C}^\pi T^2 / L. \quad (4.1)$$

In this sense \mathbf{C}^π characterizes the state. Consider the Wilson loops $\exp(\oint \mathbf{C}^\pi \cdot d\mathbf{x})$, and define

$$\mathcal{L}_{ij}^\pm = \frac{1}{2} \left[\text{Tr} \left\{ \exp[T^2(C_i^\pi \pm C_j^\pi)] \right\} - 1 \right] = \cos \left\{ \frac{1}{2}(C_i^\pi \pm C_j^\pi) \right\}. \quad (4.2)$$

Up to trivialities \mathcal{L}_{ij}^\pm describes a loop that wraps around the C -periodic box first in the i direction and then in the $\pm j$ direction, evaluated at $\mathbf{C} = \mathbf{C}^\pi$.

Table 1 contains the values of the \mathcal{L}_{ij}^\pm at all points in Π for $d = 3$. At (π, π, π) one sees that the stability subgroup is $\mathcal{G} = S_3$, the permutation group of the $d = 3$ axes,¹ because any rotation that is *not* a permutation transforms a negative \mathcal{L} into a positive one. At the other points \mathcal{G} is isomorphic to S_3 . There are three irreducible representations of S_3 , whose Young tableaux are $\square\square$, $\square\bar{\square}$, and $\bar{\square}$. If ρ is one of these representations, let R_ρ be the reducible representation of $O(3, \mathbf{Z})$ generated by combining $2^{d-1} = 4$ representations ρ , one from each minimum, each with the same (perturbative) energy. Using character tables it is easy to work out the irreducible content of R_ρ . One finds

$$\begin{aligned} R_{\square\square} &= A_1 \oplus T_2, \\ R_{\square\bar{\square}} &= E \oplus T_1 \oplus T_2, \\ R_{\bar{\square}} &= A_2 \oplus T_1. \end{aligned} \quad (4.3)$$

Eqs. (4.3) have the following consequences on the spectrum. In small volumes, where tunneling effects are suppressed, each A_1 state is degenerate with a T_2 state, each A_2 state is degenerate with a T_1 state, and each E state is degenerate with a T_1 and a T_2 state. As with periodic boundary conditions, there is presumably a tunneling transition for some intermediate value of the coupling (i.e. volume) splitting these degeneracies.

At larger volumes $L \gtrsim 1$ fm rotational invariance ought to be restored. That means that an E and a T_2 ought to combine into a spin 2 multiplet, an A_2 , a T_1 , and a T_2 ought to combine into a spin 3 multiplet, etc. In addition, there may be other features of the spectrum arising from tunneling to the local minimum at $\mathbf{C} = \mathbf{0}$.

¹In d dimensions, $\mathcal{G} = S_d$.

5 The Effective Hamiltonian

At $\mathbf{C} = (\pi, \pi, \pi)$ the quadratic term in $-H$ vanishes for two modes,

$$\begin{aligned} & (\mathbf{q}^1(\mathbf{0}) - i\mathbf{q}^3(\mathbf{0}))/\sqrt{2}, \\ & (\mathbf{q}^1(\mathbf{0}) - i\mathbf{q}^3(\mathbf{0}))^*/\sqrt{2} = (\mathbf{q}^1(-l) + i\mathbf{q}^3(-l))/\sqrt{2}, \end{aligned} \quad (5.1)$$

cf. eq. (2.2). The fluctuations of these modes and \mathbf{C} are still damped by quartic terms, so they are called quartic modes. Since perturbation theory develops a series of corrections to quadratic interactions, one must obtain an effective Hamiltonian for *all* quartic modes and solve it non-perturbatively, as in refs. [8, 6].

