



Fermi National Accelerator Laboratory

FERMILAB-PUB-91/347-T

EUCLIDEAN ASYMPTOTIC EXPANSIONS
OF GREEN FUNCTIONS OF QUANTUM FIELDS.
(I) EXPANSIONS OF PRODUCTS OF SINGULAR FUNCTIONS.

F. V. Tkachov¹

Fermi National Accelerator Laboratory
P.O.Box 500, Batavia, Illinois 60510 USA

ABSTRACT

The problem of asymptotic expansions of Green functions in perturbative QFT is studied for the class of Euclidean asymptotic regimes. Phenomenological applications are analyzed to obtain a meaningful mathematical formulation of the problem. It is shown that the problem reduces to studying asymptotic expansions of products of a class of singular functions in the sense of the distribution theory. Existence, uniqueness and explicit expressions for such expansions (*As-operation for products of singular functions*) in dimensionally regularized form are obtained using the so-called *extension principle*.

In memory of S.G. Gorishny (1958-1988).

¹on leave from Institute for Nuclear Research of the USSR Academy of Sciences, Moscow 117312, USSR; address after Dec. 14, 1991: Physics Dept., Penn State Univ., 104 Davey Lab., University Park, PA 16802, USA; E-mail address: tkachov@psuphys1.phys.psu.edu.



1 Introduction

... the perturbative version of the Wilson short distance expansion is completely clear since more than ten years.

... A new theory is not needed.

A referee for NUCLEAR PHYSICS B.

1.1 The problem of asymptotic expansions in perturbative QFT.

Approximations and asymptotic methods pervade applications of mathematics in natural sciences: nothing simplifies a problem more than reduction of the number of its independent parameters—and the idea of asymptotic expansion is one of the two most useful in this respect.

In applied quantum field theory, one deals with amplitudes—or, more generally, with Green functions of elementary or composite operators, which depend on momentum and mass parameters. Detailed investigation of dynamics of physical processes is rarely possible with more than just a few independent parameters. Therefore, one normally considers asymptotic regimes in which almost all the momenta and masses are much larger (or smaller) than the chosen few. Since the most informative dynamical framework is currently provided by the perturbation theory, Green functions are represented as infinite sums over a hierarchy of Feynman diagrams, and the initial problem falls into two parts.

First, one has to expand individual Feynman diagrams with respect to external parameters—masses and momenta. This is the analytical part of the problem. Empirical evidence indicated long ago that diagrams can be expanded in powers and logarithms of masses and momenta. For a wide class of the asymptotic regimes a formal proof of this fact was given by D. Slavnov [1]. The reasoning of [1] was based on a modification of the techniques used in the proof of the Weinberg theorem (which established the power-and-log nature of leading asymptotics in high-energy regimes; for a detailed discussion of the theorem see [2]). That result was extended to other asymptotic regimes using various techniques [3]–[6].

The main difficulty with asymptotic expansions of multiloop diagrams is that formal Taylor expansions of the integrand result in non-integrable singularities. This indicates that the integrals depend on the expansion parameter non-analytically. The papers [1]–[6] followed, with variations, the old idea of splitting integration space—whether in momentum or parametric representations—into regions in such a way as to allow one to extract the non-analytic (usually logarithmic) contribution from each region, eventually, by explicit integration. However, complexity of multiloop diagrams exacerbated by ultraviolet renormalization made obtaining convenient explicit expressions for coefficients of such expansions unfeasible within a framework of elementary integral calculus. Nevertheless, expansions that involve simplest functions of the expansion parameter (e.g. powers and logarithms) are exactly what is needed in the final respect in applications, and even such limited information on the analytical form of expansions

can be useful.¹

The second part of the perturbative expansion problem is combinatorial. The observation that asymptotic expansions of non-perturbatively defined objects should have a non-perturbative form is due to K.Wilson [8] who also discovered that such operator product expansions (OPE) can be highly useful in phenomenology. W.Zimmermann [9], using what became known as the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) techniques, was first to show that it is indeed possible to construct an OPE for a class of short-distance asymptotic regimes.²

The main achievements of [9] were, first, the demonstration of how the expansion in a “global” OPE form is combinatorially restored from terms corresponding to separate Feynman diagrams; second, the required smallness of the remainder of the expansion was proved in presence of UV renormalization. However, unlike the expansions obtained in [1], the coefficient functions of the OPE of [9] were not pure powers and logarithms of the expansion parameter (i.e. short distance or large momentum transfer) but also contained a non-trivial dependence on masses of the particles (apparently because masses were needed as regulators for infrared divergences). Therefore, although the results of [9] are firmly established theorems, they fell short of providing an adequate basis for applications (for a more detailed discussion see subsect. 2.2 below): first, the expressions for coefficient functions were unmanageable from purely calculational point of view³; second, infinite expansions could not be obtained in models with massless particles, e.g. QCD; third, the logarithms of masses contained in coefficient functions are—as became clear later [11]—non-perturbative within QCD in the sense that they cannot be reliably evaluated within perturbation theory using asymptotic freedom. All the above drawbacks have the same origin: a lack of the property that is now known as *perfect factorization* (we will discuss this point later on in subsect. 2.2).

Lastly, although the BPHZ techniques may be fine as an instrument of verification of the results discovered by other methods, its heuristic potential turned out to be inadequate: it proved to be of little help in finding *new* results. Indeed, both the formula of the Bogoliubov *R*-operation [24] and the OPE were discovered using heuristics that are foreign to the BPHZ method, while the attempts to obtain full-fledged OPE-like results for a wider class of asymptotic regimes (non-Euclidean, or Minkowskian regimes) have so far largely failed.

A fully satisfactory solution of the expansion problem—which it is the aim of the present and companion publications to describe—has been found only for the class of the so-called *Euclidean asymptotic regimes* (for precise definitions see sect. 3) in a series of publications [12]-

¹cf. an early attempt to develop calculational algorithms for coefficient functions of operator product expansions [7].

²It was also realized that the short-distance OPE is closely related to other expansion problems e.g. the problem of decoupling of heavy particles and low-energy effective Lagrangians—see a discussion and references below in subsect. 2.3; in fact, short-distance OPE and related problems constitute a subclass of Euclidean regimes studied in the present paper.

³This explains why the first large-scale calculations of OPE beyond tree level [10] were performed by “brute force”: the coefficient functions of an OPE were found by straightforward calculation of asymptotics of the relevant non-expanded amplitudes and then by explicit verification of the fact that the asymptotics have the form which agrees with the OPE ansatz. The more sophisticated methods of [14] were discovered [13] outside the BPHZ framework—using the ideas described in the present paper.

[22], including efficient calculational formulae for coefficient functions of OPE. The derivation in [12]-[22] employed a new mathematical techniques based on a novel concept of asymptotic expansions in the sense of distribution. Algorithmically, the key technical instrument here is the so-called *asymptotic operation* (*As-operation*). As has become clear [38], the techniques based on the *As-operation* offers a comprehensive and powerful alternative to what is known as the BPHZ theory.

1.2 The lesson of the *R-operation*.

N.Bogoliubov's 1952 derivation of the fundamental formula for subtraction of UV divergences (the *R-operation*) [24] provides a fine example of a highly non-trivial reasoning which led to a highly non-trivial result—a reasoning that has nothing to do with how the *R-operation* is treated (i.e. “proved”) within the framework of the BPHZ method. By examining Bogoliubov's reasoning one can exhibit the central dilemma of the theory of multiloop diagrams, and the lessons learnt thereby have a direct bearing on the theory of asymptotic expansions as well.

The dilemma is as follows. On the one hand, Feynman diagrams are objects whose complexity increases infinitely with the order of perturbation theory. But that complexity is not amorphous, it has a structure: Feynman diagrams can be generated, e.g., by iterating Schwinger-Dyson equations or via some other equivalent and orderly procedure (see e.g. the construction of perturbation theory in [24] from microcausality condition). To put it shortly, multiloop diagrams are organised in a recursive fashion. For definiteness, here is an example of a causality condition of the kind that was used by Bogoliubov:

$$T[\mathcal{L}(x)\mathcal{L}(y)\mathcal{L}(z)\mathcal{L}(u)] = T[\mathcal{L}(x)\mathcal{L}(y)] \times T[\mathcal{L}(z)\mathcal{L}(u)], \quad \text{for } x^0, y^0 > z^0, u^0, \quad (1.1)$$

which expresses the fact that chronological product of four Lagrangians is expressed as a simple product of *T*-products involving lesser number of Lagrangians, taken in a special order. (Each Lagrangian will correspond, in the final respect, to a vertex in a diagram.) On the other hand, the elements that participate in the recursion (1.1), are singular—their formal nature can best described by qualifying them as distributions—and their products must not be treated formally: products of distributions do not, generally speaking, exist (if one insists that they do, one runs into UV divergences).

Nevertheless, Bogoliubov's realization consists in that if all *T*-products of lower order are known, the totality of such relations defines $T[\mathcal{L}(x) \dots \mathcal{L}(u)]$ everywhere except for the point $x = y = z = u$.

The last step in the definition of *T*-product can be best described using the language of distribution theory as a procedure of “extension of a functional” (see below sect. 5). Without entering into detail, we only note that at a practical level, such an extension consists in adding to the r.h.s. of (1.1) of a counterterm localized at the point $x = y = z = u$. It can only be a linear combination of derivatives of $\delta(x - y)\delta(y - z)\delta(z - u)$. The number of derivatives to be added is determined by the leading singularity of the r.h.s. at the point $x = y = z = u$, which can be determined by, essentially, power counting, after which there are simple ways to fix the coefficients in order to ensure finiteness of the resulting *T*-product.

The R -operation now emerges as a straightforward iteration of the same elementary step in situations with increasing number of Lagrangians on the l.h.s.: addition of a counterterm corresponding to the singularity at an isolated point.

The net effect of such a reasoning is, in a profound sense, “organizational”: it allows one to use the inherent recursive structures in order to reduce the reasoning to just one simple step. It is no secret (see e.g. the proof of the Weinber theorem [2]) that the problems involving multiloop diagrams are reduced, essentially, to a very cumbersome (if done in a straightforward way) power counting. A proper “organization” of the problem requires to do power counting only for a simple subproblem (isolated point in the above example) so that the solution becomes rather obvious.

However, Bogoliubov in the 50s was not familiar with the techniques of the modern version of the distribution theory [23], which explains the decision not to try to formalize the underlying heuristic derivation (such a formalization can be found in [20]) but rather to treat the formula for the R -operation as given and simply “rigorously prove” it using the simplest available method—reducing the renormalized diagram to an absolutely convergent integral by resolving all recursions in the framework of a parametric representation. Thus the BPHZ method was born.

1.3 As -operation: distributions and “perfect” expansions.

It turns out [12] that the problem of expansions of multiloop diagrams can be considered in a way similar to Bogoliubov’s treatment of UV divergences. Since the main body of the present paper is devoted to explanation and clarification of this fact, only a few general remarks are offered here.

First, the entire set of Feynman diagrams is structured in a recursive fashion, even if in more difficult problems (e.g. expansions) it is may not be easy to notice the recursion and its relevance to the problem at hand.

If the second part of the expansion problem (restoration of expansions in a global OPE-like form) is to be successful, then the expansions of individual diagrams should be done so as to preserve the recursive structure. the problem gets complicated if properties like gauge invariance are involved which connect sets of diagrams. Thus, an efficient handling of recursions is a key to successful organization of any work with multiloop diagrams.

A scrutiny of the problem of singularities encountered in formal expansions of integrands from the point of view of the expansion problem in its entirety—e.i. taking into account the recursion aspect—reveals that the underlying fundamental mathematical problem is that of expansion of products of singular functions in the sense of distributions (sect. 4). Its solution involves counterterms (similarly to the theory of R -operation described above); each counterterm corresponds, as we will see, to a subgraph, and the underlying recursive structure of multiloop diagrams is naturally reflected in the expansions, allowing simple exponentiation at the second, combinatorial stage [18].

On the other hand, in order to save effort by avoiding proving useless theorems, the expan-

sions to be obtained must run in powers and logarithms of the expansion parameter (since we know the analytical structure of expansions; otherwise one would have to determine it in the process).

Now we come to the most important point: once the analytical structure of expansions is determined, a conclusion follows: *if such an expansion exists, it is unique* (see subsect.5.2). An immediate corollary is that *all structural properties of the original collection of diagrams prior to expansion are inherited by the expansion in an orderly manner*.

Thus, the two key ideas of our method are: expansions in the sense of distributions and the obligatory requirement of “perfection” of expansions to be obtained.

One can say that, in the final respect, the two ideas offer a constructive heuristic framework to bridge the gap between the lowest level of the problem (the underlying power counting, division of integration space into subregions etc.) and its highest level (global non-perturbative structure of the OPE-like expansions): reasoning using the language of distributions and “relentlessly pursuing perfection” in expansions at each step, we are guaranteed that the resulting formulae will possess all the desirable properties.

The recipes of [14] and the multiloop calculations performed in [28] confirm that a good organization has its advantages.

It remains to note that neither one of the above two ideas is specific to the case of Euclidean asymptotic regimes.

1.4 Purpose and plan.

The aim of the present paper is to explain—at a heuristic level comparable to that of the derivation of the R -operation in the classic book [24]—both the current understanding of the expansion problem and the analytical ideas behind the solution [12]-[22], presenting the results in a form that proved to be most convenient for applications in applied QFT. We begin by analyzing phenomenological applications in order to arrive at a meaningful formulation of the asymptotic expansion problem and reviewing the results previously known in order to compare them with ours. we identify the problem of asymptotic expansions of Feynman integrands in the sense of distributions as a central one, and present its explicit solution in the form of the $A\mathfrak{s}$ -operation. In this paper we do not aim at achieving a complete mathematical formalization but rather concentrate on the key motivations, notions and ideas. We will use the dimensional regularization [32] (for a review see [26]) as a familiar technical framework to deal with various singularities one encounters in this kind of problems.⁴ However, the use of dimensional regularization is by no means essential—a regularization-independent account of the proofs can be found in [20]-[22] where a fully formalized treatment of asymptotic expansion of products of

⁴The better known rigorous definition of the dimensional regularization is in terms of the α -parametric representation [33] while our reasoning is essentially based on momentum space picture. Formal constructions of dimensional regularization in terms of position/momentum representations exists [34], [35], but the original definition of [32] is the most useful one in applications (which means, by the way, that it is this definition that should be a preferred subject of investigation; indeed, it is not difficult to adapt it for the purposes of formal proofs [36], but an in-depth discussion of this point goes beyond the scope of the present publication).

singular functions is presented..

In the companion publications [18] and [19] the As -operation is applied to two problems, respectively: deriving and studying OPE and its generalizations in the MS-scheme, and studying UV renormalization and related issues. In [18] the combinatorics of the transition from expansions for single diagrams to OPE in a global form is studied. To this end the so-called As -operation for integrated diagrams is defined (it was first introduced in [16]). Its combinatorial structure is fully analogous to that of the R -operation, which fact allows one to obtain exponentiation easily and derive full asymptotic expansions of perturbative Green functions in any of the so-called Euclidean asymptotic regimes which comprise the familiar short-distance OPE, the low-energy effective Lagrangians to all orders of inverse masses of heavy particles etc. The derivation automatically yields the convenient calculational formulae that have been presented in an earlier publication [14] and extensively used in applications (see e.g. [28]). It should be noted that the treatment of combinatorics of the ordinary R -operation given in [18] is, probably, one of the simplest in the literature.

In the third paper [19] a new approach to studying UV-divergent integrals will be described which is based on the use of the As -operation of the present work, and a new representation for the R -operation will be given which generalizes the results of sect. 8. (Ref.[19] is a simplified version of the more formal text [21].) It should be noted that one of the major difficulties in studies of asymptotic expansions of Feynman diagrams used to be the presence of a subtraction procedure for eliminating UV divergences [9]. The representation [21] (see also sect. 8 below) trivializes the problem by reducing it to study of double asymptotic expansions [22], which can be accomplished by exactly the same methods as in the case of one-parameter expansions. It should be stressed that such a simplification mechanism is not limited to Euclidean problem, it is completely general. An informal description of our treatment of UV divergences (presented in full mathematical detail in [21]) will be done in [19]. We will also explain how the new representation and the As -operation work together to produce the algorithm for calculations of renormalization counterterms and renormalization group functions known as the R^* -operation [29].

The present paper is organized as follows. In the introductory sect. 2, we review some phenomenological problems of applied QFT falling under the general heading of Euclidean asymptotic expansions, and analyze the properties which the corresponding theoretical expansions must possess in order that their applications to phenomenology make sense. We also review the most important earlier results on the subject and explain what improvements and generalizations are offered by the new theory. In sect. 3 we explicitly formulate the Euclidean asymptotic expansion problem which we wish to study.

In sects.4–8, the basic ideas of our method are discussed. Heuristic motivations for studying expansions of products of singular functions in the sense of the distribution theory (the Master problem) are presented in sect. 4. Sect. 5 contains some basic background mathematical definitions. The extension principle—our key to solving the Master problem—is presented and discussed in sect. 6. In sect. 7 we demonstrate how the ideas of the Extension principle work with a simple but representative example. Sect. 8 deals with expansions of one-loop Feynman integrals and introduces some ideas concerning the treatment of UV-divergent graphs, which are generalized in [21] and [19].

In sects.9-13, we solve the Master problem in the general Euclidean case. First in sect. 9 a system of notations is developed which allows one to compactly represent and conveniently manipulate Feynman integrands, without using integral representations - which is important if the benefits of the viewpoint of the distribution theory are to be used to the full. sect. 10 describes and classifies singularities of the formal Taylor expansion of singular functions with respect to a parameter, and the important notion of IR-subgraph is introduced. In sect. 11 we derive the general formulae for the As -operation for products of singular functions, which provides the solution to the Master problem. Explicit expressions for the counterterms of the As -operation are derived in sect. 12. sect. 13 contains a detailed discussion of an example with a non-trivial pattern of singularities.

THE PROBLEM OF EUCLIDEAN ASYMPTOTIC EXPANSIONS

2 Euclidean expansions in applications.

Since there are important aspects in which formulation of the expansion problem within the standard BPHZ theory had been lacking, and since in publications with emphasis on applications the exact mathematical nature of the problems is often not understood clearly, it is appropriate to review at least those phenomenological problems that are reduced to special cases of asymptotic expansions in Euclidean regimes. This will provide motivations for our formulation of the problem of Euclidean asymptotic expansions, and give us an opportunity to review the key results on the subject as well as to explain the improvements which are contained in our results as compared to those obtained by other authors. Some important new notions will also be introduced.

2.1 Short-distance operator-product expansions.

K.Wilson [8] considered expansions of operator products at short distances of the form:

$$T[A(x)B(0)] \approx \sum_i c_i(x)O_i(0), \quad x \rightarrow 0. \quad (2.1)$$

However, despite the undeniable heuristic value of the point of view of position space representation, it is momentum representation picture that is immediately relevant for analysis of phenomenological problems while the transition to coordinate representation somewhat obscures the Euclidean nature of the corresponding asymptotic regime.

In fact, the generic problem which is essentially equivalent to short-distance expansions and

immediately related to phenomenology, is to expand expressions like:

$$\int dx e^{iqx} \langle T[A(x)B(0) \prod_i J_i(y_i)] \rangle_0 \quad (2.2)$$

as $Q^2 = -q^2 \rightarrow +\infty$.

It should be stressed that q goes to infinity along Euclidean (space-like) directions. The following examples are intended to illustrate this point.

(i) Within the QCD sum rules method [37] (for a review see [45]), when one studies the spectrum of hadrons built of light quarks, one considers expansions of the correlator of two local operators:

$$\Pi(Q^2) = i \int dx e^{iqx} \langle T[A(x)B(0)] \rangle_0 \quad (2.3)$$

at Q^2 much greater than $m_{u,d,s}^2$, the masses of light quarks.

