



# String Theory, Black Holes, and $SL(2, \mathbb{R})$ Current Algebra

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We analyse in detail the  $SL(2, \mathbb{R})$  black hole by extending standard techniques of Kac-Moody current algebra to the non-compact case. We construct the elements of the ground ring and exhibit  $W_\infty$  type structure in the fusion algebra of the discrete states. As a consequence, we can identify some of the exactly marginal deformations of the black hole. We show that these deformations alter not only the spacetime metric but also turn on non-trivial backgrounds for the tachyon and *all* of the massive modes of the string.

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## 1. Introduction

String theory is widely advertised as the only viable candidate for a complete, consistent theory of quantum gravity coupled to matter. As such, it must give unambiguous answers to the plethora of important questions left unresolved by traditional semiclassical treatments of gravitating systems. Indeed, this is the arena in which string theory *must* deliver profound insights, or else give up its claim to be the fundamental theory of Planck scale physics. Many of the most interesting and important questions in quantum gravity concern the physics of black holes. These questions involve quantum decoherence, Hawking radiation, the endpoint of black hole evaporation, the existence of naked singularities, and the quantum numbers (“hair”) of black hole solutions.

Some insight into black hole physics has already been obtained from general string arguments as well as string-inspired methods[1],[2],[3],[4]. At the same time, there has been substantial progress in developing new and better techniques to handle strings in nontrivial spacetime backgrounds, beyond the beta function methods introduced some years ago by Callan et al[5]. At least in two spacetime dimensions, matrix model techniques seem to offer hope of obtaining rigorous nonperturbative results for strings in nontrivial backgrounds. Collective field and string field theory approaches also have great promise in the long term. Another strategy is to develop conformal field theories that describe strings in nontrivial spacetime backgrounds. This approach is less rigorous and complete than matrix models or string field theory, but has the advantages of being somewhat more familiar and closer to the physics. On the other hand, this approach is more rigorous and complete than beta function or string-inspired methods, but has the disadvantage of being more abstract and unwieldy.

In [6], Witten suggested that a gauged  $SL(2, R)$  Wess-Zumino-Witten (WZW) sigma model can provide a conformal field theory model of a two dimensional bosonic string in a black hole background. This model, as well as variations of it, was further analyzed in [7],[8],[9], and many other papers. It should be emphasized that the conformal field theory approach to black holes is in no way limited to two dimensions; indeed extensions of Witten’s model to three and four dimensional black holes have already been exhibited[10],[11],[12],[13].

In this paper we will analyze the  $SL(2, R)$  black hole in more detail, by extending standard techniques of Kac-Moody current algebra. While we have not resolved all of the sticky issues which appear in noncompact coset models, we are able to pin down the

physical content of the theory to a considerable extent. We show that the discrete state operators, previously exhibited by Distler and Nelson, form a  $W_\infty$  type algebra, and may thus be regarded as generating infinite quantum “ $W$  hair” of the stringy black hole. From this structure we are able to identify some of the exactly marginal deformations of the black hole. These deformations not only alter the spacetime metric, but also turn on nontrivial backgrounds for the tachyon (which is really a massless scalar) and *all the massive modes of the string*.

Our results appear to confirm previous speculation[4][14] concerning the fundamental interplay of string physics with black hole physics. We also make contact with previous results from beta function calculations[15],[16],[17], the ground ring approach of Witten[18], as well as results from two dimensional Liouville theory.

## 2. The $SL(2, R)$ String

Following Witten[6] we consider an  $SL(2, R)$  WZW sigma model with action

$$\frac{k}{8\pi} \int d^2z \sqrt{h} h^{ij} Tr (\mathbf{g}^{-1} \partial_i \mathbf{g} \mathbf{g}^{-1} \partial_j \mathbf{g}) + ik\Gamma, \quad (2.1)$$

where  $h$  is the worldsheet metric,  $\Gamma$  is the Wess-Zumino term, and the field  $\mathbf{g}(z, \bar{z})$  is an element of  $SL(2, R)$ . This action is invariant (up to a surface term) under global  $SL(2, R)_L \times SL(2, R)_R$  transformations. It is convenient to parametrize the components of  $\mathbf{g}$  as follows

$$g_{\pm\pm} = Tr (R_\pm \mathbf{g}), \quad g_{\pm\mp} = Tr (S_\pm \mathbf{g}), \quad (2.2)$$

where  $R_\pm$  and  $S_\pm$  are defined in terms of the unit matrix and the Pauli matrices:

$$R_\pm = \frac{1}{2} (1 \pm \sigma_2), \quad S_\pm = \frac{1}{2} (\sigma_1 \mp i\sigma_3).$$

The components  $g_{++}$ ,  $g_{--}$ ,  $g_{+-}$ , and  $g_{-+}$  also parametrize  $SU(1, 1)$  (which is isomorphic to  $SL(2, R)$ ) and in fact transform as four spin  $(\frac{1}{2}, \frac{1}{2})$  components under  $SU(1, 1)_L \times SU(1, 1)_R$ . Note that  $g_{--} = g_{++}^*$ ,  $g_{+-} = g_{-+}^*$ , and we have the constraint

$$g_{++}g_{--} - g_{+-}g_{-+} = 1. \quad (2.3)$$

While the parametrization above is convenient for current algebra, it is better when discussing target space physics to use Euler angles:

$$\mathbf{g}(z, \bar{z}) = e^{\frac{1}{2}i\theta_L\sigma_3} e^{r\sigma_1} e^{\frac{1}{2}i\theta_R\sigma_3}. \quad (2.4)$$

Translation between the two parametrizations is given by

$$g_{\pm\pm} = \cosh r \exp \pm \frac{1}{2}i(\theta_L + \theta_R), \quad g_{\pm\mp} = \sinh r \exp \mp \frac{1}{2}i(\theta_L - \theta_R). \quad (2.5)$$

As in [6], we introduce an abelian gauge field  $A(z, \bar{z})$  to gauge the axial diagonal  $U(1)$  subgroup of  $SL(2, R)_L \times SL(2, R)_R$ . If we then fix to unitary gauge:  $\theta_L = -\theta_R = \theta$ , and integrate out the gauge field  $A$ , the action reduces to the Euclidean black hole background

$$\frac{k}{4\pi} \int d^2z (\partial_z r \partial_{\bar{z}} r + \tanh^2 r \partial_z \theta \partial_{\bar{z}} \theta). \quad (2.6)$$

Also, by evaluating the determinant that arises in integrating out the gauge field, one obtains (to lowest order) the dilaton term[19]

$$\Phi = 2 \ln \cosh r. \quad (2.7)$$

For  $k=9/4$ , this gauged WZW model appears to be a conformal field theory description of a bosonic string embedded in a two dimensional target space with a black hole background metric. There are difficulties involved in the consistent quantization of gauged noncompact WZW models[20], but in this paper we will simply assume that Witten's  $SL(2, R)$  model exists as a consistent unitary conformal field theory. Our purpose will be to analyze the content of this theory by employing  $SL(2, R)$  (or rather, equivalently,  $SU(1, 1)$ ) Kac-Moody current algebra.

The holomorphic and antiholomorphic  $SU(1, 1)$  currents are the composite operators[21]

$$\begin{aligned} J^\pm(z) &= \kappa ( g_{\pm\mp} \partial_z g_{\pm\pm} - g_{\pm\pm} \partial_z g_{\pm\mp} ), \\ J^3(z) &= \kappa ( g_{++} \partial_z g_{--} + g_{+-} \partial_z g_{-+} ), \\ \bar{J}^\pm(\bar{z}) &= \kappa ( g_{\mp\pm} \partial_{\bar{z}} g_{\pm\pm} - g_{\pm\pm} \partial_{\bar{z}} g_{\mp\pm} ), \\ \bar{J}^3(\bar{z}) &= \kappa ( g_{--} \partial_{\bar{z}} g_{++} + g_{-+} \partial_{\bar{z}} g_{+-} ), \end{aligned} \quad (2.8)$$

where  $\kappa = k-2=1/4$ . Note that, here and below, normal ordering is implied in operator composites. The operator product expansions of these composites with  $\mathbf{g}$  are given by [21]

$$\begin{aligned}
J^a(z)\mathbf{g}(w, \bar{w}) &= \frac{\mathbf{g}\tau^a}{z-w} + :J^a(z)\mathbf{g}(w, \bar{w}):, \\
\bar{J}^a(\bar{z})\mathbf{g}(w, \bar{w}) &= \frac{\tau^a\mathbf{g}}{\bar{z}-\bar{w}} + :\bar{J}^a(\bar{z})\mathbf{g}(w, \bar{w}):,
\end{aligned}
\tag{2.9}$$

where  $\tau^1 = i\sigma_1/2$ ,  $\tau^2 = \sigma_2/2$ ,  $\tau^3 = i\sigma_3/2$ , are the generators of  $SU(1,1)$ .

Each set of currents obeys an  $SU(1,1)$  current algebra. For example, the conformal modes of the holomorphic currents satisfy[22]

$$\begin{aligned}
[J_n^3, J_m^3] &= -\frac{1}{2}kn \delta_{n+m,0}, \\
[J_n^3, J_m^\pm] &= \pm J_{n+m}^\pm, \\
[J_n^+, J_m^-] &= kn \delta_{n+m,0} - 2J_{n+m}^3.
\end{aligned}
\tag{2.10}$$

The Virasoro generators are given by the Sugawara construction:

$$T(z) = \frac{1}{\kappa} J^i(z)J^i(z), \tag{2.11}$$

with  $c = 3k/\kappa = 27$ . Acting on states in a Kac-Moody module,  $L_0$  is simply related to the  $SU(1,1)$  zero-mode Casimir:

$$L_0 = \frac{1}{\kappa} \mathbf{J}^2 + N, \tag{2.12}$$

where  $N$  is a nonnegative integer called the *grade* of a state in a Kac-Moody module, while the zero-mode Casimir is given by

$$\begin{aligned}
\mathbf{J}^2 &= \frac{1}{2} (J_0^+ J_0^- + J_0^- J_0^+) - (J_0^3)^2, \\
&= -j(j+1).
\end{aligned}
\tag{2.13}$$

Gauging the diagonal  $U(1)$  in the WZW model is equivalent to modding out by all  $U(1)$  descendant states, both holomorphically and antiholomorphically, in every Kac-Moody module. The  $SU(1,1)/U(1)$  coset has  $c = 26$ . Acting on states in a coset module,  $L_0$  has eigenvalues

$$L_0 = -4j(j+1) + \frac{4}{9}m^2 + N, \tag{2.14}$$

where  $m$  is the  $J_0^3$  eigenvalue of a state in the coset module.