Expanding around $\mathbf{C} = (\pi, \pi, \pi)$ is a nuisance, especially since the quartic modes in eq. (5.1) prefer a complex basis for the Lie algebra. The gauge transformation

$$h(\mathbf{x}) = \exp(\pi T^2 \mathbf{x} \cdot \mathbf{l}/L) \quad (5.2)$$

eliminates these problems, but induces a non-unit twist: ${}^h\Omega_i = \Omega$, where

$$\Omega = \exp(\pi T^2) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.3)$$

as required by eq. (1.3). The transformed gauge potential obeys a new boundary condition,

$$\mathbf{A}(\mathbf{x} + L\mathbf{e}_i) = \Omega \mathbf{A}^*(\mathbf{x}) \Omega^{-1}, \quad (5.4)$$

or, in explicit Lie-algebra components,

$$\begin{aligned} \mathbf{A}^A(\mathbf{x} + L\mathbf{e}_i) &= +\mathbf{A}^A(\mathbf{x}), \quad A \in \{1, 2, 3\}, \\ \mathbf{A}^8(\mathbf{x} + L\mathbf{e}_i) &= -\mathbf{A}^8(\mathbf{x}), \\ (\mathbf{A}^5 \pm i\mathbf{A}^7)(\mathbf{x} + L\mathbf{e}_i) &= \pm i(\mathbf{A}^5 \pm i\mathbf{A}^7)(\mathbf{x}), \\ (\mathbf{A}^4 \mp i\mathbf{A}^6)(\mathbf{x} + L\mathbf{e}_i) &= \pm i(\mathbf{A}^4 \mp i\mathbf{A}^6)(\mathbf{x}). \end{aligned} \quad (5.5)$$

With eq. (5.4) the torons are still real, constant, abelian fields, parameterized as in eq. (2.3). Splitting the Hamiltonian into pieces according to the boundary conditions of eq. (5.5) and repeating the analysis of sect. 3 shows that the absolute minima of \mathcal{V}_{eff} are now at $\mathbf{C} = \mathbf{0}$ and three other points $2\pi\mathbf{e}_i$. Furthermore, at $\mathbf{C} = \mathbf{0}$ the additional quartic modes are the constant modes of \mathbf{A}^1 and \mathbf{A}^3 . Hence, the split between quartic and quadratic modes is simply

$$L\mathbf{A}^A(\mathbf{x}) = \mathbf{c}^A + g_0\mathbf{q}^A(\mathbf{x}), \quad A \in \{1, 2, 3\}, \quad (5.6)$$

with $\int d^d\mathbf{x} \mathbf{q}^A(\mathbf{x}) = 0$.

The components \mathbf{A}^a , $a \in \{4, 5, 6, 7, 8\}$, are always quadratic. \mathbf{A}^8 is anti-periodic and has half-integer momenta $\mathbf{p} = 2\pi(\boldsymbol{\nu} + \frac{1}{2}\mathbf{l})/L$. However, it does not couple to the quartic modes \mathbf{c}^A , so it plays no role in the calculations below. The $\pm i$ -periodic modes have quarter-integer momenta $\mathbf{p} = 2\pi(\boldsymbol{\nu} \pm \frac{1}{4}\mathbf{l})/L$. These are not artifacts of the boundary condition eq. (5.4); instead, they would have appeared with the standard

C -periodic boundary condition, when expanding around $\mathbf{C} = (\pi, \pi, \pi)$, as a glance at the eigenvalues in eqs. (3.8) and (3.9) verifies.

The effective Hamiltonian can be worked out by computing the amplitude to propagate from one constant field \mathbf{c}^A to another. It is defined by

$$\langle \mathbf{c}^A(T) | e^{-H_{\text{eff}}T} | \mathbf{c}^A(0) \rangle = \langle \mathbf{c}^A(T) | e^{-HT} | \mathbf{c}^A(0) \rangle, \quad (5.7)$$

where the right-hand side is computed in the full theory, and the left-hand side is an expression involving \mathbf{c}^A and its conjugate momentum $\mathbf{e}^A = -i\partial/\partial\mathbf{c}^A$. In the path integral formalism

$$\langle \mathbf{c}^A(T) | e^{-HT} | \mathbf{c}^A(0) \rangle = \int \mathcal{D}A_\mu^a e^{-\int_0^T dt \mathcal{L}} \quad (5.8)$$

and

$$\langle \mathbf{c}^A(T) | e^{-H_{\text{eff}}T} | \mathbf{c}^A(0) \rangle = \int \mathcal{D}c_j^A e^{-\int_0^T dt \mathcal{L}_{\text{eff}}}, \quad (5.9)$$

where \mathcal{L} and \mathcal{L}_{eff} are Euclidean Lagrangians.