(ii) For the deep-inelastic lepton-hadron scattering (for a review see [46]) the moments of the observable structure functions are directly expressed (see [47]) as integrals of the quantity

$$T(Q^2, 2pq_N, p_N^2) = \int dx e^{iqx} \langle N|T[A(x)B(0)]|N \rangle. \quad (2.4)$$

One can see from the expressions presented in [47] that the behaviour of the n -th moment at $Q^2 \rightarrow +\infty$ and fixed n is determined by the behaviour of $T(Q^2)$ in the deeply Euclidean region

$$Q^2 \gg |2qp_N| p_N^2.$$

(This does not contradict to the popular "light-cone" philosophy [48], where one studies the behaviour of the structure functions $F(x, Q^2)$, $x = Q^2/2qp_N$, at $Q^2 \rightarrow +\infty$ and fixed x , which is equivalent to the light-cone expansion of the operator product. Indeed, reconstruction of functions from their moments is a mathematically "non-correct" problem [49] and requires additional information for its solution. This agrees with the fact that the short-distance limit is apparently more stringent than the light-cone one. However, from the point of view of phenomenological usefulness the two approaches are rather complementary [50].)

(iii) Further examples are provided by the problem of hadronic formfactors (usually, electromagnetic ones; for a general review see [51]) within the QCD sum rules approach [37]. Here one deals with the three-point correlator of the form (cf. Fig. 1)

$$\int dx dy \exp i(q_1x - q_2y) \langle T[J(x)J(y)j(0)] \rangle_0, \quad (2.5)$$

where the current J corresponds to the hadron, and j is the electromagnetic current. Denote $Q_i^2 = -q_i^2$, $Q^2 = -(q_1 - q_2)^2$. The case of "intermediate momentum transfer" [52]

$$Q_i^2 \sim Q^2 \gg m_{u,d,\dots}^2$$

presents an example of a problem with more than one heavy momentum. However, this case still belongs to OPE problems (x and y tend to 0 simultaneously); the complication due to the fact that there are more than two operators to be merged at one space-time point is inessential.

(iv) The case of “low momentum transfer” [53] is described by the asymptotic regime

$$Q_i^2 \gg Q^2 \sim m_{u,d}^2.$$

This also is essentially an OPE-like problem: $J(x)$ and $J(y)$ are separated by a short distance $x - y \rightarrow 0$. Note that here the additional spectator current j is present within the same T -product along with the two operators $J(x)$ and $J(y)$ which are to be merged at a point. This fact precludes a straightforward substitution of the expansion (2.1) into (2.5) and gives rise to extra terms similar to those that appear in the example (iii). However, such complications are easily treated within the formalism which we will describe.

It is convenient here to point out the difference between the Euclidean and Minkowskian asymptotic regimes. Suppose the momenta q_i in (2.5) are time-like and one considers the so-called Sudakov asymptotic regime

$$Q^2 \gg q_i^2 \sim m_{u,d}^2.$$

The crucial point is that this does not imply automatically that $q_i \rightarrow 0$ componentwise but only that $q_i \rightarrow p_i$ where p_i are some light-like momenta whose components are of order $\sqrt{Q^2}$. Such asymptotic regimes differ drastically from the purely Euclidean ones considered in the present paper. Although the basic ideas of our technique are sufficiently general to make it applicable to the Minkowskian case, specific implementation should take into account many new features—e.g. non-linearity of manifolds on which singularities of integrands are localized. One can get a flavour of what to expect in Minkowskian case from the review [27].

2.2 Technical aspects of OPE. Perfect factorization.

Within perturbation theory, a first version of OPE was obtained by Zimmermann [9] in the following form:

$$T[A(x + \xi)B(x)] = \sum_i c_{N,i}(\xi, m) O_{N,i}(x) + o(|\xi|^N). \quad (2.6)$$

This result, however, despite the great theoretical significance of [9], had a somewhat theoretical rather than practical relation to calculations, e.g. within pQCD. (Note, however, that for justice’s sake, it should be stressed the result (2.6), which in any case is a theorem, dates back to pre-QCD times.)

First, the above version of OPE relies on momentum subtractions for UV renormalization, while all practical calculations beyond one loop are always carried out within the MS-scheme introduced by ’t Hooft [25] (for a review see [26]).

Second, for models with massless Lagrangian fields (e.g. for QCD with massless gluons) the dependence of the coefficient functions in (2.6) on N can not be got rid of (this is connected with the fact that in [9] the MOM-schemes for UV renormalization were used and masses in fact played the role of infrared cutoffs)—a somewhat awkward property from the viewpoint of practical calculations.

Third, the applicability of OPE to phenomenology of QCD rests on the assumption that the coefficient functions are calculable within perturbation theory. But the coefficient functions in

(2.6) depend on the masses m of elementary fields non-analytically and contain large $\log(m^2\xi^2)$ contributions. A renormalization group analysis [11] shows that in pQCD the renormalization-group resummation of such terms takes one outside the region of applicability of perturbative methods. Presence of non-perturbative contributions defeats attempts to extract the non-perturbative values of matrix elements of the composite operators by fitting the r.h.s. of the above expansion against experimental data—a procedure typical of phenomenological studies of deeply inelastic scattering, the QCD sum rules method described above etc.

The notion of *perfect factorization*—which had been lacking in, and seems to be inherently foreign to, the BPHZ theory—was introduced in [13]⁵. Its importance has by now been fully realized. At the level of operators/Green functions, perfect factorization means that all the non-analytical dependences on m in the expansions like (2.6) are localized within the matrix elements of the operators O ; while the coefficient functions c_i depend on m analytically. It follows immediately that such an expansion would allow taking the limit $m \rightarrow 0$ and, therefore, should be expected to be valid for models with massless particles. Moreover, since the masses m are no longer needed as IR regulators, one can expect to get rid of the dependence on N in (2.6). Furthermore, perfect factorization, implying that no non-analytical dependences are contained in the OPE coefficient functions, guarantees that extraction of the non-perturbative matrix elements from data is a mathematically meaningful procedure.

At the level of individual diagrams, perfect factorization means that the asymptotic expansion should explicitly run in powers and logarithms of the expansion parameter. Such expansions are unique (see below subsect. 11.2) which fact is of paramount technical importance; for example, it allows one to conclude (cf. subsects.11.2, 12.5) that such asymptotic expansions commute with multiplication by polynomials and other algebraic operations on diagrams—the property that greatly simplifies combinatorial study of OPE etc. in non-scalar models; in particular, one need not worry about things like gauge invariance of the expansions since such properties are inherited by expansions from initial integrals in a well-defined and orderly manner⁶. Uniqueness of such expansions also allows one to exhibit the recursive structure of the problem of expansion of Feynman diagrams which can be effectively used in derivation of OPE (see below subsects.11.3-11.4).

It turned out that the OPE derived directly within the MS-scheme [13]⁷, [40], possess the property of perfect factorization so that: the dependence of the coefficient functions on masses is analytical (which was earlier shown to be a necessary condition of existence of the OPE in the MS scheme [7]); the expansion is a full asymptotic series; it is valid in models with massless particles; and due to greatly simplified dependence of coefficient functions on dimensional parameters, there exist algorithms for analytical evaluation of the relevant integrals

⁵In the context of the theory of R -operation perfect factorization corresponds to renormalization using the so-called massless schemes among which the MS scheme is by far the most useful one.

⁶The techniques for studying gauge properties was extended to the framework of the Euclidean A_s -operation in [66]. In this respect recall that gauge invariance plays a central role in studies of non-Euclidean asymptotic expansions which represent a major unsolved problem with important applications to physics of hadrons (see [67]).

⁷which result came about as a special case of the general theory of EA-expansions developed in [12]-[19], [20]-[22]. Efficiency of the techniques developed therein allowed us to obtain very general formulae first made public in [16]. The results of [16] were widely discussed in the literature [39], [41], [42].

through three loops [54]. All these properties are interrelated, as is stressed above and as will be seen from our derivation.

More conventional derivations of OPE in the MS-scheme à la BPHZ were presented by Llewellyn Smith and de Vries [40] and Gorishny [39]. The latter work clarified the connection of the results of [12]-[16] with the BPHZ method⁸.

Theoretically speaking, one need not use specifically the MS-scheme to obtain OPE with the property of perfect factorization. Any of the so-called massless schemes (or *generalized MS-schemes* [21]) will do [22].

It should also be clearly understood: the most important thing about the new results is *not* the fact that, say expansions were proved “in the MS scheme”, but that such expansions possess the property of perfect factorization. This should not be surprising in view of the fact that the MS scheme has the property of polynomiality of its counterterms in masses [26] which itself is a special case of perfect factorization—this point is best understood in the context of the definition of generalized MS schemes in [21] by way of subtraction of UV-dangerous asymptotics from momentum-space integrands, so that evaluation of those asymptotics is done using the same apparatus of the *As*-operation as used in obtaining asymptotic expansions of diagrams. This similarity is not so much technical as ideological.

2.3 Heavy mass expansions.

Consider a QFT model in which all the fields are divided into two sets: light fields collectively denoted as φ and heavy fields Φ . Their Lagrangian masses are m and M , respectively. Assume that all M are much larger than m , and consider a Green function G of light fields only. It depends on M besides other—momentum, coordinate etc.—parameters. The problem is to study the asymptotic expansion of G in inverse powers of M :

$$G(\dots, M) \underset{M \rightarrow \infty}{\simeq} ? \quad (2.7)$$

In fact the first two terms in the heavy mass expansion had been known for some time. The “decoupling” theorem [55] (for a review and references see [26], [2]) asserts that, up to $O(M^{-1})$ corrections, the effect of the virtual presence of heavy particles (in phenomenologically

⁸Still another approach, specifically designed for achieving ultimate rigour in treatment of dimensional regularization within the techniques of the α -parametric representation is sketched in [41]. It is unclear, however, how successful the latter attempt was because neither that paper nor the publications cited therein contain explicit treatment of the analytical part of the proof (recall in this respect that α -parametric integrands for MS-renormalized diagrams are rather cumbersome distributions [43] so that advantages of using such a representation are somewhat unclear). On the other hand, the combinatorial part of the proof is discussed in the original publications [16], [18] in a more straightforward manner.

The regularization independent aspects of the original derivation have been exhaustively discussed in [20], [21] and [22]. Various ways to rigorously treat dimensional regularization without parametric representations were discussed in [34], [35], although not much practical understanding is added thereby to the heuristic treatment of [32], [26]. Anyway, the variety of uses of dimensional regularization in practical calculations is such that there is little hope that everything that is being done will ever be “rigorously proved”. In the final respect, mathematics is not about rigour. Mathematics is, first and foremost, about calculations.

meaningful cases) is absorbed into a finite renormalization of the Lagrangian of the light world, whose parameters become dependent on M . This fixes the form of the leading term of the $M \rightarrow \infty$ expansion. The next-to-leading term can be described as differing from the initial Green function by one local operator insertion. (This result was first obtained in [56] by making use of Wilson's OPE.) More formally, let \mathcal{L}_{tot} be the total Lagrangian of light and heavy fields, and let $\langle T\varphi(x_1)\dots\varphi(x_n) \rangle_0^{\mathcal{L}_{\text{tot}}}$ be a Green function of the light fields in the world containing both light and heavy fields. Then the leading and next-to-leading contributions to the expansion have the form:

$$\begin{aligned} \langle T\varphi(x_1)\dots\varphi(x_n) \rangle_0^{\mathcal{L}_{\text{tot}}} &\underset{M \rightarrow \infty}{\simeq} z^n(M) \left\{ \langle T\varphi(x_1)\dots\varphi(x_n) \rangle_0^{\mathcal{L}_{\text{tot}}^{(0)}} \right. \\ &\left. + \langle T\varphi(x_1)\dots\varphi(x_n) \left[\frac{g(M)}{M} \int dy \mathcal{L}_{\text{eff}}^{(1)}(y) \right] \right\rangle_0^{\mathcal{L}_{\text{tot}}^{(0)}} \} + o(M^{-1}), \end{aligned} \quad (2.8)$$

where $\mathcal{L}_{\text{eff}}^{(0)}$ is obtained from that part of \mathcal{L}_{tot} which describes the light fields, by a finite M -dependent renormalization of its parameters; $\mathcal{L}_{\text{eff}}^{(1)}(y)$ is a local operator composed of light fields; $z(M)$ and $g(M)$ are functions of M with logarithmic leading behaviour at $M \rightarrow \infty$. (The UV renormalization in all cases is implicit.)

The notation $\mathcal{L}_{\text{eff}}^{(1)}$ is suggestive, and indeed, to all orders in M^{-1} ,

$$\langle T\varphi(x_1)\dots\varphi(x_n) \rangle_0^{\mathcal{L}_{\text{tot}}} \underset{M \rightarrow \infty}{\simeq} z^n(M) \langle T\varphi(x_1)\dots\varphi(x_n) \rangle_0^{\mathcal{L}_{\text{eff}}} \quad (2.9)$$

where

$$\mathcal{L}_{\text{eff}} = \sum_n M^{-n} g_n(M) \mathcal{L}_{\text{eff}}^{(n)}. \quad (2.10)$$

The derivation of this formula will be given within the MS-scheme, so that our results will contain as a by-product a derivation of the decoupling theorem and of the formula (2.8) directly within the MS-scheme, which has been lacking. Moreover, the expansions we will obtain possess the property of perfect factorization similar to that discussed in the preceding subsection, so that the heavy and light parameters are fully factorized, i.e. $z(M)$ and $g(M)$ do not depend on m , but only on the heavy mass M and, logarithmically, on a renormalization parameter.

The most important phenomenological applications of the heavy mass expansion are: the study of effects of heavy quarks on the low-energy light quark properties, especially on the parameter Λ_{QCD} (for a review see [26]); the low-energy quark Lagrangian of the electro-weak interactions [56]; evaluation of muon contribution to the electron anomalous magnetic moment [57]. The latter problem is interesting as an example of a problem where a very high precision—and, consequently, taking into account many terms in the expansion is required.

2.4 Effects of heavy masses on OPE.

The QCD sum rules approach to studying the heavy quark bound system [37], [45] presents an example of a problem where a short-distance OPE is modified by presence of a heavy mass. Here one should study the correlator (2.3) but for such an asymptotic regime that the squared

mass M^2 of the heavy quark were of the same order of magnitude as Q^2 . (This point was stressed in [58].)

It follows from our general results that the expansion in this case has the form similar to ordinary OPE:

$$\text{eq.(2.3)} \quad Q^2 \sim M^2 \gg m_{u,d,s}^2 \quad \sum_i c_i(M, Q) \langle O_i(0) \rangle_0, \quad (2.11)$$

but with coefficient functions depending on the heavy mass M . To our knowledge, a derivation of the expansion (2.11) has heretofore been lacking.

2.5 Light-by-light scattering. Linear restrictions on heavy momenta.

Consider the problem of deeply inelastic light-by-light scattering with both photons deeply virtual [59]. Here one studies the four-point correlator of hadronic electromagnetic currents with the pattern of external momenta as depicted in Fig. 2a, in the asymptotic regime

$$Q_1^2 \sim Q_2^2 \gg m_{u,d,s}^2 \quad (Q_i^2 = q_i^2).$$

A new feature in this case is that along with the general momentum conservation an additional linear restriction is imposed on the heavy momenta (cf. Fig. 2b). Motivated by this example, we will allow arbitrary restrictions of this kind in our general treatment of Euclidean asymptotic expansions.

Note that the Green function to be expanded may not exist at the restricted momenta owing to infrared singularities. On the other hand, if it does, its expansion can be obtained via our technique in the same way as in the non-restricted case. However, the Q -dependent coefficient functions of the expansion of the non-restricted Green function (cf. the one corresponding to Fig. 2b) will in general be singular at the restricted momenta and have no relation to the coefficient functions for the restricted case. In other words, restricting heavy momenta and performing expansion are in general non-commuting operations.

Allowing restrictions on the heavy momenta has no bearing on our formalism for expansion of a single multiloop Feynman graph but only on the subsequent combinatorial analysis of the global structure of the resulting expansions of Green functions as a whole.

2.6 Contact terms.

In our treatment of Euclidean asymptotic expansions we will be considering Green functions as distributions with respect to the external momenta.⁹ Apart from fundamental meta-level arguments (finite energy resolution of measuring devices etc.), there are more specific motivations for this.

⁹The fact that expansions in this case run in powers and logs was obtained in [5], without explicit expressions for the contact terms. There are many other examples in the literature where distributions and asymptotic expansions appeared together—see e.g. eq. (46) in [68]. *

The first one is of a technical nature: we will in any case make an essential use of the theory of distributions in order to expand integrands of multiloop Feynman graphs so that considering Green functions as distributions over external momenta brings uniformity to our argument (cf. subsect. 4.2 below) and is therefore inherently natural.

Another—rather typical—motivation comes from the QCD sum rules method [37]. The starting point here is a spectral representation for the correlator (2.3), e.g.

$$\Pi(Q^2) = \int_0^\infty ds \frac{\rho(s)}{s + Q^2}. \quad (2.12)$$

An important element of the techniques of [37] is the so-called Borelization procedure which is applied to both sides of (2.12) in order to suppress contributions from large s in the spectral sum on the r.h.s. of (2.12). It is assumed in [37] without proof that Borelization procedure commutes with the asymptotic expansion of $P(Q^2)$ at $Q^2 \rightarrow +\infty$. However, this problem can be avoided (at the cost of some complication of formulae) by the following modification of the arguments of [37]. Consider the relation:

$$F(s/\lambda) = \int d^D Q \frac{\varphi(Q^2/\lambda)}{s + Q^2} = \text{const} \int_0^\infty dx x^{D/2} \frac{\varphi(x/\lambda)}{s + x}. \quad (2.13)$$

One can find—e.g. using tables of integral transforms—pairs of $\varphi(Q^2)$ (related to the Bessel functions $J_\nu(\sqrt{Q^2})$) and $F(s)$ (related to the McDonald functions $K_\nu(s)$) such that:

- (i) eq.(2.13) is satisfied;
- (ii) $\varphi(Q^2)$ is smooth in Q everywhere including $Q = 0$;
- (iii) $\varphi(Q^2)$ is bounded from above by some negative power of Q^2 at $Q^2 \rightarrow \infty$;
- (iv) $F(s)$ decreases at $s \rightarrow \infty$ faster than any power s^{-n} (namely, as $\exp(-\sqrt{s})$ times a negative power of s).

Then using (2.12) and (2.13) one gets an exact sum rule:

$$\int d^D Q \varphi(Q^2/\lambda) \Pi(Q^2) = \int_0^\infty ds \rho(s) F(s/\lambda). \quad (2.14)$$

The weight function F on the r.h.s. provides as good a suppression of larger s contributions as the simple exponent used in [37]. At the same time, the l.h.s. is the value of the distribution $\Pi(Q^2)$ on the test function $\varphi(Q^2/\lambda)$ (the fact that $\varphi(Q^2)$ does not decrease at $Q^2 \rightarrow +\infty$ as fast as is required of the Schwartz's test functions [23] is immaterial since we do not have to deal with arbitrary tempered distributions but with a specific one, $\Pi(Q^2)$, whose behaviour at large Q^2 is sufficiently good to make it well-defined on the $\varphi(Q^2)$).

The next step in the recipe of [37] would be to use the expansion at $Q^2 \rightarrow +\infty$ for $\Pi(Q^2)$. Let us rewrite the l.h.s. of (2.14) as

$$\lambda^{D/2} \int d^D Q \varphi(Q^2) \Pi(Q^2/l).$$

Now it is clear that $\Pi(Q^2)$ should be expanded not as a function but as a distribution “in the sense of distributions”. Let us explain the meaning of this phrase in our specific case (cf. subsect. 5.1).

