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MEASUREMENTS OF A HAMILTONIAN SYSTEM AND THEIR DESCRIPTION BY A DIFFUSIVE MODEL

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ABSTRACT

The dynamics of transverse particle oscillations in the Fermilab Tevatron, artificially made nonlinear, is studied experimentally. With a beam that preferentially populated regions of phase space in which nonlinearity was important, observations of the beam profile were made over periods up to an hour. These profiles are analysed in terms of a purely "diffusive" model having an empirically-fit, amplitude-dependent, diffusion coefficient. This gives both a qualitative physical picture and a quantitatively accurate, analytic, description of beam evolution. The evolution is influenced strongly by nonlinear resonance.

This letter describes a beam dynamics experiment, performed in the Fermilab Tevatron, intended to study the effect of nonlinearity on transverse particle distributions. As well as being the study of an almost ideally Hamiltonian system, this investigation was motivated by planning for future accelerators, such as the SSC (Superconducting Super-Collider), for which nonlinearity in the transverse motion will be of special importance. The effects of such nonlinearity can be studied by adding artificial nonlinearity to the most similar existing accelerator, the Tevatron. Emphasis here is on "stochastic" effects, due to the particle dynamics, that cause "diffusive" evolution of the beam distribution, even in the absence of external sources of "noise" or random scattering from residual gas molecules. The evolution of the transverse particle distribution, or profile, is observed and compared with the evolution predicted by a type of diffusion equation. This description is phenomenological in that the parameters of an amplitude-dependent, diffusion coefficient are empirically determined by fitting to the data. Still there is enough redundancy to give qualitative physical insight into the nature of the process, as well as yielding a compact, quantitative, representation of the observations.

For this experiment the Tevatron can be regarded as a linear system on which known nonlinearity in the form of sixteen sextupole magnets has been artificially superimposed. Two configurations, labeled A and B, were investigated. For them, individual sextupoles had the same strengths but the distributions of their polarities were selected to lead to strikingly different phase space structure. At the start of each observation period all particles in a needle-shaped beam consisting of some 10^{10} circulating stored protons, were kicked horizontally to yield displacement x_k (typically 3 mm) as observed at a downstream observation

point. There the resulting transverse beam profile was measured repeatedly every minute or so, using a "flying wire"[1] apparatus. During a run of (typical) duration 30 minutes, each proton circulates some 10^8 times and executes some 2×10^9 transverse "betatron" oscillations. The purpose of the kick is to generate a beam in which all the particles are in a region of measurably large diffusion. (Simple beam "stochastic heating" would not accomplish this as it would yield a "solid beam" with only the sparsely populated tails subject to strong diffusion.)

In the absence of nonlinear elements, it is known that an individual particle of arbitrary amplitude executes simple harmonic oscillation at constant amplitude with damping time equal to many tens of hours. The assumptions of Hamiltonian mechanics are probably as valid for this system as for any laboratory-observable phenomenon. One consequence of this is the certain validity of Liouville's theorem, according to which microscopic particle densities in phase space are conserved. The phase space referred to here describes motion in one transverse dimension (horizontal), with position and momentum coordinates being (x, p). In this paper the momentum is scaled to have the same units as x (millimeters), so that phase space trajectories are circles of radius $r \equiv \sqrt{J} = \sqrt{x^2 + p^2}$. Because of Liouville's theorem, an initial beam distribution depending only on r would remain invariant. In this experiment deviations from this behavior due to nonlinearity are observed by measuring the evolution of the beam distribution.

With nonlinear elements present in the accelerator lattice, phase space trajectories become progressively more distorted with increasing amplitude. Liouville's theorem continues to apply to the microscopic particle density, but void regions can be "mixed in" causing the distribution to develop a "foam" of reduced macroscopic density. In an earlier experiment using the same apparatus[2],

various of these phase space features were measured quantitatively in a nonlinear configuration similar to the present one. For two "tune" selections at each of the two configurations A and B, the predicted phase space trajectories are shown in Fig. 1. These are "Poincaré plots" taken at a fixed location ("surface of section") in the lattice, with a point plotted at the values (x, p) on each successive turn. The tune is the fraction of 360 degrees that successive points advance on each turn in such a plot. The plots in Fig. 1 were generated by a symplectic numerical tracking program TEAPOT[3] using the known linear lattice and sextupole configuration in the Tevatron. As well as trajectory distortion, there is the possibility of "low-order" resonance islands caused by "frequency entrainment" (cases $A_>$ and $B_>$), and, at very large amplitude, particle loss due to the vacuum chamber wall or other material object. To make the physical aperture definite, a scraper was placed at $x_\infty = 9\,\mathrm{mm}$. This value was chosen small enough to be certain to catch essentially all "doomed" particles, but large enough to make the observations insensitive to the actual value of x_∞. Both of these properties were verified experimentally. Though the precise location of this aperture is not critical, the loss of particles it causes is crucial to the evolution and survival of the beam.