To the order of interest \mathcal{L}_{eff} is given by the two- and four-point diagrams in fig. 2. We use Feynman perturbation theory to evaluate them. The familiar Feynman rules apply, but with a few modifications. The external lines denote wave functions c_i^A/g_0 , and an n -point function acquires a combinatorial factor $1/n!$. In general the c_i^A are time or, after Fourier transforming, frequency dependent; $p_0\mathbf{c}^A$ corresponds to $\dot{\mathbf{c}}^A$. We shall only need terms up to p_0^2 in the two-point diagrams, and the p_0 -independent piece of the four-point function. The loop frequency is integrated as usual, but the spatial loop momentum is summed over the values allowed by the boundary condition. In particular, the loop momentum is $\mathbf{p} = 2\pi\nu/L$ for $A \in \{1, 2, 3\}$ and $\mathbf{p} = 2\pi(\nu \pm \frac{1}{4}\mathbf{l})/L$ for $a \in \{4, 5, 6, 7\}$. Since $f^{aB8} = 0$ the eighth component drops out.

Taking permutation and gauge symmetry into account, the effective Lagrangian takes the form

$$L\mathcal{L}_{\text{eff}} = g_0^{-2}(\delta_{ij} - g_0^2 p_{ij}) \frac{1}{2} \dot{c}_i^A \dot{c}_j^A + V_{\text{eff}}(\mathbf{c}^A), \quad (5.10)$$

where $V_{\text{eff}}(\mathbf{c}^A)$ is given below in eq. (5.24). In the Hamiltonian the kinetic term is $g_0^2(\delta_{ij} + g_0^2 p_{ij}) \frac{1}{2} e_i^A e_j^A$, or after rescaling $c_i^A \mapsto g_0^{2/3} c_i^A$ and $e_i^A \mapsto g_0^{-2/3} e_i^A$,

$$\begin{aligned} LH_{\text{eff}} = & g_0^{2/3}(\delta_{ij} + g_0^2 p_{ij}) \frac{1}{2} c_i^A c_j^A + g_0^{2/3}(\delta_{ij} + g_0^2 q_{ij}) \frac{1}{4} F_{ik}^A F_{jk}^A \\ & + g_0^{4/3} m_{ij} c_i^A c_j^A + g_0^{8/3} \mathcal{S}_{ijkl} s^{ABDE} c_i^A c_j^B c_k^D c_l^E, \end{aligned} \quad (5.11)$$

where $F_{ij}^A = f^{ABD} c_i^B c_j^D$, and

$$s^{ABDE} = \frac{2}{3} \left(\delta^{AB} \delta^{DE} + \delta^{BD} \delta^{EA} + \delta^{AD} \delta^{BE} \right). \quad (5.12)$$

In an arbitrary covariant gauge with gluon propagator

$$\frac{1}{p^2} \left(\delta_{\mu\nu} - (1 - \alpha) \frac{p_\mu p_\nu}{p^2} \right) \quad (5.13)$$

explicit evaluation of the Feynman diagrams yields

$$p_{ij} = \frac{1}{8} \left(2 \sum' + \sum'' \right) \left(2(3 - \alpha) \frac{\delta_{ij}}{k^3} - (d + 2 - 3\alpha) \frac{k_i k_j}{k^5} \right), \quad (5.14)$$

$$q_{ij} = -\frac{1}{24} (2\Sigma' + \Sigma'') \left((d+11-12\alpha) \frac{\delta_{ij}}{k^3} - 6(d+2-3\alpha) \frac{k_i k_j}{k^5} \right), \quad (5.15)$$

$$m_{ij} = \frac{(d-1)}{4} (2\Sigma' + \Sigma'') \left(\frac{\delta_{ij}}{k} - \frac{k_i k_j}{k^3} \right), \quad (5.16)$$

and

$$S_{ijkl} = -\frac{(d-1)}{16} \left(\Sigma' + \frac{1}{8}\Sigma'' \right) \left(\frac{\delta_{\{ij}\delta_{kl}}}{k^3} - 6 \frac{k_{\{i}k_j\delta_{kl}}}{k^5} + 5 \frac{k_i k_j k_k k_l}{k^7} \right). \quad (5.17)$$

The notation $\{\dots\}$ implies complete symmetrization of the embraced indices. The