Although coset conformal field theories based on compact groups, e.g.  $SU(2)/U(1)$ , are very well understood[23], noncompact cosets are poorly understood and are not even

known to exist as unitary modular invariant theories except in special cases[22],[24],[25]. Furthermore, for purposes of doing string theory with  $SU(1,1)/U(1)$  cosets, we must be very general about introducing all Kac-Moody coset modules which contain string physical states, i.e. coset states which obey the Virasoro highest weight and mass-shell conditions

$$\begin{aligned} L_n |j, m, N\rangle &= 0, \quad \text{for all } n > 0, \\ (L_0 - 1) |j, m, N\rangle &= 0, \end{aligned} \tag{2.15}$$

and thus correspond to dimension (1,1) operators in the coset theory. While we should require that the totality of such operators form a closed local operator algebra, and that the corresponding physical states have all relatively nonnegative norms, this in no way implies that we can restrict our attention to unitary  $SU(1,1)/U(1)$  modules. In fact it turns out[9] that to obtain all the physical states one needs to consider even Kac-Moody modules which have negative norm states at the base (i.e. grade zero)!

With this in mind we now briefly review the representation theory of classical  $SU(1,1)$ [26] and of  $SU(1,1)/U(1)$  Kac-Moody[22]. The  $SU(1,1)$  Kac-Moody modules can be characterized by the states which occur at the base, the Kac-Moody primaries. These states are annihilated by all the  $J_n^\pm$  and  $J_n^3$  for all  $n > 0$ , and together constitute a module of the zero-mode  $SU(1,1)$ . Thus the Kac-Moody primaries correspond to representations (reps) of classical  $SU(1,1)$ . As shown by Bargmann, the irreducible representations (irreps) of classical  $SU(1,1)$  are either *double-sided*, *highest weight discrete series*, *lowest weight discrete series*, or *continuous series*. The double-sided representations are the only finite dimensional reps of  $SU(1,1)$ , and are isomorphic to the standard unitary irreps of  $SU(2)$ . However for  $SU(1,1)$  these reps are all nonunitary, excepting only the trivial identity representation.

The highest weight and lowest weight discrete series irreps contain a state annihilated by  $J_0^+$  or  $J_0^-$ , respectively. They are thus one-sided and infinite dimensional. A highest weight state has  $m = j$  or  $m = -j-1$ , while a lowest weight state has  $m = -j$  or  $m = j+1$ . If we apply no constraints from unitarity or single-valuedness, then  $j$  can be any real number, although in the cases where  $j$  is a nonnegative integer or half-integer the corresponding highest and lowest weights reps degenerate to the double-sided reps. In addition, because of the equivalence of irreps related by  $j \rightarrow -j-1$ , we can restrict  $j$  to values either  $\geq$  or  $\leq -1/2$ . In this paper we will restrict to negative  $j$  values

$$j \leq -1/2, \tag{2.16}$$

for all discrete series representations. With this convention we may observe that the  $m = j$  highest weight and  $m = -j$  lowest weight irreps are unitary, while the  $m = \mp(j+1)$  irreps are nonunitary.

The continuous series irreps of classical  $SU(1,1)$  have neither a highest nor a lowest weight state, which merely requires that neither  $j+m$  nor  $j-m$  is an integer. While  $m$  is real,  $j$  can be complex. In fact the unitary continuous series reps have either  $j = -1/2 + i\rho$  or  $-1/2 < j < 0$ , i.e.  $\mathbf{J}^2 > 0$ . However, for our purposes we will need the nonunitary continuous series reps with  $\mathbf{J}^2 < 1/4$ . Indeed it will suffice to consider only continuous series reps which have  $j$  real and restricted to (2.16).

Given a set of Kac-Moody primaries which form an irrep of the zero-mode  $SU(1,1)$ , the Kac-Moody module is built up by applying the raising operators  $J_{-n}^{\pm}$  and  $J_{-n}^3$ . The number of independent states with a particular  $m$  value at a particular grade in any module (excepting the special cases described below) is given by counting the number of ways such states can be obtained by the free action of the raising operators- i.e., by acting with strings of raising operators on all states in the base, modulo strings which differ only in their ordering. For example in Fig. 1 we show the multiplicities of a generic lowest weight module.

The reason why the multiplicities of the discrete and continuous series  $SU(1,1)$  Kac-Moody modules are so simply determined is that *these modules generically contain no nontrivial null states*. This is easily seen by applying the "pseudospin" analysis used by Gepner and Witten[27] to study affine  $SU(2)$ . Like  $SU(2)$ , affine  $SU(1,1)$  has an external automorphism symmetry corresponding to permuting the weights of the extended Dynkin diagram. As a result, an arbitrary module can be decomposed either into reps of the zero-mode  $SU(1,1)$  or into reps of the  $SU(1,1)$  pseudospin defined by

$$\begin{aligned} [\hat{j}_0^3, J_1^+] &= J_1^+, \\ [\hat{j}_0^3, J_{-1}^-] &= -J_{-1}^-, \\ [J_1^+, J_{-1}^-] &= -2\hat{j}_0^3, \end{aligned} \tag{2.17}$$

where

$$\hat{j}_0^3 \equiv J_0^3 - \frac{1}{2}k. \tag{2.18}$$

Now consider, for example, an  $m = -j$  lowest weight Kac-Moody primary of a lowest weight module. This state is obviously also a highest weight state with respect to pseudospin, with  $\hat{m} = -j - k/2$ . Thus, excepting the special cases where  $\hat{m}$  is a nonnegative

integer or half-integer, this lowest weight state also defines a highest weight irrep of pseudospin, which contains all of the states on the diagonal boundary of the module (see Fig. 1). It follows from the automorphism symmetry that every state in the module belongs to both a lowest weight irrep of the zero-mode algebra and a highest weight irrep of the pseudospin algebra. Thus the null state structure is trivial. It is amusing to note that, for  $k = 9/4$ , the double-sided  $SU(1,1)$  modules will look nothing like affine  $SU(2)$  modules, even though they have an  $SU(2)$ -like representation at the base. This is because for  $k = 9/4$  each state at the base of a double-sided module necessarily generates a highest weight irrep of  $SU(1,1)$  pseudospin, not a double-sided one.

$SU(1,1)/U(1)$  coset modules are obtained from  $SU(1,1)$  modules by modding out all  $U(1)$  descendant states, i.e. keeping only states which satisfy the  $U(1)$  highest weight condition:

$$J_n^3 |j, m, N\rangle = 0, \quad \text{for all } n > 0. \quad (2.19)$$

The coset modules can also be described using parafermions[22],[28],[29].

### 3. Physical States of the $SL(2, R)$ String

Using BRST cohomology, Distler and Nelson[9] have done a complete holomorphic classification of the maximal set of independent physical states in the  $SL(2, R)$  coset string theory (the actual physical states may be some truncation of this set). They are of three types: tachyon states, discrete states, and Virasoro null states. We will defer discussion of the Virasoro null states until the next section. The tachyon states consist of Kac-Moody primaries of dimension 1; they have arbitrary real  $j \leq -1/2$ , and have  $m = \pm(3j + 3/2)$ . These states appear to correspond to the normalizable tachyon states of the  $c=1$  Liouville theory[9], with  $j$  playing the role of the Liouville momentum.

The discrete states are physical states appearing at higher grade in certain  $SU(1,1)$  Kac-Moody modules for discrete values of  $j$ . Their  $(j, m)$  quantum numbers can be parametrized by two positive integers  $s$  and  $r$ . There are discrete states occurring in highest weight modules which have unitary ( $m=j$  type) representations at the base. Their quantum numbers are:

$$j = -\frac{1}{4}(s + 2r + 1), \quad m = \frac{3}{4}(s - 2r + 1). \quad (3.1)$$

There are discrete states occurring in lowest weight modules which have unitary ( $m = -j$  type) representations at the base. Their quantum numbers are:

$$j = -\frac{1}{4}(s + 2r + 1), \quad m = -\frac{3}{4}(s - 2r + 1). \quad (3.2)$$

In addition, there are discrete states appearing in continuous series modules which have nonunitary irreps at the base:

$$j = -\frac{1}{2}(s + r + 1), \quad m = \frac{3}{2}(s - r). \quad (3.3)$$

For each of these states, the grade is trivially computed from (2.15).

We should note that the discrete states listed above are only half of the states given in [9]. However the remaining discrete states map into the states above under the external automorphism. Our attitude will be to ignore this duplication, as well as the probably infinite cloning of states due to nontrivial winding sectors. Some insight on these issues can be found in [24],[7],[9].

As is done for the Liouville theory, it will be useful to augment the discrete states by a certain subset of the tachyon states. This is because the operator algebra of the discrete states cannot close on itself without the addition of these “discrete” tachyon states. With our conventions these additional states are the dimension 1 Kac-Moody primaries which have  $j = -r/4$ ,  $r = 2, 3, 4, \dots$

With the above caveats and additions, Table 1 gives a listing of the first 34 discrete states, paired with Liouville states in the manner suggested in [9]. The notation  $W_{s,n}^+$  for Liouville states is as in [18]. Our notation for  $SL(2, R)$  holomorphic discrete states is  $W_{j,m}^{hw}$  for states in highest weight modules,  $W_{j,m}^{lw}$  for states in lowest weight modules, and  $W_{j,m}^{cont}$  for states in continuous series modules. Also we denote Kac-Moody primaries by  $\Phi_{j,m}^{hw}$ ,  $\Phi_{j,m}^{lw}$ , and  $\Phi_{j,m}^{cont}$ . This table should not be taken very seriously. As pointed out in [9], the correspondence suggested between Liouville and  $SL(2, R)$  is not one-to-one, due to “extra”  $SL(2, R)$  discrete states. We will see later, when we discuss the algebra of  $SL(2, R)$  discrete state operators, that the correspondence to Liouville is in fact quite a bit more complicated than suggested by Table 1.

So far, the discussion of physical states has been purely holomorphic. However in the WZW model physical state operators are composites of the components of  $\mathfrak{g}(z, \bar{z})$  (and  $A$  in the gauged version) which are not holomorphic fields. Holomorphic physical states as described above do not actually exist in this theory. We can, however, explicitly construct

operators as composites of  $g_{\pm\pm}(z, \bar{z})$  and  $g_{\pm\mp}(z, \bar{z})$  whose holomorphic content matches that above. We find from this explicit construction that the rules for tying together holomorphic and antiholomorphic sectors in the  $SL(2, R)$  WZW model are rather simple. Modulo questions of normalizability and single-valuedness, the only constraint is  $j = \bar{j}$ . Furthermore, modules of different type but the same  $j$  can be tied together. Thus for example a highest weight module with fixed  $j$  can tie up with another highest weight module, or a lowest weight, or a continuous series, or a doubled-sided module (for suitable  $j$ ).

