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The Applicability of Diffusion Phenomenology to Particle Losses in Hadron Colliders

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THE APPLICABILITY OF DIFFUSION PHENOMENOLOGY TO PARTICLE LOSSES IN HADRON COLLIDERS

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Abstract

An analytic approach is developed for solving the inhomogeneous diffusion equation with the diffusion intensity being a fast-growing function of the betatron energy. When applied to the survival data of particle tracking for LHC and SSC, the method shows that these data are inconsistent with (any) diffusion phenomenology. A similar inconsistency is observed in the data from CERN diffusion experiment.

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1. INTRODUCTION

Understanding the limits of the stability of particles at large betatron amplitudes presents one of the major accelerator physics challenges in the design of new supercolliders such as SSC and LHC [1, 2]. The reliability of the tracking codes that are used for this purpose needs to be checked in experiments with the existing colliders. In this conjunction it is also important to understand the nature of the slow transport of particles below the dynamical aperture due to lattice nonlinearities and/or power supply ripples. The natural description of such transport is the diffusion process with amplitude-dependent diffusion coefficient, as used in the Fermilab [3] and CERN [4] diffusion experiments.

In the present paper, we analyze the properties of this diffusion model as applied to three different studies:

I) density profiles measurements in Fermilab experiment [3],

II) beam intensity time dependence in CERN scraper retraction experiment [4],

III) escape time spreads versus initial amplitude as produced in tracking for SSC [5] and LHC.

The basis of the analysis is the strong inhomogeneity, i.e. fast growth with the amplitude, of the diffusion observed in I and III.

2. ASYMPTOTIC ANALYSIS OF INHOMOGENEOUS DIFFUSION

The diffusion model that was used for the data analysis of the Fermilab diffusion experiment and which we will use to model both the tracking data and CERN scraper retraction data, has the following form. [3]:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial I}\left(D(I) \frac{\partial \rho}{\partial I} \right) \tag{1}$$

where I is the betatron action (energy) $I = (x^2/2) + (x'^2/2)$, D(I) is the action-dependent diffusion coefficient, and ρ is the density distribution function. We will develop now an asymptotic approach to solving the diffusion equation (1) when the diffusion is strongly inhomogeneous, i.e. grows very fast with increasing $I : D'/D \to \infty$*. Apart from assuming that (D'/D) is large, we will consider only the case when the logarithm of D'(I) does not change much over the range of I, corresponding to one decade of variation of D. This restriction, which can be explicitly written as $ff'' \ll (f')^2$ (where $f(I) = \ln D(I)$), is quite mild and as we will see in a number of examples later, is usually satisfied quite well. Within the class of functions D(I) thus defined we can always use the "local exponential approximation"

$$D(I_o + \Delta I) \approx D(I_o) \exp\left(\frac{D'(I_o)}{D(I_o)} \Delta I \right) \tag{2}$$

which corresponds to the linear term of the Taylor expansion of the function ln D(I), implying $\Delta I \lesssim (D / D')$. In the reference example $D = AI^n$ with $n \gg 1$, that proved to provide a good model for the Fermilab diffusion data [3], one gets $D \approx AI_o^n \exp\left(\frac{n}{I_o} \Delta I \right)$.

The basic idea in the solution of the diffusion equation (1) for the class of functions D(I) (2) is that for the fast growing functions D(I) and at each given moment of time, the particles with initial conditions above a certain value $I_e(t)$ are all lost at the absorbing boundary $I_{ab} > I_e(t)$, while those below the value $I_e(t)$ are not affected by diffusion at all. The distribution

* In the following, we reserve the prime for differentiation by action I, and the dot • for differentiation by time t.

function in the transition area should have then some universal properties due to the narrowness of that region. Thus, we will seek the general time-dependent solution in the form:

$$\rho(I,t) = \rho_o(I)\ g\left(\frac{I - I_e(t)}{\lambda}\right) \qquad (3)$$

where ρ_o (I) is an (arbitrary) initial distribution ρ (I, o) = ρ_o (I) and g is a fixed function of one variable $S = (I - I_e(t))/\lambda(t)$ qualitatively shown in Fig. 1. We expect λ to be asymptotically small when D'/D tend to infinity.