The asymptotic expansion of $\Pi(Q^2)$ considered as an ordinary function of Q^2 at $Q^2 \rightarrow +\infty$ in the usual sense can be written as

$$\Pi(\lambda Q^2) \underset{\lambda \rightarrow \infty}{\simeq} \sum_{\mathfrak{i}} c_{\mathfrak{i}}(\lambda Q^2) O_{\mathfrak{i}}, \quad (2.15)$$

where $O_{\mathfrak{i}}$ are coefficients independent of Q^2 , and $c_{\mathfrak{i}}(Q^2)$ are ordinary functions, e.g. $c_{\mathfrak{i}}(Q^2) = Q^{2(-n)}$. One can not integrate (2.15) against a test function because, first, $c_{\mathfrak{i}}(Q^2)$ may be too singular at $Q \rightarrow 0$; second, there is no guarantee that termwise integration of the r.h.s. results in a correct asymptotic expansion of the integral on the l.h.s.

On the other hand, for an expansion

$$\Pi(\lambda Q^2) \underset{\lambda \rightarrow \infty}{\simeq} \sum_{\mathfrak{i}'} c_{\mathfrak{i}'}^{\text{distr}}(\lambda Q^2) O'_{\mathfrak{i}'}, \quad (2.16)$$

to hold in the sense of distributions, all $c_{\mathfrak{i}'}^{\text{distr}}(Q^2)$ must be well-defined distributions over the whole range of Q , and its termwise integration against any test function $\psi(Q)$ must generate a correct asymptotic expansion at $\lambda \rightarrow \infty$ of the λ -dependent function

$$f(\lambda) = \int d^D Q \Pi(\lambda Q^2) \psi(Q). \quad (2.17)$$

The relation between (2.15) and (2.16) can be established by noticing that both expansions allow integration with test functions localized in small neighbourhoods of the points $Q \neq 0$. It follows that $c_{\mathfrak{i}'}^{\text{distr}}$ at $Q \neq 0$ should be either zero, or it should coincide with some $c_{\mathfrak{i}}$; in the latter case $O'_{\mathfrak{i}'} = O_{\mathfrak{i}}$. Therefore,

$$\sum_{\mathfrak{i}'} c_{\mathfrak{i}'}^{\text{distr}}(Q) O'_{\mathfrak{i}'} = \sum_{\mathfrak{i}} c_{\mathfrak{i}}^{\text{R}}(Q) O_{\mathfrak{i}} + \sum_{\mathfrak{i}'} \Delta c_{\mathfrak{i}'}(Q) O'_{\mathfrak{i}'}, \quad (2.18)$$

where $\Delta c_{\mathfrak{i}'}(Q)$ are localized at $Q = 0$, which implies that they are constructed of $\delta(Q)$ and its derivatives, while $c_{\mathfrak{i}}^{\text{R}}(Q)$ are distributions which by definition coincide with $c_{\mathfrak{i}}(Q)$ at $Q \neq 0$. For example, if

$$\Pi(Q) \simeq \frac{1}{Q^2} O_1 + o(Q^{-4}), \quad (2.19)$$

then the expansion in the sense of distributions will have the form

$$\Pi(Q) \simeq \left\{ \frac{1}{Q^2} \right\}^{\text{R}} O_1 + \delta(Q) O'_1 + o(Q^{-4}), \quad (2.20)$$

where

$$\left\{ \frac{1}{Q^2} \right\}^{\text{R}} = \lim_{\epsilon \rightarrow 0} \left[\frac{1}{Q^2 + \epsilon} + Z_{\epsilon} \delta(Q) \right], \quad (2.21)$$

with Z_{ϵ} chosen so as to render finite the integrals of (2.21) with arbitrary test functions. Note that the expression (2.20) is unique—cf. below subsect. 5.2—so that a change of Z_{ϵ} by a finite

constant is compensated by an appropriate change of O' . And, of course, one can use any other regularization in (2.21).

We see that the expansion in the sense of distributions differs from the expansion in the usual sense by δ -functional contributions (compare (2.19) with (2.20) and (2.21)). Such contributions can be conveniently denoted as *contact terms*. Ref. [60] contains an example which demonstrates that such contact terms can be numerically significant in applications.

Our formulae for general Euclidean asymptotic expansions [16]–[18] include explicit expressions for the contact terms.

3 The formal problem of Euclidean expansions.

Motivated by the examples considered above, let us now present a formal description of the general problem of Euclidean asymptotic expansions, following [16]. We will stay within the framework of perturbative QFT [24] and assume that the MS-scheme [25] is always used for UV renormalization.

3.1 Green functions.

There are two types of fields collectively denoted as φ (light fields) and Φ (heavy fields), with Lagrangian masses m (some of which may be zero) and M . Let $h_j(x_j)$ be local monomials built of φ , and $H_l(y_l)$ built of both φ and Φ . (One could allow h to be also built of Φ but such cases seem to never occur in applications.) Denote the full Lagrangian of the model as

$$\mathcal{L}_{\text{tot}}(\varphi, \Phi, m, M). \quad (3.1)$$

Consider the Green function

$$\begin{aligned} G(Q, M, k, m, \mu) = & \int \left(\prod_i dx_j \right) \left(\prod_l dy_l \right) \exp i \left[\sum_l Q_l y_l + \sum_j k_j x_j \right] \\ & \times \langle \text{vac} | RT \left[\prod_l H_l(y_l) \prod_j h_j(x_j) \right] | \text{vac} \rangle^{\mathcal{L}_{\text{tot}}}, \end{aligned} \quad (3.2)$$

where R is the UV R -operation and μ the renormalization parameter; T denotes the chronological product. It is assumed that the correlator is evaluated within the model described by \mathcal{L}_{tot} ; in the functional notation:

$$\langle \text{vac} | T[\dots] | \text{vac} \rangle^{\mathcal{L}_{\text{tot}}} \implies \int d\varphi d\Phi [\dots] \exp i \int dx \mathcal{L}_{\text{tot}}(x). \quad (3.3)$$

The Green function (3.2) as defined should be proportional to the δ -function which expresses the overall momentum conservation, but we will ignore it, just assuming that

$$\sum_l Q_l + \sum_j k_j = 0. \quad (3.4)$$

The momenta Q and masses M will be called *heavy* while k and m *light*, and it will be assumed that the ratio of scales of light over heavy parameters (denoted as κ) is vanishing, and G is to be expanded with respect to κ . More precisely, we wish to expand in κ the expression

$$G(Q, M, \kappa k, \kappa m, \mu) \underset{\kappa \rightarrow 0}{\simeq} ? \quad (3.5)$$

A remark is in order. We have implicitly assumed that the renormalization parameter μ belongs to the heavy parameters. However, since we always use the MS-scheme for UV renormalization, each Feynman graph contributing to the Green function G is a polynomial in $\log \mu$. Therefore, if we take μ to be proportional to κ , the additional dependence on κ can be easily taken into account. At the level of Green functions, the renormalization group can be used to get rid of the κ -dependence of μ . The net effect is that it is irrelevant whether we consider μ as a heavy or light parameter. Having this in view and using the homogeneity of G with respect to its dimensional parameters:

$$G(Q, M, \kappa k, \kappa m, \mu) = \kappa^{\text{const}} G(Q/\kappa, M/\kappa, k, m, \mu/\kappa),$$

one sees that the problem (3.5) is equivalent to studying expansions of G in Q and $M \rightarrow \infty$, which is more readily associated with phenomenological problems. The form (3.5) of the expansion problem is more convenient for our purposes.

3.2 Perfect factorization.

The most important requirement which asymptotic expansions should satisfy in the framework of perturbative QFT is that the dependence on heavy and light parameters should be fully factorized (see the discussion in subsect. 2.2). Thus, if the expansion has the form¹⁰

$$G(Q, M, \kappa k, \kappa m) \underset{\kappa \rightarrow 0}{\simeq} \sum_i c_i(Q, M) G_i(\kappa k, \kappa m), \quad (3.6)$$

then not only G_i should be independent of Q and M , but also $c_i(Q, M)$ of k and m .

At the level of individual diagrams, the requirement of perfect factorization means that individual terms on the r.h.s. of (3.6) should be polynomials of κ and $\log \kappa$. This automatically implies that c_i are sums of powers and logs of Q and M , while G_i , of k and m . In a more general context, one should aim at obtaining expansions (3.6) in such a form that the scaling properties of, say, $c_i(Q, M)$ with respect to Q and M be as simple as possible.¹¹

The requirement of perfect factorization is completely universal and not limited to the case of Euclidean regimes.

¹⁰It has become customary to refer to c_i as to “coefficient functions” and to G_i as to “matrix elements”—the terminology inherited from the theory of short-distance OPE.

¹¹A priori the analytical form of dependence on the expansion parameter may not be known. In more general contexts—e.g. beyond the framework of perturbation theory—the language of dilatation group and its irreducible representations might be useful here.

3.3 Euclidean regimes.

The problem (3.5) is yet too general. However, the examples considered in section 2 demonstrate that it will remain sufficiently meaningful if we limit ourselves to the special case of *Euclidean asymptotic expansions*. Namely, we assume that

$$Q_l^2 < 0, \quad \text{for each } Q_l \in Q. \quad (3.7)$$

The problem (3.5) with the restriction (3.7) can be called *Euclidean asymptotic expansion problem*. It allows a full and explicit solution [12]–[22].

The meaning of the restriction (3.7) is as follows. Perform the Wick rotation of all integration momenta in Feynman diagrams into Euclidean region. Then only the light external momenta k will remain non-Euclidean. However, since one performs expansions with respect to k at $k = 0$, the fact of k being non-Euclidean is inessential from technical viewpoint. Therefore, from the very beginning one can assume to work in a purely Euclidean theory. This drastically simplifies the geometry of IR singularities¹² and, in the final respect, allows one to obtain an explicit and complete solution of the general Euclidean expansion problem.

Note also that it is for the Euclidean version of the general expansion problem that most clean phenomenological predictions can be made within QCD where asymptotic freedom allows one to perform renormalization-group improved calculations of the Q - and M -dependent coefficient functions of OPE-like expansions.

From now on, we are going to work in a purely Euclidean theory, so that the condition (3.7) is satisfied automatically.

3.4 Linear restrictions on heavy momenta.

For the heavy momenta on the l.h.s. of (3.2) the following restriction due to momentum conservation holds:

$$\sum_l Q_l = -\kappa \sum_j k_j \underset{\kappa \rightarrow 0}{\simeq} 0, \quad (3.8)$$

i.e. the sum of all heavy momenta is not itself heavy. A less trivial example of linear restrictions is provided by the light-by-light scattering problem (subsect. 2.5).

In general, one can allow any number of linear restrictions of the form

$$\sum_l c_l Q_l = O(\kappa) \quad (3.9)$$

to be imposed on the heavy momenta. In such a case we may assume that there exist linearly-independent momentum variables \bar{Q}_i which are independent of κ , and each Q_l is expressed as a linear combination of \bar{Q}_i and k with coefficients independent of κ :

$$Q_l = Q_l(\bar{Q}, k). \quad (3.10)$$

¹²For example, consider the singularity generated by the massless propagator $1/p^4$. In Minkowskian region it is smeared over the *light cone* $p^2 = 0$ while in Euclidean region it is localized at the *point* $p = 0$.

The restrictions (3.8), (3.9) should thus be automatically satisfied.

One should take care not to impose restrictions such as would make the Green functions to be expanded ill-defined. However, even if the restricted Green function develops a singularity at $\kappa = 0$ but is otherwise well-defined for all $\kappa \neq 0$, then the expansion problem still makes sense (the singularity will then show up as a contribution proportional to a non-positive power of κ) and can be treated by our methods¹³.

Among all possible linear restrictions, the so-called *natural restrictions* are of immediate physical interest. Such restrictions have the form

$$\sum_{l \in \mathcal{L}_\alpha} Q_l = O(\kappa) \underset{\kappa \rightarrow 0}{\simeq} 0, \quad (3.11)$$

where \mathcal{L}_α is a subset of the set of all l , and \mathcal{L}_α for different α are pairwise non-intersecting. In other words, the external lines corresponding to heavy momenta are organized into non-intersecting bunches, and within each such bunch the heavy momentum conservation holds separately.

One could also impose similar linear restrictions besides conservation on the light momenta k . We will not do this, however, because in all interesting cases the final formulae will not be affected thereby (unlike the case of restrictions on heavy momenta).

3.5 Contact terms.

As was discussed in subsect. 2.6, it is natural to consider Green functions as distributions over momentum variables; the expansions in the sense of the distribution theory are characterized, from the pragmatic viewpoint, by presence of contact terms proportional to δ -functions of heavy momenta Q in c_i on the r.h.s. of (3.6).

More specifically, let $F(\bar{Q})$ be a test function of the independent variables \bar{Q}_i . Then what one has in fact to expand is the expression

$$\int \left(\prod_i d\bar{Q}_i \right) F(\bar{Q}) G(Q(\bar{Q}, \kappa k), M, \kappa k, \kappa m, \mu) \underset{\kappa \rightarrow 0}{\simeq} ? \quad (3.12)$$

for arbitrary F . This means, first, that all c_i in (3.6) are distributions and, second, that (3.6) retains its asymptotic nature after termwise integration against $F(\bar{Q})$.

When there is no need to take into account contact terms, it is sufficient to consider Q as parameters analogous to heavy masses. Then we will say that one deals with the simplified expansion problem without contact terms. One should only be careful to fix Q at "non-exceptional" values at which the Q -dependent factors of the expansion are smooth in \bar{Q} . The exact criteria here are of little practical usefulness.

Note that in most cases of phenomenological interest the Green functions are integrable (although singular at some points) functions of the light momenta k . Therefore for the sake

¹³It should be noted that in this case the singular dependence on κ in the expansion will be localized within the matrix elements G_i in (3.6) while c_i will be insensitive to what value κ is set to.

of simplicity we do not introduce test functions for k ; it will be sufficient to assume that k are fixed at some non-exceptional values. Normally, if the Green function and all terms of the asymptotic expansion are integrable functions of k , then the expansion allows termwise integrations against k -dependent test functions; or, to put it formally, the operation of asymptotic expansion commutes with such integrations.

To summarize: we work within the perturbative QFT, always use the MS scheme for UV renormalization, consider arbitrary models, and wish to obtain full asymptotic expansions for arbitrary Green functions for the class of Euclidean asymptotic regimes. We allow linear restrictions to be imposed on the heavy momenta of the Green functions, and consider and expand the latter in the sense of distributions. In the expressions to be obtained, the heavy and light parameters must be fully factorized.

BASIC IDEAS

4 Why expand products of singular functions?

The central notion of the theory of As -operation is that of asymptotic expansion in the sense of distributions. In this section we present motivations for studying expansions of products of singular functions in the sense of distributions. Although we have assumed to work in Euclidean theories, our reasoning here remains valid in the most general Minkowskian case as well.

4.1 Local study of multiloop integrands.

We start with studying expansions of an arbitrary multiloop Feynman diagram G which can be taken to be one-particle irreducible. Let $G(p, \kappa)$ be its integrand (prior to UV renormalization) where p is the set of all loop momenta, and we have also explicitly shown the dependence on the expansion parameter κ . The final goal is to obtain an expansion in κ of the expression

$$\mathbf{R} \int dp G(p, \kappa), \quad (4.1)$$

where \mathbf{R} is the UV R -operation, while integration over p is performed in infinite limits. A scrutiny reveals, however, that the starting point should be the expansion not of the integral (4.1) but of the integrand considered as a distribution over p .

Indeed, whatever prescription for the R -operation were chosen, one would not be relieved of the necessity to study contributions to the above integrals from finite regions of the integration space. Therefore, it is natural first to replace the R -operation by a cut-off at large integration momenta. This allows one to strip the expansion problem of the collateral complications due

to UV renormalization—anyway, many diagrams are not UV divergent at all. The cut-off can be chosen in the form of a smooth function $H(p/\Lambda)$ equal to zero at all sufficiently large p :

$$\int dp G(p, \kappa) H(p/\Lambda). \quad (4.2)$$

A straightforward approach to expanding such integrals in κ would be to first expand their integrands. However, the formal Taylor expansion of $G(Q, \kappa)$ in powers of κ results, as a rule, in non-integrable singularities. For example, for a scalar propagator with a light mass $m = O(\kappa)$ one has:

$$\frac{1}{p^2 + m^2} = \frac{1}{p^2} - m^2 \frac{1}{p^4} + \dots \quad (4.3)$$

The singularities of the formal expansions like those on the r.h.s. of (4.3) will be called *Euclidean infrared*—or simply IR—*singularities*. They correspond to the so-called soft singularities which appear in studies of IR divergences in QED and QCD, while collinear singularities have no analogue in the Euclidean case.

The non-integrable singularities on the r.h.s. of (4.3) preclude substitution of (4.3) into (4.2). Emergence of such singularities also means that the integrals depend on κ non-analytically. From the technical point of view, it is our major task to develop systematic algorithms for treating such singularities in this and more complicated cases and to construct correct asymptotic expansions starting from the formal series like (4.3).

The traditional approach to the problem is to split the integration region into two parts: a neighbourhood \mathcal{O} of the point $p = 0$, and the complement of \mathcal{O} , and then to extract the non-analytical part of the expansion coming from the integral over \mathcal{O} by e.g. explicit integrations or using special tricks. Although it is the most straightforward way to establish the power-and-log nature of the expansion (cf. [1], [3]), it is practically impossible to obtain explicit expressions that were suitable for restoring the OPE-like combinatorial structures out of pieces coming from different diagrams. There is, nevertheless, another approach based on the ideas and notions of modern mathematics, namely, the theory of distributions [23], which we now proceed to describe.

Our key observation is that the same propagator (4.3) can appear within different diagrams and even in different places in the same diagram, so that it is natural to regard it as a distribution, the rest of $G(p, \kappa)$ playing the role of a test function. The same is also true of the groups of factors in $G(p, \kappa)$ containing more than one propagator, and of $G(p, \kappa)$ itself. More precisely, let \mathcal{O} be a small region of the integration space. Let $G^{\text{sing}}(p, \kappa)$ be the product of those factors from $G(p, \kappa)$ whose formal expansions are singular within \mathcal{O} . Denote the rest of G as G^{reg} . There exist many Feynman diagrams leading to the same G^{sing} but different G^{reg} 's. Therefore, G^{sing} can be regarded as a distribution in \mathcal{O} , while G^{reg} plays the role of a test function. (Such a reasoning leads to a very important localisation property of the As -operation considered below in subsect. 11.3.)

One can also observe that besides troublesome propagators like (4.3), there are factors—e.g. propagators with a heavy mass,

$$\frac{1}{M^2 + (p - \kappa k)^2}, \quad (4.4)$$

or vertex polynomials—whose expansion in κ is harmless. Denote the product of all such factors in G as G^{reg} and the rest of G as G^{sing} . Again, G^{reg} may vary for the same G^{sing} , so that the latter plays the role of a distribution and the former of a test function.

Summarizing our observations, we arrive at the conclusion that the fundamental technical problem is to learn to expand in κ the integrals like

$$\int dp \varphi(p) G(p, \kappa) \quad (4.5)$$

with arbitrary test functions $\varphi(p)$; an equivalent way to put it is that one has to obtain an expansion of $G(p, \kappa)$ in the sense of distributions [23] (for a precise definition see below subsect. 5.1).

The problem of studying expansions of products of singular functions with respect to parameters in the sense of the distribution theory will be referred to, in view of its paramount importance within our formalism, as *the Master problem*. It plays the crucial role in our theory, and all mathematical difficulties of analytical nature are concentrated in it.

