Fermi National Accelerator Laboratory

NSF-ITP-92-81
FNAL-Pub-92/142-A
May, 1992

Semi-classical corrections to thermal activation

Edward W. Kolb

Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106
and NASA/Fermilab Astrophysics Center
Fermi National Accelerator Laboratory, Batavia, IL 60510
and Department of Astronomy and Astrophysics, Enrico Fermi Institute
The University of Chicago, Chicago, IL 60637

Igor I. Tkachev

Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106
and Institute for Nuclear Research of the Academy of Sciences of Russia, Moscow 117312

Abstract

The decay of a meta-stable phase through nucleation of bubbles of the true-vacuum phase can occur at non-zero temperature through classical thermal activation, with the rate per volume $\mathcal{P} \propto \exp(-F_C/k_BT)$, where F_C is the free energy for a critical bubble and T is the temperature. In this paper we calculate order $\hbar$ corrections to this rate. These corrections represent processes where the field tunnels through the potential barrier starting from a state of free energy $F < F_C$, and provide a smooth interpolation between the high-temperature and zero-temperature decay rates. We confirm that the quantum corrections are of the same order as the classical results at large T.

PACS numbers: 98.80.Dr, 11.30.Qc, 04.60.+n, 04.20.Fy

I. INTRODUCTION

Since the discovery of symmetry restoration at high temperatures[1] and the possibility of phase transitions in the early Universe, problems related to the decay of a metastable state in quantum field theory have received a lot of attention in view of the many different cosmological implications. Such is the case for first order-phase transitions, which have the most interesting and drastic influence on the history of the early Universe.[2] In this case a metastable state is separated from the ground state of a theory by a potential barrier, and the decay is due to the creation and subsequent expansion of bubbles filled by the true ground state. The probability of the formation of bubbles of the new phase at zero temperature was found in Refs. 3 and 4, while in the high temperature limit it was found in Ref. 5. However, there has never been a complete description for field theory valid in all ranges of temperatures comparable to the description for the quantum mechanical problem of tunnelling as discussed by Affleck and Langer.[6]

At zero temperature bubble creation is due to the subbarrier tunnelling of the relevant mode describing the order parameter. Boundary conditions dictate that the total energy change in this process is precisely zero. The gain in volume energy has to be compensated for by the surface tension. At high temperatures the decay is due to classical fluctuations over the barrier. There is no restrictions on the total energy of the bubble, so the dominant contribution is from the saddle point configuration corresponding to the lowest point on the barrier. Clearly, at moderate temperatures both processes should be present. Namely, not only the fluctuations with change in the free energy sufficiently large that the fluctuation can classically overcome the barrier, but bubbles with smaller free energy that may tunnel through the barrier. The smaller bubbles are present in the thermodynamical ensemble in even greater proportion in accordance with the Hibbs distribution. In general, even in the semiclassical treatment of the problem, we have to integrate over all subcritical bubbles. By subcritical we mean bubbles which are classical in all respects and differ only in the value of their free energy. It is the interplay between the Hibbs distribution and the tunnelling exponent that eventually determines the field configuration dominating the decay rate.

The full treatment would be a complicated problem, so we reduce it to the case where the thin-wall approximation for the field configurations describing the bubbles is valid. Then, in the semiclassical approach the only dynamical variable which remains is the radius of the bubble, and as we shall show at finite temperatures the tunnelling process can be described in the framework of one-dimensional quantum mechanics of bubbles similar to the zero temperature case of the Ref. 4. The main difference turns out to be that it is the free energy, not the energy, of the bubble that is the integral of the motion in the thermal bath. The negative of the pressure in the two phases (or the finite-temperature effective potential) enters the relevant equation in the same way as the energy density at zero temperature. We believe the assumptions made in deriving these results are valid when considering subbarrier motion of the bubble. It is interesting to note that in the thin-wall approximation for some range of temperatures all subcritical bubbles give compatible contributions to the decay probability.

While our work was in progress, we became aware of work by Hsu,[7] in which he also considers quantum tunnelling at finite temperature. Our main conclusions agree with Hsu. However in some important point our results are different. In estimating the tunnelling rate Hsu integrates over the energies of subcritical bubbles, rather than the free energies as in our approach. He does not calculate the tunnelling amplitude at a given finite energy, but uses the relevant rate for a $1+1$-dimensional Abelian Higgs model found in Ref. 8, while our results are valid in any model in the thin-wall approximation in $(3+1)$ dimensions.

In Sec. II we review the zero temperature tunnelling problem and find the tunnelling exponent for the bubble of non-zero total energy. Such a bubble can not appear spontaneously in vacuum, but can be relevant to some processes at finite energies. In Sec. III we construct the quantum mechanics of thin-wall bubbles at finite temperatures and calculate the decay probability in the semiclassical approximation. In the appendix we apply our the results to a model situation that reflects the case when the thin-wall approximation is valid at zero temperature, and consequently is an even better approximation at high temperatures.

II. ZERO-TEMPERATURE TUNNELLING

Our goal is to calculate the nucleation rate (per volume) of bubbles of true vacuum within a homogeneous region of false vacuum. In this section we review the derivation of the zero-temperature tunnel action in the thin-wall limit. The types of models we consider have a meta-stable "false-vacuum" state at $\phi = \phi_-$, a "true-vacuum" state at $\phi = \phi_+$, and a barrier separating the minima with a local maximum at $\phi = \phi_M$.

A. ZERO ENERGY TUNNELLING

As is well known, the probability per unit volume for the nucleation of a bubble of true vacuum is $\mathcal{P} = Ae^{-B/\hbar}$, where A is a prefactor of mass dimension 4 and B is dimensionless. Transition to the true vacuum state by quantum tunnelling occurs through the nucleation of bubbles of the energetically favored phase ($\phi = \phi_+$), which then expand outward, asymptotically approaching the velocity of light.

Tunnelling is associated with a classical motion in imaginary time, with B given by the Euclidean action S_E:

$$B = S_E = \int dt_E \int d^3x \left[\frac{1}{2} \left(\frac{d\phi}{dt_E} \right) + \frac{1}{2} (\vec{\nabla}\phi)^2 + V(\phi) \right], \tag{2.1}$$

where t_E is Euclidean time, and ϕ is a solution to the Euclidean equations of motion with boundary conditions $d\phi(0, \vec{x})/dt_E = 0$, and $\phi(\pm\infty, \vec{x}) = \phi_-$.