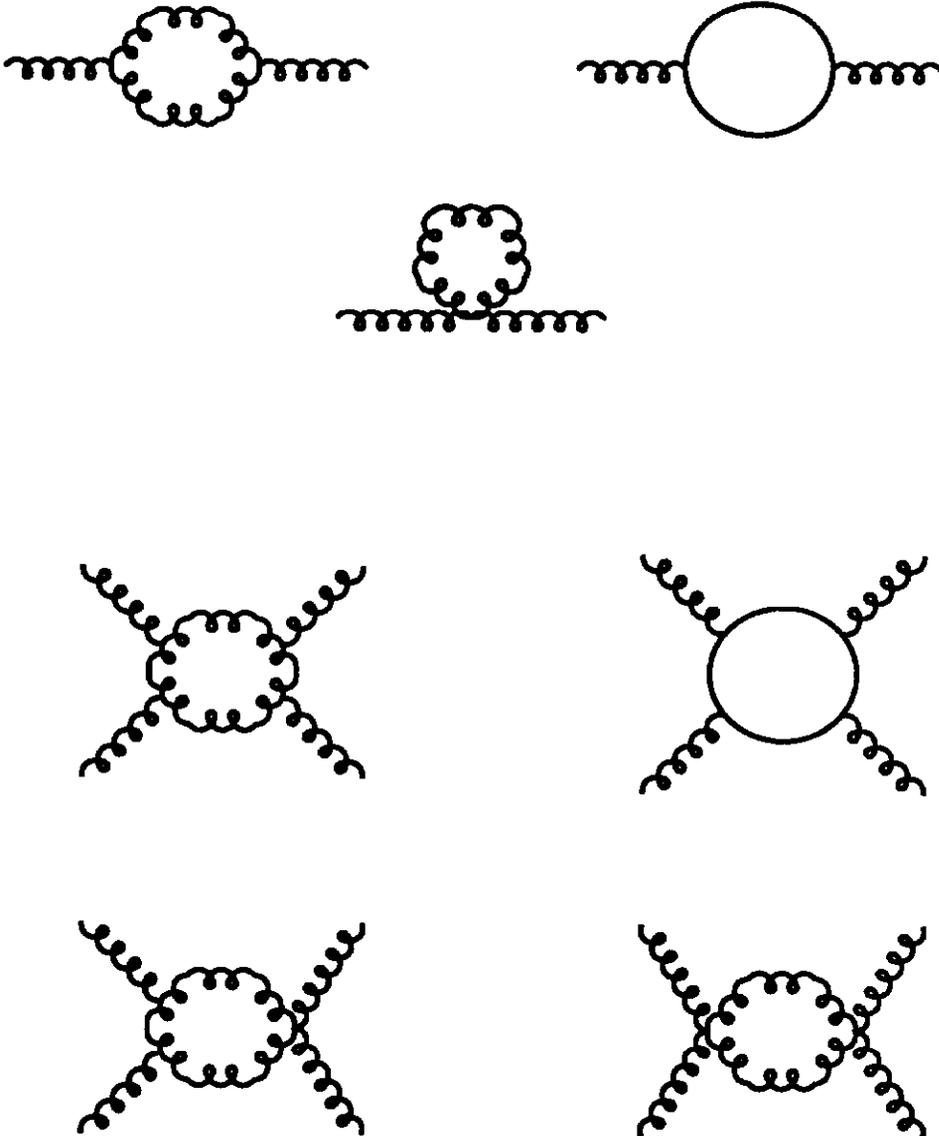


Figure 2: One loop Feynman diagrams for the two- and four-point functions.

summation symbols and their associated loop momentum values are

$$\sum' := \sum_{\nu \neq 0}, \quad k = 2\pi\nu \quad (5.18)$$

from the periodic modes in the loops, and

$$\sum'' := \sum_{\nu}, \quad k = 2\pi(\nu - \frac{1}{4}l) \quad (5.19)$$

from the $\pm i$ -periodic modes. The difference in the sums is an infrared effect; in particular, the poles as $d \rightarrow 3$ are identical, and are absorbed by the usual SU(3) coupling constant renormalization.

It would have been much too tedious to obtain these results by hand for arbitrary α , so we used the symbolic manipulation language FORM. We checked the FORM programs in several ways. For example, it was easy to reproduce the known result for the infinite-volume two-point function in an arbitrary gauge, since the Feynman rules for vertices are the same.