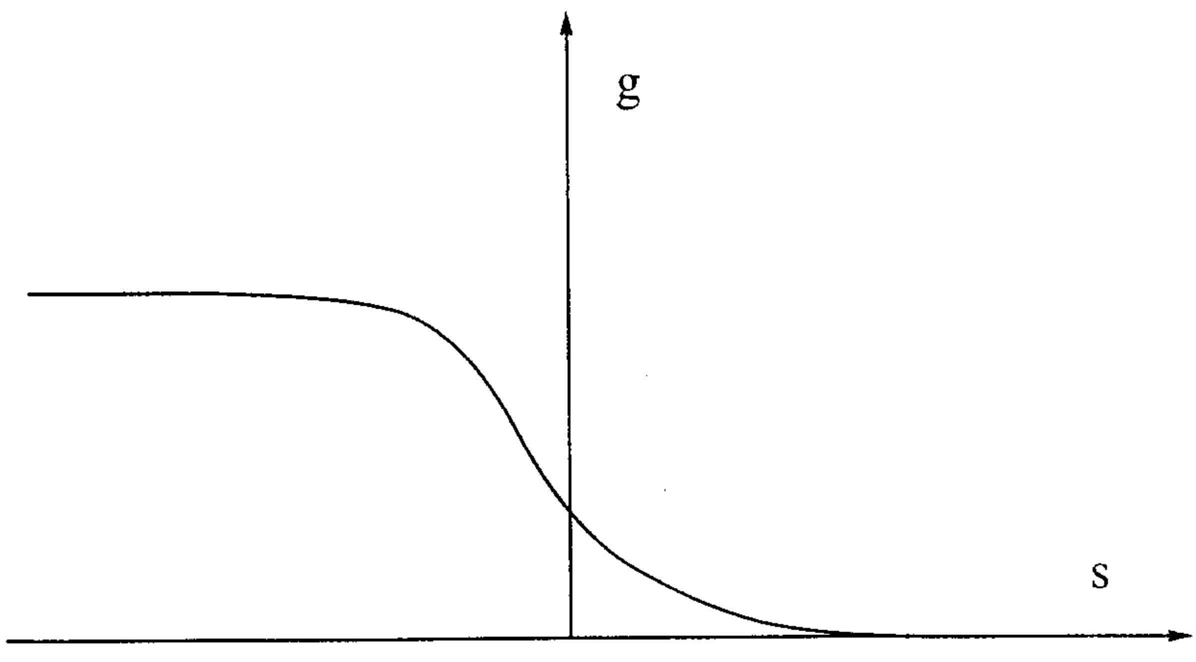

Fig. 1

Qualitative sketch of the function g in the solution (3).

The distribution (3) therefore is a product of initial distribution ρ_o and the "kink" function g, with moving center $I_e(t)$ and varying width $\lambda(t)$.

To find the three unknown functions g(s), $I_e(t)$, $\lambda(t)$ we plug the ansatz (3) in the equation (1), using the local approximation (2) with $I_o = I_e(t)$ and do not differentiate the slow-varying function g (I). The resulting equation is:

$$-g'(s)\left[s\dot{\lambda}(t) + \dot{I}_e(t) + D'(I_e(t))\exp\left(k(I_e(t))\lambda(t)s\right)\right] = \frac{D(I_e(t))}{\lambda(t)}\exp\left(k(I_e(t))\lambda(t)s\right)g''(s) \qquad (4)$$

where we introduced the notation $k(I_o) = D'(I_o)/D(I_o)$

Looking at equation (4) more closely, it is not difficult to realize that the only way of "untangling" the variables s and t and obtaining separate equations for the functions of different arguments is to assume that $\lambda(t) = 1/k(I_e(t))$. Note that an arbitrary constant that can be put in the denominator can be shown to simply rescale g and thus disappears from the final result. Having this expression for $\lambda(t)$, one immediately realizes that the first term in the square brackets in equation (4) is much smaller than the second one. This follows from the inequality $k'/k^2 \ll 1$ that is equivalent to the condition of applicability of the "local exponential approximation" (2) $ff'' \ll (f')^2$. Dropping thus that term, we arrive at:

$$-\frac{g''(s)}{g'(s)} = 1 + \frac{\dot{I}_o(t)}{D'(I_o(t))} e^{-s} \tag{5}$$

The final untangling of variables s and t can be achieved now only through the choice of the function $I_e(t)$ to satisfy the equation $\dot{I}_e(t) = -D'(I_e(t))$ An arbitrary constant that could be multiplying the right-hand side can be shown to disappear from the final result. The initial condition for that equation is $I_e(o) = I_{ab}$.