To be more specific, we have already given explicit expressions for the WZW currents in (2.8). One can then construct any state in the theory using  $SL(2, R)_L \times SL(2, R)_R$  current algebra, provided one has an explicit construction of the Kac-Moody primaries. The action of the zero mode currents on the spin half primaries (2.2) is given by the singular terms in (2.9), and their descendants are obtained from the Taylor expansion of the second (regular) term. Using Wick's theorem we can extend this construction to arbitrary primaries, some examples of which are given in Section 6. In the appendix we derive a general expression for an arbitrary Kac-Moody primary in the (ungauged)  $SL(2, R)$  WZW theory:

$$\begin{aligned} \Phi_{j,m,\bar{m}}(z, \bar{z}) &= [\Gamma(j+m+1)\Gamma(j-m+1)\Gamma(j+\bar{m}+1)\Gamma(j-\bar{m}+1)]^{1/2} \\ &\times \sum_{n=-\infty}^{\infty} \frac{(g_{++})^{j+m}(g_{--})^{j-m}(g_{-+})^{\bar{m}-m}(-g_{+-}g_{-+}/g_{++}g_{--})^n}{\Gamma(j+m-n+1)\Gamma(j-\bar{m}-n+1)\Gamma(\bar{m}-m+n+1)\Gamma(n+1)}. \end{aligned} \quad (3.4)$$

For primaries which transform like continuous-continuous under  $SL(2, R)_L \times SL(2, R)_R$ , this expression reduces to a hypergeometric function [30], [7]. For example:

$$\Phi_{-1/2,0,0}^{c-c}(z, \bar{z}) = (g_{++}g_{--})^{-1/2} F\left(\frac{1}{2}, \frac{1}{2}; 1; g_{+-}g_{-+}/g_{++}g_{--}\right), \quad (3.5)$$

which is equivalent to the integral expression given in [7]. Note  $\Phi_{-1/2,0,0}^{c-c}$  is (roughly speaking) the  $SL(2, R)$  analog of the cosmological constant operator in the Liouville theory.

For other primaries (3.4) reduces to a simple expression. For example:

$$\Phi_{-1,-1,-1}^{hw-hw}(z, \bar{z}) = \frac{1}{g_{++}^2}, \quad \Phi_{-1,1,1}^{lw-lw}(z, \bar{z}) = \frac{1}{g_{--}^2}. \quad (3.6)$$

We should warn the reader that it is not clear whether all of the formal composites implied by (3.4) really exist as well-defined operators creating normalizable states (or even

non-normalizable states in the sense of [31]). The normalization convention chosen in (3.4) is reasonable for  $\Phi^{hw-hw}$ ,  $\Phi^{lw-lw}$ ,  $\Phi^{hw-lw}$ , and  $\Phi^{lw-hw}$  states, but certainly not for more exotic states. Some of the physical states are definitely not square integrable with respect to the classical  $SL(2, R)$  invariant measure. For example (3.5) is not square integrable, though the rest of the continuous-continuous physical states are. The identity is also not square integrable[32].

#### 4. The Ground Ring

In addition to the physical states already discussed, the  $SL(2, R)/U(1)$  string theory has an infinite number of physical Virasoro null states. These correspond to physical states of ghost number zero in BRST language, and are the analogues of the “ground ring” operators introduced in [18]. The existence of these null states is a direct result of a “pathological” property of the  $k=9/4$   $SU(1, 1)/U(1)$  Kac-Moody coset modules, namely, the presence of an infinite number of operators with conformal dimension zero. By contrast, in a typical conformal field theory only the identity operator has zero dimension. These dimension zero operators are a result of the fact that the zero-mode  $SU(1, 1)$  Casimir makes a negative contribution to the mass operator (2.14) for  $j < -1$ . Liouville theory, of course, has similar properties.

Let us consider a coset state which is Virasoro highest weight and has dimension zero. Then the coset Virasoro algebra implies that  $L_{-1}$  of this state is simultaneously Virasoro highest weight and a Virasoro descendant. Thus it either vanishes identically, or it is a Virasoro null state. Since  $L_{-1}O = \partial_z O$  when  $O$  is Virasoro primary,  $L_{-1}$  of the identity is the only case in which null states of this construction vanish identically.

The  $(j, m)$  quantum numbers of the independent dimension zero physical states are given in [9]. In Table 2 we list the first 12 dimension zero coset operators. Our notation is  $O_{j,m}^{hw}$ ,  $O_{j,m}^{lw}$ ,  $O_{j,m}^{cont}$ , and  $O_{j,m}^{double}$  for operators in highest weight, lowest weight, continuous series, and double-sided modules. We have grouped the operators by their  $j$  values, and listed in addition on the same line all of the non-null physical states with the same  $j$ . Up to questions of normalizability, single-valuedness, and closure, the  $k=9/4$  gauged  $SL(2, R)$  WZW theory will contain operators that behave like any listed operator of given  $j$  holomorphically and any other listed operator of the same  $j$  antiholomorphically. Of particular interest are the dimension  $(1, 0)$  and  $(0, 1)$  operators of the form:

$$J_{j,m,\bar{m}}(z, \bar{z}) \equiv W_{j,m} \bar{O}_{j,\bar{m}}, \quad \bar{J}_{j,m,\bar{m}}(z, \bar{z}) \equiv O_{j,m} W_{j,\bar{m}}. \quad (4.1)$$

As discussed in [18],[33] for the Liouville theory, such operators are purely holomorphic/antiholomorphic up to string null states. They may therefore be regarded as an infinite set of conserved physical currents, with corresponding conserved charges.

In the Liouville theory there is a chiral “ground ring” of dimension zero operators  $O_{j,m}^+$  generated by  $X = O_{1/2,1/2}^+$  and  $Y = O_{1/2,-1/2}^+$ . A somewhat analogous structure appears for the string in a black hole background. We may consider the dimension zero coset Virasoro primaries

$$X = O_{-5/4,3/4}^{lw}, \quad Y = O_{-5/4,3/4}^{hw}. \quad (4.2)$$

These operators correspond to grade 1 states built up from the lowest/highest weight Kac-Moody primaries  $\Phi_{-5/4,-1/4}^{lw}$  and  $\Phi_{-5/4,1/4}^{hw}$ , and thus live in modules which are nonunitary even at the base. They have the explicit form

$$\begin{aligned} X &= 2\sqrt{\frac{2}{11}} \left[ J_{-1}^+ + \frac{16}{3} J_{-1}^3 J_0^+ - 5 J_{-1}^- J_0^+ J_0^+ \right] \Phi_{-5/4,-1/4}^{lw}, \\ Y &= 2\sqrt{\frac{2}{11}} \left[ J_{-1}^- + \frac{16}{3} J_{-1}^3 J_0^- - 5 J_{-1}^+ J_0^- J_0^- \right] \Phi_{-5/4,1/4}^{hw}. \end{aligned} \quad (4.3)$$

The corresponding null states are obtained by applying  $L_{-1}^{coset}$ , which we can effectively write in terms of the  $SU(1,1)$  Virasoro operators and currents as

$$L_{-1}^{coset} X = \left[ L_{-1} + \frac{3}{2k} J_{-1}^3 \right] X. \quad (4.4)$$

Actually  $X$  and  $Y$  should be regarded as shorthand for the holomorphic content of the four operators

$$a_1(z, \bar{z}) = X \bar{X}, \quad a_2(z, \bar{z}) = Y \bar{Y}, \quad a_3(z, \bar{z}) = X \bar{Y}, \quad a_4(z, \bar{z}) = Y \bar{X}, \quad (4.5)$$

which are built up, respectively, from the four Kac-Moody primaries

$$\begin{aligned} \Phi_{-5/4,-1/4,-1/4}^{lw-lw}(z, \bar{z}) &= \sqrt{g_{--}}, & \Phi_{-5/4,1/4,1/4}^{hw-hw}(z, \bar{z}) &= \sqrt{g_{++}}, \\ \Phi_{-5/4,-1/4,1/4}^{lw-hw}(z, \bar{z}) &= \sqrt{g_{-+}}, & \Phi_{-5/4,1/4,-1/4}^{hw-lw}(z, \bar{z}) &= \sqrt{g_{+-}}. \end{aligned} \quad (4.6)$$

Let us now consider the operator product of  $X$  and  $Y$ . As in [18], we suppress operators of zero or negative integer dimension which, by the analysis of [9], correspond to BRST exact states. Then we may write

$$\lim_{z \rightarrow w} X(z)Y(w) = \sum_i O_{j(i),0}(w), \quad (4.7)$$

i.e.  $X$  fused with  $Y$  gives back other dimension zero  $m=0$  physical state operators.

The fusion rules for  $SU(1,1)/U(1)$  coset operators are greatly simplified due to the absence of nontrivial  $SU(1,1)$  null states in generic coset modules. Usually in a Kac-Moody conformal field theory zeroes of the 3-point function arise from three distinct sources: vanishings of the Wigner (Clebsch-Gordan) coefficients associated with tensor products of the underlying Kac-Moody primaries, vanishings due to null states in the modules, and vanishings due to the conformal and global  $SL(2, R)$  Ward identities[21]. One writes:

$$O_{j_1, m_1} \circ O_{j_2, m_2} = \sum_j N(j, j_1, j_2) C_{m_1 m_2 m_1 + m_2}^{j_1 j_2 j} O_{j, m_1 + m_2}, \quad (4.8)$$

where the  $C$ 's are the Wigner coefficients. Now in the case at hand we have not solved for the complete 3-point functions (in part because we do not know the proper definition of the 2-point functions!) but since we can compute Wigner coefficients we can determine most of the fusion rules for the physical states of the string theory.

Thus, to determine which dimension zero operators appear on the right-hand side of (4.7), we consider the classical  $SU(1,1)$  tensor product

$$\Phi_{j,0} = \sum_{n=-1}^{\infty} C_n \Phi_{-5/4, n+3/4}^{lw} \otimes \Phi_{-5/4, -n-3/4}^{hw}, \quad (4.9)$$

where we have abbreviated the Wigner coefficients by  $C_n$ . Since  $X$  and  $Y$  are each descended from three distinct Kac-Moody primaries, there are several other relevant tensor products, but consideration of these does not alter the conclusion reached below.

Although the Wigner coefficients for tensor products of various unitary representations of  $SU(1,1)$  can be found in the mathematical literature[34],[32], we have here a tensor product of lowest weight and highest weight *nonunitary* representations. Thus it is safest to proceed from first principles. A general discussion can be found in appendix B.

