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Abstract

We present a simple and intuitive derivation of the track parameter covariance matrix due to multiple Coulomb scattering for use in track fitting with the Kalman filter. We derive all the covariance matrix elements for two experimentally relevant track parameterizations (i.e. x and y slopes and intercepts, and direction cosines and intercepts) in the presence of thin scatterers and absence of magnetic fields. We further comment on how to account for thick and/or continuous scattering centers.

1 Introduction

Track fitting (including multiple Coulomb scattering) with the Kalman filter technique[1][2] has recently proven to be a powerful alternative to the traditional Wiener or "global" method, as well as to the breakpoint and various other ad hoc methods[3]–[12].

The Kalman filter is a recursive or stepwise procedure for estimating the state parameters of a linear, discrete dynamic system (e.g. a track in a particle physics detector). The Kalman method was originally introduced in 1960[1], and subsequently led to a revolution in various engineering fields, including

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optimal signal processing, navigation, spacecraft tracking etc[13]. It was not until Fruhwirth's 1987 paper[2], however, that the Kalman filter technique became generally known in particle physics. Prior to this Billoir[4] independently derived equations equivalent to the Kalman filter equations without realizing it. With Fruhwirth's paper, however, the full power of the Kalman technique became available, especially the smoother algorithm.

The Kalman filter solution reproduces the result of the global least squares fit, but avoids the $N \times N$ matrix inversion required by the latter method (where N is the total number of measurements). This matrix is in general non-diagonal due to correlations (caused by multiple scattering) between measurements. The largest matrix inversion required by the Kalman technique is the smaller of $m \times m$ (gain matrix formalism, see below) and $p \times p$ (weighted means formalism), where m is the number of measurements per point, and p is the number of parameters to be estimated (i.e. dimension of the state vector).

In contrast to the Kalman filter, the breakpoint method[7] reproduces the results of the global fit through actual fitting of the multiple scattering angles at each scattering plane. Ad hoc methods often only approximate the result of the global fit, and are generally used simply to avoid the costly matrix inversion.

An essential ingredient in the Kalman technique is the covariance matrix of the track parameters due to multiple Coulomb scattering. Although some partial results are available in the current literature[3][12], results are inconsistant, and derivations are often absent. In the following we offer a concise and general method for evaluating the covariance matrices in the presence of thin scattering centers and absence of magnetic fields.

The organization of our discussion is as follows. In section 2 we review the Kalman filter technique, and present a simplified derivation of the filter equations. In section 3 we define our notation and coordinate systems, and derive all terms of the covariance matrix for two experimentally relevant track parameterizations. We then comment on how our formalism can be used with thick and/or continuous scattering centers. In section 4 we discuss recent results concerning the variance of the projected scattering angle due to multiple scattering. Section 5 contains a summary and conclusions.

2 The Kalman Filter

The Kalman technique focuses on a $p \times 1$ "state vector" that contains the p state parameters to be estimated, and on a model that extrapolates the state vector from point to point. These points can either be real points in space or time, or can simply be dimensionless integers (e.g. the track number in vertex fitting).

In the following, we use $\bar{x}_k$ to denote the $p \times 1$ state vector containing the true state parameters at point k. The state vector extrapolation model in the linear case is

$$\bar{x}_k = F_{k-1}\bar{x}_{k-1} + w_{k-1},$$

where F_{k-1} is a matrix that extrapolates the state vector from point $k-1$ to point k, and w_{k-1} represents "process noise" that corrupts the state vector (in track fitting, for example, the process noise is due to multiple scattering). The process noise is assumed to be unbiased and to have finite variance, and its covariance matrix is Q_k.

The main result of this paper, given in section 3, is the evaluation of Q_k for two experimentally relevant track parameterizations.

The components of the state vector $\bar{x}_k$ are not measured directly. The actual m measurements m_k at point k are linear functions of the state vector $\bar{x}_k$ such that,

$$m_k = H_k\bar{x}_k + \epsilon_k,$$

where m_k is a $m \times 1$ vector, H_k is a $m \times p$ matrix, and ϵ_k represents measurement noise, or measurement errors. In analogy to the process noise, the measurement noise ϵ_k is assumed to be unbiased and to have finite variance, and its covariance matrix is V_k.

Following Gelb[14], we define $\bar{x}_k^{k-1}$ to be the best estimate of the true state vector $\bar{x}_k$ using all measurements up to but *not* including the kth measurement, and $\bar{x}_k^k$ to be the best estimate of the state vector *including* the kth measurement. In the language of Kalman filtering, $\bar{x}_k^{k-1}$ is the *predicted estimate* of $\bar{x}_k$, and $\bar{x}_k^k$ is the *filtered estimate* of $\bar{x}_k$. Further, for $n > k$, $\bar{x}_k^n$ is called the *smoothed estimate* of $\bar{x}_k$.