All possible solutions to the Euclidean equations of motion satisfying the above boundary conditions contribute to the transition. However, the solution with the least action makes the largest contribution to the transition. At zero temperature, in the absence of seeds for nucleation, the least-action solution has $O(4)$ symmetry, in which case ϕ is a function only of $r^2 = t_E^2 + |\vec{x}|^2$, and the Euclidean action is

$$S_E = 2\pi^2 \int_0^\infty r^3 dr \left[\frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 + V(\phi) \right], \tag{2.2}$$

with ϕ a solution to the O(4)-symmetric Euclidean equations of motion

$$\frac{d^2\phi}{dr^2} + \frac{3}{r}\frac{d\phi}{dr} - \frac{dV(\phi)}{d\phi} = 0, \tag{2.3}$$

with boundary conditions $d\phi(0)/dr = 0$ and $\phi(+\infty) = \phi_-$.

Closed-form analytic solutions to the Euclidean equation of motion cannot be found. However a simple approximation can be found in the thin-wall approximation where the difference in potential between the false and true vacuum states are small compared to the maximum height of the barrier separating them. In the thin-wall limit the "friction" term in the equation of motion, $r^{-1}d\phi/dr$, can be neglected, and the solution is $d\phi/dr = -\sqrt{2V_0(\phi)}$; $r = \int^\phi d\phi'/\sqrt{2V_0(\phi')}$, where $V_0(\phi)$ is the potential in the limit of exact degeneracy. In the thin-wall limit the solution has the form

$$\phi(r) = \begin{cases} \phi_+ & r \ll R \\ \phi_{TW}(r) & r \simeq R \\ \phi_- & r \gg R \end{cases} \tag{2.4}$$

where $\phi_{TW}(r)$, the solution to the Euclidean equations of motion neglecting friction, depends upon $V_0(\phi)$.

Thus, in the thin-wall limit the $O(4)$ Euclidean action can be expressed in terms of the bubble radius R:

$$S_E = 2\pi^2 R^3 \int_{\phi_-}^{\phi_+} d\phi \sqrt{2V_0(\phi)} - 2\pi^2 \frac{R^4}{4} \Delta V. \tag{2.5}$$

The origin of the two contributions to the action are clear: The second term is a volume term and the first term is a surface term with

$$S_1 \equiv \int_{\phi_-}^{\phi_+} d\phi \sqrt{2V_0(\phi)} \tag{2.6}$$

playing the role of the surface tension. The radius of the bubble that results in the least action with $O(4)$ symmetry, R_4, can be found by extremizing the action with respect to R:

$$R_4 = 3S_1/\Delta V. \tag{2.7}$$

This results in a Euclidean action

$$S_E = \pi^2 S_1 R_4^3/2 = 27\pi^2 S_1^4/2(\Delta V)^3 = \pi^2 \Delta V R_4^4/6. \tag{2.8}$$

At non-zero temperatures the $O(4)$ invariance of the least-action solution will be broken. The easiest way to handle this situation and calculate quantum corrections to thermal activation is to consider the (Lorentzian) quantum mechanics of the motion of thin-wall bubbles. The quantum nucleation process will be treated by considering a one-dimensional tunnelling problem with the bubble radius R playing the role of the dynamical variable.

B. FINITE ENERGY TUNNELLING

The Lorentzian equation of the motion of the bubble in vacuum is[4]

$$E = -\frac{4\pi}{3} \Delta V R^3 + \frac{4\pi S_1 R^2}{\sqrt{1 - R'^2}}, \tag{2.9}$$

where $R' \equiv dR/dt$ is the derivative of the bubble radius with respect to the time coordinate in the rest frame, and E is an integral of the motion representing the total energy of the bubble. $E = 0$ for the bubble described in Sec. IIA, but we will consider a generalization of the description to include non-zero E. We will take Eq. (2.9) as the Hamiltonian constraint $H(R, R') = E$. The corresponding Lagrangian is given by[9] $L = R' \int (R')^{-2} H(R, R')\, dR'$, or

$$L = (4\pi/3)R^3 \Delta V - 4\pi S_1 R^2 \sqrt{1 - R'^2}\,. \tag{2.10}$$

Now, the standard relations complete the definition of the Hamiltonian system:

$$\begin{aligned} P_R &= dL/dR' = 4\pi S_1 R^2 R'(1 - R'^2)^{-1/2}, \\ H(P_R, R) &= -(4\pi/3)\Delta V R^3 + \sqrt{P_R^2 + (4\pi S_1 R^2)^2}\,. \end{aligned} \tag{2.11}$$

We can consider the function $\mathcal{V}(R) \equiv H(P_R = 0, R) = -(4\pi/3)\Delta V R^3 + 4\pi S_1 R^2$ as the potential in a one-dimensional bubble motion problem. This potential is shown in Fig. 1. At $0 \leq E < \mathcal{V}_{\max}$ there are two branches of classical evolution corresponding to $0 < R < R_1$ and $R_2 < R < \infty$, where R_1 and R_2, the classical turning points, are solutions to Eq. (2.9) with $R' = 0$. The tunnelling probability between these states is given by $\mathcal{P}(E) \propto e^{-B(E)/\hbar}$, with $B(E) = 2\int_{R_1}^{R_2} P_E dR$ along the bubble wall trajectory with P_E the Euclidean momentum. Making use of Eq. (2.11) and substituting R' from Eq. (2.9) rotated to imaginary time, we find

$$B(E) = 2\int_{R_1}^{R_2} \sqrt{[4\pi S_1 R^2]^2 - [E + (4\pi/3)\Delta V R^3]^2}\, dR. \tag{2.12}$$

It is convenient to introduce the dimensionless variables

$$R_0 \equiv \frac{3S_1}{\Delta V}, \qquad \omega \equiv \frac{E}{4\pi S_1 R_0^2}, \qquad x = R/R_0. \tag{2.13}$$