Note that m_{ij} and \mathcal{S}_{ijkl} are gauge invariant, but p_{ij} and q_{ij} depend on α . However, the wave-function renormalization

$$c_i^a \rightarrow Z_{ij}^{1/2} c_j^a, \quad e_i^a \rightarrow Z_{ij}^{-1/2} e_j^a, \quad (5.20)$$

where

$$Z_{ij} = \delta_{ij} + g_0^2 z_{ij}, \quad z_{ij} = \frac{1}{3}(p_{ij} - q_{ij}), \quad (5.21)$$

removes the gauge dependence, and sets the coefficients of the kinetic term and the tree-level potential term equal. In addition, after renormalization the $k_i k_j / k^5$ terms cancel:

$$p_{ij} - z_{ij} = q_{ij} + 2z_{ij} = \delta_{ij} \alpha_2, \quad (5.22)$$

where

$$\alpha_2 = \frac{25-d}{72} (2\sum' + \sum'') \frac{1}{k^3} = \frac{11}{3(4\pi)^2} \frac{1}{\epsilon} + a_2, \quad (5.23)$$

with $d = 3 - 2\epsilon$. This is a pleasant surprise. After this field renormalization and MS-scheme coupling constant renormalization

$$\begin{aligned} LH_{\text{eff}} = & g^{2/3}(1 + g^2 a_2) \left(\frac{1}{2} e_i^A e_i^A + \frac{1}{4} F_{ij}^A F_{ij}^A \right) \\ & + g^{4/3} m_{ij} c_i^A c_j^A + g^{8/3} \mathcal{S}_{ijkl} s^{ABDE} c_i^A c_j^B c_k^D c_l^E, \end{aligned} \quad (5.24)$$

where $g^2 = g_{\text{MS}}^2(1/L)$. Numerical values for a_2 , m_{ij} , and \mathcal{S}_{ijkl} are in Table 2. Notice how the quarter-momentum sums lead to terms in H_{eff} that break the cubic symmetry down to the permutation symmetry, as discussed in sect. 4.

After renormalization, these formulae straightforwardly reproduce previous calculations of Lüscher [4] when the momenta are color blind. Moreover, we also checked the final results by repeating the calculations using Lüscher's technique of Bloch perturbation theory.

Some qualitative properties of the spectrum follow from eq. (5.24) immediately. To lowest order in $g^{2/3}$, which should pertain to the smallest of volumes, the effective

Table 2: The coefficients in the effective Hamiltonian. The tensors m_{ij} and \mathcal{S}_{ijkl} are completely symmetric.

coefficient	Σ'	Σ''	total
$3(4\pi)^2 a_2/11$	-0.2727019	0.5986198	0.3259179
$6\pi m_{ii}$	-5.6745750	-0.2004840	-5.8750590
$6\pi m_{ij}$		-1.4383214	-1.4383214
$60\pi^2 \mathcal{S}_{iiii}$	-0.4289976	0.2502256	-0.1787720
$60\pi^2 \mathcal{S}_{ijjj}$		0.0446557	0.0446557
$60\pi^2 \mathcal{S}_{iijj}$	0.6311655	-0.0730294	0.5581361
$60\pi^2 \mathcal{S}_{ijkk}$		-0.0039135	-0.0039135

Hamiltonian is invariant under the full rotation group $O(3)$. Indeed, it is identical to the effective Hamiltonian of $SU(2)$ in a periodic box. (This is because the quartic modes at any absolute minimum of \mathcal{V}_{eff} form an $\mathfrak{su}(2)$ subalgebra.) From ref. [8] we know, therefore, that the vacuum has quantum numbers $J^P = 0^+$, where P refers to parity. Furthermore, the 2^+ glueball lies lower than the 0^+ glueball [8]. For periodic $SU(2)$ all these states (including the vacuum) exhibit an eight-fold degeneracy, associated with central conjugations. For C -periodic $SU(3)$ they exhibit a four-fold degeneracy, associated with the rotations connecting the four minima of \mathcal{V}_{eff} . Of course, in $SU(3)$ one must specify charge conjugation; all states built out of the quartic modes c^A have $C = +$.

At $O(g^{4/3})$ the m_{ij} -terms in eq. (5.24) break the $O(3)$ symmetry down to the permutation group S_3 . The 0^{++} states are simply labeled as $\square\square$ states, but the 2^{++} states split into a $\square\square$ and two \square states with different energies. Higher order terms respect this pattern. All states are in addition 4-fold degenerate, but as discussed in sect. 4 this degeneracy is lifted non-perturbatively by tunneling between different perturbatively degenerate states.