4.2 Remarks.

(i) If φ in (4.5) is localized within a small region \mathcal{O} , then one can study local structure of the expansion of $G(p, \kappa)$ in each \mathcal{O} . Then an expansion valid for all p can be easily obtained using standard techniques of the distribution theory. Such a localization trick proves to be a powerful instrument in obtaining expansions of products of singular functions—see below sect. 11 and [20].

(ii) Transition from localized φ to $\varphi = 1$ requires an analysis of asymptotics of the corresponding distributions in the UV region $p \rightarrow \infty$. For example, as will be shown below, studying UV renormalization of the graph G is equivalent to expansion of $G(\lambda p, \kappa)$ considered as a distribution in $p \neq 0$, at $\lambda \rightarrow \infty$. One can show that this is a special case of the Master problem with a special choice of the expansion parameter. Studying how asymptotic properties of an expansion are affected by taking the limit $p \rightarrow \infty$ reduces to studying double As -expansions which is done in essentially the same manner as in the case of expansions with respect to one parameter [22].

(iii) The expansion problem as formulated above requires to consider Feynman diagrams as distributions in heavy external momenta. Such a formulation is very natural in the context of the Master problem. Indeed, if the diagram is integrated over the heavy momenta Q with the test function $\chi(Q)$, then instead of the integrals (4.5) one should expand integrals of the form

$$\int dp dQ \varphi(p) \chi(Q) G(p, Q, \kappa). \quad (4.6)$$

Combine both sets of momenta into one: $p' = (p, Q)$, consider G as a distribution in p' , and expand the expression

$$\int dp' \varphi'(p') G(p', \kappa). \quad (4.7)$$

One can see that, essentially, there is nothing new here as compared with (4.5). Choosing $\varphi'(p, Q) = \varphi(p) \chi(Q)$, we return to (4.5).

4.3 Analogy with UV renormalization.

As was pointed out in [12] there exists an analogy between the problem of UV divergences in Bogoliubov's interpretation [24] and the problem of singularities in expansions like (4.3): in both cases the divergences spring up as a result of formal manipulations with objects which by their nature are distributions. Therefore, Bogoliubov's solution of the UV problem provides an insight into how the divergences in expansions (4.3) should be handled.

Let us reason as follows (for definiteness consider (4.3)). An important observation is that the expansion (4.3) (with 4-dimensional p) allows (i.e. retains its meaning as an asymptotic expansion after) a termwise integration with any test function which is equal to zero in any arbitrarily small neighbourhood of the point $p = 0$. Therefore, the only way to improve upon the formal expansion (4.3) is to add terms localized at $p = 0$ to the r.h.s. So, it is to be expected that an expansion valid in the sense of distributions will look like

$$\frac{1}{p^2 + m^2} \underset{m \rightarrow 0}{\simeq} \frac{1}{p^2} - \frac{m^2}{p^4} + c(m) \delta(p) + \dots, \quad (4.8)$$

where $c(m)$ is a scalar coefficient.

The role of the new "counterterm" on the r.h.s. is two-fold. First, it should provide an infinite contribution to counterbalance the divergence due to the singularity of p^{-4} and ensure integrability of the r.h.s. as a whole against arbitrary test functions. Second, it should ensure the asymptotic character of the expansion. Note that the dependence of $c(m)$ on m can not be trivial, for there is no other place for the expected non-analyticity in m to show up.

Thus, heuristically speaking, our problem is to obtain explicit expressions for the coefficients $c(m)$ in (4.8) and in more complicated cases.

5 Asymptotic expansions of distributions.

Although asymptotic expansions and distributions did appear simultaneously in various contexts in the literature, it seems that no systematic study of the corresponding mathematical notions has ever been undertaken. Below are presented background mathematical definitions and some results of a general character. Since we are not aiming at attaining full mathematical formalization in the present paper, some technical details are omitted and others treated at a heuristic level. A substantial formal treatment of the subject can be found in [20].

5.1 Definition.

Let $F(p, \kappa)$ be a distribution over p depending on the parameter κ . We say that the series

$$F(p, \kappa) \underset{\kappa \rightarrow 0}{\simeq} \sum_{\pi} \kappa^{\pi} F_{\pi}(p, \kappa), \quad (5.1)$$

represents an asymptotic expansion in the sense of distributions if a termwise integration of (5.1) with any test function $\varphi(p)$ results in a correct asymptotic expansion in the usual sense, i.e.

$$\int dp F(p, \kappa) \varphi(p) \underset{\kappa \rightarrow 0}{\simeq} \sum_{n \leq N} \kappa^n \left[\int dp F_n(p, \kappa) \varphi(p) \right] + o(\kappa^N). \quad (5.2)$$

This implies that each F^n is a well-defined distribution over p . We say in such cases that the expansion (5.1) *allows integration* with arbitrary test functions.

We will usually have to deal with expansions such that each F_n is a polynomial in $\log \kappa$:

$$F_n(p, \kappa) = \sum_{i=0}^{I_n} \log^i \kappa F_{n,i}. \quad (5.3)$$

This form of κ -dependence will be called *soft* (see also below subsect. 5.3).

5.2 Uniqueness.

If an expansion like (5.1), (5.3) exists, it is unique. Indeed, integrating (5.1) with test functions we get asymptotic expansions in the usual sense, for which the uniqueness is an elementary fact [61]. It is important to understand that there is no point in discussing uniqueness unless the form of dependence on the expansion parameter is explicitly fixed, as in (5.1), (5.3).

The property of uniqueness is extremely important. First, it allows one to immediately obtain useful technical results like the localization property of the As -operation for products of singular functions (subsect. 11.3) or its commutativity with multiplications by polynomials (see below subsect. 12.5). Second, it implies that the expansions which we will derive cannot, in a sense, be improved upon.

5.3 Regularization.

It is often convenient to introduce a regularization into both sides of (5.1). We assume that it is controlled by a parameter ϵ , and taking it off corresponds to $\epsilon \rightarrow 0$. Denote the regularized version of the distribution f as $\{f\}^\epsilon$, so that

$$f(p) = \lim_{\epsilon \rightarrow 0} \{f(p)\}^\epsilon \quad (5.4)$$

on each test function φ . Then instead of (5.1), one can write

$$\{F(p, \kappa)\}^\epsilon \underset{\kappa \rightarrow 0}{\simeq} \sum_n \kappa^n \{F_n(p, \kappa)\}^\epsilon. \quad (5.5)$$

We stress that the expression (5.5) need not be a true asymptotic expansion for $\epsilon \neq 0$, but must only become such upon taking the limit $\epsilon \rightarrow 0$. Still, the dimensionally regularized expansions that we obtain for the standard Feynman integrands are asymptotic ones even for $\epsilon \neq 0$. This peculiar fact, however, seems to have no practical implications.

Within the dimensional regularization, (5.3) should be modified. Indeed, the dependence on κ takes the form

$$\{F_n(p, \kappa)\}^\epsilon = \sum_{i=0}^{I_n} \kappa^{\epsilon i} \{F'_{n,i}\}^\epsilon, \quad (5.6)$$

which will also be qualified as *soft*.

An important fact is that although the expansion (5.6) as a whole does have a well-defined finite limit at $\epsilon \rightarrow 0$ by construction, the quantities $\{F'_{n,i}(p)\}^\epsilon$, in general, do not: if one integrates such an object with a test function then the resulting expression will contain pole singularities as $\epsilon \rightarrow 0$. It is, of course, always possible to rearrange the r.h.s. in such a way as to make each term have a finite limit at $\epsilon \rightarrow 0$. To this end it is sufficient to form special linear combinations of the braced expressions. However, such a representation is of little practical interest, at least as long as one stays within the calculational framework of dimensional regularization and perturbative QFT. This is because in applications one has to perform calculations not with individual diagrams but with their sums. Summations over full sets of relevant diagrams result in considerable cancellations between them, especially in gauge models like QCD, so that the efforts spent on separate treatment of individual diagrams will be lost. On the other hand, there exist formulae [14] connecting the phenomenological quantities that one needs to calculate in the final respect, with the corresponding full sums of diagrams. Such formulae are compact and, as experience shows [28], very convenient, which explains our use of expansions in the dimensionally regularized form.

5.4 κ -dependent test functions.

As is clear from the motivations presented in subsect. 4.1, “natural” test functions that emerge after localization of the expansion problem, depend on the expansion parameter κ , so that a typical problem is to expand expressions like

$$\int dp \varphi(p, \kappa) F(p, \kappa). \quad (5.7)$$

Under certain conditions the expansion of (5.7) can be obtained by substituting the expansion (5.1) for F and a similar expansion for φ and reordering the terms in the product in ascending powers of κ . There are two such conditions [20].

(i) The first one is of little practical significance and is included for completeness' sake. Both $F(p, \kappa)$ and all $F_n(p, \kappa)$ must be distributions in the precise meaning of the word, i.e. (a) be linear functionals defined on all Schwartz test functions and (b) possess certain properties of continuity. However, there is no practical chance to encounter a functional satisfying (a) but not (b), as such functionals can only be proved to exist, by making use of the notorious Axiom of Choice, outside the realm of practical mathematics. There would be no need to mention (b), but there has not yet been enough time for the alternative Axiom of Determination due to Myczelsky and Steinhaus (see e.g. [62]) to purge the mathematical textbooks of pathological counterexamples.

(ii) The really important condition imposes restrictions on the asymptotic expansion of the

test function. Namely, if φ is expanded as

$$\varphi(p, \kappa) \simeq \sum_n \kappa^n \varphi_n(p, \kappa) \quad (5.8)$$

(where the dependence of $\varphi_n(p, \kappa)$ on κ is soft, as in (5.3)), then the expression

$$[\varphi(p, \kappa) - \sum_{n \leq N} \kappa^n \varphi_n(p, \kappa)] / \kappa^N \quad (5.9)$$

should tend to zero as $\kappa \rightarrow 0$ in the following sense. The expression in the square brackets must be non-zero only within a compact region K of the integration space for all $\kappa \neq 0$; the expression (5.9), as well as its derivatives in p , must tend to zero uniformly with respect to p .

For the Euclidean expansion problem within the standard perturbation theory one has to consider only κ -dependent test functions of the form (cf. (7.41))

$$\varphi(p, \kappa) = \psi(p) f(p, \kappa), \quad (5.10)$$

where $\psi(p)$ is a Schwartz test function, while $f(p, \kappa)$ is a rational function of p and κ and such that all its singularities are localized at points p where $\psi(p) = 0$, for all κ . It is not difficult to understand that for such $\varphi(p, \kappa)$ condition (ii) is satisfied.

6 The extension principle.

Having developed a suitable language, we can turn to the expansion problem proper. Our purpose now is to offer a very general and abstract framework for studying asymptotic expansions of distributions. We will introduce some notions and present simple but important propositions under a general heading of *extension principle* which constitutes, essentially, a context in which to work with explicit problems. To make it work for a particular problem, specific estimates and bounds must be established, which may technically be the most cumbersome part of the problem. But the importance of the conceptual framework of the extension principle consists in the fact that it allows one to guess the structure of the expansions to be obtained, shows what kind of estimates are required, and is, therefore, a powerful heuristic tool.

6.1 Motivations.

Let $F(p, \kappa)$ be a distributions in $p \in P$ (where P is a Euclidean space) depending on the expansion parameter κ . For simplicity assume that $F(p, \kappa)$ is an integrable function of p for all $\kappa \neq 0$.

A typical situation which we will have to deal with can be described as follows. One can

obtain (e.g. using the Taylor theorem) an approximation¹⁴

$$F(p, \kappa) \underset{\kappa \rightarrow 0}{\simeq} \bar{F}_N(p, \kappa) + o(\kappa^N), \quad (6.1)$$

where $\bar{F}_N(p, \kappa)$ is an integrable function for $p \in \mathcal{O}$ where \mathcal{O} is an open region in P , and has non-integrable singularities at the points from $P \setminus \mathcal{O}$; the notation¹⁵ $o(\kappa^N)$ stands for the terms that vanish faster than κ^N as $\kappa \rightarrow 0$. Suppose that the expansion (6.1) is such that it is valid pointwise for each fixed $p \in \mathcal{O}$, and allows termwise integration with arbitrary test functions localized within \mathcal{O} . It is required to obtain an approximation for $F(p, \kappa)$ to the same order $o(\kappa^N)$ valid in the sense of the distribution theory on the entire P .

Let us look at the problem from a different angle. Let L be the vector space of all test functions $\varphi(p)$, $p \in P$, and L_N the set of the test functions φ localized within \mathcal{O} . L_N forms a vector subspace in L . The distribution $F(p, \kappa)$ is, by definition, a linear functional on L , which we will denote as F_κ . On the other hand, $\bar{F}_N(p, \kappa)$ generates a linear functional $\bar{F}_{\kappa, N}$ defined on L_N , and the restriction of F_κ to L_N can be approximated as

$$[F_\kappa]_{L_N} \underset{\kappa \rightarrow 0}{\simeq} \bar{F}_{\kappa, N} + o(\kappa^N), \quad (6.2)$$

which is to be understood so that (6.2) should be valid in the usual sense if one evaluates both sides of (6.2) on arbitrary vectors $\varphi \in L_N$. The problem is to construct an approximation

$$F_\kappa \underset{\kappa \rightarrow 0}{\simeq} F_{\kappa, N} + o(\kappa^N), \quad (6.3)$$

which would be valid on the entire L . Therefore, one can say that the problem consists in extending the approximating functional defined on the subspace onto the entire space in such a way as to preserve its approximation properties. Its solution is provided by the so-called *extensions principle* [12].¹⁶

6.2 The extension principle.

Let L be a linear space and F_κ a linear functional on it which depends on the expansion parameter κ . Assume there exists a subspace $L_N \subset L$ and a functional $\bar{F}_{\kappa, N}$ defined on L_N , such that $\bar{F}_{\kappa, N}$ approximates F_κ to order $o(\kappa^N)$ on L_N . More precisely, for each $\varphi \in L_N$

$$\langle F_\kappa - \bar{F}_{\kappa, N}, \bar{\varphi} \rangle = o(\kappa^N). \quad (6.4)$$

¹⁴In this section we consider not full asymptotic expansions, but approximations to a given order $o(\kappa^N)$. Transition to infinite asymptotic series can also be treated in an abstract manner but there is not much wisdom to be gained from it, so we postpone the discussion of this point till the next section where an example is analyzed in detail.

¹⁵The more familiar notation $O(\kappa^N)$ is often used in theoretical physics to denote terms behaving as κ^N , possibly with logarithmic corrections. Note that a priori we can say nothing about the behaviour of the remainder of the full expansion we wish to construct except that it should vanish faster than the last term retained. That is why the o -notation is preferable to the O -notation in our case.

¹⁶Our extension principle belongs to the Hahn-Banach type results. The classical Hahn-Banach theorem (see any textbook on functional analysis) considers extension of functionals preserving the property of being bounded by a seminorm. In our case, the extensions should preserve the approximation property.

We wish to construct a functional $F_{\kappa,N}$ defined and approximating F_κ to order $o(\kappa^N)$ on the entire L , and coinciding with $\bar{F}_{\kappa,N}$ on L_N .

To this end, let φ_α be a set of vectors transverse to L_N and such that each $\varphi \in L$ can be uniquely expanded as

$$\varphi = \sum k_\alpha \varphi_\alpha + \bar{\varphi}, \quad (6.5)$$

where $\varphi \in L_N$. It is sufficient to define $F_{\kappa,N}$ on φ_α , because on $\bar{\varphi}$ its value is known. Now, the desired definition is:

$$\langle F_{\kappa,N}, \varphi_\alpha \rangle = \langle F_\kappa, \varphi_\alpha \rangle + o(\kappa^N), \quad (6.6)$$

where we have indicated that the definition allows an arbitrariness of order $o(\kappa^N)$, which should not be surprising.

It is not difficult to check that $F_{\kappa,N}$ thus defined is unique within accuracy $o(\kappa^N)$.

6.3 Counterterms and consistency conditions.

The above construction can be represented as the following two-step procedure. Suppose it is possible to find an extension $\bar{F}_{\kappa,N}^R$ of $\bar{F}_{\kappa,N}$ from L_N onto the entire L , which does not necessarily approximate F_κ on L but has, perhaps, some other nice properties. Let δ_β be a set of functionals on L such that

$$\langle \delta_\beta, \varphi \rangle = 0, \quad \text{for all } \varphi \in L_N, \quad (6.7)$$

and

$$\langle \delta_\beta, \varphi_\alpha \rangle = \delta_{\alpha,\beta} \quad (\text{the Kronecker symbol}). \quad (6.8)$$

(Instead of (6.8), one can simply require that δ_β be a maximal linearly independent set of functionals satisfying (6.7).) Then $F_{\kappa,N}$ should be representable as

$$F_{\kappa,N} = \bar{F}_{\kappa,N}^R + \sum_\alpha c_\alpha^R(\kappa) \delta_\alpha, \quad (6.9)$$

where $c_\alpha^R(\kappa)$ are unknown coefficients. $F_{\kappa,N}$ thus defined reduces to $\bar{F}_{\kappa,N}$ on each $\bar{\varphi} \in L_N$, so owing to (6.4) it is sufficient to require that (6.9) approximate F_κ to $o(\kappa^N)$ on each φ_α :

$$\langle [F_\kappa - F_{\kappa,N}], \varphi_\alpha \rangle = o(\kappa^N). \quad (6.10)$$

This *consistency condition* [12], [15] allows one to find the unknown coefficients in (6.9):

$$c_\alpha^R(\kappa) = \langle [F_\kappa - \bar{F}_{\kappa,N}^R], \varphi_\alpha \rangle + o(\kappa^N). \quad (6.11)$$

It is not difficult to check the equivalence of this construction and that of subsect. 6.2. Note that the recipe (6.9) is more in the spirit of the distribution theory, for one can find a correct form of the required extension (6.9), up to some coefficients, practically without reference to test functions that only make their appearance in the consistency conditions (6.10).

6.4 Remark.

Assume that φ_α in (6.10) depend on a parameter Λ without violating (6.8): $\varphi_\alpha = \varphi_\alpha^\Lambda$, and assume that there exists a finite limit:

$$\lim_{\Lambda \rightarrow \infty} \langle [F_\kappa - \bar{F}_{\kappa,N}^R], \varphi_\alpha^\Lambda \rangle \quad (6.12)$$

even if φ_α^Λ itself does not converge to any vector in L . Now, if performing the limit $\Lambda \rightarrow \infty$ does not take one outside the $o(\kappa^N)$ uncertainty, i.e. if the limiting value (6.12) and

$$\langle [F_\kappa - \bar{F}_{\kappa,N}^R], \varphi_\alpha^\Lambda \rangle \quad (6.13)$$

are $o(\kappa^N)$, then instead of (6.11) one may take:

$$c_\alpha^R(\kappa) = \lim_{\Lambda \rightarrow \infty} \langle [F_\kappa - \bar{F}_{\kappa,N}^R], \varphi_\alpha^\Lambda \rangle + o(\kappa^N). \quad (6.14)$$

The option to take such limits proves to be very important—see below subsect. 7.5.

