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Abstract

A closed form for the parameterization of the error matrix that arises due to Multiple Coulomb Scattering is described. The errors depend on only one angle, rather than the two quasi-independent projected angles which are commonly used.

Introduction

Multiple Coulomb Scattering (MS) introduces small deviations into the track parameters compared with those of an unscattered track (i.e a particle traversing the vacuum). The effect is usually described by an angle, Θ^{MS} [1] and a corresponding lateral shift in the position, ϵ [2]. It is usually assumed that the error on the physical process of measurement (the resolution) and the MS errors are independent. Also note that the MS process can be decoupled from energy losses and thus, does not affect the momentum.

MS is a stochastic process, namely, the probability for a scattering event (denote by the state $X(t)$ in the phase space) to take place at time $t_\circ$ (site k_i) depends only on the physical condition in the immediate past at time $t < t_\circ$ (site k_{i-1}). The stochastic nature of MS is described as a convolution of local probability density functions satisfying the Chapman - Kolmogorov identity,

$$w_2(X,t|Y,s) = \int w_2(X,t|\xi,u)w_2(\xi,u|Y,s)d\xi,$$

where $w_2(X,t|Y,s)dX$ is the probability that the event $X < X(t) \leq X + dX$ occurs at time t, given that $X(s) = Y$ for $t > s$. The subscript "2" emphasizes the fact that only the state in the immediate past matters. Following the spirit of the Chapman - Kolmogorov identity, it is most suitable to treat MS errors in a local way, convoluting their probability density functions along the particles trajectory.

This article deals with the estimation of the errors on track parameters due to MS, in the milieu of track reconstruction for high energy physics (HEP) detectors. The article is organized in two sections. In the first section we describe the concept of the local tracking method [5] which is lately used in some of the largest HEP experiments, such as DELPHI and ZEUS. In the second section we describe a parameterization of the MS error and outline construction of the error matrix.

1 The concept of local tracking

A track is locally defined (at a fixed plane of measurement), by five parameters; two coordinates (i.e. two measurements at that fixed plane), two direction cosines (or angles), and the radius of curvature (when there is a magnetic field), which is proportional to the momentum. In Cartesian coordinates, one has a five dimensional vector, $\bar{V} = (x, \hat{x}, y, \hat{y}, \frac{1}{p})$. Note however, that the parameterization of the track and the errors of its parameters in one system can always be transformed to another system appropriate to the detector geometry. It is thus sufficient to evaluate the errors for one set of parameters, for example in the Cartesian parameterization.

The track model, $F(\bar{V})$, is a function of the track parameters, and describes the trajectory of the particle in the detector. In the case of nonlinearities (the presence of a magnetic field), the analytical function $F(\bar{V})$ can be linearized at a given point in space $a_\circ$. The linearized track model, $f(\bar{V})$, is expressed as [3]

$$f(\bar{V}) = F(\bar{V}|_{a_\circ}) + \frac{\partial F(\bar{V})}{\partial V_i}|_{a_\circ}(\bar{V}|_a - \bar{V}|_{a_\circ}).$$

Propagation of the track from a measured location a to the next measured location b can be described by a matrix derived from the track model, $\Phi_{ij}^{ab} = \frac{\partial V_i|_a}{\partial V_j|_b}$. In the case of nonlinearities, it may be approximated by the linearized track model, $f(\bar{V})$.

In a similar fashion, errors that occur locally in location a are propagated to location b using the same propagation matrix, Φ_{ij}^{ab} [3]. The propagation of errors is done using the following relation:

$$\Sigma_P^b = \Phi_{ij}^{ab} \Sigma^a \Phi_{ji}^{ab} \tag{1}$$

where Φ_{ji}^{ab} is the transpose of Φ_{ij}^{ab}. The resulting matrix, Σ_P^b, contains the error variance and covariance, estimated at location a, as they are propagated to location b.

In order to estimate (locally) the MS effect on the track parameters, it is necessary to evaluate the error variance and covariance in a given plane of measurement (location b) due to the traversing of a scattering material with a given thickness, L, and radiation length $L_\circ$ that is located between the b and a planes of measurement. The propagation of these variance and covariance can be done using equation 1. For a complete estimation of the error matrix, the MS error variance and covariance are added to the measurement error matrix and the resulting matrix is propagated across the measurement planes.