The function g is found then to be

$$g(s) = 1 - \exp\left[-e^{-s}\right] \tag{6}$$

Thus, by determining the functions g, I_e and λ we completely defined the solution (3). Due to the smallness of λ, the qualitative image of that solution is that of moving boundary $I_e(t)$, with the distribution ρ_o vanishing above this value.

3. Mean escape time analysis

One of the common ways of representing the tracking data for long-term stability of particles is through so-called "survival plots", where escape times to a certain boundary for particles started at different amplitudes are shown. One example of "survival plots" for the tracking study of SSC [5] is shown in Fig. 2. We will address the issues of whether such distributions of escape times can appear in the diffusion model (1) and how to extract the diffusion intensity D(I) from the survival data. The idea of using the magnitudes of the spreads of escape times in the survival plots as a compatibility test with the diffusion model was originally introduced by J. Cary [7]. Our approach to the realization of this idea differs though from that of J. Cary [8]: we take advantage of approximately 'locally exponential' character of the escape time dependence on the action τ (I) and employ the analytic method of Section 2 to calculate the escape time spreads.

According to the more general theory of escape in diffusion processes with drifts [6], the probability G(I,t) of surviving, or not escaping to an absorbing boundary, within time t starting from the initial condition I for the diffusion process (1) satisfies the diffusion equation:

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial I}\left(D(I)\frac{\partial G}{\partial I}\right) \tag{7}$$

with the initial condition G(I,0) = 1 and boundary condition $G(I_{ab},t) = 0$ at the absorbing wall. The mean escape time defined as $T(I) = -\langle t(\partial G/\partial t)\rangle = \langle G \rangle$ (average is over t), can then be shown [6] to be the solution of the equation:

$$\frac{d}{dI}\left(D(I)\frac{dT}{dI}\right) = -1 \tag{8}$$

with the boundary condition $T(I_{ab}) = 0$.