The requirement that the right-hand side of (4.9) be an eigenstate of the Casimir (2.13) produces the following recursion relation:

$$(n + \frac{1}{2})(n + 2)C_{n+1} + (n - \frac{1}{2})(n + 1)C_{n-1} - (j(j + 1) + 2n^2 + 3n + \frac{1}{2})C_n = 0. \quad (4.10)$$

This can be converted into a hypergeometric differential equation by introducing the function

$$f(t) = \sum_{n=0}^{\infty} C_{n-1} t^n. \quad (4.11)$$

Then (4.10) is solved by

$$f(t) = (1 - t)^j F(j - 1/2, j + 1; -1/2; t). \quad (4.12)$$

The coefficients  $C_n$  are obtained from (4.12) by expanding the hypergeometric function in the standard hypergeometric series, which converges at  $t=1$  for  $j < -1/2$ . To determine for what values of  $j$  these Wigner coefficients are nonvanishing, we first assume that the original tensor product states can be normalized (i.e. that  $X$  and  $Y$  really exist in the WZW model). Then the Wigner coefficients can self-consistently be taken as nonvanishing for those  $j$  values  $\leq -1/2$  such that

$$\sum_{n=-1}^{\infty} |C_n|^2, \quad (4.13)$$

is a convergent series. This constraint has the unique solution

$$j = -3/2. \quad (4.14)$$

Actually the tensor product (4.9) also contains the identity, which is missed in the above argument because the identity is not normalizable expressed in a normalized tensor product basis (i.e.  $C_n = (-1)^n$ ). This exception has been noted in the literature[32]. So our final result is the fusion rule

$$X \circ Y \sim I + O_{-3/2,0}^{cont}. \quad (4.15)$$

It is interesting to compare this result with the ground ring geometry of the Liouville theory[18]. For the Liouville theory compactified to the  $SU(2)$  radius, this is a three dimensional cone coming from the relation

$$a_1 a_2 - a_3 a_4 = 0 . \quad (4.16)$$

Here we seem to have an analogous relation

$$a_1 a_2 + a_3 a_4 = 1 . \quad (4.17)$$

up to undetermined numerical coefficients. This is similar to what is expected for the Liouville theory with nonzero cosmological constant.

By considering arbitrary tensor products of  $X$ 's and  $Y$ 's one sees that the fusion algebra generates all of the physical dimension zero operators in the cohomology. For example, the fusion of  $X$  with itself gives

$$X \circ X \sim O_{-3/2,3/2}^{lw} + O_{1/2,3/2}^{double}, \quad (4.18)$$

where these grade two operators are defined modulo normalizations by the expressions

$$O_{-3/2,3/2}^{lw} = \left[ \frac{3}{10} J_{-2}^3 - \frac{1}{5} J_{-1}^3 J_{-1}^3 + \frac{3}{2} J_{-1}^+ J_{-1}^- \right. \\ \left. - \frac{27}{10} J_0^+ J_{-2}^- + \frac{39}{5} J_0^+ J_{-1}^3 J_{-1}^- + \frac{99}{8} J_0^+ J_0^+ J_{-1}^- J_{-1}^- \right] \Phi_{-3/2,3/2}^{lw}, \quad (4.19)$$

and

$$O_{1/2,3/2}^{double} = [8J_{-1}^3 J_{-1}^+ - 21J_{-2}^+ - 15J_{-1}^+ J_{-1}^+ J_0^-] \Phi_{1/2,1/2}^{double}. \quad (4.20)$$

This result follows from the tensor product relations

$$\Phi_{-5/4,-1/4}^{lw} \otimes \Phi_{-5/4,-1/4}^{lw} = \Phi_{1/2,-1/2}^{double}, \quad (4.21)$$

(with an unusual normalization) and

$$\Phi_{-5/4,-1/4}^{lw} \otimes \Phi_{-5/4,7/4}^{lw} - i\sqrt{2}\Phi_{-5/4,3/4}^{lw} \otimes \Phi_{-5/4,3/4}^{lw} - \Phi_{-5/4,7/4}^{lw} \otimes \Phi_{-5/4,-1/4}^{lw} \\ = \Phi_{-3/2,3/2}^{lw}, \quad (4.22)$$

(note that for this nonunitary tensor product the Wigner coefficients cannot all be made real).

Not surprisingly, the  $SL(2, R)$  ground ring is more complicated than it's counterpart in  $SU(2)$  Liouville. As we shall see, this is also true of the symmetry algebra of physical currents.

## 5. $W_\infty$ Algebra of the Black Hole $W$ hair

### 5.1. The charge algebra of the Liouville theory

For the  $SU(2)$  Liouville theory Klebanov and Polyakov[35] and Witten[18] have computed the algebra of the physical conserved chiral charge operators

$$Q_{s,n}^\pm = \oint dz W_{s,n}^\pm(z). \quad (5.1)$$

The algebra is remarkably simple:

$$\left[ Q_{s,n}^+, Q_{s',n'}^+ \right] = (ns' - n's) Q_{s+s'-1, n+n'}^+, \quad (5.2)$$

and

$$\begin{aligned} \left[ Q_{s,n}^+, Q_{-s'-1, n'}^- \right] &= -(ns' - n's) Q_{-s-s', n+n'}^-, \\ \left[ Q_{-s-1, n}^-, Q_{-s'-1, n'}^- \right] &= 0. \end{aligned} \quad (5.3)$$

If we truncate the algebra to charges with  $s$  integer, then (5.2) is precisely the wedge subalgebra of  $w_{1+\infty}$ , the contraction of  $W_\infty(\lambda)$  for  $\lambda=1/2$  [36],[37]. When a cosmological constant is added to the Liouville theory, this symmetry algebra is modified to a much more complicated uncontracted  $W_\infty$  structure, which does not appear to correspond to any of the previously catalogued algebras[38],[39].

We may take these facts as indications of what to expect for the holomorphic part of the algebra of conserved charges in the  $SL(2, R)$  black hole background. Since these charges represent stringy “hair” of the black hole, what we are seeking may be termed the algebra of  $W$  hair of the two dimensional black hole[4].

In (5.2) the  $SU(2)$   $J_0^3$  eigenvalue  $n$  is simply conserved and plays the role of “conformal” mode number of the generators of  $w_{1+\infty}$ . The same will be true for the  $SL(2, R)$  charges

$$Q_{j,m} = \oint dz W_{j,m}(z), \quad (5.4)$$

and thus we must obtain a wedge algebra from these charges as well. Furthermore, upon shifting the  $SU(2)$  spin  $s$  by 1 in (5.2) it becomes equivalent to the “spin” of the corresponding  $w_{1+\infty}$  generator; thus for example  $Q_{1,\pm 1}^+$ ,  $Q_{1,0}^+$  are the “spin two” Virasoro generators  $L_{\pm 1}$ ,  $L_0$  in the sense of  $W_\infty$ . One might expect that a similar relationship

between  $j$  and  $s$  would hold for the  $SL(2, R)$  charge algebra, though this is not obvious. It will turn out that for  $Q_{j,m}$ 's corresponding to continuous series representations there is a simple relation between  $j$  and the  $W_\infty$  "spin"  $s$ :

$$s = -2j, \quad (5.5)$$

but  $Q_{j,m}$ 's corresponding to discrete series representations do not have definite  $s$ .

Before delving into the details of the  $W$  hair algebra, we want to point out that (5.2) may actually be a contraction of a more general family of algebras than  $W_\infty(\lambda)$ . To see this, consider the *super*  $W_\infty(\lambda)$  algebra of Bergshoeff et al[40]. For  $\lambda=0$  or  $1/2$ , it takes the form[40]:

$$\begin{aligned} [V_n^s, V_m^{s'}] &= \sum_{l=0}^{\infty} q^{2l} g_{2l}^{ss'}(n, m) V_{n+m}^{s+s'-2l-2} + \text{c.t.}, \\ [\tilde{V}_n^s, \tilde{V}_m^{s'}] &= \sum_{l=0}^{\infty} q^{2l} \tilde{g}_{2l}^{ss'}(n, m) \tilde{V}_{n+m}^{s+s'-2l-2} + \text{c.t.}, \\ \{G_n^{(+s)}, G_m^{(-s')}\} &= \sum_{l=0}^{\infty} q^l (b_l^{ss'}(n, m) V_{n+m}^{s+s'-l-1} + \tilde{b}_l^{s,s'}(n, m) \tilde{V}_{n+m}^{s+s'-l-1}) + \text{c.t.}, \\ [V_n^s, G_m^{(\pm)s'}] &= \sum_{l=0}^{\infty} q^{l-1} (\mp 1)^{l+1} a_l^{ss'}(n, m) G_{n+m}^{(\pm)s+s'-l-1}, \\ [\tilde{V}_n^s, G_m^{(\pm)s'}] &= \sum_{l=0}^{\infty} q^{l-1} (\mp 1)^{l+1} \tilde{a}_l^{ss'}(n, m) G_{n+m}^{(\pm)s+s'-l-1}, \end{aligned} \quad (5.6)$$

where c.t. stands for central terms. This algebra can be contracted by defining[40]

$$L_n^s = V_n^s + \tilde{V}_n^s, \quad \tilde{L}_n^s = q(V_n^s - \tilde{V}_n^s), \quad (5.7)$$

then taking the limit  $q \rightarrow 0$ . The contracted algebra in this case is the  $N=2$  super  $w_\infty$  algebra[41]:

$$\begin{aligned} [L_n^s, L_m^{s'}] &= [n(s' - 1) - m(s - 1)] L_{n+m}^{s+s'-2}, \\ [L_n^s, G_m^{(\pm)s'}] &= [n(s' - 1) - m(s - 1)] G_{n+m}^{(\pm)s+s'-2}, \\ [L_n^s, \tilde{L}_m^{s'}] &= [n(s' - 1) - m(s - 1)] \tilde{L}_{n+m}^{s+s'-2}, \\ [\tilde{L}_n^s, G_m^{(\pm)s'}] &= \pm G_{n+m}^{(\pm)s+s'-1}, \\ \{G_n^{(+s)}, G_m^{(-s')}\} &= -2L_{n+m}^{s+s'-1} - 2[n(s' - 1) - m(s - 1)] \tilde{L}_{n+m}^{s+s'-2}, \end{aligned} \quad (5.8)$$

( $\tilde{L}$  is usually written as a current:  $\tilde{L}_n^s = J_n^{s+1}$ ).

We speculate (but have not proven) that there exists a *bosonic* version of the super  $W_\infty(\lambda)$  algebra in which all generators are bosonic and anticommutators are replaced by commutators. Further, we imagine that the structure constants are unaltered except for those changes necessary to recover an *automorphism* symmetry under  $\lambda \rightarrow \frac{1}{2} - \lambda$ , rather than the *anti-automorphism* symmetry that determines much of the structure of super  $W_\infty(\lambda)$  [40]. Similar bosonic versions of superalgebras have been discussed in the literature [42],[43],[44]. Such an algebra, of course, has only a superficial connection with genuine supersymmetry.