In the Kalman scheme $\bar{x}_k^k$ is taken to be an arbitrary linear function of the extrapolated (or predicted) value of the state vector, $\bar{x}_k^{k-1}$, and the actual measurement m_k made at point k:

$$\bar{x}_k^k = K_k^1\bar{x}_k^{k-1} + K_k^2 m_k,$$

where K_k^1 and K_k^2 are arbitrary matrices. The Kalman prediction equation is taken to be

$$\bar{x}_k^{k-1} = F_{k-1}\bar{x}_{k-1}^{k-1}.$$

Requiring $\bar{x}_k^k$ to be unbiased (i.e. expectation value of $(\bar{x}_k^k - \bar{x}_k) = 0$) yields

$$\bar{x}_k^k = \bar{x}_k^{k-1} + K_k(m_k - H_k\bar{x}_k^{k-1})$$

where K_k is called the Kalman gain matrix. Further,

$$K_k^1 = (I - K_k H_k)$$

and

$$K_k^2 = K_k.$$

K_k is determined by requiring that it minimize the sum of the squares of the standard deviations of the estimated parameters,

$$\frac{\partial Tr(C_k)}{\partial K_k} = 0,$$

where $Tr(C_k)$ is the trace of C_k and C_k is the $p \times p$ covariance matrix of the estimated parameters at point k. Solving for K_k yields

$$K_k = C_k^{k-1} H_k^T (V_k + H_k C_k^{k-1} H_k^T)^{-1},$$

where

$$C_k^{k-1} = F_{k-1} C_{k-1}^{k-1} F_{k-1}^T + Q_{k-1}$$

and

$$C_k^k = (1 - K_k H_k) C_k^{k-1}$$

(the superscript T denotes the transpose). In correspondence to our earlier notation, C_k^{k-1} is the best estimate of C_k excluding the k^{th} measurement, and C_k^k is the best estimate of C_k including the k^{th} measurement.

We note that our equations correspond to the "gain matrix" formulation of the Kalman filter method. Fruhwirth[2] further explores the "weighted means" formulation, which focuses on the K_k^1 and K_k^2 matrices instead. The two methods are mathematically equivalent; the gain matrix formulation is generally preferred if $m < p$.

Full discussions of the Kalman filter, Kalman smoother, and other approaches to track fitting including multiple scattering can be found in references [2]–[21].

3 Calculation of Covariance Matrices

3.1 Notation and Outline of Calculation

Multiple scattering is conveniently parameterized by two mutually orthogonal, uncorrelated scattering angles θ_1 and θ_2[3][12][22], which are assumed to be small. We ignore the transverse displacement of the track as it crosses the scattering plane[23], and assume that the scattering takes place at a single point in the center of the scattering plane.

The covariance matrix elements $\langle \theta_i, \theta_j \rangle$ for the scattering angles θ_1 and θ_2 can be written as

$$\langle \theta_i, \theta_j \rangle = \sigma^2(\theta_{proj})\ \delta_{ij}, \tag{1}$$

where $\sigma^2(\theta_{proj})$ is the variance of the projected multiple scattering angle (see section 4 for an explicit form for $\sigma(\theta_{proj})$).

The covariance matrix elements $\langle P_i, P_j \rangle$ for any two arbitrary functions $P_i(\theta_1, \theta_2)$ and $P_j(\theta_1, \theta_2)$ can be evaluated by using the propagation of errors formula[22]

$$\langle P_i, P_j \rangle = \sum_{m,n=1}^{2} \frac{\partial P_i}{\partial \theta_m} \frac{\partial P_j}{\partial \theta_n} \langle \theta_m, \theta_n \rangle. \tag{2}$$

In the present context, $\langle P_i, P_j \rangle$ represents the elements of the covariance matrix Q_k, defined in section 2. Substitution of Eq. 1 into equation Eq. 2 yields

$$\langle P_i, P_j \rangle = \sigma^2(\theta_{proj}) \left(\frac{\partial P_i}{\partial \theta_1} \frac{\partial P_j}{\partial \theta_1} + \frac{\partial P_i}{\partial \theta_2} \frac{\partial P_j}{\partial \theta_2} \right). \tag{3}$$

Hence, calculation of the covariance matrix elements for any track parameterization requires evaluation of the partial derivatives of the track parameters with respect to the two orthogonal scattering angles (in the limit $\theta_1, \theta_2 \to 0$).