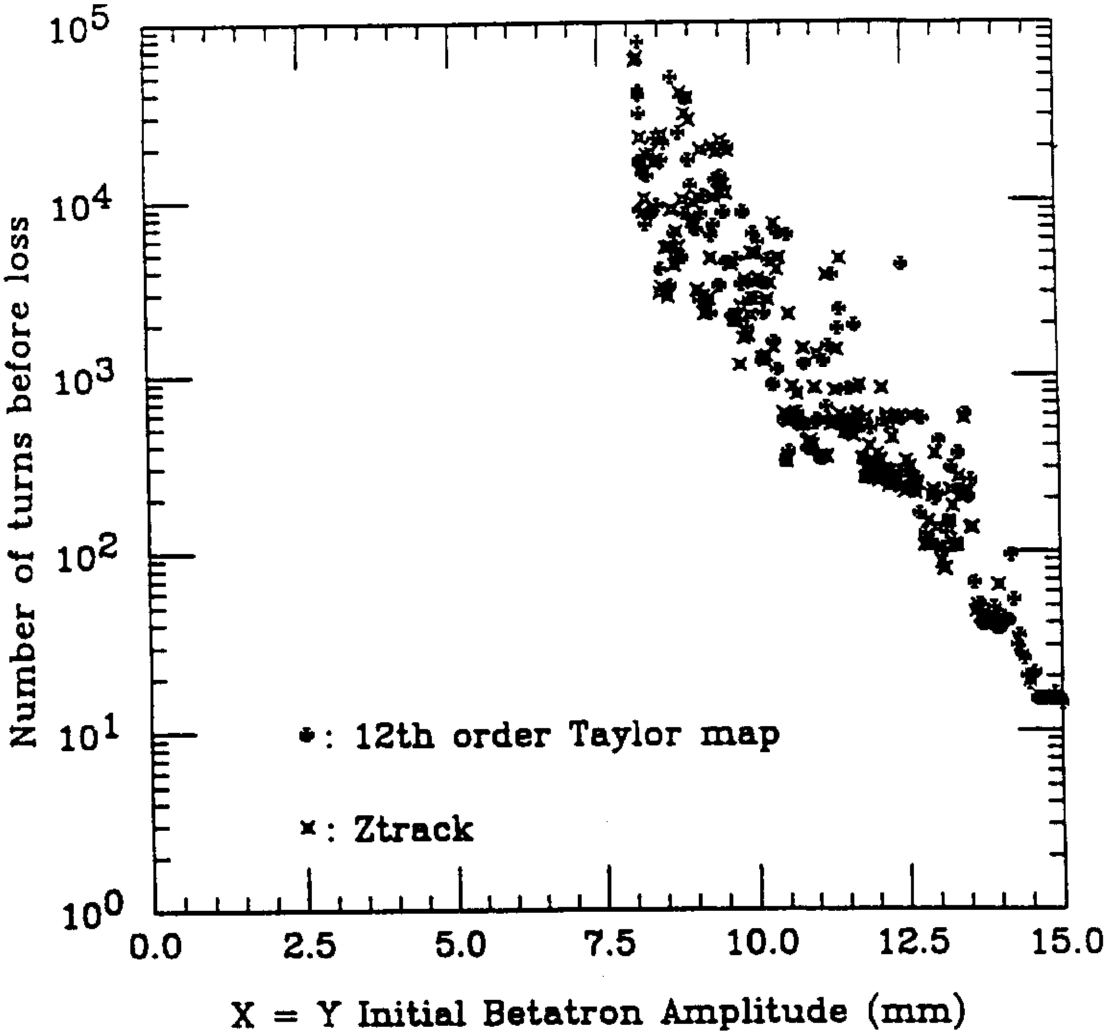

Fig. 2

100.000 turns survival plots for 2 TeV SSC injection lattice /5/.

The second boundary condition at I = 0 in our case of positive-definite coordinate I is that of a reflective wall yielding [6] $D(I)\ (dT/dI)|_{I=0} = 0$.

The escape time is found then to be

$$T(I) = \int_{I}^{I_*} dI_1 \frac{I_1}{D(I_1)} \ . \tag{9}$$

The escape time data in the survival plots provide the dependence of the escape time on the action T(I), which defines thus the diffusion intensity $D(I) = -(I/(dT/dI))$. An important question then is to understand whether the observed spreads of escape times, which are quite large (Fig. 2), are compatible with the diffusion model as deduced from the mean escape time.

The natural measure of these spreads is the r.m.s. width of the escape time distribution $\Delta T = \sqrt{\langle t^2 \rangle - \langle t \rangle^2}$. While the first moment $\langle t \rangle$ is just T, the second moment $T_2 = \langle t^2 \rangle = -\langle t(\partial G / \partial t) \rangle$ is known from the general theory [6] to satisfy the equation:

$$-T(I) = \frac{d}{dI}\left(D(I) \frac{dT_2}{dI} \right) \tag{10}$$

with the same boundary conditions as for T(I). The second moment thus is explicitly found to be

$$T_2(I) = \int_I^{I_{ab}} \frac{dI_1}{D(I_1)} \int_\delta^{I_1} T(I_2) dI_2 \tag{11}$$

where the reflecting wall was artificially placed at $I = \delta$ rather than at $I = 0$.

Let us try now to extract the quantity $T_2(I)$ from the survival plots of Fig. 2. The mean escape time T(I) on this plot decreases (with an increasing I) exponential-like, so that the dominant contributions to both internal and external integrals in (11) come from the lower limits of integration. It appears also that the function T(I) and the corresponding diffusion intensity $D(I) = -(I/(dT/dI))$ satisfy the "local exponential approximation" (2), since the slope of the curve $f(I) = \ln T(I)$ doesn't change much when f changes by about unity. From this approximation, one can explicitly find the quantity T_2 (11) to be

$$T_2(I) = -\frac{1}{D'(I)} \frac{T^2(\delta)}{T'(\delta)} = -\frac{T(I)}{I} \cdot \frac{T^2(\delta)}{T'(\delta)} \tag{12}$$

where the condition of the distances I-I_{ab} and I_1-δ being large relative to the characteristic scale $T(I)/T'(I)$ was used. This formula indicates that the second moment T_2 at the point I is very sensitive to the behaviour of diffusion intensity at small actions. In particular, the quantity T_2 (I) diverges whenever the escape time T_2 (I_1) behaves at small $I_1 \rightarrow 0$ as $T(I_1) \sim 1/I_1^k$ with any $k > 1$.

The quantity T_2 is thus inconvenient for comparison with the "survival plots" since large T_2 in many cases can (and in fact does) account only for the long "tail" of the distribution of escapes times $f(I,t) = -(\partial G(I,t)/\partial t)$. The preferable quantity of choice then is the width of this distribution Δ defined as the half-max width. To find it, one can use the same asymptotic approach of Section 2 for the solution of the diffusion equation (7) for the function G(I,t). The estimates of the half-max width then can be obtained by requiring that the argument of the function g in the solution (3) change by unity, yielding

$$\Delta(I) \sim \frac{\lambda(I)}{D'^2(I)} = \frac{D(I)}{D'^2(I)} = \frac{T^2(I)}{I(dT/dI(I))}. \tag{13}$$

The dimensionless quantity $\Delta/T = \lambda(I)/I$ (where I is the characteristic scale of growth of the function T(I): $\lambda = T/T'$) for the curve T(I) in Fig. 2 can be easily seen then to be much smaller than unity. This clearly contradicts the wide spreads (over one decade at least) of the escape times observed in the survival plot in Fig 2.