Altogether one may distinguish four regimes: very small volumes, where lowest order perturbation theory applies, and the states are labeled by representations of the full rotation group $O(3)$; somewhat larger volumes, where higher order perturbative contributions become non-negligible and the S_3 multiplets are resolvable; intermediate volumes, where tunneling restores the cubic rotation group $O(3, \mathbf{Z})$; and, finally, large volumes, where the full $O(3)$ rotation group reappears. This demonstrates that also with C -periodic boundary conditions there is no simple connection between the small volume perturbative regime and the large volume non-perturbative regime.

6 Conclusions

A preliminary write-up of these results [11] drew the incorrect conclusion that the absolute minimum of \mathcal{V}_{eff} was unique. If that had been the case, the volume dependence of glueball masses in a C -periodic box may have matched more smoothly onto large volume results than in a periodic box. Since we now know that there are *four* degenerate

minima of the toron effective potential, tunneling transitions are expected. However, the $SU(3)$ C -periodic effective Hamiltonian is similar (at lowest order identical) to its $SU(2)$ periodic counterpart. Consequently, it might be simpler to solve than the periodic $SU(3)$ case [9, 10]. Of course, there may be technical complications owing to the local minimum.

A complete spectrum calculation goes beyond the scope of this paper, which concentrates on the derivation of the effective Hamiltonian and some of its qualitative features. A spectrum calculation would nevertheless be interesting, especially when confronted with numerical simulations of the glueball spectrum in a C -periodic box. In simulations one can also probe the physically most interesting large volume regime. A comparison of periodic and C -periodic systems may reveal how sensitive the gluon system is to the infrared cutoff, and how large the volume must be before we see non-perturbative large volume results.

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A Torons are Real, Constant and Abelian

The potential energy is

$$V[A] = \frac{1}{4g_0^2} \int d^d x (F_{ij}^a)^2, \quad (\text{A.1})$$

where $0 \leq x_i < L$,

$$F_{ij} = \nabla_i A_j - \nabla_j A_i + [A_i, A_j] = F_{ij}^a T^a \quad (\text{A.2})$$

is the field strength, and g_0 is the bare coupling. Obviously, if \mathbf{A} is constant ($\nabla_i A_j = 0$) and abelian ($[A_i, A_j] = 0$), F_{ij} and V vanish. Ref. [1] showed that any C -periodic boundary condition is gauge-equivalent to eq. (2.1). With those boundary conditions, however, a constant \mathbf{A} must also be real. On the other hand, suppose the field strength (and therefore the potential energy) vanishes for some \mathbf{A} . We want to show that \mathbf{A} is gauge equivalent to a real, constant, abelian configuration.

Let a be the gauge transformation that brings \mathbf{A} into the complete axial gauge:

$$\begin{aligned} {}^a A_1(\mathbf{x}) &= 0 \quad \text{everywhere,} \\ {}^a A_2(\mathbf{x}) &= 0 \quad \text{for } x_1 = 0, \\ {}^a A_3(\mathbf{x}) &= 0 \quad \text{for } x_1 = x_2 = 0, \dots \end{aligned} \quad (\text{A.3})$$

This gauge transformation induces non-trivial transition functions ${}^a \Omega_i$. The field strength vanishes everywhere if and only if parallel transport around any contractable path is trivial. Starting with infinitesimal paths near $\mathbf{x} = \mathbf{0}$, it is easy to see that this is so only if ${}^a \mathbf{A}(\mathbf{x}) = 0$ and only if ${}^a \Omega_i(\mathbf{x}) = V_i$, where V_i are constant matrices obeying

$$V_i V_j^* = V_j V_i^*. \quad (\text{A.4})$$

The condition on the ${}^a \Omega_i$ comes from considering paths that cross two boundaries but are still contractable.

There is a gauge transformation b , constructed in Appendix B, such that

$${}^{ba}\Omega_i = {}^bV_i = \exp(T^2C_i), \quad (\text{A.5})$$

but still preserving the complete axial gauge. Now let $c = \exp(-T^2C \cdot \mathbf{x}/L)$ and $g = cba$. Then ${}^gA = CT^2/L$ is real, constant and abelian as claimed. Moreover, the transition functions have been gauged away again, ${}^g\Omega_i = 1$.