The outline of our derivation is as follows. We express the unscattered track parameters in a *reference* coordinate system, let the track to scatter at the point (x_0, y_0, z_0), and recalculate the scattered track parameters in the reference system. The scattering angles θ_1 and θ_2 are defined in a *track* coordinate system whose origin is (x_0, y_0, z_0), and whose z axis is aligned with the unscattered track (note that although the z axis of the track system is aligned with the unscattered track, the orientation of the other two axes is arbitrary). We then calculate the partial derivatives of the scattered track parameters with respect to θ_1 and θ_2 in the limit $\theta_1, \theta_2 \to 0$, and use the propagation of errors formula to obtain our final result.

The angles between the projections of the unscattered (scattered) track onto the x-z and y-z planes of the reference system and the z-axis of the reference system are θ_x and θ_y (θ_x^* and θ_y^*) respectively. Similarly, the angles between the projection of the scattered track onto the x-z and y-z planes of the track system and the z-axis of the track system are θ_1 and θ_2. In the limit $\theta_1, \theta_2 \to 0$, $\theta_x^*, \theta_y^* \longrightarrow \theta_x, \theta_y$.

Finally, the direction cosines of the x, y, and z axes of the track system in the reference system are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, and $(\alpha_3, \beta_3, \gamma_3)$ respectively.

3.2 Direction Tangent Case

For the direction tangent case, we define the 4×1 state vector $\bar{x}$ as:

$$\bar{x} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = \begin{pmatrix} x_0 - z_0 \tan\theta_x \\ y_0 - z_0 \tan\theta_y \\ \tan\theta_x \\ \tan\theta_y \end{pmatrix}. \tag{4}$$

That is, $(P_1, P_2, 0)$ is the intercept of the unscattered track with the reference x-y plane at $z = 0$, and P_3 and P_4 are the direction tangents (or slopes, dx/dz and dy/dz) of the unscattered track in the reference system. In the track coordinate system the incoming (unscattered) track unit vector is simply

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

while after scattering the unit vector becomes

$$\frac{1}{\sqrt{1 + \tan^2\theta_1 + \tan^2\theta_2}} \begin{pmatrix} \tan\theta_1 \\ \tan\theta_2 \\ 1 \end{pmatrix}. \tag{5}$$

The 3×3 rotation matrix connecting the track and reference systems is

$$R = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix},$$

with elements obeying the relations

$$\begin{aligned} \alpha_i\alpha_j + \beta_i\beta_j + \gamma_i\gamma_j &= \delta_{ij}, \\ \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 &= 0, \\ \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 &= 0, \\ \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 &= 0. \end{aligned} \tag{6}$$

Thus the scattered unit vector in the reference system is

$$a \begin{pmatrix} \tan\theta_x^* \\ \tan\theta_y^* \\ 1 \end{pmatrix} = a' \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} \begin{pmatrix} \tan\theta_1 \\ \tan\theta_2 \\ 1 \end{pmatrix}, \tag{7}$$

where a and a' are normalization constants for the unit vectors. Solving Eq. 7 for the direction tangents of the scattered vector in the reference system yields

$$P_3 = \tan\theta_x^* = \frac{\alpha_1 \tan\theta_1 + \alpha_2 \tan\theta_2 + \alpha_3}{\gamma_1 \tan\theta_1 + \gamma_2 \tan\theta_2 + \gamma_3}, \tag{8}$$

$$P_4 = \tan\theta_y^* = \frac{\beta_1 \tan\theta_1 + \beta_2 \tan\theta_2 + \beta_3}{\gamma_1 \tan\theta_1 + \gamma_2 \tan\theta_2 + \gamma_3}. \tag{9}$$

Taking the partial derivatives of P_3 and P_4 with respect to θ_1 and θ_2 in the small angle limit gives

$$\left.\frac{\partial P_3}{\partial \theta_1}\right|_{\theta_1,\theta_2 \longrightarrow 0} = \frac{\alpha_1\gamma_3 - \alpha_3\gamma_1}{(\gamma_3)^2}, \tag{10}$$