Thus, we arrive at the negative and conceptually important conclusion: the statistics of the escape times as observed in tracking simulation is not compatible with any diffusion process. Therefore more refined models are needed to describe the dynamical processes involved.

4. Scraper Manipulation Analysis

The measurements of the transport of the particles in CERN SPS were performed [4] by kicking the beam to rather large amplitudes, moving the scraper in to the edge of the core of the beam, and retracting it in a short time by just a millimeter or two. The subsequent evolution of the beam intensity was observed over a period of time of about 10–20 minutes.

The characteristic result of such measurement appears as shown in Fig.3. The basic features of this curve are:

1) the presence of a flat section with zero loss
2) rather sharp shoulder of transition to non zero loss
3) quite linear behavior after the shoulder for the time at least just as long as the length of the flat section.

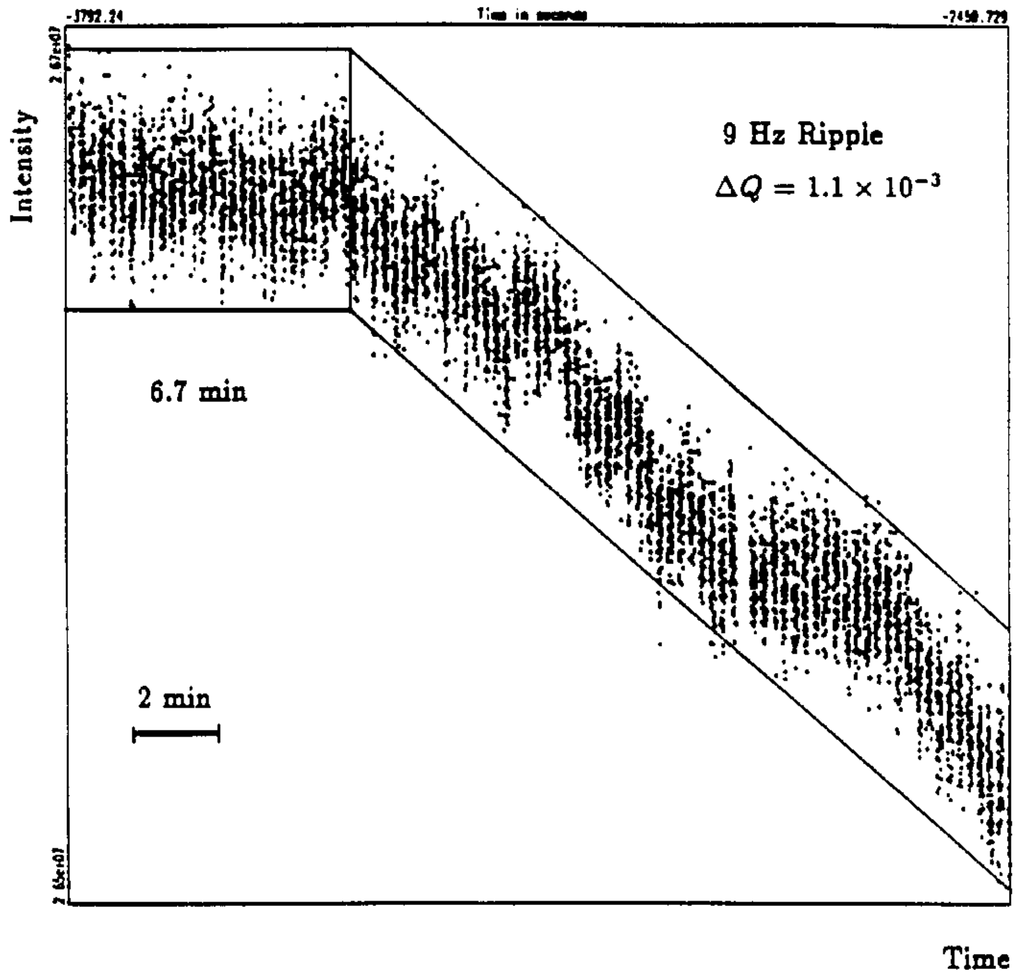

Fig. 3

After the retraction of both scrapers in the horizontal and vertical plane the particles need 6.7 minutes to reach the first (vertical) scraper. Thereafter a constant loss in intensities sets in.