All measurements were performed in the vicinity of the " $\frac{2}{5}$-resonance" (horizontal, fractional, tune ν_x close to 0.4). This means that the motion approximately repeats every fifth turn. Configurations were intentionally chosen in pairs, such as $A_>$, just above resonance with $\nu_x = 0.408$, and $A_<$, just below resonance with $\nu_x = 0.392$; everything else held constant. For both sextupole distributions under study the tune decreases with increasing amplitude. This causes entrainment to occur, with the tune locking onto exactly $\frac{2}{5}$, only in the $>$ cases. This is

confirmed by Fig. 1. With the tune so close to 0.4, resonances at other rational tune values are expected to have minor importance.

The particle distribution in phase space can be approximately described by a symmetric probability distribution $\rho(r;t)$ depending on r, not separately on x and p. For a brief interval after the beam is first kicked this is not true, but within less than a thousand turns (negligibly short on the time scale of our observations) "filamentation" leads to a distribution that uniformly fills a ring of radius x_k in phase space. This is due to shear tangential motion in phase space resulting from the dependence of tune on amplitude. We call a beam formed this way a "hollow beam". The same rapid shear motion in phase space continues to preserve azimuthal symmetry even if the stochastic forces are asymmetric. Consistent with this, to develop a model for the evolution of $\rho(r;t)$ we could conjecture the existence of a diffusion current equal to $-D_r(r)\nabla\rho(r;t)$ where $D_r(r)$ is an amplitude-dependent diffusion coefficient to be determined. Applying the continuity equation to this description yields the heat equation, the most basic diffusion equation. We have analysed our data on the basis of that equation and have obtained a self-consistent phenomenological description, very comparable with the following, better justified by the physics, alternative.

It was shown by Landau[4] that the influence of stochastic forces on a distribution of Hamiltonian oscillators can be described by a one dimensional diffusion equation

$$\frac{\partial \rho_J(J;t)}{\partial t} = \frac{\partial}{\partial J}\Big(D_J(J)\frac{\partial \rho_J(J;t)}{\partial J}\Big), \tag{1}$$

similar to, but not identical to, the radial heat equation in a cylindrically symmetric situation. The relation among distribution functions can be inferred from

the normalization condition

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\ \rho(x,p;t) = 2\pi \int_{0}^{\infty} \rho(r;t) r dr = \int_{0}^{\infty} \rho_J(J;t) dJ = N(t) \qquad (2)$$

where $N(t)$, the total number of particles surviving at time t, is called the beam intensity, a directly measurable quantity. Before the beam is kicked its distribution is given by

$$\rho(r;0_-) = \rho(x,p;0_-) = \frac{N(0)}{2\pi\sigma_0^2} e^{-\frac{x^2+p^2}{2\sigma_0^2}}, \qquad (3)$$

where the r.m.s. width σ_0 was measured at the start of each run. Our apparatus measures the projection of this distribution onto the x-axis; this can be matched directly to distribution (3) which is in factorized form. Later comparisons require numerical projection. Just after the kick the beam distribution is given by[5]

$$\rho(r;0_+) = \frac{N(0)}{2\pi\sigma_0^2} e^{-\frac{r^2+x_k^2}{2\sigma_0^2}} I_0\left(\frac{r x_k}{\sigma_0^2}\right), \qquad (4)$$

where I_0 is the modified Bessel Function of order zero. The condition for the beam to be truly hollow is $\sigma_0 << x_k$. In actual cases $\sigma_0 = 1.1 \pm 0.1$ mm and x_k was never less than 2.4 mm.