6.5 Regularized form of the extension principle.

There is a third way to represent the extended approximating functional $F_{\kappa,N}$, which will be widely used in what follows. Define a *regularization* of $\bar{F}_{\kappa,N}$, i.e. a functional $\bar{F}_{\kappa,N}^\epsilon$ defined on the entire L which depends on an additional *regularization parameter* ϵ and such that $\bar{F}_{\kappa,N}^\epsilon \rightarrow \bar{F}_{\kappa,N}$ on L_N as $\epsilon \rightarrow 0$, i.e.

$$\lim_{\epsilon \rightarrow 0} \langle [\bar{F}_{\kappa,N}^\epsilon - \bar{F}_{\kappa,N}], \bar{\varphi} \rangle = 0, \quad \text{for each } \bar{\varphi} \in L_N. \quad (6.15)$$

It is easy to check that the definition (6.9) can now be rewritten as

$$F_{\kappa,N} = \lim_{\epsilon \rightarrow 0} \left[\bar{F}_{\kappa,N}^\epsilon + \sum_\alpha c_\alpha^\epsilon(\kappa) \delta_\alpha \right] + o(\kappa^N), \quad (6.16)$$

where

$$c_\alpha^\epsilon(\kappa) = \langle [F_\kappa - \bar{F}_{\kappa,N}^\epsilon], \varphi_\alpha \rangle + o(\kappa^N). \quad (6.17)$$

Here one can also use parameterized sequences of φ_α as in (6.14).

6.6 Summary.

The procedure of constructing an approximating functional now is as follows. One has the linear functional F_κ on L which is to be expanded in κ , and one is given another functional $\bar{F}_{\kappa,N}$ which approximates F_κ to a given accuracy on many but not all vectors from L . The key point is to identify the maximal subspace L_N on which $\bar{F}_{\kappa,N}$ approximates F_κ . After this is done, the rest consists essentially in adding to $\bar{F}_{\kappa,N}$ a linear combination of functionals δ_α which vanish on L_N , with coefficients determined by the consistency conditions.

In the context of subsect. 6.1, the fact that δ_α must be zero on L_N can be interpreted as that the corresponding distributions are localized in the complement $P \setminus \mathcal{O}$ of the open set \mathcal{O} .

Let us now turn to an example.

7 Example: expansion of $(p^2 + \kappa^2)^{-1}$.

The abstract considerations of the previous section provide us with a general approach to constructing expansions of distributions. Now we wish to show how it works in practice. There are also some important details—e.g. the transition from approximations to a given order to full asymptotic expansions—which can be more conveniently explained using an example.

7.1 Formal expansion.

Let P be a 4-dimensional Euclidean space; its elements will be denoted as p . Consider the scalar propagator

$$F(p, \kappa) = \frac{1}{p^2 + \kappa^2}. \quad (7.1)$$

We wish to obtain its expansion for $\kappa \rightarrow 0$ valid in the sense of the distribution theory.

First consider the formal expansion to order $o(\kappa^2)$:

$$\frac{1}{p^2 + \kappa^2} = \frac{1}{p^2} - \kappa^2 \frac{1}{p^4} + o(\kappa^2). \quad (7.2)$$

It holds pointwise for any $p \neq 0$. Moreover, it allows integration with any test function which is equal to zero in some small neighbourhood of $p = 0$. But most important is that although it does not allow integrations with arbitrary test functions owing to the non-integrable singularity in the $O(\kappa^2)$ term on the r.h.s., still it allows integration with test functions $\bar{\varphi}$ which only satisfy the condition $\bar{\varphi}(0) = 0$, apparently because then $\bar{\varphi}(p) \sim p$, $p \rightarrow 0$, and the singularity is effectively suppressed.

7.2 Approximation properties of the formal expansion.

More precisely, one has to prove that

$$\int d^4p \bar{\varphi}(p) \left[\frac{1}{p^2 + \kappa^2} - \frac{1}{p^2} + \kappa^2 \frac{1}{p^4} \right] = o(\kappa^2). \quad (7.3)$$

The simplest way¹⁷ to do this is as follows. One splits the integral as

$$\int = \int_{|p| < c\kappa} + \int_{|p| > c\kappa}. \quad (7.4)$$

For $|p| < c\kappa$, one rescales $p \rightarrow \kappa p$, uses the scaling properties of the bracketed expression and the fact that $\bar{\varphi}(p) \sim p$, $p \rightarrow 0$, and finds that

$$\int_{|p| < c\kappa} \sim O(\kappa^3) = o(\kappa^2). \quad (7.5)$$

¹⁷pointed out by S.Yu.Khlebnikov.

For $|p| > c\kappa$, one uses the Taylor theorem to estimate the bracketed expression by an expression similar to the first discarded term in (7.2), namely, $\text{const} \times \kappa^4/p^6$, estimates $\bar{\varphi}(p)$ by $\text{const} \times |p|$ and integrates over $c\kappa < |p| < M < +\infty$ where M depends on $\bar{\varphi}$. The result is

$$\int_{|p|>c\kappa} \sim O(\kappa^4 \log \kappa) = o(\kappa^2). \quad (7.6)$$

The desired result (7.3) follows immediately from (7.4)-(7.6).

We have discussed the proof in detail because the same steps are performed in the most general case as well.

Now we are going first to implement the recipe of subsect. 6.3. After this is done, we will cast the results into a regularized form corresponding to subsect. 6.4, using the dimensional regularization.

7.3 Intermediate operation $\tilde{\mathbf{R}}$.

Before proceeding further, let us construct a distribution $\tilde{\mathbf{R}}[p^{-4}]$ defined on all test functions and such that the equality

$$\frac{1}{p^4} = \tilde{\mathbf{R}} \left[\frac{1}{p^4} \right]. \quad (7.7)$$

be valid on the test functions $\bar{\varphi}$ defined above. An explicit expression for such a distribution can be found, if one fixes any one test function $\psi(p)$ such that $\psi(0) = 1$, and integrates both sides of (7.7) with $\varphi(p) - \varphi(0)\psi(p)$. Then one finds:

$$\int d^4p \tilde{\mathbf{R}} \left[\frac{1}{p^4} \right] \varphi(p) = \int d^4p \left\{ \frac{1}{p^4} [\varphi(p) - \varphi(0)\psi(p)] + c\delta(p)\varphi(p) \right\}, \quad (7.8)$$

where

$$c = \int d^4p \tilde{\mathbf{R}} \left[\frac{1}{p^4} \right] \psi(p) \quad (7.9)$$

is a single constant to be fixed—which can be done arbitrarily—in order that $\tilde{\mathbf{R}}[p^{-4}]$ were fully defined. Introducing a regularization into the r.h.s. of (7.8), one gets a representation of $\tilde{\mathbf{R}}[p^{-4}]$ “in terms of infinite counterterms”:

$$\tilde{\mathbf{R}} \left[\frac{1}{p^4} \right] = \lim_{\epsilon \rightarrow 0} \left[\left\{ \frac{1}{p^4} \right\}^\epsilon + Z_\epsilon \delta(p) \right], \quad (7.10)$$

where $\{\dots\}^\epsilon$ denotes the regularization, e.g.

$$\left\{ \frac{1}{p^4} \right\}^\epsilon = \frac{1}{p^4 + \epsilon^4}, \quad \text{or} \quad \left\{ \frac{1}{p^4} \right\}^\epsilon = \Phi(|p| > \epsilon) \frac{1}{p^4} \quad \text{etc.}, \quad (7.11)$$

where we have introduced a convenient function

$$\Phi(x) \equiv 1, \quad \text{if the logical expression } x \text{ is “true”,} \quad (7.12)$$

0, otherwise.

It might be helpful to note that the distribution $\tilde{\mathbf{R}}[p^{-4}]$ defined above is similar to the so-called “+”-distribution,

$$\int_0^1 \varphi(x) \left[\frac{1}{x} \right]_+ dx \equiv \int_0^1 \frac{[\varphi(x) - \varphi(0)]}{x} dx, \quad (7.13)$$

which is widely used in the QCD parton model calculations.

7.4 Consistency condition.

According to the recipe of subsect. 6.3, we should first redefine the r.h.s. of (7.2) in such a way as to make it well-defined on all test functions. To this end we replace p^{-4} in (7.2) by $\tilde{\mathbf{R}}[p^{-4}]$ without changing values of the r.h.s. on $\bar{\varphi}$, and obtain an expression

$$\frac{1}{p^2 + \kappa^2} = \frac{1}{p^2} - \kappa^2 \tilde{\mathbf{R}} \left[\frac{1}{p^4} \right] + o(\kappa^2), \quad (7.14)$$

which holds, as (7.2), on $\bar{\varphi}$, but unlike (7.2), the r.h.s. is well-defined on all $\varphi(p)$.

The only distribution that can be added to the r.h.s. of (7.14) without violating the asymptotic character of (7.14) on $\bar{\varphi}$ is $\delta(p)$. The immediate conclusion is that the expansion of $(p^2 + \kappa^2)^{-1}$ to order $o(\kappa^2)$ in the sense of the distribution theory can only have the form

$$\frac{1}{p^2 + \kappa^2} = \frac{1}{p^2} - \kappa^2 \tilde{\mathbf{R}} \left[\frac{1}{p^4} \right] + c_0^{\mathbf{R}}(\kappa) \delta(p) + o(\kappa^2), \quad (7.15)$$

and is defined up to a single numeric-valued (non-analytic) function of κ , $c_0^{\mathbf{R}}(\kappa)$.

In order to determine $c_0^{\mathbf{R}}(\kappa)$, recall the prescription of subsect. 6.3. One notes that any test function $\varphi(p)$ can be represented as

$$\varphi(p) = \varphi(0)\psi(p) + \bar{\varphi}(p), \quad (7.16)$$

where $\psi(p)$ has already been defined and $\bar{\varphi}(0) = 0$. On such $\bar{\varphi}$ (7.15) degenerates into (7.2) and, therefore, is valid. Hence, in order that (7.15) be valid on all test functions, one only has to ensure that it holds on $\psi(p)$. Let us take this as a definition of $c_0^{\mathbf{R}}(\kappa)$ (a “consistency condition”, cf. (6.10)). Then, integrating both sides of (7.15) with $\psi(p)$, one gets:

$$c_0^{\mathbf{R}}(\kappa) = \int d^4p \psi(p) \left\{ \frac{1}{p^2 + \kappa^2} - \frac{1}{p^2} + \kappa^2 \tilde{\mathbf{R}} \left[\frac{1}{p^4} \right] \right\} + o(\kappa^2). \quad (7.17)$$

Note that $c_0^{\mathbf{R}}(\kappa)$ is determined at this stage only up to $o(\kappa^2)$. This agrees with the fact that ψ has been chosen arbitrarily, because a change in ψ preserving the condition $\psi(0) = 1$ will change $c_0^{\mathbf{R}}(\kappa)$ by a term of order $o(\kappa^2)$.

7.5 Dependence of counterterms on κ .

Although eqs.(7.15), (7.17) provide a correct expansion of the propagator in the sense of the distribution theory to $o(\kappa^2)$, one further important refinement is possible in the spirit of subsect. 6.4. It is natural to try to find a $\psi(p)$ such that would simplify (7.17) as much as possible. To this end, notice that the square-bracketed expression in (7.17) is bounded by $\text{const} \times \kappa^4/p^6$ at $p \rightarrow \infty$. Therefore, $c_0^R(\kappa)$ will be changed by a finite $o(\kappa^2)$ contribution if one simply takes $\psi(p) \equiv 1$. More precisely, let $\psi(p) = \Phi(p/\Lambda)$ and let $\Lambda \rightarrow \infty$. The resulting expression for $c_0^R(\kappa)$ is:

$$c_0^R(\kappa) = \int d^4p \left\{ \frac{1}{p^2 + \kappa^2} - \frac{1}{p^2} + \kappa^2 \tilde{\mathbf{R}} \left[\frac{1}{p^4} \right] \right\}. \quad (7.18)$$

The distribution $\tilde{\mathbf{R}}[p^{-4}]$ has a simple scaling behaviour:

$$F(\lambda p) = \lambda^{-4} [F(p) + \text{const} \times \log \lambda \delta(p)] \quad (7.19)$$

(which can be easily deduced from the definition (7.8)). Therefore, replacing $p \rightarrow \kappa p$ in (7.18) and using (7.19), one gets:

$$c_0^R(\kappa) = \kappa^2 [K_1 + K_2 \log \kappa], \quad (7.20)$$

where K_1 and K_2 are constants independent of κ . The fact of a simple dependence of $c_0^R(\kappa)$ as defined in (7.18) on κ singles out this definition from the family (7.17).

7.6 The remainder.

Another way to check the validity of the expansion (7.15) with $c_0^R(\kappa)$ given by (7.18), is to notice that the remainder can be represented as:

$$\begin{aligned} & \int d^4p \varphi(p) \left\{ \frac{1}{p^2 + \kappa^2} - \frac{1}{p^2} + \kappa^2 \tilde{\mathbf{R}} \left[\frac{1}{p^4} \right] - c_0^R(\kappa) \delta(p) \right\} \\ &= \int d^4p [\varphi(p) - \varphi(0)] \left\{ \frac{1}{p^2 + \kappa^2} - \frac{1}{p^2} + \kappa^2 \frac{1}{p^4} \right\}. \end{aligned} \quad (7.21)$$

(By the way, this representation verifies that the r.h.s. of (7.15) is independent of the arbitrary constant in the definition of $\tilde{\mathbf{R}}[p^{-4}]$ (7.8).)

To check that the r.h.s. is $o(\kappa^2)$ one splits the integration region into two subregions: $|p| < K\kappa$ and $|p| > K\kappa$ with K appropriately chosen. For the first subregion one represents $\varphi(p) - \varphi(0)$ as $\sum p_i \tilde{\varphi}_i(p)$ where all $\tilde{\varphi}_i(p)$ are smooth everywhere including $p = 0$ [63], and then performs the scaling $p \rightarrow \kappa p$. In the second region, one bounds the first factor in the integrand on the r.h.s. by a constant, and uses the Taylor theorem to estimate the second factor. Then one easily obtains the desired estimate. (For more details see [20].)

7.7 IR counterterms as UV renormalized integrals.

Let us discuss the expression (7.18). Introduce again the cut-off $\Phi(p/\Lambda)$ explicitly into the integrand of (7.18). Then one can rewrite (7.18) as:

$$c_0^R(\kappa) = \lim_{\Lambda \rightarrow \infty} \left\{ \int_{|p| < \Lambda} d^4 p \frac{1}{p^2 + \kappa^2} - K'_1 \Lambda^2 - \kappa^2 (K'_2 + K'_3 \log \Lambda) \right\}, \quad (7.22)$$

and the limit is finite by construction.

The first term on the r.h.s. is the vacuum average of the operator $T[\varphi^2(x)]$ (without normal ordering), while the rest play the role of its UV counterterms.

We take note of this fact for two reasons. First, the coefficients of IR counterterms analogous to c_0^R in more complicated cases will also admit simple interpretation as contributions to Green functions with local operator insertions. Second, reversing the course of thought, one can view (7.18) as a peculiar¹⁸ representation of an UV-renormalized one-loop Feynman graph in terms of subtraction of asymptotics of the integrand at large integration momenta. Such a representation will play an important role in our further study of EA-expansions, effectively reducing the case of UV-divergent diagrams to that of UV-convergent ones (cf. below subsect. 8.3 and [21]).

7.8 Dimensionally regularized form.

The form (7.15), (7.18) for the expansion is not the only one possible, and, definitely, not the most convenient one. If one uses a regularization, then, taking into account (7.18), one has the following expression instead of (7.15) and (7.18):

$$\frac{1}{p^2 + \kappa^2} = \lim_{\epsilon \rightarrow 0} \left[\frac{1}{p^2} - \kappa^2 \left\{ \frac{1}{p^4} \right\}^\epsilon + c_0^\epsilon(\kappa) \delta(p) + o(\kappa^2), \right] \quad (7.23)$$

where

$$c_0^\epsilon(\kappa) = \int d^4 p \left[\frac{1}{p^2 + \kappa^2} - \frac{1}{p^2} + \kappa^2 \left\{ \frac{1}{p^4} \right\}^\epsilon \right]. \quad (7.24)$$

Much simpler expressions emerge within the dimensional regularization. (Note that in this case one regularizes also the expression to be expanded as well as those terms of the expansion which actually do not need to be regularized, but this can do no harm. For simplicity of notation we will omit the limit signs.) Thus, instead of (7.23)–(7.24) one gets:

$$\frac{1}{p^2 + \kappa^2} = \frac{1}{p^2} - \kappa^2 \frac{1}{p^4} + c_0(\kappa) \delta(p) + o(\kappa^2), \quad (7.25)$$

where

$$c_0(\kappa) = \int d^D p \left[\frac{1}{p^2 + \kappa^2} - \frac{1}{p^2} + \kappa^2 \frac{1}{p^4} \right] = \int d^D p \frac{1}{p^2 + \kappa^2} \quad (7.26)$$

¹⁸or natural—depending on one's prejudices.

(we take $\epsilon = (4-D)/2$). The terms on the r.h.s. vanished owing to the fact that the dimensional regularization preserves the formal scale invariance, so that e.g.

$$\int d^D p \frac{1}{p^{2\alpha}} \implies p \rightarrow \lambda p \implies \lambda^{D-2\alpha} \int d^D p \frac{1}{p^{2\alpha}} = 0. \quad (7.27)$$

Another consequence of this fact is:

$$c_0(\kappa) = (\kappa^2)^{D/2-1} c_0(1), \quad (7.28)$$

i.e. the dependence on κ takes the form discussed in subsect. 5.3.

Of the three forms of the expansion ((7.15) and (7.18); (7.22) and (7.23); (7.25) and (7.26)), the most important are: the first one since it does not use artificial regularizations and contains only explicitly convergent terms, and the third one which owing to its simplicity is most convenient in applications. Note, however, that the representation in terms of dimensional regularization (7.25), (7.26) is not devoid of some mysticism: indeed, the IR divergence of the term p^{-4} on the r.h.s. of (7.25) is compensated by a divergent part of $c_0(\kappa)$ which, judging from the r.h.s. of (7.26), has an UV origin. However, one should firmly remember that that is due to the fact that the dimensional regularization nullifies certain types of integrals (cf. (7.27)) and formally preserves scale invariance which establishes a connection between IR and UV contributions. Therefore, it becomes possible for an expression to contain both types of divergences balanced so as to render the whole finite. In more conventional regularizations there occur only compensations of divergences of the same nature (cf. [22]).

7.9 Complete expansion.

Now let us turn to higher terms in the expansion (7.15). To derive them, instead of following the recipe of the extension principle, we will use a trick motivated by the representation of the remainder of the expansion in the form of (7.21).¹⁹ The advantage of it is that it provides a short-cut for the reader familiar with the dimensional regularization to arrive at the final formulae. However, this trick cannot be extended to more complicated situations of Minkowskian asymptotic regimes.

Consider the expression

$$\int d^4 p A(p) B(p) = o(\kappa^{2N}), \quad (7.29)$$

where

$$A(p) = \frac{1}{p^2 + \kappa^2} - \sum_{n=0}^N (-\kappa^2)^n \frac{1}{p^{2(n+1)}}, \quad (7.30)$$

$$B(p) = \varphi(p) - \mathbf{T}_{\omega_N} \varphi(p), \quad (7.31)$$

$$\omega_N = 2(N-1), \quad (7.32)$$

¹⁹ An essentially similar trick was used in [39].

and

$$\mathbf{T}_{\omega_N} \varphi(p) \equiv \sum_{n=0}^{\omega_N} \frac{p^n}{n!} \varphi^{(n)}(0) \quad (7.33)$$

is a partial Taylor series. The expression (7.29) will turn out to be the remainder of an expansion to order $o(\kappa^{2N})$.