The question then is whether these features can be accounted for in a diffusion model. The setup of the initial distribution in the scraper retraction experiment is shown in Fig 4. I_s is the position of the scraper as it was moved in, and I_r is where it was retracted to.

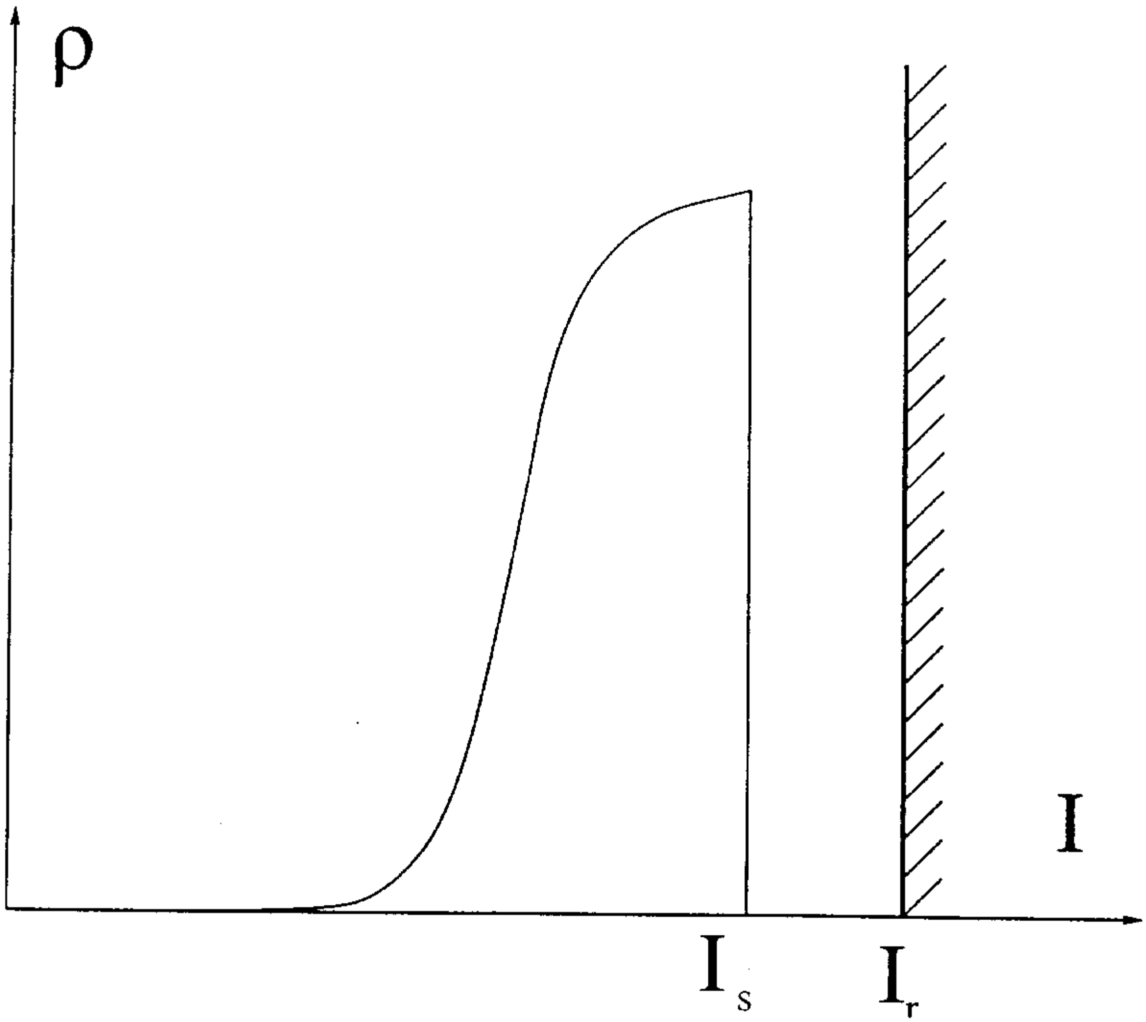

Fig. 4

Initial distribution after the scraper retraction.

We will consider the loss of particles at the wall $I = I_r$ in two extreme situations: first, when the diffusion intensity D(I) changes very little at the distances of the order I_r-I_S around the point $I = I_S$ and second, when it changes by several decades, so that $\lambda = D(I_s)/D'(I_s) \ll I_r - I_S$.