Note, that R_0 is simply the bubble radius for the $O(4)$ symmetric bubble found above [cf. Eq. (2.7)]. With these definitions the integral Eq. (2.12) can be rewritten as

$$
\begin{aligned}
B(E) &= 8\pi S_1 R_0^3 I(\omega), \\
I(\omega) &\equiv \int_{x_1}^{x_2} \sqrt{(x^2+x^3+\omega)(x^2-x^3-\omega)}\, dx,
\end{aligned}
\tag{2.14}
$$

where x_1 and x_2 are roots of the equation $x^3-x^2+\omega=0$. For tunnelling with zero energy, $I(0)=\pi/16$, and $B(0)=S_E=\pi^2 S_1 R_0^3/2$, which reproduces the result of Eq. (2.8). This gives the probability of the creation of the true-vacuum bubble from "nothing" (note that $x_1=0$ in this case). At sufficiently high energy, corresponding to $\omega=\omega_C=4/27$, we have $x_1=x_2$ so that $I(\omega_C)=0$, and the transition may be realized by a purely classical motion. For $0<E<E_C\equiv 16\pi S_1 R_0^2/27$, $\mathcal{P}(E)\sim\exp[-B(E)/\hbar]$ gives the probability of tunnelling between the two possible branches of classical evolution $R<R_1$ and $R>R_2$.

We have calculated numerically the integral $I(\omega)$, Eq. (2.14). The result is presented in Fig. 2. We see that with sufficient accuracy it can be represented by a linear function[10]

$$
I(\omega)\simeq\frac{\pi}{16}\left(1-\frac{27}{4}\omega\right). \tag{2.15}
$$

This gives (recall that $E_C=16\pi S_1 R_0^2/27$)

$$
B(E)=\frac{\pi^2}{2}S_1R_0^3-\frac{27\pi}{32}ER_0=S_E\left(1-\frac{E}{E_C}\right). \tag{2.16}
$$

This expression for $B(E)$ can be extended beyond the thin-wall approximation with E_C equal to the sphaleron energy:

$$
B(E)=S_E[1-f(E)E/E_C], \tag{2.17}
$$

where $f(E_C)=1$. What we have shown here is that in the thin-wall approximation $f(E)\simeq 1$ for all $E<E_C$.

III. BUBBLE NUCLEATION IN A THERMAL BACKGROUND

Finite-temperature tunnelling will be similar to finite-energy tunnelling, with thermal energy playing the role of the non-zero initial energy. In calculating the tunnelling rate at zero temperature but with non-zero total energy of the field configuration, one must specify the probability of the initial configuration. However, in thermal equilibrium the probability for any configuration is simply proportional to the Boltzmann factor. Now let us consider tunnelling at finite-temperature, starting with the standard prescription and including the effect of mixing classical thermal activation with quantum tunnelling.

A. THERMAL ACTIVATION

The formal generalization of the zero-temperature results to finite temperature is straightforward. The quantum statistics of bosons at finite temperature is equivalent to the theory in Euclidean space with fields periodic in Euclidean time with period in $\hbar/k_B T$ (antiperiodic boundary conditions for fermions). Thus, the thermal background in general can break the $O(4)$ symmetry of the minimal action because the solution becomes periodic in the Euclidean time direction.

With the above boundary conditions the Euclidean action becomes

$$S_E = \int_0^{\hbar/k_B T} dt_E \int d^3x \left[\frac{1}{2} \left(\frac{d\phi}{dt_E} \right)^2 + \frac{1}{2} \left(\vec{\nabla}\phi \right)^2 + V_T(\phi) \right], \tag{3.1}$$

where $V_T(\phi)$ is finite temperature effective potential which accounts for interactions of the ϕ field with the thermal background. To find the least-action bubble we have to replace the $O(4)$ symmetric solution of Sec. II.A by a generic $O(3)$-symmetric bubble. In the limit that the radius of the bubble is much larger than $\hbar/k_B T$, ϕ will be independent of t_E, and $B/\hbar \longrightarrow F/k_B T$ where F is the free energy of a spherical bubble:

$$F = 4\pi \int_0^\infty x^2 dx \left[\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 + V_T(\phi) \right], \tag{3.2}$$

with ϕ the solution to the Euclidean equation of motion $d^2\phi/dx^2+(2/x)d\phi/dx-dV_T/d\phi=0$ with boundary conditions $\phi(\infty)=0$ and $d\phi(0)/dx=0$.

In the thin-wall approximation

$$F=4\pi R^2 S_1(T)-\frac{4\pi}{3}R^3\Delta V_T, \tag{3.3}$$

where again $S_1(T)$, given by Eq. (2.6) but with $V_0(\phi)$ being replaced by $V_T(\phi)$, playing the role of the surface tension, and the second term is a volume term. Note also that even in thin-wall approximation at small temperatures ΔV_T receives two contributions. Not only does it account for the difference in minima in the zero temperature potential energy, but it will also include a contribution if there are different numbers of massless particles in the two phases (which in fact is the generic result). To see this, recall that the finite-temperature result for the standard potential $V_0(\phi)=(\lambda/4)(\phi^2-\phi_0^2)^2$ is

$$\begin{aligned} V_T(\phi) &= V_1(\phi) \\ &+\frac{T^4}{2\pi^2}\int_0^\infty dx\, x^2 \ln\left\{1-\exp\left[-\sqrt{x^2+\left(-\lambda\sigma^2+3\lambda\phi^2\right)/T^2}\right]\right\}, \end{aligned} \tag{3.4}$$

where $V_1(\phi)$ is the zero-temperature 1-loop potential. At high temperature, the T-dependent part contains a term $-\pi^2T^2/90$, which is just the negative of the pressure of a massless particle. If particles have different masses in the two phases, or different numbers of massless particles in the two phases, there will be a pure "thermal" pressure difference. In general, $V_T(\phi_+)$ is equal to the negative of the pressure in the favored phase, p_+, accordingly $V_T(\phi_-)=-p_-$, so that $\Delta V_T=p_+-p_-\equiv\Delta p>0$.

Minimalization of the free energy with respect to R in the case when (3.2) is valid yields the radius, R_C, of the $O(3)$ "time-independent" bubble (sphaleron) and the corresponding value for F_C:

$$\begin{aligned} R_C &= 2S_1(T)/\Delta V_T \\ F_C &= (4\pi/3)R_C^2 S_1(T)=(16\pi/3)S_1(T)^3/(\Delta V_T)^2, \end{aligned} \tag{3.5}$$

Thus, in the high-temperature limit

$$\mathcal{P} \propto e^{-F_C/k_B T}, \tag{3.6}$$

where F_C is the activation energy of the critical bubble, or in modern parlance, the free energy of the sphaleron.