The  $\lambda=1/2$  version of this bosonic superalgebra is obtained from (5.6) by replacing anticommutators with commutators and letting

$$\tilde{g} \rightarrow -\tilde{g}, \quad \tilde{a} \rightarrow -\tilde{a}. \quad (5.9)$$

This algebra can be contracted by defining

$$L_n^s = q(V_n^s + \tilde{V}_n^s), \quad \tilde{L}_n^s = V_n^s - \tilde{V}_n^s. \quad (5.10)$$

then taking the limit  $q \rightarrow 0$ . The contracted algebra has the same form as (5.8), but with anticommutators replaced by commutators, and with  $L_n^s \leftrightarrow \tilde{L}_n^s$ . If we now define

$$Q_n^s = \frac{1}{2}(G_n^{(+s)} + G_n^{(-s)}), \quad (5.11)$$

we find the following contracted subalgebra:

$$\begin{aligned} [\tilde{L}_n^s, \tilde{L}_m^{s'}] &= [n(s' - 1) - m(s - 1)] \tilde{L}_{n+m}^{s+s'-2}, \\ [\tilde{L}_n^s, Q_m^{s'}] &= [n(s' - 1) - m(s - 1)] Q_{n+m}^{s+s'-2}, \\ [Q_n^s, Q_m^{s'}] &= [n(s' - 1) - m(s - 1)] \tilde{L}_{n+m}^{s+s'-2}. \end{aligned} \quad (5.12)$$

Comparing (5.12) with (5.2), we see that (5.2), including the half-integer spin generators, is precisely the wedge of (5.12). Furthermore, the  $Q^-$  generators of (5.3) are precisely analogous to the *negative* spin  $L_n^s$ 's in this contracted algebra.

## 5.2. The $W$ hair algebra

If we now examine the  $SL(2, R)$   $W$  hair algebra, we will find an uncontracted wedge of something similar to this bosonic superalgebra. To determine commutators of the  $SL(2, R)$  charges (5.4), we need the  $1/(z-w)$  terms in the operator products of the corresponding  $W_{j,m}(z)$ 's (we ignore null states in this discussion). Only the antisymmetric parts of these terms contribute to the commutators of charges, however the symmetric parts are also needed in constructing operators which are exactly marginal (see the following section). Thus we are interested in the full "lone star" algebra[45] not just the  $W$  algebra.

We look first at operator products of continuous series  $W_{j,m}$ 's with continuous series  $W_{j,m}$ 's fusing to other continuous series  $W_{j,m}$ 's. These operator product coefficients are determined by the Wigner coefficients for the appropriate tensor products of Kac-Moody primary states at the base. Thus it suffices to examine tensor products like

$$\Phi_{j,m_1+m_2}^{cont} = \sum_{n=-\infty}^{\infty} C_n \Phi_{j_1,m_1+n}^{cont} \otimes \Phi_{j_2,m_2-n}^{cont}. \quad (5.13)$$

The sum is unrestricted since the tensor product is of two continuous series reps. Furthermore since we are producing another continuous series rep from the tensor product, there are no boundary conditions to impose on solutions to the Wigner recursion relations. In principle this means that there can be two independent solutions, but in fact it suffices to exhibit the one given by

$$C_n = \left[ \frac{\Gamma(n-j_1+m_1)\Gamma(n+j_1+m_1+1)}{\Gamma(n-j_2-m_2)\Gamma(n+j_2-m_2+1)} \right]^{1/2} D_n, \quad (5.14)$$

where  $D_{n-1}$  is the coefficient of  $t^n$  in the expansion of

$$f(t) = t^{(j_1-m_1+2)}(1-t)^{(j+m_1+m_2)} F(j+j_1+j_2+2, j+j_1-j_2+1; 2j_1+2; t). \quad (5.15)$$

One can easily see that the resulting Wigner coefficients define a convergent series if and only if  $j$  takes one of the values

$$j = j_1 + j_2 + 1, j_1 + j_2 + 2, j_1 + j_2 + 3, \dots \quad (5.16)$$

But in fact we have no solution at all, since for these values  $j$  and  $m_1 + m_2$  are either both integers or both half integers, and thus do not parametrize a continuous series representation. So there are no continuous series  $W_{j,m}$ 's at all appearing in the product of two continuous series  $W_{j,m}$ 's.

There is one exception to this result, involving the operator  $\Phi_{-1/2,0}^{cont}$ . Recall that this is the only non-square integrable discrete state, and also corresponds to the only continuous series state in a unitary representation. If we consider the operator product of  $\Phi_{-1/2,0}^{cont}$  with itself, the Wigner coefficients of the classical tensor product are given by the expansion of

$$f(t) = t^{3/2}(1-t)^j F(j+1, j+1; 1; t), \quad (5.17)$$

which for  $j=-1/2$  gives

$$f(t) = t^{3/2}(1-t)^{-1/2} F(\frac{1}{2}, \frac{1}{2}, 1, t). \quad (5.18)$$

Comparing with (3.5), we see that although (5.18) gives a divergent Wigner series, this appears to simply reflect the fact that  $\Phi_{-1/2,0}^{cont}$  is not square integrable, while the original tensor product basis is, by definition, normalized.

Now let us determine which discrete series  $W_{j,m}$ 's appear in the product of two continuous series  $W_{j,m}$ 's. The discrete series  $W_{j,m}$ 's are always at sufficiently high grade that one of the relevant tensor products will involve the lowest/highest weight Kac-Moody primary at the base of the module. We write this state as a tensor product of continuous series primaries:

$$\Phi_{-m_1-m_2, m_1+m_2}^{lw} = \sum_{n=-\infty}^{\infty} C_n \Phi_{j_1, m_1+n}^{cont} \otimes \Phi_{j_2, m_2-n}^{cont}. \quad (5.19)$$

The Wigner coefficients are then trivially determined by the condition that  $J_0^{total}$  annihilate the state. The unique solution is

$$C_n = (-1)^n \left[ \frac{\Gamma(n+j_2-m_2+1)\Gamma(n-j_2-m_2)}{\Gamma(n+j_1+m_1+1)\Gamma(n-j_1+m_1)} \right]^{1/2}. \quad (5.20)$$

The asymptotic behavior of these coefficients is

$$\lim_{n \rightarrow \infty} C_n \sim n^{-m_1-m_2} = n^j, \quad (5.21)$$

which thus give a convergent series for all discrete series reps with  $j < -1/2$ . Thus we reproduce the result of Pukánszky[34] that *all* unitary discrete series reps appear in the product of *any two* (Hermitian but not necessarily unitary) continuous series reps. Of course, since all of our continuous series  $W_{j,m}$ 's have  $m$  values which are integer or half integer, we will only produce the discrete series  $W_{j,m}$ 's which have  $j$  integer or half integer.

Before interpreting these results in the language of  $W$  algebras, we treat one more case. Let us determine which continuous series  $W_{j,m}$ 's appear in the product of a lowest weight  $W_{j,m}$  with a continuous  $W_{j,m}$  (the case of highest weight  $\times$  continuous is precisely analogous). Thus consider

$$\Phi_{j,-j_1+m_2}^{cont} = \sum_{n=0}^{\infty} C_n \Phi_{j_1,-j_1+n}^{tw} \otimes \Phi_{j_2,m_2-n}^{cont}. \quad (5.22)$$

As before we solve for the Wigner coefficients:

$$C_n = \left[ \frac{\Gamma(n-2j_1)\Gamma(n+1)}{\Gamma(n-j_2-m_2)\Gamma(n+j_2-m_2+1)} \right]^{1/2} D_n, \quad (5.23)$$

where the  $D_n$  are obtained by expanding a dummy function  $f(t)$ . Normally one would have to impose a boundary condition on  $f(t)$  due to the fact that the  $C_n$  series in (5.22) terminates at the lower end (i.e.  $n=0$ ); this boundary condition would be  $f(t) \rightarrow t$  as  $t \rightarrow 0$ . However in the actual recursion relation that defines the  $C_n$ , the  $C_n$  for  $n \geq 0$  are decoupled from the  $C_n$  with  $n < 0$ . Thus a consistent solution results from taking any  $f(t)$  (there are two independent solutions) to define the  $C_n$  with  $n \geq 0$ , and taking the trivial solution  $C_n=0$  for  $n < 0$ . We may thus find a solution from

$$f(t) = t^{(2j_1+2)}(1-t)^{(j-j_1+m_2)} F(j+j_1+j_2+2, j+j_1-j_2+1; 2j_1+2; t). \quad (5.24)$$

It is now easy to see that the resulting  $C_n$ 's define a convergent series provided  $j$  takes one of the values

$$j = j_1 + j_2 + 1, j_1 + j_2 + 2, j_1 + j_2 + 3, \dots \quad (5.25)$$

We can now recast these results in the language of  $W_\infty$ . We use the correspondence (5.5) to write the continuous series  $W_{j,m}$ 's as  $W_\infty$  generators  $L_n^a$ :

$$\begin{aligned}
L_{\pm 1}^2 &= \Phi_{-1, \pm 3/2}^{\text{cont}} \\
L_{\pm 2}^3 &= \Phi_{-3/2, \pm 3}^{\text{cont}} \\
L_0^3 &= W_{-3/2, 0}^{\text{cont}} \\
L_{\pm 3}^4 &= \Phi_{-2, \pm 9/2}^{\text{cont}} \\
L_{\pm 1}^4 &= W_{-2, \pm 3/2}^{\text{cont}} \\
&\dots
\end{aligned} \tag{5.26}$$

Note that these are all of the integer spin  $W_\infty$  generators in the wedge which have  $s+n$  odd. The exception is  $L_0^1$ , which is a special case; if we defined this to be simply  $\Phi_{-1/2, 0}^{\text{cont}}$ , then it would not behave at all like a  $U(1)$  current, since

$$\Phi_{-1/2, 0}^{\text{cont}} \circ \Phi_{-1/2, 0}^{\text{cont}} \sim \Phi_{-1/2, 0}^{\text{cont}} + W_{-1, 0}^{\text{hw}} + W_{-1, 0}^{\text{lw}} + \dots \tag{5.27}$$

So instead we *define*  $L_0^1$  to be the exactly marginal operator which can be constructed as an infinite series beginning with  $\Phi_{-1/2, 0}^{\text{cont}}$ :

$$\begin{aligned}
L_0^1 &= \Phi_{-1/2, 0}^{\text{cont}} + W_{-1, 0}^{\text{hw}} - W_{-1, 0}^{\text{lw}} + \dots, \\
L_0^1 \circ L_0^1 &= 0.
\end{aligned} \tag{5.28}$$

Note in the above expression (and many that follow) we suppress overall numerical coefficients which could anyway be absorbed into normalizations; however an important relative minus sign was made explicit in (5.28).