Our model of the diffusive evolution is based on a 4-parameter expression for the diffusion coefficient:

$$D_J(J) = \begin{cases} D_f, & \text{if } r < r_0, \\ D_f + d_0(J - J_0)^K, & \text{if } r_0 < r\ . \end{cases} \qquad (5)$$

The parameter $r_0 = \sqrt{J_0}$ can be regarded as a phase space radius inside which there is no dynamics-induced diffusion (a kind of experimentally determined radius of the largest KAM[6] surface). Since the particle density is tailored initially

to be large only in a reasonably slender radial ring the diffusion coefficient is best determined near there. At larger amplitudes, other than being required by the data to be large, the diffusion coefficient is poorly determined. In the particular parameterization chosen, the rate at which diffusion increases as r exceeds r_0 is controlled by the coefficient d_0 and the exponent K. The parameter D_f represents uniform diffusion, such as could be caused by external sources of noise, but it was allowed to vary from configuration to configuration. Model profiles were obtained by solving Eq. (1) with diffusion coefficient given by Eq. (5), with boundary conditions that forbade a sink at $J = 0$ and forced the vanishing of $\rho(r)$ at $r = x_\infty$. Resulting profiles are shown in Fig. 2. The smooth curves are profiles predicted at the times indicated. The jagged curve is one raw profile measured by the flying wire system. It is typical of the hundreds of profiles measured. To suppress the jagged nature of these profiles they were somewhat smoothed before comparison with the model; this accounts for the jagged curve being somewhat narrower than the smooth curve predicted at the same time.

In performing fits to determine the parameters in D_J, the only features of the measured distributions used were the total area, proportional to $N(t)$, and the full width at half maximum. For each of the configurations studied, as well as for the bare Tevatron (sextupole strengths set to zero), data sets like those in Fig. 3 were taken. Pairs of small-kick and large-kick runs were taken differing only in the value of x_k, which was adjusted by controlling a kicker voltage V_k [x_k(mm) $\simeq$ 0.47 V_k(kV)]. Values are indicated on Fig. 3. The idea was to preferentially populate a low diffusion region in one run and a high diffusion region in the other. A more narrowly spaced data collection grid was neither permitted by time, nor justified by the inherent slenderness of the pre-kicked

beam. "Best-fit" parameters were determined that minimized the sum of the squares of deviations on these four plots. They are listed in Table 1, and the resultant diffusion coefficients are plotted in Fig. 4. For comparison with Fig. 1, the final column gives the ratio r_0/Δ; $\Delta = (x_{max} - x_{min})/2$, where x_{min} and x_{max} are the extremes of regular motion along the x-axis.

TABLE 1. DIFFUSION COEFICIENT PARAMETERS

Parameter	Kicks	ν_x	d_0	r_0	K	D_f	r_0/Δ
Units	kV			mm		mm^4/min	
$A_<$	6, 9	0.392	185.0	4.88	1.192	4.7	0.42
$A_>$	6, 9	0.408	117.9	3.9	1.683	3.0	0.44
$B_<$	6, 9	0.392	10.0	3.4	1.76	0.87	0.51
$B_>$	6, 9	0.408	110.0	3.4	1.70	0.8	0.50
Bare Tevatron	12, 14	0.405	1.03	6.54	3.63	30.8	

A kind of consistency check made toward the end of most runs was to reduce x_{∞} suddenly. This was intended to confirm that the subsequent beam evolution would be well described by the same diffusion coefficient and the profile in existence at that time. Unfortunately, these fits are very sensitive to small uncertainties in the absolute scraper location. An offset between nominal (4.0 mm) and best-fit value (2.7 mm) was measured; this was judged to be consistent with absolute position uncertainties in the set-up. With this adjustment, the excellent fits to the 6 kV data of Fig. 3 result. However, holding all parameters constant, the 9 kV full width fit is not very good. The source of this discrepancy is not understood. It is also not understood why the best-fit value of D_f varies from

configuration to configuration. The very different value obtained for the bare Tevatron undoubtedly reflects the very different nonlinearity environment and beam evolution at very large amplitudes.