B. QUANTUM CORRECTIONS

Now we include the quantum corrections to the classical thermal activation picture. The basic idea is that in addition to the classical probability that a thermal fluctuation will have enough free energy to make a critical bubble, thermal fluctuations with smaller free energy can tunnel through the barrier. Thus, tunnelling is a combination of thermal energy to get part way up the barrier, and quantum tunnelling to get through the remaining barrier. To calculate this we generalize the zero-temperature, finite-energy tunnelling formalism developed in the previous section.

Let us start by deriving the equations for the bubble motion in a thermal background. The motion of the bubble wall specifies a certain (three-dimensional) hypersurface Σ in the four-dimensional space time. Let us introduce the coordinates $\{x^i\}$ on this hypersurface, together with the coordinate n measured in the direction of the outward normal, so that

$$ds^2 = -dn^2 + {}^3g_{ij}dx^i dx^j. \tag{3.7}$$

These coordinates are called Gaussian coordinates. Let T^α_β be the components of the stress-energy tensor in terms of the Gaussian coordinates. We can write the total stress-energy tensor as

$$T^\alpha_\beta = S^\alpha{}_\beta \delta(n) + T^\alpha_\beta(\text{out})\theta(n) + T^\alpha_\beta(\text{in})\theta(-n). \tag{3.8}$$

One may interpret this relation as the definition of the bubble wall tension tensor $S^{\alpha}{}_{\beta}$. Conservation of the energy momentum tensor, $T^{\beta}{}_{\alpha\,;\beta} = 0$, gives

$$
\begin{aligned}
S^{n}{}_{\alpha} &= 0, \\
S^{\beta}{}_{\alpha\,;\beta} + [T^{n}{}_{\alpha}] &= 0,
\end{aligned}
\tag{3.9}
$$

where $[T] = T(\text{out}) - T(\text{in})$. The first equation in (3.9) tells us that only the three dimensional tensor $S^{j}{}_{i}$ could be non-zero. The second equation results in the system of equations:

$$
\begin{aligned}
S^{j}{}_{i\,;j} + [T^{n}{}_{i}] &= 0, \\
-S^{j}{}_{i}\Gamma^{i}_{nj} + [T^{n}{}_{n}] &= 0.
\end{aligned}
\tag{3.10}
$$

The Christoffel symbols Γ^{i}_{nj} can be found in the following way. Let N^{α} be the components of the outward normal in Gaussian coordinates: $N^{n} = 1$, $N^{i} = 0$. Then $\Gamma^{i}_{nj} = N^{i}{}_{;j}$. Let the hypersurface Σ be defined in any other coordinate system $\{y^{\mu}\}$ by the equation $n(y^{\mu}) = 0$. The components of the outward normal in this system are $N_{\mu} = -n_{,\mu}|_{y^{\mu}\in\Sigma}$. Then the expression for Γ^{i}_{nj} can be found in terms of the function $n(y^{\mu})$; see, e.g., Ref. (11). Equations (3.10) determine the form of the hypersurface Σ and, consequently, the evolution of the two-dimensional phase separation boundary of arbitrary shape for arbitrarily specified energy-momentum tensors.

For our purposes we shall make following assumptions: Both inside and outside the bubble the energy-momentum tensor of the medium is that of a perfect fluid:

$$
T^{\nu}{}_{\mu} = (\rho + p)u^{\nu}u_{\mu} - p\delta^{\nu}{}_{\mu}. \tag{3.11}
$$

When the shell moves through the medium, the contribution of particles bound to the bubble wall leads to $S^{2}{}_{2} \neq S^{0}{}_{0}$ and $\dot{S}^{0}{}_{0} \neq 0$,[11] and there is a non-trivial outflow and inflow of particles. Both effects can be very important in different applications. However, when considering the process of bubble creation, i.e., the motion of the virtual bubble, we can assume that the structure of the stress energy tensor of the bubble wall is the same as in vacuum, $S^i{}_j = S_1(T)\delta^i{}_j$ (one can show that this structure is a property of a wall constructed only from the classical field ϕ[11]) and that there is no coherent motion of the medium in the rest frame. That is, for the velocity of the medium relative to the shell we have $u_\mu(\text{in}) = u_\mu(\text{out})$. Also, assume that there is no entropy creation and the temperature both inside and outside of the bubble is the same, i.e. $(\rho+p)|_{\text{in}} = (\rho+p)|_{\text{out}}$. With these assumptions we have $[T^\nu{}_\mu] = \Delta p\,\delta^\nu{}_\mu$. Then the first equation of Eq. (3.10) gives $S_1(T) = \text{constant}$, and the second becomes

$$-S_1(T)N^\mu{}_{;\mu} + \Delta p = 0. \tag{3.12}$$

For the spherically symmetric bubble we can define the coordinate n by

$$n = [r - R(t)]/(1 - \dot{R}^2)^{1/2}, \tag{3.13}$$

where r is the radial coordinate in spherical polar coordinates, y^μ, in which the 4-interval takes the form $ds^2 = dt^2 - dr^2 - r^2 d\Omega^2$. With this definition $N_\mu = -\partial_\mu n|_{r=R(t)}$ is the unit normal vector, $N^\mu N_\mu = -1$. We have $N^\mu{}_{;\mu} = \partial_t N_0 - 2N_r/R$, where $N_r = -1/(1-\dot{R}^2)^{1/2}$, $N_0 = \dot{R}/(1-\dot{R}^2)^{1/2}$. Using the identity $\dot{R}\partial_t N_0 = -\partial_t N_r$, we can rewrite the equation of motion (3.12) in the following way:

$$S_1(T)\partial_t(R^2 N_r) + \Delta p R^2 \dot{R} = 0. \tag{3.14}$$

This equation admits the first integral

$$F = -\frac{4\pi}{3}R^3\Delta p + \frac{4\pi S_1(T)R^2}{\sqrt{1-R'^2}}\ . \tag{3.15}$$