Furthermore, the results derived above imply that we can obtain all of the remaining integer spin  $W_\infty$  generators in the wedge as *infinite sums* of discrete series  $W_{j,m}$ 's. For example, we can define  $L_0^2$  from the operator product

$$\begin{aligned}
L_1^2 \circ L_{-1}^2 &= \Phi_{-1, 3/2}^{\text{cont}} \circ \Phi_{-1, -3/2}^{\text{cont}} \\
&\sim W_{-1, 0}^{\text{hw}} + W_{-1, 0}^{\text{lw}} + W_{-2, 0}^{\text{hw}} + W_{-2, 0}^{\text{lw}} + \dots \\
&\equiv L_0^2.
\end{aligned} \tag{5.29}$$

Given this we may then define  $L_0^4$  from the operator product

$$\begin{aligned}
L_{-1}^4 \circ L_{-1}^4 &= W_{-2, 3/2}^{\text{cont}} \circ W_{-2, -3/2}^{\text{cont}} \\
&\sim W_{-1, 0}^{\text{hw}} + W_{-1, 0}^{\text{lw}} + W_{-2, 0}^{\text{hw}} + W_{-2, 0}^{\text{lw}} + \dots \\
&\equiv L_0^4 + L_0^2.
\end{aligned} \tag{5.30}$$

The symmetries of the Wigner coefficients under Weyl reflection are such that the series of operators defining  $L_0^4$  will begin with  $W_{-2,0}^{hw}$  and  $W_{-2,0}^{lw}$ .

Continuing with our examples:

$$\begin{aligned} L_2^3 \circ L_{-1}^2 &= \Phi_{-3/2,3}^{cont} \circ \Phi_{-1,-3/2}^{cont} \\ &\sim W_{-3/2,3/2}^{hw} + W_{-3/2,3/2}^{lw} + W_{-5/2,3/2}^{hw} + W_{-5/2,3/2}^{lw} + \dots \\ &\equiv L_1^3 \end{aligned} \quad (5.31)$$

We will assume that the infinite series appearing in this construction produce normalizable and orthogonal states (it would be better to prove this). In that case, given our results and the simple rules for products of unitary discrete series representations[32], we have obtained an uncontracted integer spin  $W_\infty$  wedge algebra:

$$\left[ L_n^s, L_m^{s'} \right] = \sum_{l=0}^{\infty} g_{2l}^{s,s'}(n, m) L_{n+m}^{s+s'-2l-2}, \quad (5.32)$$

where the structure constants  $g_{2l}^{s,s'}(n, m)$  are computable –with rapidly increasing difficulty– from conformal field theory techniques. It should be emphasized that the difficulty in computing structure constants derives solely from the fact that the current algebraic construction of the higher grade discrete states themselves becomes rapidly very complicated.

We have not yet accounted for all of the independent combinations of discrete series  $W_{j,m}$ 's with  $j$  integer or half integer. These can be interpreted once we consider the  $W_{j,m}$ 's with  $j$  quarter integer. These operators are all discrete series, and correspond to half-integer spins  $s=3/2, 5/2, 7/2, \dots$  in our mapping to  $W_\infty$ . The spin 3/2 generators have the following products:

$$\begin{aligned} \Phi_{-3/4,3/4}^{cont} \circ \Phi_{-3/4,3/4}^{cont} &= W_{-3/2,3/2}^{lw}, \\ \Phi_{-3/4,-3/4}^{cont} \circ \Phi_{-3/4,-3/4}^{cont} &= W_{-3/2,-3/2}^{hw}, \\ \Phi_{-3/4,3/4}^{cont} \circ \Phi_{-3/4,-3/4}^{cont} &= 0. \end{aligned} \quad (5.33)$$

The full product algebra generated from these spin 3/2 operators appears to account precisely for the “extra” half-integer and integer spin generators. Furthermore, comparing (5.33) with (5.8), we see that this extra structure gives our  $W$  hair algebra the form of a bosonic superalgebra as described in general terms above. Since the generators of the  $W$  hair algebra act on the ground ring, which itself is generated by  $j=1/4$  operators  $X$  and  $Y$ , we speculate that our algebra may be related to the “symplecton” of Biedenharn and Louck[46].

In closing this section, we consider the question of why the embedding of the  $W_\infty$  discrete state algebra is so much more complicated for the  $SL(2, R)$  black hole than for flat space Liouville theory. The answer is that  $SU(2)$  Liouville is a very special case. The  $w_{1+\infty}$  algebra is simply related to an  $SU(2)$  enveloping algebra, and thus fits very neatly into the underlying  $SU(2)$  Kac-Moody current algebra. For the black hole the  $W_\infty$  algebra of the discrete states has no apparent relation to the underlying  $SL(2, R)$  Kac-Moody current algebra, even though this current algebra has its own  $W_\infty$  structure[47]. On the other hand, we can now, in hindsight, understand better why the discrete states obtained by Distler and Nelson had to be such a motley assortment of  $SL(2, R)$  states. Nonunitary continuous series reps appear because unitary continuous reps would fuse back onto the continuum of unitary continuous reps[34]. Unitary discrete series reps appear because these fuse onto an infinite series of other unitary discrete reps with increasing  $|j|$ , just as we get from fusing two continuous series reps. Nonunitary discrete series reps appear in the ground ring states because they fuse to a series of discrete reps with decreasing  $|j|$ .

## 6. Deformations of the Black Hole Background

The dimension  $(1, 1)$  (marginal) operators of the  $SL(2, R)/U(1)$  conformal field theory

$$O(z, \bar{z}) = W_{jm} \bar{W}_{j'm'} , \quad (6.1)$$

are the infinitesimal moduli for the black hole background and can generate deformations of the theory that preserve the central charge. If such an operator retains its conformal dimension in the *deformed* theory it generates a one-parameter family of conformally invariant backgrounds of fixed central charge and is called *exactly* marginal.

The maximal set of mutually commuting currents of a Kac-Moody algebra are independent exactly marginal operators, since the operator product of any two such currents does not contain any simple pole pieces which would give logarithmic contributions to the two-point function and thus shift the conformal dimension of the current [48][49]. For example, in the  $c = 1$  model at the  $SU(2)$  radius there is only one exactly marginal operator,  $J_3 \bar{J}_3$ , which changes the radius of the target space [50]. In the Liouville theory, the algebra of the discrete states is  $w_{1+\infty}$  so that the set of independent exactly marginal operators is the subset  $W_{s,0}^+ \bar{W}_{s,0}^+$ ,  $s=0, 1, \dots$ . In the case at hand, the discrete series  $W_{j,m}$  do not behave like “currents” in that their operator products with themselves produces an infinite number of other discrete series  $W_{j,m}$ ’s. As discussed in the previous section, we

expect rather that the physical moduli are constructed from an abelian subset of the  $W_\infty$  generators. In particular,

$$\begin{aligned} L_0^1 \bar{L}_0^1, \\ L_0^2 \bar{L}_0^2, \end{aligned} \tag{6.2}$$

are exactly marginal deformations of the  $SL(2, R)$  black hole background.

It is possible to compute explicitly, in the semi-classical ( $k \rightarrow \infty$ ) limit[6], the back reaction on the black hole background from such an operator. We construct the appropriate composite field in the ungauged  $SL(2, R)$  WZW model using the results of Section 3, and gauge the axial diagonal  $U(1)$ . Such a gauge invariant operator can be added to the action of the gauged  $SL(2, R)$  WZW model, and one can compute the infinitesimal deformation of the background to lowest order in  $1/k$ . These deformations may then be compared with approximate solutions of the beta function equations for the black hole in non-trivial tachyon backgrounds [17],[15],[51].

To discuss target space physics we use the Euler angle parametrization of the WZW fields (2.5). In unitary gauge,  $\theta_L = -\theta_R = \theta$ , this gives

$$g_{\pm\pm} = \cosh r, \quad g_{\pm\mp} = \sinh r e^{\mp i\theta}, \tag{6.3}$$

and the action of the gauged  $SL(2, R)$  WZW model is

$$L = \frac{k}{2\pi} \int d^2 z \partial_z r \partial_{\bar{z}} r + \sinh^2 r (\partial_z \theta \partial_{\bar{z}} \theta + 2A_z \partial_{\bar{z}} \theta - 2A_{\bar{z}} \partial_z \theta) - 4 \cosh^2 r A_z A_{\bar{z}} \tag{6.4}$$

The Kac-Moody primaries (3.4)-(3.6) are readily translated into unitary gauge and the currents (2.8) can be written as

$$\begin{aligned} J^\pm(z) &= \pm \kappa \left( -\partial_z r \pm \frac{i}{2} \sinh 2r \partial_z \theta \right) e^{\mp i\theta} \\ \bar{J}^\pm(\bar{z}) &= \mp \kappa \left( -\partial_{\bar{z}} r \mp \frac{i}{2} \sinh 2r \partial_{\bar{z}} \theta \right) e^{\pm i\theta} \end{aligned} \tag{6.5}$$

The discrete states come in pairs related by a *duality* symmetry which flips the sign of the right-handed (holomorphic)  $U(1)$  current. Although the string effective action is expected to be duality invariant to all orders, the deformations of the background need not respect duality symmetry.

Using the results of Section 3, it is easy to show that the composite fields

$$\begin{aligned} \psi^{\pm\pm} &=: (\bar{J}^\pm)^N (J^\pm)^N (g_{\pm\pm})^{j+m-N} : \\ \psi^{\pm\mp} &=: (\bar{J}^\pm)^N (J^\mp)^N (g_{\pm\mp})^{j+m-N} : \end{aligned} \tag{6.6}$$

are discrete states lying on the boundary of a highest/lowest weight module at grade  $N$  with coset dimension  $(1,1)$ , and which obey the physical state conditions (2.15). Their  $(j,m)$  values correspond to  $s=1, r=1, 2, \dots$  in (3.1)-(3.2). For grade three and upwards, there are additional discrete states living in the bulk of the module which can be constructed with more difficulty. They correspond to  $s > 1$  in (3.1)-(3.2).