The discussion up to now has assumed that the motion is purely horizontal. In fact, individual particles are also undergoing appreciable vertical oscillation ($\nu_y = 0.45$). Before being kicked the beam is approximately round, and the kick does not alter the vertical motion. As a result the beam is ribbon shaped after the kick with vertical motion small compared to horizontal. It cannot, however, be assumed that horizontal diffusion is uninfluenced by this motion. Our diffusion coefficient has to be regarded as including its influence. The same comments can be made about longitudinal motion ($\nu_s = 0.001$). In addition, because the flying wire is located at a point in the ring, HE11, having non-zero dispersion, the momentum oscillations accompanying this motion have the further effect of broadening the observed profiles slightly. To study this effect we performed a complete analysis of the data, taking this into account approximately. This involved "deconvoluting" the before-kick distribution; to subtract the contribution to the width of the momentum spread, and then improving upon the analytic expression (4) by performing a numerical, azimuthal, phase space integration. This did not alter the results appreciably—the alteration was judged to be too small to justify the extra complexity. Summarizing, our diffusion coefficient is intended to account phenomenologically for the evolution of horizontal profiles in the presence of small vertical and longitudinal motion.

Certain tentative, qualitative, inferences about the nature of the diffusion process can be drawn from comparison of Fig. 1 and Fig. 4. It might be expected that diffusion, like multiple scattering, would cause beam broadening;

the opposite is observed. This behavior is caused by the sink at x_∞ which devours large amplitude particles, reducing the beam intensity, depleting the tail of the distribution, and causing net narrowing. The feature of the data most naturally demanding the diffusive description is the enormous range of time scales involved. At small amplitudes the time scale is so long that diffusion can be neglected; the time scale at large amplitudes is negligibly short. The full range is analytically encompasssed by the curves of Fig. 4.

After formation the beam rapidly (millions of turns) sheds its large amplitude particles and the periphery of the distribution approaches what appears to be a universal distribution. The distribution would then be stable indefinitely except that weak diffusion occasionally brings particles to the "edge" from where they can be swept away quickly.

The calculations of Fig. 1 can be compared with the experimental data, parameterized according to Eq. (5), and listed in Table 1. The near constancy of the ratio r_0/Δ shows that theory and observation are consistent. As a rule of thumb, based on this ratio, one can conjecture that diffusion causes particles outside roughly $(0.47 \pm 0.04)\Delta$ to be lost rapidly. The more rapid diffusion observed in configurations $A_>$ and $B_>$ suggests strongly that the diffusion rate is enhanced by proximity to resonance. Certainly it is routinely observed in accelerator operations that beam loss is aggravated by resonance.[7]

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REFERENCES

1. J. Gannon et. al. Proc. of 1989 IEEE Part. Acc. Conf., p68.
2. A. Chao, *et. al.*, *Phys. Rev. Lett.* **61**, 2752, 1988.
3. L. Schachinger and R. Talman, *Particle Accelerators*, **22**, 35 (1987).
4. L. D. Landau; ascribed to *Zh. Eksper. Theor. Fiz.* **7**, 203, 1937, by A.J. Lichtenberg and M.A. Lieberman, *Regular and Stochastic Motion*, Springer-Verlag, New York.
5. H. S. Carslaw and J. C. Jaeger, *Operational Methods in Applied Mathematics*, Oxford, 1953.
6. Kolmogorov, Arnold, and Moser.
7. L. R. Evans, *Experimental Study of Resonances*, in *Frontiers of Particle Beams*, ..., M. Month and S. Turner, Eds., Springer-Verlag, Berlin, 1989.