The structure of this expression is precisely the same as the structure of Eq. (2.9), but with Δp now including the thermal pressure. Since the pressure is the negative of the "free energy density," it is the free energy of the bubble, F, which is the integral of the motion in a thermal background, not the energy E. At any fixed F the picture of classical and sub-barrier motion of the bubbles will be the same as was described in Sec. IIB, with the replacements

$$\begin{aligned} E &\to F, \\ \Delta V &\to \Delta V_T \equiv \Delta p, \\ S_1 &\to S_1(T). \end{aligned} \tag{3.16}$$

Let us now consider the decay of a metastable state. The number density of bubbles with F is proportional to e^{-F/k_BT}, and the total probability of producing a new phase bubble is

$$\mathcal{P} \propto \int_0^\infty e^{-F/k_BT}\mathcal{P}(F) \Big/ \int_0^\infty e^{-F/k_BT} dF\ . \tag{3.17}$$

The bubbles may be of two types, sub-critical bubbles with $F \leq F_C$, which must tunnel through the barrier $R_1 < R < R_2$ with probability $\mathcal{P}(F < F_C) = A_\hbar e^{-B(F)/\hbar}$, or critical bubbles with $F > F_C$, which may classically evolve with a probability $\mathcal{P}(F > F_C) = A_T \simeq 1$ (independent of $\hbar$). Following the discussion in Sec. IIB, $F_C = 16\pi S_1(T)R_T^2/27$ and $R_T = 3S_1(T)/\Delta V_T$. Note that R_T need not be the radius of the $O(4)$ zero-temperature bubble ($R_4 = 3S_1/\Delta V$), nor is it the radius of the $O(3)$ bubble ($R_C = 2S_1(T)/\Delta V_T$). Since $R_T = 3R_C/2$, it is simple to express F_C in terms of R_C: $F_C = (4\pi/3)R_C^2 S_1$, exactly the result of Eq. (3.5).

The probability is naturally divided into a quantum part for $F \leq F_C$, and a classical part for $F \geq F_C$: $\mathcal{P} \propto \mathcal{P}_\hbar + \mathcal{P}_T$, where

$$\begin{aligned} \mathcal{P}_\hbar &= A_\hbar \int_0^{F_C} e^{-F/k_BT} e^{-B_T(F)/\hbar}\, dF \Big/ \int_0^\infty e^{-F/k_BT} dF \\ \mathcal{P}_T &= A_T \int_{F_C}^\infty e^{-F/k_BT} dF \Big/ \int_0^\infty e^{-F/k_BT} dF = A_T e^{-F_C/T}. \end{aligned} \tag{3.18}$$

$\mathcal{P}_T$ is simply the standard result of Eq. (3.6), a purely classical expression (independent of $\hbar$), while $\mathcal{P}_\hbar$ represents the probability that a sub-critical fluctuation tunnels through the barrier. It is inherently quantum mechanical, as it vanishes in the $\hbar \to 0$ limit. The function $B_T(F)$ can be obtained from $B(E)$, defined in Eq. (2.16), after the replacements of Eq. (3.16).

We have

$$\begin{aligned} F/k_BT + B_T(F) &= F/k_BT + B_T(0)\,(1 - F/F_C) \\ &= B_T(0) + F\left(\frac{1}{k_BT} - \frac{B_T(0)}{F_C}\right), \end{aligned} \tag{3.19}$$

where again $B_T(0) = S_T \equiv \pi^2 S_1(T) R_T^3/2$. The integral for $\mathcal{P}_\hbar$ can then be evaluated directly, with result

$$\mathcal{P}_\hbar \propto \frac{F_C/k_BT}{F_C/k_BT - S_T/\hbar}\left[e^{-S_T/\hbar} - e^{-F_C/k_BT}\right]. \tag{3.20}$$

As expected, this is a quantum result; it vanishes in the $\hbar \to 0$ limit.

Notice that $S_T/(F_C/T) = (27\pi/32)R_TT$, so $T = 32/27\pi R_T$ sets the scale for the high-temperature approximation. We will examine $\mathcal{P}_\hbar$ in three different regimes, depending upon the value of R_TT.

Low temperatures ($R_TT \ll 1$): The bubble with $F = 0$ dominates the integral, and $\mathcal{P}_\hbar \propto \exp(-S_T/\hbar)$. From Eq. (3.15) we see that the trajectory of this bubble is defined by $R'^2 = 1 - (R_T/R)^2$, i.e., it is $O(4)$ invariant in imaginary time. This agrees with Linde's prescription to use the $O(4)$-invariant solution with the temperature corrected effective potential until the bubbles nearly start to overlap in imaginary time.[5] These bubbles are in all respects the same as the zero temperature $O(4)$-invariant bubbles except ΔV is replaced by $\Delta V_T = \Delta p$. This part of the expression survives in the low temperature limit and smoothly goes over to the zero-temperature result.

High temperatures ($R_TT \gg 1$): The bubble with $F \sim F_C$ dominates the integral, and $\mathcal{P}_\hbar \propto \hbar \exp(-F_C/k_BT)$. Thus, in the high-temperature limit, the quantum contribution is as large as the classical thermal activation. This result can be easily understood. Namely, at high temperatures both thermal activation and quantum tunnelling contributions are saturated by the sphaleron solution.

Intermediate case ($R_T T \sim 1$): All subcritical bubbles with $0 < F < F_C$ are nearly equally important, see Eq. (3.19), and must be accounted for if $B(0) \lesssim 10^2$. This is a consequence of the fact that in the thin-wall approximation $f(F) \approx 1$. It is in this intermediate regime where our results may significantly change previous results.

Actually, $f(F) \neq 1$. In some models where the thin-wall approximation is invalid, $f(F)$ can differ significantly from $f(F) \approx 1$. If $d^2[F/k_B T + B(F)]/d^2 F < 0$ (which is in fact the case in the thin-wall approximation but with a small deviation[10] of $B(F)$ from the linear law, Eq. (2.15)), then either $O(4)$ invariant bubbles will dominate the decay probability, or the sphaleron does. The transition between those two regimes now is even steeper than in the thin-wall case when in the intermediate regime all subcritical bubbles were important. If $d^2[F/k_B T + B(F)]/d^2 F > 0$, then at some intermediate range of temperatures there will exist field configuration which will dominate the decay probability, which are neither $O(4)$-invariant, nor strictly time independent. Rather, as a function of temperature it will interpolate in between these two limits. This configuration would be a solution to the equation $1/k_B T + dB/dF = 0$. Since $dB/dF = -\tau$, where τ is the period of subbarrier motion between turning points, we see that the extremal configuration would obey the standard periodicity requirements on the fields in a thermal bath.