The fact that the r.h.s. of (7.29) is indeed $o(\kappa^{2N})$ can be explained as follows. $B(p)$ is $O(p^{\omega_N+1})$ at $p \rightarrow 0$ and $O(p^{\omega_N})$ at $p \rightarrow \infty$, and is independent of κ . $A(p)$ is $o(\kappa^{2N})$ for $p \neq 0$, has a non-integrable singularity $O(p^{2/(N+1)})$ at $p \rightarrow 0$, and is bounded by $O(\kappa^{2/(N+1)}/p^{2(N+2)})$ at $p \rightarrow \infty$. One can see that the non-integrable singularity of $A(p)$ at $p \rightarrow 0$ is suppressed by $B(p)$, while the growth of $B(p)$ at $p \rightarrow \infty$ is not rapid enough to spoil integrability of $A(p)$ at infinity. A more formal proof proceeds along the lines of subsect. 7.6.

There are two remarks worth making here. First, the interplay of asymptotics in (7.29) is only possible owing to the uniformity of $(p^2 + \kappa^2)^{-1}$ with respect to simultaneous scaling in p and κ . Second, the expression on the l.h.s. of (7.29) contains no cut-offs, and is therefore devoid of arbitrariness.

To represent the final result in a more convenient form, introduce a full basis of homogeneous polynomials $\mathcal{P}_\alpha(p)$, such that

$$\mathcal{P}_\alpha(\lambda p) = \lambda^{|\alpha|} \mathcal{P}_\alpha(p), \quad (7.34)$$

and the dual set of derivatives of the δ -function :

$$\int d^D p \mathcal{P}_\alpha(p) \delta_\beta(p) = \delta_{\alpha,\beta} \quad (7.35)$$

(cf. (6.8)). Then the Taylor expansion can be represented as:

$$\mathbf{T}_\omega \varphi(p) = \sum_{|\alpha| \leq \omega} \mathcal{P}_\alpha(p) \left[\int d^D p' \delta_\alpha(p') \varphi(p') \right]. \quad (7.36)$$

For the example under consideration, one can take

$$\mathcal{P}_\alpha(p) \rightarrow p^{2\alpha}, \quad \delta_\alpha(p) \rightarrow \partial^{2\alpha} \delta(p), \quad a = 0, 1, 2, \dots, \quad (7.37)$$

owing to the $O(D)$ -invariance.

Finally, introduce the dimensional regularization into the (convergent) expression on the l.h.s. of (7.29). Then integrate it using (7.27). Simple transformations using (7.36) lead to the following final result, wherein $\int d^D p \varphi(p)$ has been omitted from both sides:

$$\frac{1}{p^2 + \kappa^2} = \sum_{n=0}^N (-\kappa^2)^n \frac{1}{p^{2(n+1)}} + \sum_{|\alpha| \leq \omega_N} c_\alpha(\kappa) \delta_\alpha(p) + o(\kappa^{2N}), \quad (7.38)$$

where

$$c_\alpha(\kappa) = \int d^D p \mathcal{P}_\alpha(p) \frac{1}{p^2 + \kappa^2} = \kappa^{D-2+|\alpha|} c_\alpha(1), \quad (7.39)$$

and ω_N is given by (7.32). One can easily find from (7.39), which order in κ each counterterm belongs to, and one can see that the additional dependence on κ in each order in κ is soft in the sense of subsect. 5.3.

To avoid misunderstanding, we stress once more that the expansion (7.38) is to be understood in the following sense (cf. the remarks in subsect. 5.3). One should first integrate both sides termwise against a test function, and take off the regularization (i.e. take the limit $D \rightarrow 4$). The resulting expression will represent a correct expansion of a κ -dependent function in powers and logarithms of κ to order $o(\kappa^{2N})$.

Concerning cancellations of the IR divergences in the first sum on the r.h.s. of (7.38) by the apparently UV divergent expressions in (7.39), see the end of subsect. 7.8.

Note also that (7.38) is a true infinite asymptotic expansion of $(p^2 + \kappa^2)^{-1}$ in the sense of distributions, and after the regularization is taken off, one only has powers and logarithms of κ on the r.h.s. of (7.38), and such an expansion is unique (cf. subsect. 5.2).

7.10 Several κ -dependent factors.

Let us consider an example with several factors, such that the singularities of their expansions are located at different points. For example, consider the following integrand corresponding to a self-energy contribution:

$$\frac{1}{(p^2 + \kappa^2)} \times \frac{1}{[(p - Q)^2 + \kappa^2]}. \quad (7.40)$$

We regard the external momentum Q as a heavy parameter and fix it at a non-zero value, which corresponds to the simplified version of the expansion problem without contact terms (subsect. 3.5). The final recipe for expanding (7.40) in the sense of distributions will be to simply multiply the expansions for the two factors. But the reasoning behind it is more instructive than the result itself (cf. a similar reasoning in a more complicated case below in subsects.13.2 and 13.3).

The key observation here is that the expansion of the second factor is regular around the singular point of the first factor; more precisely, if φ_1 is such that it is equal to zero around the singular part of the second factor, then the κ -dependent test function

$$\bar{\varphi}_1(p, \kappa) = \varphi_1(p) \frac{1}{(p - Q)^2 + \kappa^2} \quad (7.41)$$

satisfies the condition (ii) of subsect. 5.4. Therefore, one can formally expand $\bar{\varphi}_1$ and use (7.38) for $(p^2 + \kappa^2)^{-1}$, and we conclude that for $p \neq Q$, the expansion of (7.40) is given by the Taylor expansion of the second factor times the full expansion of the first factor. Swapping the factors and considering test functions φ_2 which are identically zero around the singular point of the first factor, we arrive at a similar conclusion. The final observation is that since the singular points of the two factors are separated, any $\varphi(p)$ can be represented as $\varphi_1(p) + \varphi_2(p)$, whence the desired result can be easily deduced. (Another way to see it is to notice that products of δ -functions from expansions of the two products result in zeros.)

To summarize, we have demonstrated with a simple example how the ideas of the extension principle discussed in the precedent section work in practice. We have also demonstrated a short-cut way to derive our final result, the expansion (7.38), represented in the dimensionally regularized form. It can be represented in an explicitly finite form with the regularization taken off (for more on this see [20]), but the form (7.38) is more convenient for our purposes.

8 Applications to one-loop integrals.

The results of the previous sections can now be applied to one-loop integrals. We first consider the UV-convergent case, and then show how the case of UV-divergent MS-renormalized integrals can be reduced, essentially, to the convergent case. The most important non-technical element of our reasoning is a new definition of the MS-scheme in terms of subtractions of UV-asymptotic part directly from the integrand, prior to any momentum integrations (subsect. 8.4 and [21]).

8.1 UV-convergent one-loop integral.

Consider the following UV-convergent one-loop integral:

$$\int d^4 p \frac{1}{p^2 + \kappa^2} \times \frac{1}{(p^2 + M^2)^2}. \quad (8.1)$$

One would expect (we discuss this in more detail below) that although the “test function” $(p^2 + \kappa^2)^{-2}$ is not a true Schwartz test function, its decrease at $p \rightarrow \infty$ is sufficiently rapid to allow one to substitute the expansion (7.38) for $(p^2 + \kappa^2)^{-1}$ into (8.1) in order to get the asymptotic expansion for $\kappa \rightarrow 0$ of the integral (8.1) as a whole. Doing this and integrating the δ -functions, one obtains:

$$\begin{aligned} \text{eq.(8.1)} = \lim_{D \rightarrow 4} \left\{ \int d^D p \frac{1}{p^2} \times \frac{1}{(p^2 + M^2)^2} - \int d^D p \frac{\kappa^2}{p^4} \times \frac{1}{(p^2 + M^2)^2} \right. \\ \left. + \int d^D p \frac{1}{p^2 + \kappa^2} \times \frac{1}{M^4} \right\} + o(\kappa^2). \end{aligned} \quad (8.2)$$

The validity of (8.2) can be easily checked by explicit integrations.

In order to make connection with what follows, let us briefly discuss why the rapid decrease of the “test function” $(p^2 + M^2)^{-2}$ in (8.1) allows one to substitute (7.38) directly into (8.1). By definition, integrations over infinite regions involve a limiting procedure. Let us make it explicit in (8.1). Take a smooth function $H(p)$ such that

$$\begin{aligned} H(p) = 1, \quad \text{for } |p| < 1, \\ 0, \quad \text{for } |p| > 2. \end{aligned} \quad (8.3)$$

We can introduce the cut-off $H(p/\Lambda)$ into (8.1):

$$\text{eq.(8.1)} = \lim_{\Lambda \rightarrow \infty} \int d^4 p \frac{1}{p^2 + \kappa^2} \left\{ \frac{H(p/\Lambda)}{(p^2 + M^2)^2} \right\}. \quad (8.4)$$

Now it is fully correct to use (7.38) in order to expand the expression under the limit sign to order, say, $o(\kappa^2)$. Then one has to check that (i) the limit $\Lambda \rightarrow \infty$ exists for each term of the expansion of $(p^2 + \kappa^2)^{-1}$ and (ii) the remainder which was $o(\kappa^2)$ prior to taking limit will be $o(\kappa^2)$ after it. While (i) is obvious, to check (ii) one can recall the representation (7.29) of the remainder term. The desired result easily follows therefrom, moreover, the proof here closely repeats the reasoning of the paragraph after (7.3).

The conclusion is that for absolutely convergent 1-loop integrals the asymptotic expansions are obtained by substituting the expansions in the sense of the distribution theory into the integrand, and then performing termwise integrations over the loop momentum.

8.2 Separation of UV and IR divergences.

A remark is in order here. The expansion (8.2) contains two terms that diverge as $D \rightarrow 4$: the second integral is logarithmically divergent for $p \rightarrow 0$, and the third one, for $p \rightarrow \infty$. However, by construction, the expansion as a whole is finite as $D \rightarrow 4$. Therefore, the divergences must cancel. But the peculiar feature of (8.2) is that one divergence occurs in a κ -dependent integral, while the other one in an integral depending on M but not on κ . This is not satisfactory because in the final respect in applications one has to deal with *finite* functions of κ and M .

As we have already noted, all our expansions can be represented in an explicitly convergent form which makes no use of the dimensional regularization, but this requires a special techniques (see [20]). However, it is still possible to rearrange (8.2) in terms of κ - and M -dependent contributions which are separately finite, by using only notions known to the practitioners of perturbative QFT who are familiar with the techniques of UV renormalization within the MS-scheme. The price to be paid for this is that IR divergences will be canceled by UV counterterms, but we have already explained in subsect. 7.4 why such things can happen.

Look at the third integral on the r.h.s. of (8.2). It is an UV divergent 1-loop Feynman integral which can be made finite by adding an UV counterterm which can be evaluated e.g. in the MS-scheme. Such UV counterterms are polynomials in masses and external momenta, and in our case can be taken in the form:

$$\kappa^2 \frac{\text{const}}{D-4} \mu^{D-4}, \quad (8.5)$$

where μ is introduced to preserve dimensionality (cf. (8.9) below). Now, using (8.5), the expansion (8.2) can be identically rewritten as:

$$\begin{aligned} \text{eq.}(8.1) = \lim_{D \rightarrow 4} \left\{ \left[\int d^D p \frac{1}{p^2} \times \frac{1}{(p^2 + M^2)^2} \right] \right. \\ \left. - \kappa^2 \left[\int d^D p \frac{1}{p^4} \times \frac{1}{(p^2 + M^2)^2} + \frac{\text{const} \times \mu^{D-4}}{M^4(D-4)} \right] \right. \\ \left. + \frac{1}{M^4} \left[\int d^D p \frac{1}{p^2 + \kappa^2} + \frac{\kappa^2 \text{const} \times \mu^{D-4}}{D-4} \right] \right\} + o(\kappa^2). \end{aligned} \quad (8.6)$$

Since the first and third square-bracketed terms as well as the expression as a whole are finite in the limit $D \rightarrow 4$, so finite must be the second term too, which is not difficult to verify by explicit calculations.

Note that by rescaling the integration variables, one can exhibit the dependence of various terms in (8.6) on κ and M :

$$\begin{aligned} \text{eq. (8.6)} &= \frac{1}{M^4} \int d^4 p \frac{1}{p^2} \times \frac{1}{(p^2 + 1)^2} \\ &- \frac{\kappa^2}{M^4} \lim_{D \rightarrow 4} \left\{ \left[\int d^D p \frac{1}{p^4} \times \frac{1}{(p^2 + 1)^2} + \frac{\text{const} \times (\mu/M)^{D-4}}{D-4} \right] \right. \\ &\quad \left. - \left[\int d^D p \frac{1}{p^2 + 1} + \frac{\text{const} \times (\mu/\kappa)^{D-4}}{D-4} \right] \right\} + o(\kappa^2). \end{aligned} \quad (8.7)$$

We stress once more that the numerical connection of the UV counterterm in the third term and the counterterm in the second term, whose role is to cancel the IR divergence,—this connection can be traced back to the fact that the dimensional regularization nullifies certain integrals containing both IR and UV divergences simultaneously. One could pursue the topic further and analyze such cancellations more explicitly, but as the relevant formulae are quite complicated while the results perfectly useless, we stop here.

To conclude, we have demonstrated a trick for recasting expansions that emerge from our methods into an explicitly convergent form. The trick employs specific properties of UV renormalization of Feynman integrals, and can be generalized to the multiloop case where it takes the form of an inversion of the R -operation [16]–[18].

8.3 UV divergent one-loop integrals.

Let us now turn to UV divergent one-loop integrals, e.g.

$$\int d^4 p \frac{1}{p^2 + \kappa^2} \times \frac{1}{p^2 + M^2}. \quad (8.8)$$

Its MS-renormalized version which we wish to expand in κ , is:

$$\lim_{D \rightarrow 4} \left[\int d^D p \frac{1}{p^2 + \kappa^2} \times \frac{1}{p^2 + M^2} - \frac{z}{D-4} \mu^{D-4} \right], \quad (8.9)$$

where z is a numerical constant and μ is 't Hooft's unit of mass which plays the role of the subtraction parameter in the MS-scheme.

Our purpose is to show that the expansion (7.25) (and the more general expansion (7.38)) can be directly substituted into the integral in (8.9) in order to get the asymptotic expansion for (8.9). (Correctness of this recipe can also be checked by explicit calculations.) The motivation comes from the fact that the divergent contribution to the integral in (8.9) is generated by the asymptotics of the integrand at $p \rightarrow \infty$, while replacing $(p^2 + \kappa^2)^{-1}$ by its expansion leaves that asymptotics intact. These heuristic arguments can be formalized in the following way.

The crucial step is to redefine the MS-subtraction in (8.9) in a manner independent of dimensional regularization (because the expression (8.9) does not allow one to apply the reasoning like that in subsect. 8.1). To this end recall (subsect. 7.7) that a subtraction of large- p asymptotics from the integrand is equivalent to UV renormalization. One has:

$$\frac{1}{p^2 + \kappa^2} \times \frac{1}{p^2 + M^2} \underset{(\kappa, M \rightarrow 0)}{p \rightarrow \infty} \simeq \frac{1}{p^4}, \quad (8.10)$$

so that subtracting p^{-4} from the integrand in (8.9) would produce an UV convergent integral. However, p^{-4} is ill-defined at $p = 0$. Let us modify it at $p = 0$ so as to ensure its integrability:

$$\left[\frac{1}{p^4} \right]^R = \frac{1}{p^4} + \frac{z'}{D-4} \mu^{D-4} \delta(p) \quad (8.11)$$

(cf. (7.10)). z' in (8.11) is a numeric constant, and the parameter μ has the dimension of mass and is introduced to preserve dimensionality.

Now, the expression

$$\int d^D p \left\{ \frac{1}{p^2 + \kappa^2} \times \frac{1}{p^2 + M^2} - \left[\frac{1}{p^4} \right]^R \right\} \quad (8.12)$$

is finite at $D = 4$ both at $p \rightarrow 0$ and $p \rightarrow \infty$. Owing to (7.27), it can be represented as:

$$\int d^D p \frac{1}{p^2 + \kappa^2} \times \frac{1}{p^2 + M^2} - \frac{z'}{D-4} \mu^{D-4}. \quad (8.13)$$

Recall that the constant z in (8.9) was chosen solely from the requirement of finiteness of (8.9). Hence, comparing (8.9) and (8.13) we see that $z = z'$, and eq.(8.9) = eq.(8.12). Finally, recall that the distribution (8.11) can be represented as (7.8) with a suitably chosen c , and no regularization is used in (7.8). Therefore:

$$(8.9) = \int d^4 p \left\{ \frac{1}{p^2 + \kappa^2} \times \frac{1}{p^2 + M^2} - \left[\frac{1}{p^4} \right]^R \right\}. \quad (8.14)$$

So, we have represented the MS-renormalized integral (8.9) in the form of an integral over p of the expression which differs from the original integrand (8.8) by a contribution independent of κ .

Further reasoning presents no problems, as the representation (8.14) allows one to use the arguments of subsect. 8.1 with only inessential modifications in order to obtain the expansion in κ . We leave it to an interested reader to check that the final result is such as if we had substituted the expansion (7.38) directly into (8.9).

The method for recasting the expansion into an explicitly convergent form described in subsect. 8.2 is also applicable here.

8.4 Generalized MS-schemes.

Now consider the following definition of a class of subtractions schemes. (It is shown elsewhere that this definition works in the case of multiloop Feynman integrals [21], [19].) Let $F(p)$ be the integrand of a 1-loop Feynman diagram. Evaluate its asymptotic expansion for $p \rightarrow \infty$ and discard the terms that are integrable at large p . Denote the result as $F^{\text{as}}(p)$. The terms in $F^{\text{as}}(p)$ that generate logarithmic divergences at $p \rightarrow \infty$, also have a logarithmic singularity at $p = 0$, while the rest are integrable at that point. Redefine the logarithmically divergent terms at $p = 0$ similarly to (7.8), (7.10) to make them integrable at all finite p . Denote the result as $[F^{\text{as}}(p)]^{\text{R}}$. (Note that the redefinition should only affect powers of p and commute with other dimensional parameters; in particular, the constant c in (7.8) should be independent of the masses and external momenta of the diagram.) The renormalized integral is now defined as

$$\int d^4 p \left\{ F(p) - [F^{\text{as}}(p)]^{\text{R}} \right\}. \quad (8.15)$$

It is easy to see (cf. the reasoning in subsect. 8.3) that the class of subtractions schemes thus defined includes the MS-scheme as a special case. Note also that the UV counterterms in these schemes are always polynomials in masses and external momenta of the diagram.

To take an example, consider the integral that has already been encountered above in subsect. 7.7:

$$\int d^4 p \frac{1}{p^2 + \kappa^2}, \quad (8.16)$$

then

$$F(p) = \frac{1}{p^2 + \kappa^2}, \quad F^{\text{as}}(p) = \frac{1}{p^2} - \kappa^2 \frac{1}{p^4}, \quad (8.17)$$

and

$$[F^{\text{as}}(p)]^{\text{R}} = \frac{1}{p^2} - \kappa^2 \left[\frac{1}{p^4} \right]^{\text{R}}. \quad (8.18)$$

The renormalized integral is

$$\int d^4 p \left\{ \frac{1}{p^2 + \kappa^2} - \frac{1}{p^2} + \kappa^2 \left[\frac{1}{p^4} \right]^{\text{R}} \right\}. \quad (8.19)$$

Introducing the dimensional regularization into (8.19), using (8.11) and (7.27), we arrive at an expression,

$$\int d^D p \frac{1}{p^2 + \kappa^2} + \kappa^2 \frac{z'}{D-4} \mu^{D-4},$$

where $z' = c_0 + (D-4)c_1 + \dots$ and c_i are arbitrary finite constants. If $c_i = 0$, $i \geq 1$, one recovers the standard MS-scheme.

8.5 Summary.

The representation of the MS-renormalization as subtractions from the bare integrand of those terms from its asymptotics at large loop momentum which are responsible for UV divergences,

allows one to reduce the problem of expansion of 1-loop integrals to the UV convergent case. And the reasoning of the above sections shows that the correct expansion of the renormalized integrals can be obtained by direct substitution of the corresponding expansion in the sense of distributions in place of the κ -dependent terms, and rearranging the resulting expression along the lines of subsect. 8.2, in order to obtain the expansion in an explicitly convergent form.