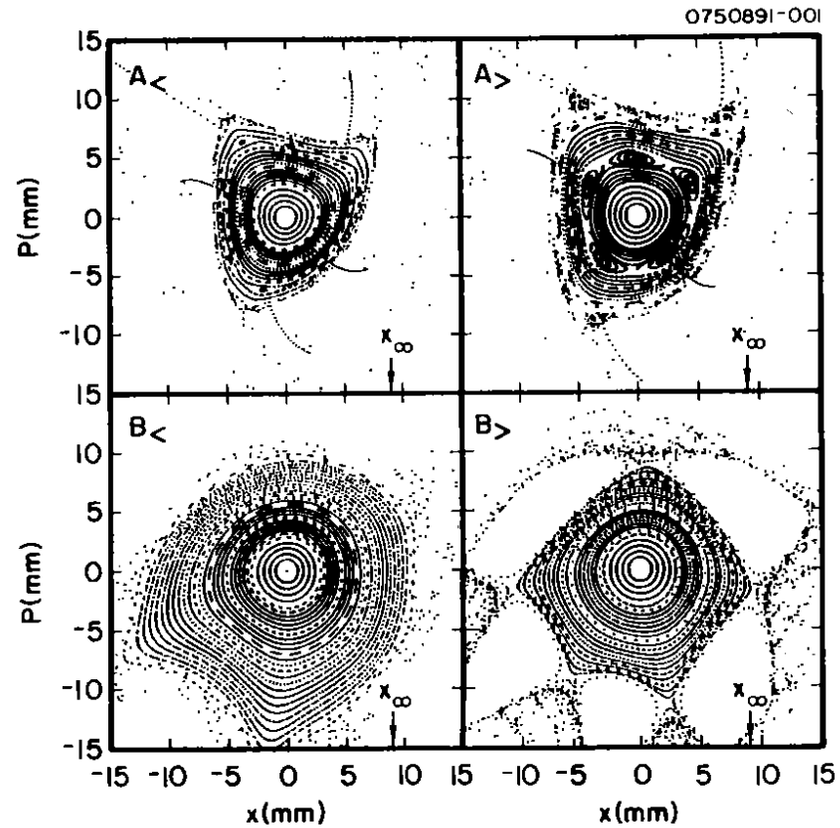
FIGURE CAPTIONS

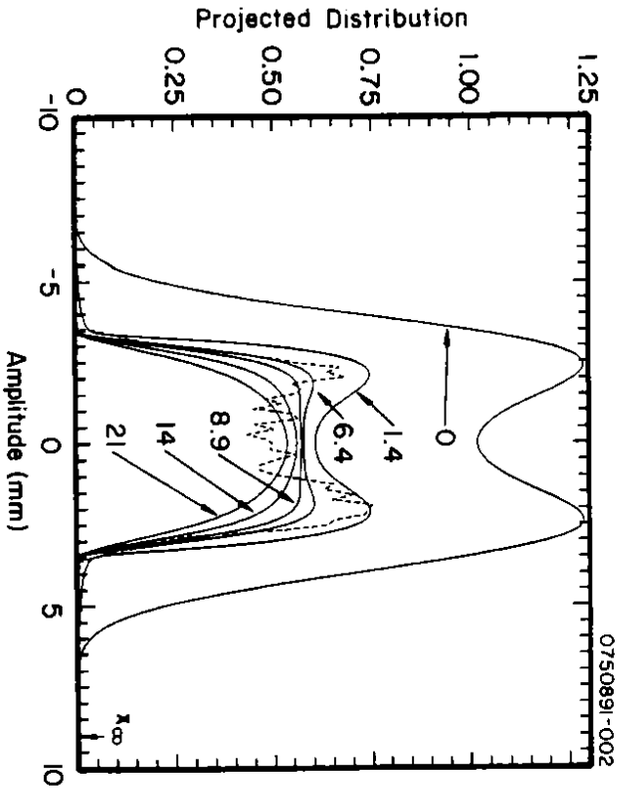
Figure 1. Predicted Poincaré plots for the four nonlinear configurations studied. Identical starting coordinates were chosen for all plots. A and B indicate different sextupole distributions; < and > refer to just below and just above the dominant resonance. Note the 5-island resonance chains in the > cases. Other parameters are listed in Table 1.

Figure 2. Evolution of the beam profile as predicted by the diffusion model: smooth curves. Times (in minutes after hollow beam formation) are indicated. The jagged curve is the raw flying wire measurement of the profile at t = 6 minutes.

Figure 3. Measured full widths and intensities (points) compared to model-derived values (smooth curve). Intensities are normalized to 1 at $t = 0$. The step discontinuities at late times are the result of sudden reductions of x_∞ to 2.7 mm.

Figure 4. Empirically determined, amplitude dependent, diffusion coefficients for four intentionally-nonlinear configurations and the "linear" bare Tevatron. The curves are best determined by the data where the diffusion constant is small. Note the suppressed zero of the horizontal scale.