$$
\begin{aligned}
\left.\frac{\partial P_3}{\partial \theta_2}\right|_{\theta_1,\theta_2 \longrightarrow 0} &= \frac{\alpha_2\gamma_3 - \alpha_3\gamma_2}{(\gamma_3)^2}, && (11)\\
\left.\frac{\partial P_4}{\partial \theta_1}\right|_{\theta_1,\theta_2 \longrightarrow 0} &= \frac{\beta_1\gamma_3 - \beta_3\gamma_1}{(\gamma_3)^2}, && (12)\\
\left.\frac{\partial P_4}{\partial \theta_2}\right|_{\theta_1,\theta_2 \longrightarrow 0} &= \frac{\beta_2\gamma_3 - \beta_3\gamma_2}{(\gamma_3)^2}. && (13)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\frac{\partial P_1}{\partial \theta_1} &= -z_0\frac{\partial P_3}{\partial \theta_1},\\
\frac{\partial P_1}{\partial \theta_2} &= -z_0\frac{\partial P_3}{\partial \theta_2},\\
\frac{\partial P_2}{\partial \theta_1} &= -z_0\frac{\partial P_4}{\partial \theta_1},\\
\frac{\partial P_2}{\partial \theta_2} &= -z_0\frac{\partial P_4}{\partial \theta_2}.
\end{aligned}
$$

Substituting the above partial derivatives into Eq. 3, and simplifying with Eqs. 6 yields

$$
\begin{aligned}
\langle P_3, P_3\rangle &= \sigma^2(\theta_{proj})\left\{\left[\frac{\alpha_1\gamma_3 - \alpha_3\gamma_1}{(\gamma_3)^2}\right]^2 + \left[\frac{\alpha_2\gamma_3 - \alpha_3\gamma_2}{(\gamma_3)^2}\right]^2\right\}\\
&= \sigma^2(\theta_{proj})\left[\frac{(\alpha_3)^2 + (\gamma_3)^2}{(\gamma_3)^4}\right]. && (14)
\end{aligned}
$$

Note that this result only depends on $(\alpha_3, \beta_3, \gamma_3)$, i.e. on the orientation of the z axis of the track system with respect to the reference system. Using the identities

$$
\begin{aligned}
\alpha_3 &= \frac{\tan\theta_x}{(1+\tan^2\theta_x + \tan^2\theta_y)^{1/2}},\\
\beta_3 &= \frac{\tan\theta_y}{(1+\tan^2\theta_x + \tan^2\theta_y)^{1/2}}, && (15)\\
\gamma_3 &= \frac{1}{(1+\tan^2\theta_x + \tan^2\theta_y)^{1/2}},
\end{aligned}
$$

$\langle P_3, P_3\rangle$ becomes (in the limit $\theta_1, \theta_2 \to 0$, $\theta_x^*, \theta_y^* \to \theta_x, \theta_y$)

$$
\begin{aligned}
\langle P_3, P_3\rangle &= \sigma^2(\theta_{proj})\,(1+\tan^2\theta_x + \tan^2\theta_y)(1+\tan^2\theta_x)\\
&= \sigma^2(\theta_{proj})\,[1+(P_3)^2][1+(P_3)^2+(P_4)^2]. && (16)
\end{aligned}
$$

Similarly,

$$
\langle P_4, P_4\rangle = \sigma^2(\theta_{proj})\,[1+(P_4)^2][1+(P_3)^2+(P_4)^2]. \qquad (17)
$$

In the same manner,

$$\begin{aligned}
\langle P_3, P_4 \rangle &= \sigma^2(\theta_{proj}) \, \frac{\alpha_3 \beta_3}{(\gamma_3)^4} \\
&= \sigma^2(\theta_{proj}) \, P_3 P_4 [1 + (P_3)^2 + (P_4)^2]. \qquad (18)
\end{aligned}$$

Finally, the full covariance matrix of the track parameters, constructed from the preceding three elements (i.e. Eqs. 16, 17 and 18), is

$$\begin{pmatrix}
z_0^2\langle P_3, P_3\rangle & z_0^2\langle P_3, P_4\rangle & -z_0\langle P_3, P_3\rangle & -z_0\langle P_3, P_4\rangle \\
z_0^2\langle P_3, P_4\rangle & z_0^2\langle P_4, P_4\rangle & -z_0\langle P_3, P_4\rangle & -z_0\langle P_4, P_4\rangle \\
-z_0\langle P_3, P_3\rangle & -z_0\langle P_3, P_4\rangle & \langle P_3, P_3\rangle & \langle P_3, P_4\rangle \\
-z_0\langle P_3, P_4\rangle & -z_0\langle P_4, P_4\rangle & \langle P_3, P_4\rangle & \langle P_4, P_4\rangle
\end{pmatrix}.$$