In order to generalize this recipe to the multiloop case we need, first, to learn to expand more complicated products of singular functions (which is done in the present paper), second, to generalize the above treatment of UV-divergent integrals to the multiloop case [21], [19], and third, to generalize the rearrangement trick of subsect. 8.2 [16]–[18].

THE MASTER EXPANSION

9 Notations for products of singular functions.

Admittedly, multiloop Feynman diagrams are cumbersome objects. Nevertheless, the analytical aspects of the problem, say, of UV renormalization or the Euclidean expansion problem are by no means as complicated as one can imagine regarding them from the point of view of the traditional techniques prevailing in rigorous studies of perturbative QFT like e.g. the α -representation which completely destroys the essential multiplicative structures of Feynman integrands. However, to reveal the underlying simplicity and help one to ignore the plethora of irrelevant detail, a careful choice of notations is crucial. Below we describe such a system of notations (discussed in a more formalized manner in [20]) which does not make use of parametric representations and applies equally well to both scalar and non-scalar theories.

9.1 Graphs and products.

Let G be a Feynman graph. Its loop momenta p_1, p_2, \dots, p_l are always Euclidean vectors of 4 dimensions. (For studying toy models and examples it is sometimes convenient to consider the 2-dimensional case.) The loop momenta are combined into a single vector of $4 \times l$ dimensions:

$$p = (p_1, p_2 \dots p_l). \tag{9.1}$$

The variable p may also incorporate the heavy external momenta—see remark (iii), subsect. 4.2. In such a case

$$p = (p_1, p_2 \dots p_l, \bar{Q}_1 \dots), \tag{9.2}$$

where \bar{Q} are defined in subsect. 3.4.

The integrand of the graph G in momentum representation is denoted as $G(p)$. Dependence on other parameters can be shown with additional arguments after p . $G(p)$ is a product of

factors corresponding to both lines and vertices. The factors are numerated by the label g . The set of values of the label is denoted also as G . The use of the same symbol for three different but related entities should not be misleading. Similarly, it should not be misleading if we denote as $g(p)$ that factor in the product $G(p)$, which corresponds to the label g . So,

$$G(p) = \prod_{g \in G} g(p). \quad (9.3)$$

We will also have to study subproducts of G . Let $\gamma \subset G$. Then

$$\gamma(p) \equiv \prod_{g \in \gamma} g(p) \quad (9.4)$$

and also

$$G \setminus \gamma(p) \equiv \prod_{g \in G \setminus \gamma} g(p) = G(p) / \gamma(p). \quad (9.5)$$

where $G \setminus \gamma$ is the standard notation for the difference of sets.

9.2 Conventions for non-scalar factors.

We allow each function $g(p)$ to carry sub- (or super-) scripts: Lorentzian, $SU(n)$ etc. Then the product (9.3) may contain implicit contractions. If, for example, the product (9.4) carries a pair of subscripts, which are contracted in (9.3), then this contraction is implied in (9.4) as well.

Now, let Γ' and Γ'' be non-intersecting subproducts from G . In G there may exist such a pair of contracted subscripts that one of them belongs to Γ' , while the other one to Γ'' . Then the product

$$\Gamma'(p) \Gamma''(p) \quad (9.6)$$

implies contraction of such a pair. Furthermore, we will build new expressions from G by replacing some subproducts γ by certain expressions, say, Z_γ , so that Z_γ will carry exactly the same subscripts as γ . Then the products like

$$\Gamma' Z_{\Gamma''}, \quad Z_{\Gamma'} Z_{\Gamma''} \quad \text{etc.} \quad (9.7)$$

imply the same contractions as (9.3).

The implicit presence of such contractions will have no complicating effect on our reasoning as compared with the scalar case.

9.3 Momentum dependence.

Let us discuss the dependence of factors on p . One should distinguish factors of two types: those corresponding to lines and to vertices of Feynman graphs.

Let g correspond to a line. The momentum flowing through this line is a linear combination of various momentum variables of our problem:

$$\left(\sum_{i=1\dots l} c_{g,i} p_i + Q_g \right) + k_g, \quad (9.8)$$

where Q_g and k_g are linear combinations of heavy and light external momenta from the sets Q and k . Then $g(p)$ is a propagator depending on (9.8) and a mass. The mass may be light, i.e. equal to zero or proportional to κ , or heavy, i.e. non-zero and independent of κ . Since k_g in (9.8) is proportional to κ and will in fact play the role of an expansion parameter, it is convenient to introduce a notation for the non-vanishing part of (9.8):

$$L_g(p) \equiv \text{the parenthesised expression in (9.8)}. \quad (9.9)$$

Now one can write:

$$g(p) = F_g(L_g(p) + k_g, \kappa). \quad (9.10)$$

In fact our arguments are true for a wide class of functions $F_g(L, \kappa)$ including any standard perturbation theory propagators raised to integer powers and multiplied by polynomials of L .

9.4 Assumptions on the properties of factors.

Let us describe those properties of $F_g(L, \kappa)$ that are essential for our theory. (For a formalized description see [20].) One should consider two cases:

(i) The function $F_g(L, \kappa)$ at $L = \kappa = 0$ is regular (which corresponds to propagators with heavy masses). Then for $\kappa \rightarrow 0$, F_g is expanded into a series of functions which are smooth in L , and the expansion is uniform in L in any bounded region and allows arbitrary termwise differentiations in L .

(ii) The function $F_g(L, \kappa)$ at $L = \kappa = 0$ is singular (which corresponds to propagators with a light mass). Here we assume that $F_g(L, \kappa)$ has the following properties:

(a) scaling:

$$F_g(\lambda L, \lambda \kappa) = \lambda^{d_g} F_g(L, \kappa);$$

(b) asymptotic expansion at each $L \neq 0$:

$$F_g(L, \kappa) = \sum_n \kappa^n F_{g,n}(L), \quad L \neq 0;$$

(c) all $F_{g,n}(L)$ are smooth at $L \neq 0$ and have simple scaling properties:

$$F_{g,n}(\lambda L) = \lambda^{d_g - n} F_{g,n}(L);$$

(d) regularity properties: the expansion in (b) allows arbitrary termwise differentiations in L , and the remainder of the expansion to order $O(\kappa^N)$ has an upper bound of the form of the

modulo of the first discarded term: $\kappa^{N+1}|L|^{d_g-N-1}$ (cf. property (c); also note that property (d) ensures that the κ -dependent test functions that arise at intermediate steps of our reasoning will possess the properties described in subsect. 5.4).

In case *i/ii* we will say that the factor is of *regular/singular type*, respectively.

It should be stressed that the described properties are sufficient for our purposes and comprise the problems within the standard perturbation theory. But our methods will work in a more general context, e.g. if the singular functions have logarithmic or power behaviour with a non-integer exponent.

9.5 Factors corresponding to vertices.

Now, let g correspond to a vertex of the graph. One can assume that $g(p)$ is a uniform polynomial of momenta entering the vertex. There are three ways to include such factors into a common framework with lines. First, all vertex polynomials can be merged with test functions. Second, each vertex polynomial can be split so that each part may be included into one of the propagators attached to the vertex (such modifications of propagators are allowed by our definitions—see subsect. 9.4). The third way—which we prefer—consists in generalizing the notations introduced for lines: it is sufficient to allow L_g to consist of several independent momentum components like (9.8), i.e.:

$$L_g = \left(\sum_{i=1\dots l} c_{g,v,i} p_i + Q_{g,v} \right)_{v=1,\dots} \quad (9.11)$$

For convenience, we assume that, by definition, the vertex functions belong to the singular type (see subsect. 9.4 above).

Equivalence of expansions obtained in all three cases follows from the uniqueness of the final result (subsect. 5.2). More explicitly, the equivalence can be checked using the property of the A -operation to commute with multiplications of the expression to be expanded by polynomials of p (see below subsect. 12.5).

9.6 Summary.

Let us summarize the entire scheme. The product (9.3) is defined on the space of the multi-component variable (9.11), and consists of factors of the form (9.10). The factors need not be scalar functions—the conventions of subsect. 9.2 are operative. The first argument in (9.10) is a set of momentum variables of the form (9.11). The functions F_g satisfy the conditions described in subsect. 9.4. Finally, the factors may be of two types—singular or regular (see the end of subsect. 9.4).

10 Formal expansions and IR subgraphs.

An important element of our technique is classification of singularities of the formal (Taylor) expansions in κ of the products (9.3). The key notion here is that of IR-subgraph, introduced in subsect. 10.2. Note that there are similarities between our Euclidean space classification of IR-singular points and the Minkowski space reasoning of Libby and Sterman [64].

10.1 Some notations.

Denote the operation of formal expansion in κ as \mathbf{T}_κ . The expansion $\mathbf{T}_\kappa g$ is the same as the one described in property (b), subsect. 9.4, up to a replacement of the momentum variable. Then

$$\mathbf{T}_\kappa g(p, \kappa) = \prod_{g \in G} \mathbf{T}_\kappa g(p, \kappa), \quad (10.1)$$

and similarly for each subset $\gamma \subset G$. The r.h.s. of (10.1) should be understood as a simple infinite series in integer powers of κ , obtained by formal multiplication of the series with reordering of terms.

It is convenient to denote the operation of partial expansion up to terms $O(\kappa^N)$ as $\mathbf{T}_{\kappa, N}$. We will also omit the subscript indicating the expansion parameter as $\mathbf{T}_\kappa \rightarrow \mathbf{T}$, $\mathbf{T}_{\kappa, N} \rightarrow \mathbf{T}_N$, if misunderstanding is excluded.

10.2 Singularities of the formal expansion.

Let us describe the structure of singularities of the formal expansion (10.1). Each factor of the singular type generates singularities at those points where $L_g(p) = 0$. We call the set of such points *singular plane of g* and denote it as

$$\pi_g = \{p \mid L_g(p) = 0\}. \quad (10.2)$$

Now consider an arbitrary subset $\gamma \subset G$ such that all its elements are of singular type. Define the *singular plane of the subset γ* :

$$\pi_\gamma = \bigcap_{g \in \gamma} \pi_g. \quad (10.3)$$

On this plane, all factors $g \in \gamma$ are singular simultaneously.

We will consider only such subsets γ , to which there correspond non-empty singular planes.

An important point is that the same singular plane may correspond to different subsets. For example, in Fig.3 the pairs of propagators with momenta p_1 and p_2 ; p_1 and $p_1 + p_2$ generate the same singular plane described by the equations $p_1 = p_2 = 0$. Further, there is one largest among the subsets with the same singular plane. This follows from the fact that if $\pi_\gamma = \pi_{\gamma'}$ then $\pi_\gamma = \pi_{\gamma \cup \gamma'}$. In Fig.3 such a subproduct consists of all three propagators. Subsets, to which new factors cannot be added without reducing their singular plane, will be described as *complete* and will be called (*complete*) *IR-subgraphs* or, if confusion is excluded, simply subgraphs.

10.3 Diagrammatic interpretation of IR-subgraphs.

Take a Feynman graph and evaluate momenta for each line. To check if a given set of singular lines γ is complete, perform the following test. Set to zero all the momenta flowing through the lines of γ , and all light external momenta. Using momentum conservation at vertices, reevaluate the momenta for the rest of the lines. If there is a line of singular type not belonging to γ , whose momentum will vanish after reevaluation, then the set γ is not complete. If there are no such lines, then γ is an IR-subgraph. For example, in Fig.4a the pair of vertical lines forms an IR-subgraph.

Note that the same subset γ in the same diagram G may be or be not an IR-subgraph, depending on whether the full Euclidean expansion problem is considered or its simplified version without contact terms (see subsect. 3.5). For example, for the graph in Fig.5 (masses in both lines are light) the entire graph will be its own IR-subgraph if the graph is considered as a distribution over Q , so that Q is included into integration momenta; the corresponding singular plane is described by the equations $p = Q = 0$. In the simplified version one considers integration only over the loop momentum p , while Q is fixed at a non-exceptional value (this means here simply that $Q \neq 0$); in this case the pair of lines cannot be an IR-subgraph because its singular plane is empty.

The property of being IR-subgraph also depends on the additional restrictions specified for the heavy momenta in the particular problem. For example, in Fig.4a which differs from Fig.4b by a linear restriction $Q_1 = Q_2$, the pair of vertical lines forms an IR-subgraph, while this is not so for Fig.4b.

When a vertex should be included into an IR-subgraph? Recall that any vertex is a singular-type factor by definition (cf. subsect. 9.5), and its singularities are localized at the points where all the momenta from the set L_j are equal to zero. Therefore, in the most general case the recipe is as follows: if all the lines incident to this vertex belong to the subgraph, then the vertex is to be included, too. However, in the simplified version of the expansion problem, an IR-subgraph may not contain a vertex with a non-zero total heavy external momentum entering into it.

Starting from the basic analytical definition, one can always enumerate IR-subgraphs in any specific case. However, we do not attempt translating the definition of the IR-subgraph into the graph-theoretic language: the result would be cumbersome and practically useless. For the purposes of doing the combinatorics of factorization in [18], we will only need the properties of complements of IR-subgraphs that can be easily derived from the above definition (cf. subsects. 5.2, 6.1 and 6.4 in [18]).

10.4 Why IR-subgraphs?

The reason for considering only IR-subgraphs among all subproducts of $G(p)$ is as follows. If γ is a subgraph, then singularities of the formal expansion

$$\mathbf{T}_\kappa G(p) = [\mathbf{T}_\kappa \gamma(p)] \times [\mathbf{T}_\kappa G \setminus \gamma(p)] \quad (10.4)$$

near almost any point at the plane π_γ (except for the intersections of π_γ with singular planes of factors not belonging to γ), are determined solely by γ .

Vice versa, let p_0 be any point from the space of p . Select from the graph the smallest subproduct γ such that $\mathbf{T}_\kappa G \setminus \gamma(p)$ have no singularities at $p = p_0$. Then γ will be an IR-subgraph.

10.5 Hierarchy of IR-subgraphs.

The set of all IR-subgraphs of a given graph has a natural ordering. Namely, $\gamma \leq \Gamma$ if and only if γ belongs to Γ . Note that if $\gamma \leq \Gamma$, then $\pi_\gamma \supset \pi_\Gamma$. Correspondingly, one can define *maximal* and *minimal* subgraphs. Note that there may be several maximal subgraphs; their singular planes do not intersect.

It is convenient to introduce the notion of *immediate precedence* for subgraphs. We say that the subgraph γ immediately precedes the subgraph Γ , and write $\gamma \triangleleft \Gamma$, if $\gamma < \Gamma$ and there are no subgraphs between γ and Γ , i.e. no subgraphs Γ' such that $\gamma < \Gamma' < \Gamma$. We describe the subgraph which immediately precedes the maximal one as *submaximal*.

10.6 Proper variables of subgraphs.

Let Γ be a subgraph of the graph G . In general, $\Gamma(p, \kappa)$ is independent of some components of p . Indeed, it is easy to check that $\Gamma(p, \kappa)$ is always invariant with respect to shifts in p parallel to π_Γ . Split p into two parts: transverse and longitudinal with respect to π_Γ : $p = (p_\Gamma, p_\Gamma^\perp)$, so that the singular plane π_Γ is described by the equation $p_\Gamma = 0$. Then $\Gamma(p, \kappa)$ depends only on p_Γ . The variables p_Γ will be referred to as the set of *proper variables* of Γ .

Although the choice of proper variables is not unique, transition to another set of proper variables is equivalent to a change of coordinates in the space of p_Γ . Here invariant coordinateless formulations are possible and useful—for more on this see [20].

The proper variables of an IR-subgraph can be easily determined using its graphical representation. Thus, for the subgraph in Fig.3 the proper variables are the set of momenta p_1 and p_2 (or p_1 and $p_2' = p_1 + p_2$ etc.).

Consider the case when the IR-subgraph has loops (see e.g. Fig.6). The loop momenta belong to the proper variables of the subgraph. Denote the set of all loop momenta as p_γ^{int} , and the rest of the proper variables as p_γ^{ext} . So,

$$p_\gamma = (p_\gamma^{\text{int}}, p_\gamma^{\text{ext}}). \quad (10.5)$$

To conclude this section, note that any IR-subgraph can be considered as a graph in its own right. Then the loop momenta of the graph should comprise all proper variables of the IR-subgraph. Graphically, this corresponds to merging all the “external” vertices of the IR-subgraph into one. (In Fig.6, the external vertices are a and b .) The resulting graph possesses

an additional property: it is its own (and the only maximal) IR-subgraph. In that case the singular plane of the entire graph is reduced to a point which can be assumed to be the zero point. It is clear that for each subgraph one can consider the problem of its asymptotic expansion. Moreover, we will see in subsect. 11.3 that construction of the *As*-operation for the graph is logically preceded by its construction for the subgraphs. It is worth noting that the *As*-operation for a subgraph is formulated entirely in terms of its proper variables.

10.7 Quantitative characteristics.

Let us introduce some quantitative characteristics for description of singularities of IR-subgraphs. For each IR-subgraph Γ define:

$$d_\Gamma = \sum_{g \in \Gamma} d_g, \quad (10.6)$$

which describes simultaneous scaling in κ and the proper variables:

$$\Gamma(\lambda p, \lambda \kappa) = \lambda^{d_\Gamma} \Gamma(p_\Gamma, \kappa) \quad (10.7)$$

(cf. property (a), subsect. 9.4);

$$\dim p_\Gamma \quad (10.8)$$

is the dimensionality of the space of proper variables of the subgraph;

$$\omega_\Gamma = -d_\Gamma - \dim p_\Gamma \quad (10.9)$$

is the *singularity index*. Its meaning is as follows. If for all $\Gamma \leq G$ one has $\omega_\Gamma + N < 0$ (strict inequality!) then the formal expansion of G to order κ^N does not contain non-integrable singularities and is a correct asymptotic expansion in the sense of distributions.

11 *As*-operation for products of singular functions.

In this section, following the recipe of the extension principle (sect. 5), we establish formulae for the expansion of products of singular functions in the sense of distributions in the form of the so-called *As*-operation whose structure is similar to that of the Bogoliubov *R*-operation. The derivation presented bears a resemblance to, and was partially inspired by the analysis of the *R*-operation in [24]. Explicit expressions for counterterms of the *As*-operation (which, unlike the *R*-operation, is determined uniquely) will be obtained in the next section.

11.1 General remarks and notations.

We are going to derive the recipe of the *As*-operation, the instrument for evaluating expansions in the form of infinite asymptotic series for any product (graph) $G(p)$ of the described type. We will use the notation:

$$G(p, \kappa) \underset{\kappa \rightarrow 0}{\simeq} \text{As } G(p, \kappa) = \sum_n \kappa^n G_n(p, \kappa). \quad (11.1)$$

Here each G_n should be a distribution in p , and the dependence on κ should be soft (see the definition in subsect. 5.1). The expansion (11.1) should be valid in the sense of distributions (see subsect. 5.1). This means that for each test function φ the following estimate should hold:

$$\int dp \varphi(p) [G(p, \kappa) - \mathbf{As}_N G(p, \kappa)] = o(\kappa^N). \quad (11.2)$$

The subscript on the *As*-operation means that in (11.2) all the terms of order $o(\kappa^N)$ are discarded, i.e.

$$\mathbf{As}_N G = \sum_{n \leq N} \kappa^n G_n. \quad (11.3)$$

Existence of such expansions is either obvious or non-trivial, depending on the point of view. It is rather obvious if one recalls the results of [1] wherefrom one concludes that expansion of G integrated with a test function runs in powers and logarithms of the expansion parameter and takes into account that such an expansion should be linear in test functions. On the other hand, from the point of view of the technique of [9] existence of such expansions is not at all trivial because the construction of [9] corresponds to an expansion similar to (11.3) for each N but with G_n depending on N , too.