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0.75
1.00
1.25
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-5
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6.4
8.9
14
21
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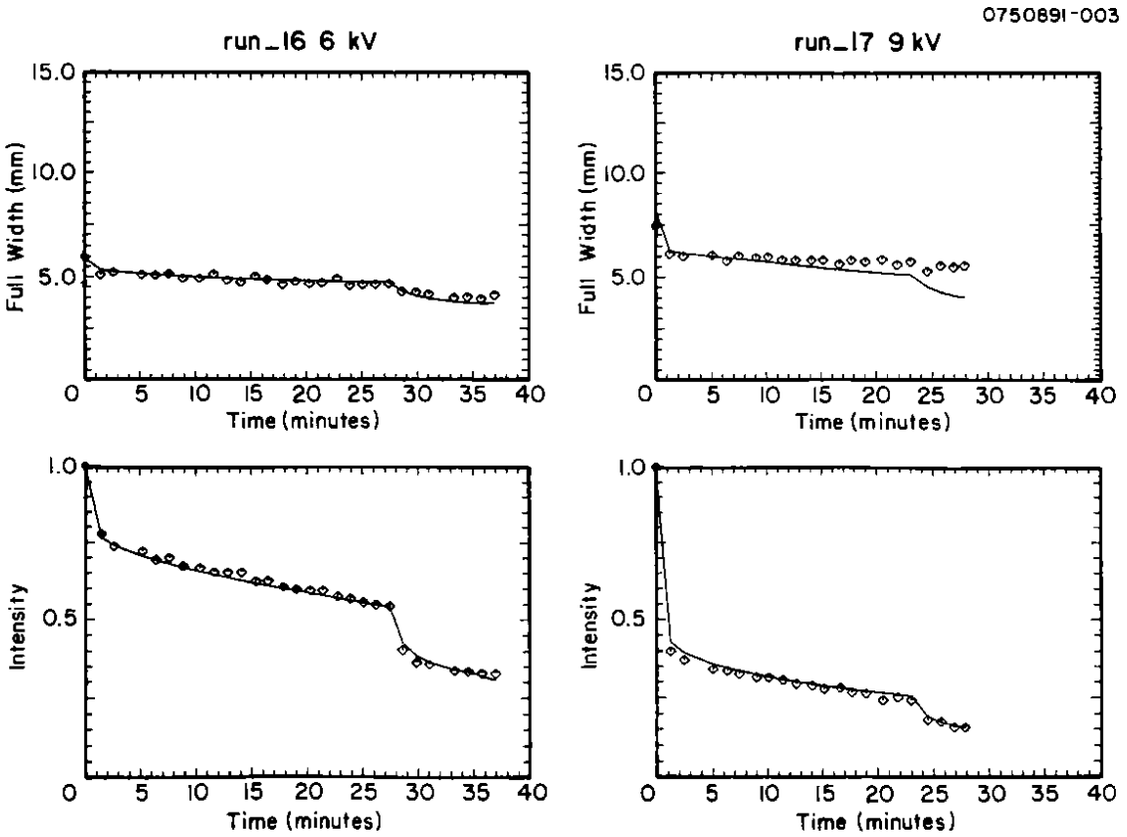
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run_17 9 kV
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Intensity
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0.5
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15
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25
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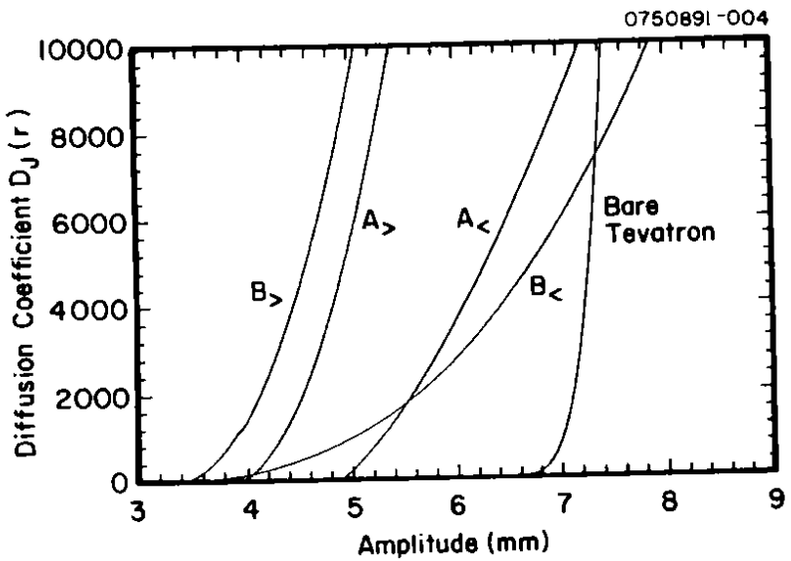
35
40
Time (minutes)


0750891-004
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8000
6000
4000
2000
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3
4
5
6
7
8
9
Diffusion Coefficient $D_J(r)$
Amplitude (mm)
$B_>$
$A_>$
$A_<$
$B_<$
Bare
Tevatron