The WZW composite fields are covariantized according to the prescription:

$$\begin{aligned} \mathbf{g}(z, \bar{z}) &\rightarrow e^{\frac{i}{2}\sigma_2 \int^z A_x dx} \mathbf{g}(z, \bar{z}) e^{\frac{i}{2}\sigma_2 \int^{\bar{z}} A_x d\bar{z}} \\ \partial_z \mathbf{g} &\rightarrow \partial_z \mathbf{g} + \frac{i}{2} A_x (\sigma_2 \mathbf{g} + \mathbf{g} \sigma_2). \end{aligned} \quad (6.7)$$

One can show that functionals of  $g_{\pm\pm}, g_{\pm\mp}$  with non-vanishing  $J^3(\bar{J}^3)$  eigenvalue are dressed by Wilson lines. Also, ordinary derivatives of functionals with  $m+\bar{m} \neq 0$  are converted into covariant derivatives. For example, the derivatives of the spinor weights transform as

$$\begin{aligned} \partial_x g_{\pm\pm} &\rightarrow e^{\pm(\frac{i}{2} \int^x A_x dx + \frac{i}{2} \int^{\bar{x}} A_x d\bar{x})} (\partial_x g_{\pm\pm} \pm 2 i A_x g_{\pm\pm}) \\ \partial_x g_{\pm\mp} &\rightarrow e^{\pm(\frac{i}{2} \int^x A_x dx - \frac{i}{2} \int^{\bar{x}} A_x d\bar{x})} (\partial_x g_{\pm\mp} \pm 2 i A_x g_{\pm\mp}). \end{aligned} \quad (6.8)$$

The currents  $J^\pm, \bar{J}^\pm$  of the ungauged  $SL(2, R)$  model become parafermions of the coset model

$$\begin{aligned} \Psi^\pm &= e^{\pm i \int^x A_x dx} (g_{\pm\mp} \partial_x g_{\pm\pm} \pm 2 i A_x g_{\pm\pm} - g_{\pm\pm} \partial_x g_{\pm\mp}) \\ \bar{\Psi}^\pm &= e^{\pm i \int^{\bar{x}} A_x d\bar{x}} (g_{\mp\pm} \partial_{\bar{x}} g_{\pm\pm} \pm 2 i A_x g_{\pm\pm} - g_{\pm\pm} \partial_{\bar{x}} g_{\mp\pm}) \end{aligned} \quad (6.9)$$

These expressions are reminiscent of the ‘‘classical parafermions’’ in [52].

There are four independent grade one moduli of the black hole. Each of them has  $m=\bar{m}=0$ . In this case the vector potential is particularly easy to integrate out of the action since it only appears in the covariant derivatives. Their explicit form is

$$\begin{aligned} \psi^{++} + \psi^{--} &= -2 \operatorname{sech}^2 r \partial_x r \partial_{\bar{x}} r - 2 \sinh^2 r (\partial_x \theta \partial_{\bar{x}} \theta + 2 A_x \partial_{\bar{x}} \theta - 2 A_x \partial_x \theta - 4 A_x A_{\bar{x}}), \\ \psi^{++} - \psi^{--} &= 2i \tanh r (\partial_x \theta \partial_{\bar{x}} r + \partial_{\bar{x}} r \partial_x \theta) - 4 i \tanh r (A_x \partial_{\bar{x}} r + A_{\bar{x}} \partial_x r), \end{aligned} \quad (6.10)$$

and their partners under the duality transformation:  $\psi^{+-} \pm \psi^{-+}$ .

We should also consider the deformation produced by (3.5):

$$\Phi_{-1/2,0,0}^{c-c}(r) = \frac{1}{\cosh r} F\left(\frac{1}{2}, \frac{1}{2}; 1; \tanh^2 r\right) = \frac{1}{\cosh r} K(\tanh^2 r) \quad (6.11)$$

where  $K$  is an elliptic function. Recall that in the Liouville theory the analogous operator is the zero mode of the tachyon, and is exactly marginal. In our case the deformation

(6.11) also corresponds to turning on a nonzero tachyon background, however (6.11) is *not* exactly marginal. Rather, as we have discussed, the exactly marginal deformation which turns on a nonzero tachyon background is

$$L_0^1 \bar{L}_0^1 = \Phi_{-1/2,0,0}^{c-c} + i(\psi^{++} - \psi^{--}) + \dots \quad (6.12)$$

which includes back-reaction on the space-time metric as well as higher order corrections involving the “massive modes” of the string. Since the Lagrangian is quadratic and non-derivative in  $A$  we can integrate out the vector potential giving

$$L = \partial_x r \partial_{\bar{x}} r (1 + 4\alpha^2 \tanh^2 r \operatorname{sech}^2 r) + 2\alpha \tanh r \operatorname{sech}^2 r (\partial_x r \partial_{\bar{x}} \theta + \partial_x \theta \partial_{\bar{x}} r) \\ + \tanh^2 r \partial_x \theta \partial_{\bar{x}} \theta + \alpha \Phi_{-1/2,0,0}^{c-c}(r), \quad (6.13)$$

where  $\alpha$  an arbitrary parameter. We diagonalize the metric via the coordinate transformation

$$\theta \rightarrow \tilde{\theta}(r, \theta), \quad \frac{\partial \tilde{\theta}}{\partial \theta} = 1, \quad \frac{\partial \tilde{\theta}}{\partial r} = 2\alpha \operatorname{csch} r \operatorname{sech} r, \quad (6.14)$$

which gives

$$G_{rr} = 1 - 4\alpha^2 \operatorname{sech}^2 r, \quad G_{\tilde{\theta}\tilde{\theta}} = \tanh^2 r. \quad (6.15)$$

To compare with the beta function results we change variables,  $\tanh^2 r = 1 - \mu e^{-2\rho}$ , so that the metric takes the form

$$ds^2 = \frac{1 - 4\alpha^2 \mu e^{-2\rho}}{1 - \mu e^{-2\rho}} (d\rho)^2 + (1 - \mu e^{-2\rho}) (d\tilde{\theta})^2, \quad (6.16)$$

and the one-loop contribution, from the integration over  $A$ , is the linear dilaton background

$$\Phi(r) = \ln \cosh r \rightarrow 2\rho - \ln \mu. \quad (6.17)$$

Now we recall that the beta function equations are only solved in the weak field approximation for the tachyon, which corresponds to large  $r$ . Then we note that, as  $r \rightarrow \infty$ , the elliptic function in (6.11) tends to a logarithm[53]

$$K(\tanh^2 r) \rightarrow \ln \cosh r + 2\sqrt{2}, \quad (6.18)$$

so that the static tachyon background is

$$T(\rho) \sim \alpha(\rho - \frac{1}{2} \ln \mu) \sqrt{\mu} e^{-2\rho}. \quad (6.19)$$

These expressions are in complete agreement with eq.(15) of [17], where the back reaction on the black hole metric was computed from the beta function equations. The parameter  $\mu$  plays the role of the mass of the black hole. The Hawking temperature can be calculated as in [17].

Finally, consider perturbing the action of the gauged WZW model by the duality invariant operator

$$\begin{aligned}\alpha O(z, \bar{z}) &= \alpha (\psi^{++} + \psi^{--} + \psi^{+-} + \psi^{-+}) \\ &= -2\alpha (\operatorname{sech}^2 r + \operatorname{csch}^2 r) \partial_x r \partial_{\bar{x}} r + 2\alpha \partial_x \theta \partial_{\bar{x}} \theta - 4\alpha (A_x \partial_{\bar{x}} \theta - A_{\bar{x}} \partial_x \theta) + 8\alpha A_x A_{\bar{x}}.\end{aligned}\tag{6.20}$$

This is the simplest part of the exactly marginal deformation  $L_0^2 \bar{L}_0^2$ . Integrating out  $A$  as before yields

$$L = \partial_x r \partial_{\bar{x}} r \left[ 1 - 2\alpha (\operatorname{csch}^2 r + \operatorname{sech}^2 r) \right] + \partial_x \theta \partial_{\bar{x}} \theta \left[ \sinh^2 r + 2\alpha - \frac{(\sinh^2 r + 2\alpha)^2}{\cosh^2 r + 2\alpha} \right].\tag{6.21}$$

The one-loop contribution to the action from the measure in the integration over  $A$  gives the target space dilaton term with

$$\Phi = \ln [\cosh^2 r + 2\alpha]\tag{6.22}$$

To compare with the original black hole metric we now make a coordinate transformation that converts the dilaton back to its original form

$$\cosh^2 r + 2\alpha \rightarrow \cosh^2 r\tag{6.23}$$

and to  $O(\alpha)$ , we obtain the target space metric

$$ds^2 = \frac{k}{2} \left( (dr)^2 + \tanh^2 r (d\theta)^2 \right)\tag{6.24}$$

Thus we find that, to lowest order in the coupling,  $\alpha$ , the perturbation simply rescales the original action by an overall constant.

## 7. Conclusion

It must certainly be possible to put the  $SL(2, R)/U(1)$  coset string theory on a firmer formal footing. We have avoided some technical issues in this paper, not because they are intractable, but rather because they can be more confidently addressed once the underlying

physical content has been made manifest. We believe that we have made substantial progress in this direction. It would be well to prove unitarity and modular invariance for this theory, and to resolve the remaining ambiguity in the definition and evaluation of physical correlators.

We would also like to get more information about the  $W$  hair algebra and its precise relation to other  $W_\infty$  structures. Although we have made some speculative remarks in this regard, a much better job can and should be done.

As we have observed, the physical  $W$  hair structure has great difficulty embedding itself in the underlying abstract  $SL(2, R)$  Kac-Moody algebra. This may be a hint that coset conformal field theories are not very well suited for describing stringy black holes, and that we will be better off in the long run turning to alternate methods. It may well be that these structures are more accessible in other formulations such as string field theory.

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## Appendix A. Explicit Construction of $SL(2, R)$ Primaries

In the  $SL(2, R)$  WZW theory the Kac-Moody primaries are composites of  $g_{++}$ ,  $g_{--}$ ,  $g_{-+}$ , and  $g_{+-}$ . Their form is completely determined up to normalizations by the requirement that they form irreducible representations of classical  $SU(1, 1)_L \times SU(1, 1)_R$ . The  $g$ 's transform like  $j=\bar{j}=1/2$  spinors, with  $(m, \bar{m})$  components  $(1/2, 1/2)$ ,  $(-1/2, -1/2)$ ,  $(-1/2, 1/2)$  and  $(1/2, -1/2)$ , respectively. We write a general ansatz for a primary in the form

$$\Phi_{a,b,c,d}(z, \bar{z}) = \sum_{n=-\infty}^{\infty} R_n (g_{++})^{a+n} (g_{--})^{b+n} (g_{-+})^{c-n} (g_{+-})^{d-n}, \quad (\text{A.1})$$

where the  $R_n$  are coefficients and the parameters  $a, b, c, d$  satisfy

$$a + b + c + d = 2j. \quad (\text{A.2})$$

Note that  $j=\bar{j}$  is automatic in this construction. Following Vilenkin[30], it is useful to define a rescaled function

$$\Phi_{a,b,c,d}(z, \bar{z}) = |g_{-+}|^{2j} \bar{\Phi}_{a,b,c,d} \left( \frac{g_{++}}{|g_{-+}|}, \frac{g_{--}}{|g_{-+}|}, e^{i\varphi} \right), \quad (\text{A.3})$$

where  $\exp i\varphi$  denotes  $g_{-+}/|g_{-+}|$ .