3.3 Direction Cosine Case

For the direction cosine case the 4×1 state vector $\bar{x}$ is

$$\bar{x} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = \begin{pmatrix} x_0 - z_0(\alpha_3/\gamma_3) \\ y_0 - z_0(\beta_3/\gamma_3) \\ \alpha_3 \\ \beta_3 \end{pmatrix},$$

where α_3, β_3, and γ_3 are the direction cosines of the incoming (unscattered) track in the reference sytem. Proceeding as before, and using the definition of direction cosines, we have

$$\begin{pmatrix} \alpha^* \\ \beta^* \\ \gamma^* \end{pmatrix} = a \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} \begin{pmatrix} \tan\theta_1 \\ \tan\theta_2 \\ 1 \end{pmatrix},$$

where a is a normalization constant, and α^*, β^*, and γ^* are the direction cosines of the scattered track in the reference system (note that in the limit $\theta_1, \theta_2 \to 0$, $(\alpha^*, \beta^*, \gamma^*) \longrightarrow (\alpha_3, \beta_3, \gamma_3)$). Evaluating the partial derivatives of the scattered track parameters in the limit $\theta_1, \theta_2 \to 0$ gives

$$\begin{aligned}
\partial\alpha^*/\partial\theta_1 &= \alpha_1, & \partial\alpha^*/\partial\theta_2 &= \alpha_2, \\
\partial\beta^*/\partial\theta_1 &= \beta_1, & \partial\beta^*/\partial\theta_2 &= \beta_2, \\
\partial\gamma^*/\partial\theta_1 &= \gamma_1, & \partial\gamma^*/\partial\theta_2 &= \gamma_2.
\end{aligned}$$

Finally, using the propagation of errors formula and the orthogonality relations of the rotation matrix, the full covariance matrix of the track parameters is

$$\sigma^2(\theta_{proj}) \times$$

$$\begin{pmatrix} \left[\frac{z_0^2(1-P_4^2)}{(1-P_3^2-P_4^2)^2}\right] & \left[\frac{z_0^2 P_3 P_4}{(1-P_3^2-P_4^2)^2}\right] & \left[\frac{-z_0}{(1-P_3^2-P_4^2)^{1/2}}\right] & 0 \\ \left[\frac{z_0^2 P_3 P_4}{(1-P_3^2-P_4^2)^2}\right] & \left[\frac{z_0^2(1-P_3^2)}{(1-P_3^2-P_4^2)^2}\right] & 0 & \left[\frac{-z_0}{(1-P_3^2-P_4^2)^{1/2}}\right] \\ \left[\frac{-z_0}{(1-P_3^2-P_4^2)^{1/2}}\right] & 0 & 1-P_3^2 & -P_3P_4 \\ 0 & \left[\frac{-z_0}{(1-P_3^2-P_4^2)^{1/2}}\right] & -P_3P_4 & 1-P_4^2 \end{pmatrix}.$$

3.4 Comments

If the transverse displacement of the track as it crosses the scattering plane cannot be ignored, two modifications must be made. First, the error matrix for the multiple scattering angles θ_1 and θ_2 (Eq. 1) must be replaced by the full 4×4 correlated error matrix of multiple scattering angles and displacements[3][12]. Second, the partial derivatives of the track parameters with respect to the two transverse displacements must be calculated, following the method given in the preceding sections; Eq. 2 (rather than Eq. 3) is then used.

If the scattering centers are too thick to be considered discrete, the formula for the mean scattering angles (see section 4) and displacements[22] must be replaced by an integral over the scattering medium[3][12].

4 Variance of Multiple Scattering Angle

Several forms of the standard deviation $\sigma(\theta_{proj})$ are available. Lynch and Dahl[22][24] obtain, with result good to approximately 11%,

$$\sigma(\theta_{proj}) = \frac{13.6}{\beta cp} z\sqrt{L/L_r}\ [1 + 0.038 \ln(L/L_r)], \tag{19}$$

where L is the path length through the material, L_r is the radiation length of the material, βc is the velocity of the particle, z is the charge, and p is the momentum in MeV. This result is a modification of the previous formulae of Rossi-Griesen[26] and Highland[27] (who introduced the logarithmic term).

Lynch and Dahl also present an alternative form, good to better than 2%, that uses Moliere's characteristic angle and screening angle[25] instead of the radiation length.

5 Conclusions

The application of the Kalman filter technique to track fitting requires evaluation of the covariance matrix of the track parameters due to multiple scattering. Results in the current literature are generally incomplete, and derivations are often absent.

We have briefly introduced the basics of Kalman filter technique, and have calculated the full covariance matrix for two experimentally relevant track parameterizations for thin scatterers in the absence of a magnetic field: direction tangents and intercepts, and direction cosines and intercepts. Where they overlap, our results agree with the partial ones quoted in reference [12].

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