This concept is best realized in the Kalman filter approach to track reconstruction in HEP detectors [6]. In the Kalman filter framework, one estimates the track parameters and their errors locally, adds to them the MS error matrix and then propagates both the track parameters and the resulting error matrix to the next plane of the detector. The track parameters are then updated by a fit procedure resulting in a new set of parameters for that plane. In this way, one optimally follows the particle trajectory in the detector and the errors associated with it.

2 Parameterization of the MS error

Next we derive a parameterization of the errors of the track parameters due to MS, in terms of the scattering angle Θ^{MS}, and outline the calculation of the error variance and covariance of the track parameters.

Let us break the trajectory of the particle traversing the material in the detector into a series of quasi straight lines, (each with an infinite radius of curvature), such that the trajectory that associates the two locations a and b can be described to first order by the direction cosines at a, $\hat{x}_i{}^a$, where i runs from 1 to 3.

The effect of MS is to scatter the track such that instead of reaching location b, the particle is most probably found in a cone with an opening angle Θ^{MS} around the original line, $\underline{ab}$. To first order, the errors of the direction cosines are $\delta\hat{x}_i{}^a$, such that the scattered line is now defined by the new (scattered) direction cosines:

$$\hat{x}_i{}^{a'} = \hat{x}_i{}^a + \delta\hat{x}_i{}^a \tag{2}$$

The intersection of the two lines in a plane defines an angle θ_s with a variance which is equal to the projected MS angle [2]:

$$\theta_s = \frac{\Theta^{MS}}{\sqrt{3}} \tag{3}$$

The cosine of the angle of intersection θ_s is given by:

$$cos(\theta_s) = \sum_i \hat{x}_i{}^a \cdot \hat{x}_i{}^{a'} = \sum_i \hat{x}_i{}^a \cdot (\hat{x}_i{}^a + \delta\hat{x}_i{}^a) = 1 + \sum_i \hat{x}_i{}^a \cdot \delta\hat{x}_i{}^a \tag{4}$$

For θ_s small enough, the cosine can be expressed as

$$cos(\theta_s) = 1 - \frac{\theta_s^2}{2} \tag{5}$$

Parameterizing the new direction cosines as a Taylor expansion of the original ones we have

$$\hat{x}_i{}^{a'} = \hat{x}_i{}^a - \delta\theta_i\sqrt{1 - \hat{x}_i{}^{a^2}} \tag{6}$$

where the parameters $\delta\theta_i$ are small angular deviations of the direction angles. In an isotropic material it is legitimate to assume that the $\delta\theta_i$ are equal on average i.e. $\delta\theta_i = \delta\theta_j = \delta\theta_k = \delta\theta$. The new direction cosines are required to satisfy orthonormality.

$$\sum_i \hat{x}_i{}^{a'} \cdot \hat{x}_i{}^{a'} = \sum_i [\hat{x}_i{}^a \cdot \hat{x}_i{}^a - 2\delta\theta\hat{x}_i\sqrt{1 - \hat{x}_i{}^{a^2}} + \delta\theta^2(1 - \hat{x}_i{}^{a^2})] = 1 \tag{7}$$

Solving equation 7 for $\delta\theta$ we have:

$$\delta\theta = \sum_i \hat{x}_i\sqrt{1 - \hat{x}_i{}^{a^2}} \tag{8}$$

Substitution of equation 8 in equation 4 with $\delta\hat{x}_i{}^a = -\delta\theta\sqrt{1 - \hat{x}_i{}^{a^2}}$, yields a parameterization of $\delta\theta$ in terms of the projected scattering angle θ_s:

$$\delta\theta = \frac{\theta_s}{\sqrt{2}} \tag{9}$$

Hence, we identify the errors on the direction cosines:

$$\delta\hat{x}_i{}^a = -\frac{\theta_s}{\sqrt{2}}\sqrt{1-\hat{x}_i{}^{a^2}} \tag{10}$$

as a function of the MS angle. Using this expression one can calculate the error variance:

$$\sigma_{\hat{x}_i} = (\frac{\partial\delta\hat{x}_i}{\partial\delta\theta}\delta\theta)^2 = \frac{\theta_s^2}{2}(1-\hat{x}_i{}^2) \tag{11}$$

and the covariance of the direction cosines:

$$\sigma_{\hat{x}_i}\sigma_{\hat{x}_j} = \frac{\partial\delta\hat{x}_i}{\partial\delta\theta}\delta\theta\frac{\partial\delta\hat{x}_j}{\partial\delta\theta}\delta\theta = \frac{\theta_s^2}{2}\sqrt{(1-\hat{x}_i{}^2)(1-\hat{x}_j{}^2)} \tag{12}$$