Now  $\tilde{\Phi}$  must be an eigenstate of both  $J_L^2$  and  $J_R^2$  with eigenvalue  $-j(j+1)$ . This gives a recursion relation for the coefficients  $R_n$ :

$$(a+n+1)(b+n+1)R_{n+1} + (c-n+1)(d-n+1)R_{n-1} + [(a+n)(b+n) + (c-n)(d-n)]R_n = 0. \quad (\text{A.4})$$

This is solved by

$$R_n = \frac{(-1)^n}{\Gamma(a+n+1)\Gamma(b+n+1)\Gamma(c-n+1)\Gamma(d-n+1)}. \quad (\text{A.5})$$

We thus obtain

$$\Phi_{a,b,c,d} = \sum_{n=-\infty}^{\infty} \frac{(g_{++})^a (g_{--})^b (g_{-+})^c (g_{+-})^d (-g_{+-}g_{-+}/g_{++}g_{--})^n}{\Gamma(a-n+1)\Gamma(b-n+1)\Gamma(c+n+1)\Gamma(d+n+1)}, \quad (\text{A.6})$$

and from the action of the raising and lowering operators we can identify the parameters  $a, b, c, d$  as

$$a+d = j+m, \quad b+c = j-m, \quad a+c = j+\bar{m}, \quad b+d = j-\bar{m}. \quad (\text{A.7})$$

It is clear by inspection of (A.6) that the parameter  $d$  is redundant and can be set to zero. We then obtain, using a simple choice of normalization, the expression (3.4).

## Appendix B. $SU(1,1)$ Tensor Products

$SU(1,1)$  tensor products have the form

$$\Phi_{j,m_1+m_2} = \sum_{n=-\infty}^{\infty} C_{m_1 m_2 m_1+m_2}^{j_1 j_2 j}(n) \Phi_{j_1, m_1+n} \otimes \Phi_{j_2, m_2-n}, \quad (\text{B.1})$$

where we assume that the individual tensor product states are normalized. The requirement that  $\Phi_{j,m_1+m_2}$  be an eigenstate of the total  $J^2$  gives a second order recursion relation for the Wigner coefficients:

$$\begin{aligned}
0 = & C_{n+1} \sqrt{(j_1 - m_1 - n)(j_1 + m_1 + n + 1)(j_2 + m_2 - n)(j_2 - m_2 + n + 1)} \\
& + C_{n-1} \sqrt{(j_1 + m_1 + n)(j_1 - m_1 - n + 1)(j_2 - m_2 + n)(j_2 + m_2 - n + 1)} \\
& + C_n \left[ -j(j+1) + (m_1 + m_2)(m_1 + m_2 - 1) \right. \\
& \left. + (j_1 - m_1 - n + 1)(j_1 + m_1 + n) + (j_2 + m_2 - n)(j_2 - m_2 + n + 1) \right].
\end{aligned} \tag{B.2}$$

In the special case that  $\Phi_{j, m_1 + m_2}$  is supposed to be either highest or lowest weight, then we may use a simpler first order recursion relation. For example, requiring that  $\Phi_{j, m_1 + m_2}$  be annihilated by  $J_1^- + J_2^-$  gives

$$C_{n+1} \sqrt{(j_1 - m_1 - n)(j_1 + m_1 + n + 1)} + C_n \sqrt{(j_2 - m_2 + n + 1)(j_2 + m_2 - n)} = 0, \tag{B.3}$$

which is solved trivially.

In the more general case of (B.2), it is useful to define new coefficients by

$$C_n = \left[ \frac{\Gamma(n - j_1 + m_1) \Gamma(n + j_1 + m_1 + 1)}{\Gamma(n - j_2 - m_2) \Gamma(n + j_2 - m_2 + 1)} \right]^{1/2} D_n. \tag{B.4}$$

The new coefficients  $D_n$  satisfy

$$\begin{aligned}
0 = & (n - j_1 + m_1)(n + j_1 + m_1 + 1)D_{n+1} + (n - j_2 - m_2 - 1)(n + j_2 - m_2)D_{n-1} \\
& - \left[ j(j+1) - (m_1 + m_2)(m_1 + m_2 - 1) \right. \\
& \left. + (n - j_1 + m_1 - 1)(n + j_1 + m_1) + (n - j_2 - m_2)(n + j_2 - m_2 + 1) \right] D_n.
\end{aligned} \tag{B.5}$$

To solve this, we introduce a dummy function  $f(t)$ :

$$f(t) = \sum_n D_{n-1} t^n. \tag{B.6}$$

One can then show that  $f(t)$  satisfies Riemann's differential equation in the form[54]

$$f(t) = P \begin{pmatrix} 0 & 1 & \infty \\ \lambda & \mu & \nu \\ \lambda' & \mu' & \nu' \end{pmatrix}, \tag{B.7}$$

where

$$\begin{aligned}
\lambda &= -j_1 - m_1 + 1, & \lambda' &= j_1 - m_1 + 2 \\
\mu &= j + m_1 + m_2, & \mu' &= -j - 1 + m_1 + m_2 \\
\nu &= j_2 - m_2, & \nu' &= 1 - \lambda - \lambda' - \mu - \mu' - \nu.
\end{aligned}
\tag{B.8}$$

One solution of this equation is

$$f(t) = t^\lambda (1-t)^\mu F(a, b; c; t), \tag{B.9}$$

where

$$\begin{aligned}
a &= \lambda + \mu + \nu \\
b &= 1 - \lambda' - \mu' - \nu \\
c &= 1 - \lambda' + \lambda.
\end{aligned}
\tag{B.10}$$

In general, there is also a second independent solution. This is given by interchanging  $\lambda$  and  $\lambda'$  in (B.9) and (B.10). Note that

$$a + b - c = 2j - 1. \tag{B.11}$$

If in addition  $c$  is not zero or a negative integer, then this implies that, for both solutions, the corresponding hypergeometric series is convergent at  $t=1$  for all  $j < -1/2$ . This means that we can evaluate the hypergeometric functions by their series expansions

$$F(a, b; c; 1) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+c)\Gamma(n+1)}. \tag{B.12}$$

We may thus determine the coefficients  $D_n$  and  $C_n$ .

To determine which  $j$  values actually occur in the tensor product, we must first impose any boundary conditions on the solution which will arise if any of  $\Phi_j$ ,  $\Phi_{j_1}$ , or  $\Phi_{j_2}$  is a discrete or double-sided representation. We must also require

$$\sum_n |C_n|^2 < \infty, \tag{B.13}$$

so that  $\Phi_{j, m_1+m_2}$  is normalizable.

In addition to the examples worked out in the text, we note below some useful results for tensor products of discrete series reps.

- The tensor product of two unitary highest weight reps contains only other highest weight reps. The only possible  $j$  values  $j_1 + j_2, j_1 + j_2 - 1, j_1 + j_2 - 2, \dots$
- The same is true for the product of two unitary lowest weight reps.
- In the tensor product of a highest weight and lowest weight representation with the same  $j$ , the only discrete rep which appears is the identity[32].

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## Figure Captions

Fig. 1. Current algebra representation built on a discrete series representation of  $SL(2, R)$ . The number in each circle indicates the multiplicity of states at that value of  $(L_0, J_0^3)$ .

Liouville	$SL(2, R)$	Grade
$W_{0,0}^+$	$\Phi_{-1/2,0}^{cont}$	0
$W_{1/2,\pm 1/2}^+$	$\Phi_{-3/4,\mp 3/4}^{hw/lw}$	0
$W_{1,0}^+$	$W_{-1,0}^{hw}, W_{-1,0}^{lw}$	1
$W_{1,\pm 1}^+$	$\Phi_{-1,\pm 3/2}^{cont}$	0
$W_{3/2,\pm 3/2}^+$	$\Phi_{-5/4,\mp 9/4}^{hw/lw}$	0
$W_{3/2,\pm 1/2}^+$	$W_{-5/4,\pm 3/4}^{hw/lw}$	2
$W_{2,\pm 2}^+$	$\Phi_{-3/2,\pm 3}^{cont}$	0
$W_{2,\pm 1}^+$	$W_{-3/2,\pm 3/2}^{hw}, W_{-3/2,\pm 3/2}^{lw}$	3
$W_{2,0}^+$	$W_{-3/2,0}^{cont}$	4
$W_{5/2,\pm 5/2}^+$	$\Phi_{-7/4,\mp 15/4}^{hw/lw}$	0
$W_{5/2,\pm 3/2}^+$	$W_{-7/4,\pm 9/4}^{hw/lw}$	4
$W_{5/2,\pm 1/2}^+$	$W_{-7/4,\mp 3/4}^{hw/lw}$	6
$W_{3,\pm 3}^+$	$\Phi_{-2,\pm 9/2}^{cont}$	0
$W_{3,\pm 2}^+$	$W_{-2,\pm 3}^{hw}, W_{-2,\pm 3}^{lw}$	5
$W_{3,\pm 1}^+$	$W_{-2,\pm 3/2}^{cont}$	8
$W_{3,0}^+$	$W_{-2,0}^{hw}, W_{-2,0}^{lw}$	9

TABLE 1

Ground Ring States	Physical States
$O_{-1,0}^{double}$ (identity)	$W_{-1,0}^{hw/lw}, \Phi_{-1,\pm 3/2}^{cont}$
$O_{-5/4,\mp 3/4}^{hw/lw}$	$\Phi_{-5/4,\mp 9/4}^{hw/lw}, W_{-5/4,\pm 3/4}^{hw/lw}$
$O_{-3/2,\mp 3/2}^{hw/lw}, O_{1/2,\pm 3/2}^{double}, O_{-3/2,0}^{cont}$	$\Phi_{-3/2,\pm 3}^{cont}, W_{-3/2,\pm 3/2}^{hw}, W_{-3/2,\pm 3/2}^{lw}, W_{-3/2,0}^{cont}$
$O_{-7/4,\mp 9/4}^{hw/lw}, O_{-7/4,\pm 3/4}^{hw/lw}$	$\Phi_{-7/4,\mp 15/4}^{hw/lw}, W_{-7/4,\pm 9/4}^{hw/lw}, W_{-7/4,\mp 3/4}^{hw/lw}$

TABLE 2