Carrying out the gauge transformation h in eq. (5.2) shifts the value of C , viz. ${}^hC = C - \pi l$. Hence, under the boundary condition of eq. (5.4) the torons are real, constant and abelian as well.

B Construction of b

The axial gauge condition does not fix gauge transformations that are constant for $0 \leq x_i < L$, but complex conjugated each time $x_i \geq L$. Under such a transformation $v : V_i \mapsto vV_iv^T$. Let $W_i = V_iV_i^*$, and note that repeated use of eq. (A.4) implies that all W_i can be simultaneously diagonalized. Denote the unitary matrix that carries out the diagonalization by u , and write

$$\tilde{V}_i = uV_iu^T, \quad \tilde{V}_i\tilde{V}_i^* = \Lambda_i = \text{diag}(\lambda_i^{(1)}, \lambda_i^{(2)}, \lambda_i^{(3)}). \quad (\text{B.1})$$

Then

$$\tilde{V}_i = \Lambda_i\tilde{V}_i^T = (\tilde{V}_i\Lambda_i)^T, \quad (\text{B.2})$$

which implies, for each i , that \tilde{V}_i falls into one of two cases: Case (0): Λ_i is the unit matrix. Then $\tilde{V}_i = \tilde{V}_i^T$ and hence $V_i = V_i^T$. Case (α, β) : $\lambda_i^{(\alpha)} = e^{i2C_i}$, $C_i \neq 0$, for some α . Then eq. (B.2) says that the other two entries of Λ_i are $\lambda_i^{(\beta)} = e^{-i2C_i}$ and $\lambda_i^{(\gamma)} = 1$, and it says that all elements of \tilde{V}_i vanish except $\tilde{V}_i^{(\alpha\beta)} = ie^{-i(C_i-\Phi)}$, $\tilde{V}_i^{(\beta\alpha)} = ie^{i(C_i+\Phi)}$, $\tilde{V}_i^{(\gamma\gamma)} = e^{-i2C_i}$.

Either there is a direction I such that \tilde{V}_I falls under case (0), or all \tilde{V}_i fall under a case (α, β) . Suppose the former, i.e. $V_I = V_I^T$. Since $V_I \in \text{SU}(N)$ there is a matrix w such that $w^2 = V_I^{-1}$, and w is symmetric, since V_I is. Now define the C -periodic gauge transformation b_1 to be w for $0 \leq x_i < L$, w^* if one $x_i > L$, etc. Then ${}^{b_1}V_I = wV_Iw^T = 1$ and eq. (A.4) implies that all wV_j are real. But if all V_i were real, eq. (A.4) would imply that they commute. Define C_i by ${}^{b_1}V_i = \exp(\omega^A T^A C_i)$, where $A \in \{2, 5, 7\}$. This is possible because the ${}^{b_1}V_i$ are real and commute. Here

$$\begin{aligned} \omega^2 &= \cos \theta, \\ \omega^5 &= \sin \theta \cos \phi, \\ \omega^7 &= \sin \theta \sin \phi \end{aligned} \quad (\text{B.3})$$

form a unit vector. The real, constant gauge transformation

$$b_2 = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \cos \theta \sin \phi & \cos \theta \cos \phi & -\sin \theta \\ \sin \theta \sin \phi & \sin \theta \cos \phi & \cos \theta \end{pmatrix} \quad (\text{B.4})$$

rotates $\omega^A T^A$ to T^2 . Hence, setting $b = b_2 b_1$, ${}^bV_i = \exp(T^2C_i)$ as in eq. (A.5).

Now suppose all \tilde{V}_i fall under a case (α, β) . Eq. (A.4) requires that Φ , α , and β be the same for all i . Without loss one may assume $(\alpha, \beta) = (1, 2)$. Then define

$$v = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\Phi/2} & ie^{-i\Phi/2} & 0 \\ ie^{-i\Phi/2} & e^{-i\Phi/2} & 0 \\ 0 & 0 & \sqrt{2}e^{i\Phi} \end{pmatrix}. \quad (\text{B.5})$$

The gauge transformation that brings V_i into the desired form is $b = vu$ for $0 \leq x_i < L$, complex conjugated each time $x_j > L$.

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