In the first case, one can obtain an explicit solution of the diffusion equation with a constant diffusion coefficient if one assumes a simple initial distribution that equals unity for $I < I_r$ and zero for $I > I_r$

$$\rho(I,t) = \frac{2}{\sqrt{\pi}} \int_{\frac{I-I_r}{\sqrt{Dt}}}^{0} e^{-s^2/2} ds \qquad (14)$$

The total "intensity of the beam" to be compared with the experimental curve of Fig.3 can be calculated (apart from the irrelevant constant contribution) as

$$L(t) = \int_0^{I_r} (\rho(I,t) - 1)\, dI \tag{15}$$

and is explicitly found to be proportional to $-\sqrt{Dt}$'

The intensity L(t) for the realistic initial distribution, which equals unity for $I < I_S$ and zero for $I > I_S$, will naturally have a transient with characteristic time scale $\Delta t \sim (I_r - I_S)^2/D(I_S)$ and then approach the asymptotic behavior $\Delta L \sim -\sqrt{Dt}$ as shown in Fig. 5.

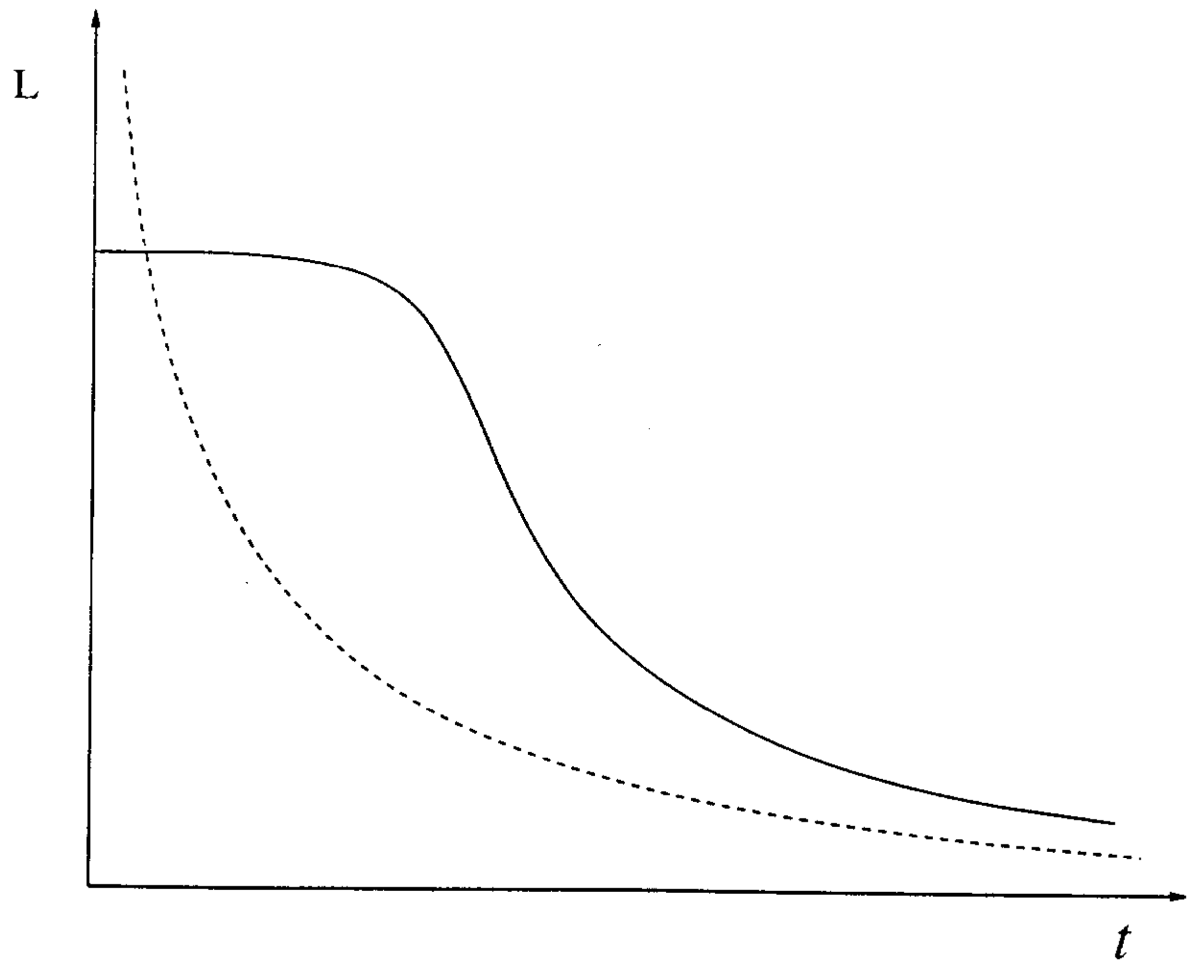

Fig. 5

Qualitative dependence of beam intensity L (15) on time in diffusion model. Dashed line is 1/t.