IV. DISCUSSION AND CONCLUSIONS

In this paper we have done several things: 1) We have shown how to calculate the decay probability of a metastable state that interpolates between the high-temperature limit and the zero-temperature limit of previous calculations. This result is given in Eq. (3.20). Note, that the result obtained indicates that in an intermediate range of temperatures there can be significant contributions to the prefactor due to subcritical bubbles, at least in the thin-wall approximation. A proper treatment must incorporate the prefactors from the very beginning, which was not done in the present paper. 2) We have shown that in agreement with Hsu, but contrary to naïve expectation, even at high temperatures the quantum contributions to tunnelling are just as important (at least in the exponent) as the classical thermal activation result. 3) We have pointed out the the correct formalism uses the free energy, not the energy, to calculate the quantum probability.

APPENDIX

As an example, let us consider a scalar field system, which at zero temperature has an effective potential with two nearly degenerate minima such that the thin-wall approximation is valid. With increasing temperature the pressure difference becomes even smaller, until at the critical temperature, T_C, both minima are degenerate. So the thin-wall approximation should be valid up to T_C if it is valid at $T = 0$. Some particles obtain mass from coupling to the scalar field acting as the order parameter of the phase transition. If the mass of the lightest of these particles is sufficiently smaller than T_C, then in both minima the only temperature dependent contribution is proportional to T^4 due to the pressure of the massless particles. Recall that the effective potential is the negative of the pressure, and if the number of massless degrees of freedom is different in the two phases, we can not neglect their contribution in calculating the decay rate. The pressure is given by $p = -V + \pi^2 N T^4/90$. It is equal in both phases at the critical temperature defined by the condition

$$-\Delta V + \pi^2 N_- T_C^4/90 = \pi^2 N_+ T_C^4/90, \tag{A.1}$$

where N_- and N_+ are numbers of effectively massless degrees of freedom in the symmetric and the broken phases respectively, and $N_- > N_+$. We have $\Delta V/T_C^4 = \pi^2 \Delta N/90$, and

$$\Delta p = \Delta V - \pi^2 \Delta N T^4/90 = \Delta V \left[1 - (T/T_C)^4\right]. \tag{A.2}$$

Consequently,

$$\begin{aligned} S_T &= B_T(0) = B(0)\left[1 - (T/T_C)^4\right]^{-3}, \\ \frac{F_C}{T} &= \frac{32}{27\pi T R_T} B_T(0) \equiv \frac{B(0)}{A}\frac{T_C}{T}\left[1 - \left(\frac{T}{T_C}\right)^4\right]^{-2}, \end{aligned} \tag{A.3}$$

where

$$A \equiv \frac{27\pi T_C R_0}{32} = \frac{27}{32}\left(\frac{540}{N_- - N_+}\right)^{1/4} B(0)^{1/4}. \tag{A.4}$$

The natural value for A would be $A \approx 5$ to 10, which corresponds to $B(0) \sim 10^2$ and $N_- - N_+ \sim 10$ to 10^2 . The temperature dependence of $\mathcal{P}_\hbar(T)/\mathcal{P}_\hbar(0)$, plotted with the use of Eq. (3.20) is shown in Fig. (3). Notice the unusual temperature dependence of the result.

Acknowledgements

It is a pleasure to thank M. Gleiser, V. A. Kuzmin, P. G. Tinyakov and S. D. H. Hsu for useful discussions. We are grateful to all participants of the workshop on Cosmological Phase Transitions for numerous discussions while this work was in progress and ITP UCSB for kind hospitality. This work was supported at the ITP by the National Science Foundation under Grant No. PHY89-04035, and at Fermilab in part by the Department of Energy and NASA (grant #NAGW-2831).

References

1. D. A. Kirzhnits and A. D. Linde, *Phys. Lett.* **42B**, 471 (1972).

2. For review see: A. D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic Publishers, 1990); E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, 1990).

3. S. Coleman, *Phys. Rev. D* **15**, 2922 (1977); C. Callan and S. Coleman, *Phys. Rev. D* **16**, 1762 (1977).

4. M. B. Voloshin, I. Yu. Kobzarev, and L. B. Okun', *Sov. J. Nucl. Phys.* **20**, 644 (1975).

5. A. D. Linde, *Nucl. Phys.* **B216**, 421 (1983).

6. I. Affleck, *Phys. Rev. Lett.* **46**, 388 (1981); J. Langer, *Ann. Phys.* **41**, 108 (1967).

7. S. D. H. Hsu, *On tunnelling at finite energies and temperatures*, Harvard University preprint HUTP–92–A009 (unpublished).

8. S. Khlebnikov, V. A. Rubakov, and P. Tinyakov, DESY preprint 91–033 (unpublished).

9. V. A. Berezin, N. G. Kozimirov, V. A. Kuzmin and I. I. Tkachev, *Phys. Lett.* **B212**, 415 (1988).

10. With the accuracy $O(10^{-5})$ the correction to $I(\omega)$, Eq. (2.15), may be approximated by $\Delta I(\omega) = \pi\omega\left(1 - \sqrt{\omega/\omega_C}\right)/10$. This is at most a 6% correction. However it must be included if $B \gtrsim 10^2$.

11. V. A. Berezin, V. A. Kuzmin and I. I. Tkachev, *Phys. Rev. D* **36**, 2919 (1987).

12. S. Coleman and E. Weinberg, *Phys. Rev. D* **8**, 1888 (1973).

13. V. A. Kuzmin, M. E. Shaposhnikov, and I. I. Tkachev, *Sov. Sci. Rev. A Phys. Rev.* **8**, 71 (1987).

14. V. A. Kuzmin, M. E. Shaposhnikov, and I. I. Tkachev, *Phys. Rev. D* **45**, 466 (1992).

FIGURE CAPTIONS

Fig. 1: The potential $\mathcal{V}(R)/4\pi S_1 R_0^2$ as a function of $x = R/R_0$.