The errors of the parameters x_i are expressed by

$$\delta x_i = \frac{\theta_s^2}{2}\delta\hat{x}_i x_i = -\frac{\theta_s}{\sqrt{2}}x_i\sqrt{1-\hat{x}_i{}^2} \tag{13}$$

Using equations 10 and 13, one can calculate the remaining error variance:

$$\sigma_{x_i} = (\frac{\partial\delta x_i}{\partial\delta\theta}\delta\theta)^2 = \frac{\theta_s^2}{2}(1-\hat{x}_i^2)x_i^2 \tag{14}$$

and covariance:

$$\sigma_{x_i}\sigma_{x_j} = \frac{\partial\delta x_i}{\partial\delta\theta}\delta\theta\frac{\partial\delta x_j}{\partial\delta\theta}\delta\theta = \frac{\theta_s^2}{2}x_i x_j\sqrt{(1-\hat{x}_i^2)(1-\hat{x}_j^2)} \tag{15}$$

$$\sigma_{x_i}\sigma_{\hat{x}_j} = \frac{\partial\delta x_i}{\partial\delta\theta}\delta\theta\frac{\partial\delta\hat{x}_j}{\partial\delta\theta}\delta\theta = -\frac{\theta_s^2}{2}x_i\sqrt{(1-\hat{x}_i^2)(1-\hat{x}_j^2)} \tag{16}$$

Let us emphasize again that the full covariance matrix, V_{ij}^{ms}, can be transformed to another set of parameters rather than the Cartesian coordinates and the direction cosines. Using the "propagation error formula" [4]

$$V_{nm}(\bar{f}) \approx \sum_{i,j}\frac{\partial f_n}{\partial x_i}\frac{\partial f_m}{\partial x_j}V_{ij}(\bar{x}) \tag{17}$$

one can express the errors on any other parameterization, $\bar{f}$, of the particle trajectory.

The propagation of the local error matrix, V_{ij}^{ms}, is straight forward for the linear case. A linear track model at a given plane $z = z_k$ can be propagated to the following plane, $z = z_{k+1}$, with the following transfer matrix:

$$\Phi(\triangle z, \hat{x}, \hat{y}) = \begin{pmatrix} 1 & \frac{\triangle z}{\sqrt{1-\hat{x}^2-\hat{y}^2}} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{\triangle z}{\sqrt{1-\hat{x}^2-\hat{y}^2}} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{18}$$

where $\triangle z = z_{k+1} - z_k$ is the distance between the $k, k+1$ planes. Assuming that the direction cosines do not change drastically along the particle trajectory from z_1 to z_n, equation 1 is applied N times along the path, $L = N \triangle z = z_n - z_1$, such that

$$\Sigma_1^n = \Phi_{ij}^N V_{ij}^{ms} \Phi_{ji}^N \tag{19}$$

Note that due to the block diagonal form of the Φ matrix the off diagonal terms are given by:

$$\Phi_{1;12,34}^n = \frac{N \triangle z}{\sqrt{1 - \hat{x}^2 - \hat{y}^2}} = \frac{z_n - z_1}{\sqrt{1 - \hat{x}^2 - \hat{y}^2}} \tag{20}$$

Equations 19 and 20 confirm the intuitive expectation of a linear dependence of the errors on the path length , L. However, if a magnetic field is present one can still use equation 19 for short enough paths i.e. cords that approximate the arc, for which the average change in the direction cosines is tolerable (small compared to the MS errors). One then propagates the error matrices of a given cord at the break point to the next cord, using the direction cosines of this cord etc.

To summarize, we have shown how the MS error may be parameterized using only one variable, the scattering angle Θ^{MS}, which is evaluated in the theory of MS. Using this parameterization, a full error matrix can be constructed locally and propagated across the detector. This approach is a natural consequence of the stochastic character of MS.

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