This type of behavior obviously does not possess the most important feature, a sharp shoulder, of the experimental graph of Fig. 3.

In the second case $\lambda = D(I_s)/D'(I_s) \gg I_s$ we will assume again that the diffusion intensity D(I) satisfies the "local exponential approximation" (2) in the range $|I - I_S| \sim I_r - I_S$. The motion of the centre of the "kink" g in the solution (3) then is $I_e(t) \approx I_S - \lambda \ln (tD_0/\lambda^2)$ (where $D_0 = D\,(I_S)$). Since the initial condition ρ_0 of Fig. 4 is not smooth as it was assumed in the derivation of Section 2, the beam intensity dependence on time will have some transient over time of the order $t_{tr} = \lambda^2/D_0$ after which the loss rate r(t) will approach its asymptotic time dependence $r(t) \approx \dot{I}_e(t) \sim (1/t)$.

Since the only time scale involved is t_{tr}, the presence of a "shoulder" as in the experiment graph of Fig. 4 is not possible, and qualitatively the beam intensity dependence on time will be similar to the one shown in Fig. 5. Thus, in the second case as well as in the first one, the loss of particles in the diffusion process is distinctly different from what is observed in the experimental graph of Fig. 3.

5. DISCUSSIONS AND CONCLUSIONS

The results of the Fermilab diffusion [3] experiment indicate that the diffusion intensity is a fast growing function of betatron energy that can be locally approximated by an exponential function. The same property is observed in the survival plots in particle tracking for LHC and SSC.

We presented an analytic approach to describing the evolution of the distributions of particles in the presence of absorbing walls in the case of such fast-growing diffusion intensities. This evolution demonstrates a major degree of universality: the initial distribution remains unaffected for betatron energies I less than a certain time-dependent value $I_e(t)$ while the density is completely depleted from the energies I larger than $I_e(t)$. The transient region of energies around I_e is narrow and the distribution function there is also universal (does not depend on initial distribution except for the normalization).

Asymptotic analysis of density evolution was applied to the "survival data" from particle tracking for SSC and LHC and it was demonstrated that the observed spreads of survival times are much larger than what they should be from the diffusion model. That proves that this survival data is not compatible with any diffusion model, and more refined statistical models of dynamics are required. One candidate for such a model can be the more general Markov process with jumps, where the evolution of the density distribution is defined by [6]

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial I}\left(D(I)\frac{\partial \rho}{\partial I} \right) + \int dI'\left[\rho(I')W(I',I) - \rho(I)W(I,I')\right] \qquad (16)$$

The last term in the right-hand side describes the effect of jumps with the probability W(I,I'). The idea of including jumps in the statistical description seems appealing also because of the commonly observed "intermittency" of slow and fast motion in tracking [2]. It would be interesting to try to fit the statistics of trajectories in tracking into some jump and diffusion model (16).

In Section 4, we discussed the character of particle loss in the scraper retraction diffusion experiment at CERN. Simple qualitative-type considerations also indicate the incompatibility of the observed decay of the beam intensity with that of diffusion models in the same setup.

In view of the incompatibilities of the diffusion models with the CERN diffusion experiment data and tracking "survival" data one could naturally ask why the Fermilab diffusion experiment data /3/ was basically quite successfully fitted with diffusion models. The answer to that question can be conjectured to be the somewhat different quantities analyzed in these different approaches. Indeed, in the Fermilab experiment, the measured quantities were the substantial changes of the density distribution (corresponding to the intensity loss of 20 to 80 per cent) as obtained by "flying wire" monitors. In the CERN diffusion experiment, the beam intensity changes very little, so the measured loss rates are defined by the small probability of escape to the retracted scraper. Similarly it is the escape process characteristics that are represented in the "survival plots" of tracking. It may very well be that the major changes in the density distributions are accounted for by the diffusive part of the random process, while the escape processes with sufficiently far removed boundaries are dominated by other properties like jumps in the model (16).

6. ACKNOWLEDGMENTS

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