Fig. 2: The integral $I(\omega)$ of Eq. (2.14). The dashed line is the analytic approximation to the integral, Eq. (2.15).

Fig. 3: The temperature dependence of $\mathcal{P}_\hbar(T)/\mathcal{P}_\hbar(0)$ plotted with the use of Eq. (3.20) in the case when both minima of the potential are almost degenerate at zero temperature.

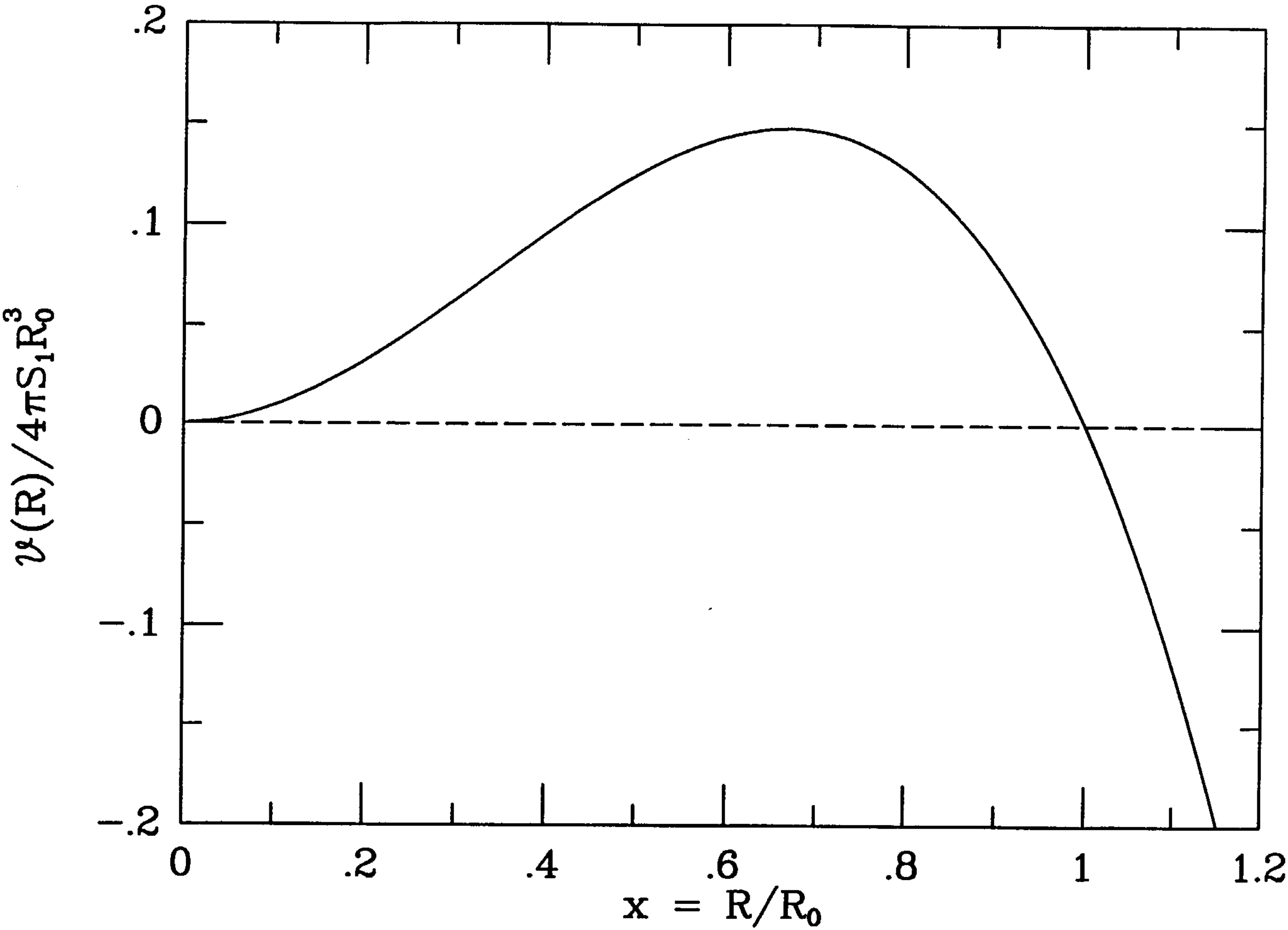


Figure 1

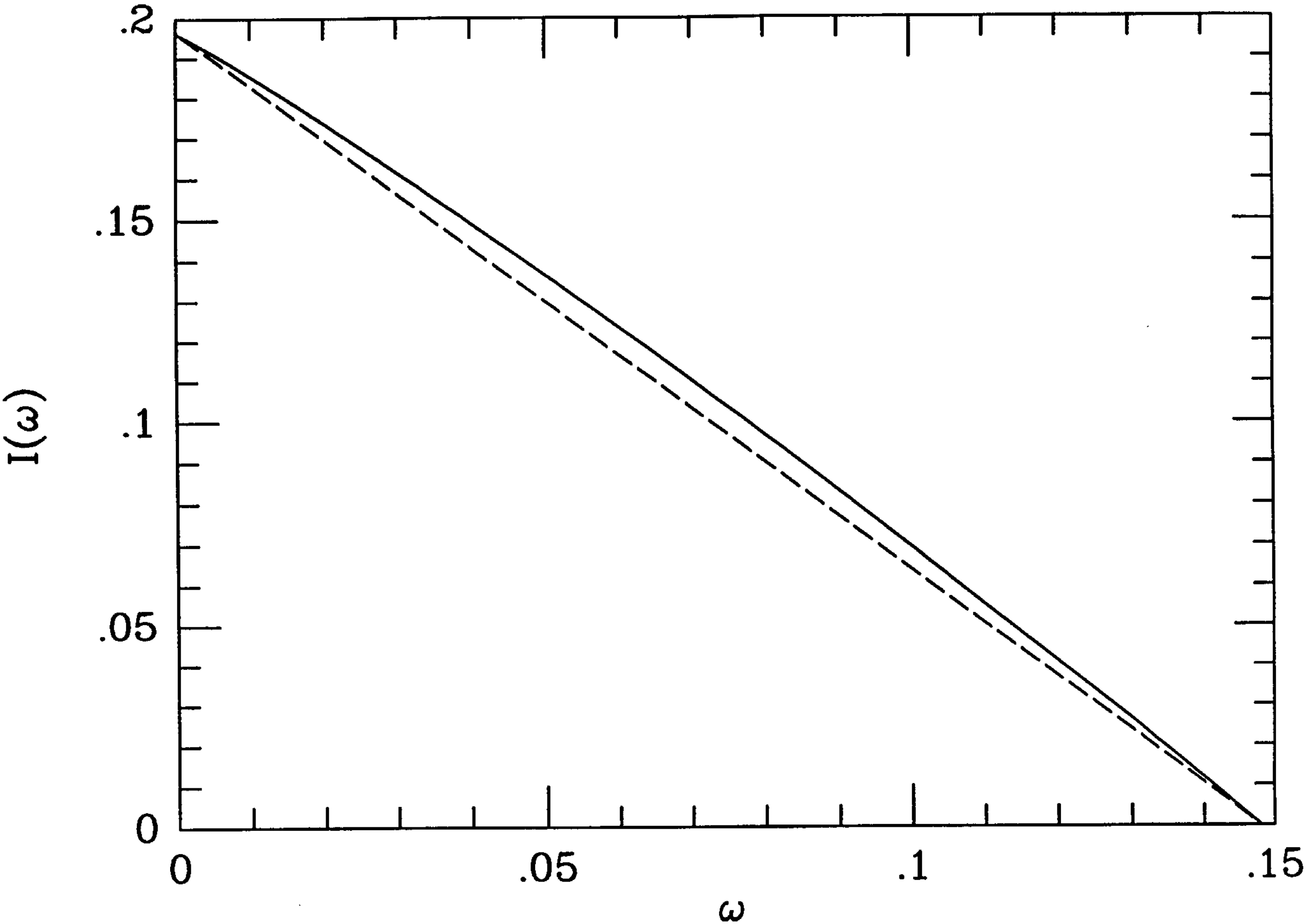
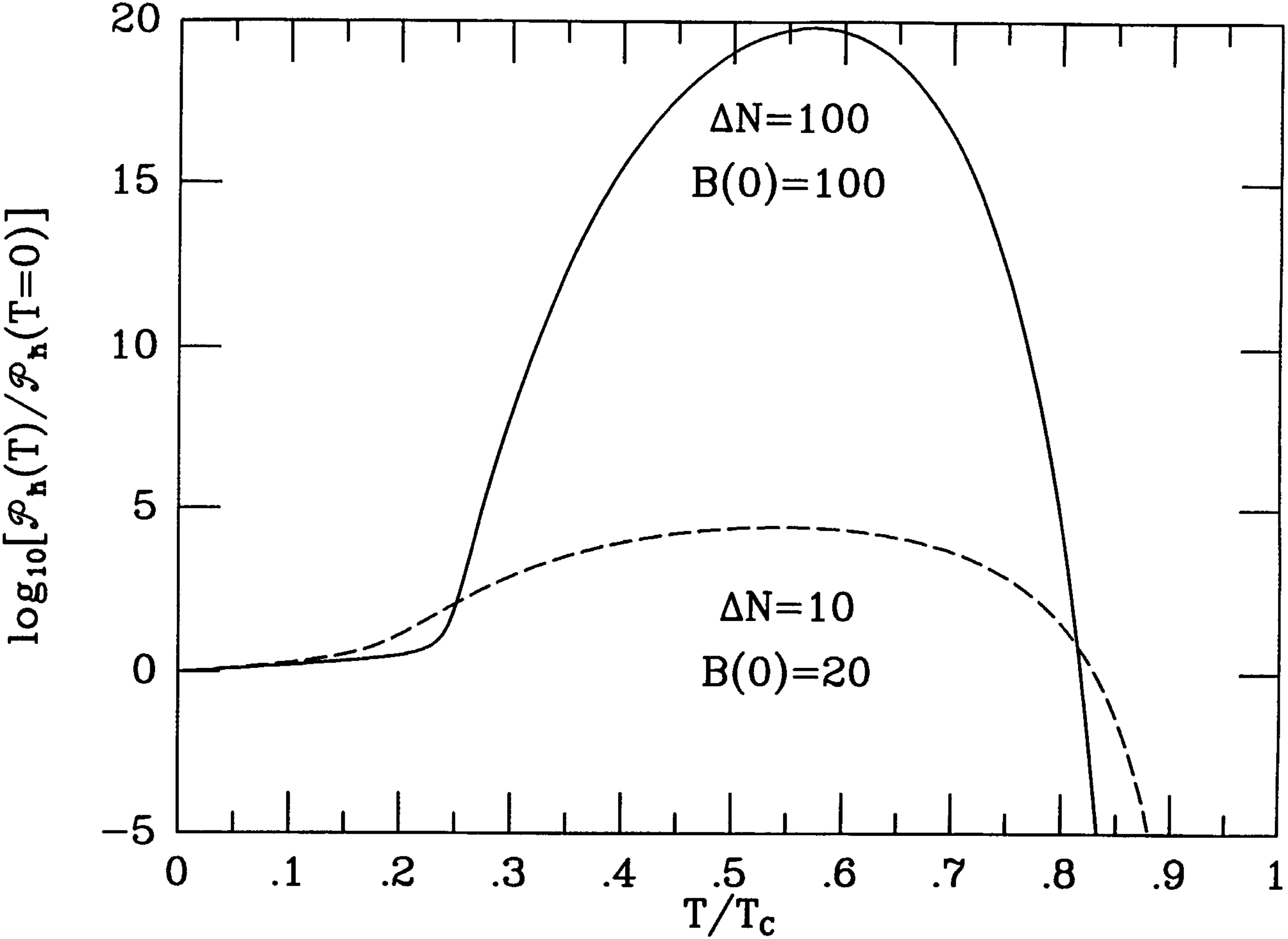


Figure 2



20
15
10
5
0
−5
log₁₀[𝒫ₕ(T)/𝒫ₕ(T=0)]
ΔN=100
B(0)=100
ΔN=10
B(0)=20
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1
T/Tc

Figure 3