11.2 Uniqueness of the expansion.

From subsect. 5.2 it follows that if an expansion of the form (11.1) exists, then it is unique (provided one always expands in powers and logarithms of the small parameter). Uniqueness of the *As*-operation will allow us in subsect. 11.3 to determine the necessary conditions which it must satisfy. This will give us a sufficient number of hints to construct it explicitly.

11.3 Localization property of the *As*-operation.

Let us study the local structure of the *As*-operation (cf. remark (i) in subsect. 4.2). First of all note that the formal expansion

$$G(p, \kappa) \sim \mathbf{T}G(p, \kappa), \quad \kappa \rightarrow 0, \quad (11.4)$$

allows integrations with test functions φ whose support does not intersect singular planes of the formal expansion:

$$\text{supp } \varphi \cap \left(\bigcup_{\gamma} \pi_{\gamma} \right) = \emptyset. \quad (11.5)$$

This follows from the properties of the functions of which G is built (cf. subsect. 9.4). The r.h.s. of (11.4) defines a functional which is an asymptotic expansion of the functional on the l.h.s., on the subspace of test functions satisfying (11.5). From uniqueness it follows that on test functions satisfying (11.5) the *As*-operation should coincide with the operation of the formal expansion \mathbf{T} :

$$\mathbf{As} G(p, \kappa) = \mathbf{T}G(p, \kappa), \quad p \notin \bigcup \pi_{\gamma}. \quad (11.6)$$

Further, consider any small region \mathcal{O} , and let Γ be a subgraph such that $\mathbf{T}G \setminus \Gamma(p)$ has no singularities within \mathcal{O} (cf. subsect. 4.1). Take an arbitrary φ such that $\text{supp } \varphi \subset \mathcal{O}$. Then

$$\int dp G(p, \kappa) \varphi(p) = \int \Gamma(p, \kappa) \varphi''(p, \kappa), \quad (11.7)$$

where

$$\varphi''(p, \kappa) = G \setminus \Gamma(p, \kappa) \varphi(p) \quad (11.8)$$

is a test function with $\text{supp } \varphi \subset \text{supp } \varphi''$, but depending on κ .

Recall (subsect. 10.6) that Γ is independent of that component of p which is longitudinal with respect to π_Γ . Therefore we may perform integration of φ'' over the longitudinal components on the r.h.s. of (11.7). So, we arrive at the problem of expansion for the subgraph in terms of its proper variables, and with test functions depending on the expansion parameter. The assumptions made in subsect. 9.4 ensure that the expansion procedure described in subsect. 5.4 will be applicable here.

So, assume that the problem of expanding $\Gamma(p, \kappa)$ is solved, i.e. that the action of the *As*-operation on Γ is already known. Then the expansion of the r.h.s. of (11.7) can be represented as

$$\int dp [\mathbf{As} \Gamma(p, \kappa)] \times [\mathbf{T} \varphi''(p, \kappa)] = \int dp [\mathbf{As} \Gamma(p, \kappa)] \times [\mathbf{T} G \setminus \Gamma(p, \kappa)] \varphi(p). \quad (11.9)$$

On the other hand, the expansion of the r.h.s. of (11.7) should be given by

$$\int dp \mathbf{As} G(p, \kappa) \varphi(p).$$

Then again from uniqueness of the *As*-operation we conclude that, given the relation described above between the region \mathcal{O} and the subgraph Γ , on all test functions localized in \mathcal{O} the following *localization property* must be valid:

$$\mathbf{As} G(p) = [\mathbf{T} G \setminus \Gamma(p)] \times [\mathbf{As} \Gamma(p)], \quad p \in \mathcal{O}. \quad (11.10)$$

This is the most important structural property of the *As*-operation. It exhibits the recursive structure of the expansion problem considered on the entire collection of Feynman diagrams.

11.4 Structure of the *As*-operation.

Let us represent the *As*-operation in terms of counterterms localized at singular points of the formal expansion, in analogy with the well-known expression of the Bogoliubov *R*-operation in terms of quasilocal counterterms [24]. $\mathbf{As} G$ may differ from $\mathbf{T}G$ only by corrections localized on singular planes of the formal expansion. Consequently, one can take the following ansatz for the full expansion:

$$\mathbf{As} G = \mathbf{T}G + \sum_{\gamma} (\mathbf{E}\gamma) K_{G,\gamma}, \quad (11.11)$$

where summation runs over all IR-subgraphs, while the distribution $(\mathbf{E}\gamma)$ is localized on π_γ . Let us determine $K_{G,\gamma}$ from (11.10), expanding the *As*-operation in accordance with (11.11):

$$\mathbf{T}G + \sum_{\gamma \leq G} (\mathbf{E}\gamma) K_{G,\gamma} = (\mathbf{T}G \setminus \Gamma) (\mathbf{T}\Gamma + \sum_{\gamma \leq \Gamma} (\mathbf{E}\gamma) K_{\Gamma,\gamma}). \quad (11.12)$$

We get the conditions:

$$K_{G,\gamma} = K_{\Gamma,\gamma}(\mathbf{T}G \setminus \Gamma). \quad (11.13)$$

Without loss of generality we may assume that for each Γ :

$$K_{\Gamma,\Gamma} = 1. \quad (11.14)$$

(Indeed, within the problem for the subgraph Γ considered in terms of its proper variables, the distribution $(\mathbf{E}\Gamma)$ is localized at the origin, so that $K_{\Gamma,\Gamma}$ must be a constant independent of momentum variables. And such a constant can always be included into $(\mathbf{E}\Gamma)$.)

Setting $\gamma = \Gamma$ in (11.13) and using (11.14), we get:

$$K_{G,\Gamma} = \mathbf{T}G \setminus \Gamma \quad (11.15)$$

for each subgraph $\Gamma \leq G$. We see that the dependence on Γ disappears from the r.h.s. of (11.13), as expected. Finally:

$$\mathbf{A}s G = \mathbf{T}G + \sum_{\gamma \leq G} (\mathbf{E}\gamma) (\mathbf{T}G \setminus \gamma). \quad (11.16)$$

The expressions $(\mathbf{E}\gamma)$ will be referred to as *counterterms for subgraphs* γ .

For (11.1) to be true, it is necessary that the counterterm had the form of an infinite series

$$(\mathbf{E}\gamma) = \sum_n \kappa^n (\mathbf{E}\gamma)_n, \quad (11.17)$$

where $(\mathbf{E}\gamma)_n$ depend softly on κ .

11.5 Summary.

So, if the *As*-operation (11.1)–(11.3) exists, it is unique and, in accordance with the extension principle, must have the form (11.16)–(11.17). However, the analysis of subsect. 11.3 indicates that the *As*-operation can be naturally defined by (11.16) using the induction based on the natural order among IR-subgraphs (see also below sect. 12). Then if the *As*-operation has been constructed for all $\gamma < \Gamma$, it only remains to determine the counterterms $(\mathbf{E}\Gamma)$ possessing all the required properties. Note that the reasoning of sect. 7 is in fact applicable to any minimal IR-subgraphs, which provides a correct starting point for the induction. The general formulae for the counterterms derived in the next section will include the case of minimal subgraphs as a simple special case.

12 Expressions for counterterms of the *As*-operation.

12.1 Structure of the *As'*-operation.

Suppose that the existence of the *As*-operation has been established for all $\gamma < \Gamma$, and explicit expressions for $(\mathbf{E}\gamma)$ of the form of (11.17) have been found. From the reasoning of the precedent

section it follows that the expansion for Γ valid on test functions which are equal to zero around $p = 0$ has the form:

$$\Gamma(p_\Gamma, \kappa) \underset{\kappa \rightarrow 0}{\simeq} \mathbf{As}'\Gamma(p_\Gamma, \kappa), \quad p_\Gamma \neq 0, \quad (12.1)$$

where \mathbf{As}' is the As -operation without the last counterterm:

$$\mathbf{As}'\Gamma = \mathbf{T}\Gamma + \sum_{\gamma < \Gamma} (\mathbf{E}\gamma)(\mathbf{T}\Gamma \setminus \gamma), \quad (12.2)$$

so that

$$\mathbf{As}\Gamma = \mathbf{As}'\Gamma + (\mathbf{E}\Gamma). \quad (12.3)$$

Indeed, let H run over all submaximal subgraphs of Γ , i.e. $H \triangleleft \Gamma$ (see subsect. 10.5). Then pairwise intersections of the singular planes π_H consist only of the point $p_\Gamma = 0$. Therefore, if $\varphi(p_\Gamma) = 0$ around $p_\Gamma = 0$ then φ can be represented as

$$\varphi = \sum_{H \triangleleft \Gamma} \varphi_H, \quad (12.4)$$

where $\varphi_H(p_\Gamma) \equiv 0$ in a neighbourhood of π'_H for any $H' \neq H$. Then

$$\langle \Gamma \varphi \rangle = \sum_{H \triangleleft \Gamma} \langle \Gamma \varphi_H \rangle = \sum_{H \triangleleft \Gamma} \langle H, \Gamma \setminus H \varphi_H \rangle \quad (12.5)$$

$$\underset{\kappa \rightarrow 0}{\simeq} \sum_{H \triangleleft \Gamma} \langle \mathbf{As} H, \mathbf{T}\Gamma \setminus H \varphi_H \rangle$$

(cf. the reasoning in subsect. 11.3, especially (11.9)). On the other hand, the operation \mathbf{As}' as defined in (12.2) inherits the localization property analogous to (11.10):

$$\langle \mathbf{As}'\Gamma, \varphi_H \rangle = \langle \mathbf{As} H, \mathbf{T}\Gamma \setminus H \varphi_H \rangle, \quad (12.6)$$

so that the r.h.s. of (12.5) is equal to

$$\sum_{H \triangleleft \Gamma} \langle \mathbf{As}'\Gamma, \varphi_H \rangle = \langle \mathbf{As}'\Gamma, \varphi \rangle, \quad (12.7)$$

whence follows (12.1).

12.2 Approximation properties of the \mathbf{As}' -operation.

Our inductive assumptions imply that all counterterms $(\mathbf{E}\gamma)$ for $\gamma < \Gamma$ have the form (11.17). Then $\mathbf{As}'\Gamma$ can be represented as an expansion in powers of κ :

$$\mathbf{As}'\Gamma(p_\Gamma, \kappa) = \sum_n \kappa^n \Gamma'_n(p_\Gamma, \kappa), \quad (12.8)$$

where $\Gamma'_n(p_\Gamma, \kappa)$ can softly depend on κ . Besides, the functions Γ'_n possess a number of natural properties following from our assumptions. First, they inherit the scaling property (10.7):

$$\Gamma'_n(\lambda p_\Gamma, \lambda \kappa) = \lambda^{d_g - n} \Gamma'_n(p_\Gamma, \kappa). \quad (12.9)$$

Second, from (12.9) and the soft dependence on κ , one immediately gets the scaling property in the momentum argument:

$$\Gamma'_n(\lambda p_\Gamma, \kappa) = \lambda^{d_g - n} (\Gamma'_n(p_\Gamma, \kappa) + \text{soft corrections}). \quad (12.10)$$

To put it differently, the dependence of the expression $\lambda^{n-d_g} \Gamma'_n(\lambda p_\Gamma, \kappa)$ on λ is soft.

Note the important parallel between what we have here and what we had in subsect. 7.1: the expansion (12.8), similarly to (7.2), contains terms progressively more singular at $p_\Gamma \sim 0$ as $n \rightarrow \infty$, and both expansions are valid in the sense of the distribution theory for $p_\Gamma \neq 0$. We can push the analogy even further. Consider the expansion (12.8) truncated at the terms of order $O(\kappa^N)$:

$$\Gamma = \mathbf{A} s'_N \Gamma + o(\kappa^N), \quad p_\Gamma \neq 0. \quad (12.11)$$

(The use of the subscript N here is analogous to (11.3).) From (12.10) it follows that the r.h.s. of (12.11) becomes non-integrable at zero for $\omega_\Gamma + N \geq 0$ (ω_Γ is defined in (10.9)). However if the test function has a zero of order $\omega_\Gamma + N + 1$ at $p_\Gamma = 0$, then the r.h.s. of (12.11) is well defined and, moreover, on such test functions the estimate $o(\kappa^N)$ of the expansion (12.11) is valid.

The last property is fully natural. Its proof is essentially not difficult, is entirely based on power counting and has the same structure as that in subsect. 7.2. One considers a neighborhood of the origin of a radius $O(\kappa)$ wherein the uniformity property (12.9) is used for explicitly extracting the factor $O(\kappa^{N+1})$ from the integral of the remainder term of the expansion over this neighbourhood. Then one uses the properties of the *As*-operation on subgraphs (the inductive assumption) to estimate the integral over the rest of the space. This last step is somewhat cumbersome though fully straightforward. A detailed discussion can be found in [20]. (See also sect. 13 below.)

12.3 Expressions for counterterms.

The above results allow one to apply here the same reasoning as in the example in sect. 7. From the extension principle it follows that the expansion (12.11) can be extended to all p_Γ by addition of counterterms localized at $p_\Gamma = 0$:

$$\Gamma = \mathbf{A} s'_N \Gamma + (\mathbf{E}_N \Gamma) + o(\kappa^N). \quad (12.12)$$

$(\mathbf{E}_N \Gamma)$ may only contain derivatives of the δ -function of order not higher than $\omega_\Gamma + N$:

$$(\mathbf{E}_N \Gamma)(p_\Gamma) = \sum_{|\alpha| \leq \omega_\Gamma + N} c_{\Gamma, \alpha}(\kappa) \delta_{\Gamma, \alpha}(p_\Gamma) \quad (12.13)$$

(here $\delta_{\Gamma, \alpha}(p_\Gamma)$ is a full basis of δ -functions and their derivatives localized at $p_\Gamma = 0$; the order of derivatives is denoted as $|\alpha|$). To find explicit expressions of the coefficients $c_{\Gamma, \alpha}$ (recall that in general they depend on N , but it will turn out possible to choose them independent of N), we use the trick of subsect. 7.9.

Consider the expression

$$\int dp_\Gamma A(p_\Gamma)B(p_\Gamma) = o(\kappa^N), \quad (12.14)$$

where

$$A = \Gamma - \mathbf{As}'_N \Gamma, \quad (12.15)$$

$$B = \varphi - \mathbf{T}_{\omega_\Gamma + N} \varphi, \quad (12.16)$$

with ω_Γ as defined in (10.9). The fact that the l.h.s. of (12.14) is indeed $o(\kappa^N)$ is proved exactly as in subsect. 7.9; one only has to note that owing to the scaling properties (10.7), (12.9) one has:

$$\begin{aligned} (I - \mathbf{As}'_N) \Gamma(\lambda p_\Gamma, \kappa) &= \lambda^{d_\Gamma} (I - \mathbf{As}'_N) \Gamma(p_\Gamma, \kappa/\lambda) \\ &\underset{\lambda \rightarrow \infty}{\simeq} \lambda^{d_\Gamma} \times o\left((\kappa/\lambda)^N\right), \quad p_\Gamma \neq 0. \end{aligned} \quad (12.17)$$

Therefore, the integral converges at $p_\Gamma \rightarrow \infty$ by power counting. (Note, however, that this is not an ordinary absolutely convergent integral because the term $\mathbf{As}'_N \Gamma$ is a distribution. A more accurate definition of the integral in (12.14) involves introducing a cut-off $H(p_\Gamma/\Lambda)$ with any H defined as in (8.3) and taking the limit $\Lambda \rightarrow \infty$. The resulting value is independent of the exact form of H . Also note that the integral (12.14) can be transformed into an absolutely convergent integral of ordinary functions by invoking the explicit expressions for the operation \mathbf{As}' and integrating out the δ -functions.)

Introduce the dimensional regularization into the l.h.s. of (12.14), multiply A and B , and perform termwise integrations using the fact that the dimensionally regularized integration nullifies integrands with simple scaling properties—cf. (7.27). Finally, we arrive at the following analogue of (7.39):

$$c_{\Gamma, \alpha}(\kappa) = \int dp_\Gamma \mathcal{P}_{\Gamma, \alpha}(p_\Gamma) \Gamma(p_\Gamma, \kappa), \quad (12.18)$$

where $\mathcal{P}_{\Gamma, \alpha}$ is the basis of polynomials of the variable p_Γ , which is dual to the basis of δ -functions $\delta_{\Gamma, \alpha}(p_\Gamma)$ —cf. (7.35). The counterterms $(\mathbf{E}\Gamma)_n$ from (11.17) now are as follows:

$$(\mathbf{E}\Gamma)_n = \sum_{|\alpha| = \omega_\Gamma + n} c_{\Gamma, \alpha}(\kappa) \delta_{\Gamma, \alpha}. \quad (12.19)$$

Using the scaling properties, we get:

$$c_{\Gamma, \alpha}(\kappa) = \kappa^{(|\alpha| - \omega_\Gamma + \dots)} c_{\Gamma, \alpha}(1), \quad (12.20)$$

where the dots denote a correction proportional to $D-4$, the deviation of the complex parameter of dimensionality from the canonical integer value. The integer part of the exponent in (12.20) shows which order in κ this term belongs to. One can also see that the dependence on κ in the expansion $\mathbf{As}\Gamma$ is as discussed in subsect. 5.3.

The formulae (11.16), (11.17), (12.19) and (12.18) present the full solution of the Master problem in the Euclidean case.

12.4 Remarks.

(i) It should be stressed that although the above formulae are represented in a form which is specific to dimensional regularization, there are no serious obstacles (except for necessity to introduce a large number of new notations) for writing analogous formulae in other regularizations (e.g. using cut-offs). But this falls outside the scope of the present paper (see, however, [20]).

(ii) Concerning the mathematical aspects of our derivation, the full details of the proof that are regularization independent are presented in [20]. As to the use of the dimensional regularization in the precedent section, it can be justified along the following lines. First, the definition of dimensionally regularized integrals of arbitrary smooth test functions over momentum variables was presented in [34]. Second, as noted after (12.17), the integral in (12.14) can be represented as an absolutely convergent integral of ordinary smooth functions. Therefore, one only has to extend the results of [34] to a wider class of functions, which seems to present no difficulties except that it is a rather boring exercise. This issue will, hopefully, be addressed in a future publication. Another approach would be to check the final formulae for expansions of Feynman diagrams using the α -parametric integral representation. But the attempts to do so [16] show that a full proof would be a straightforward, cumbersome and unilluminating procedure.

12.5 Commutativity with multiplication by polynomials.

In conclusion of the discussion of the Master problem, a remark is in order. We have proved existence of the *As*-operation and its uniqueness (subsect. 5.2). It follows that it commutes with multiplication of Γ by an arbitrary polynomial of momenta—indeed, the polynomial can either be included into the distribution to be expanded (Γ) or, equivalently, it can be considered to be a part of the test function; the final result cannot depend on which option is chosen. This means that the operation \mathbf{E} which generates the counterterm for a subgraph, commutes with multiplication by polynomials, too. This can also be checked with the help of the explicit representations obtained above.

The described property of the *As*-operation is useful in situations when one needs to transform expansions of Feynman diagrams to a convenient form.

13 Example of an *As*-expansion.

13.1 The expression to be expanded and its singularities.

Let the integration variable p consist of two 4-dimensional components:

$$p = (p_1, p_2). \tag{13.1}$$

The product to be expanded is (cf. Fig.7a):

$$G(p, \kappa) = g_1(p, \kappa)g_2(p, \kappa)g_3(p, \kappa), \quad (13.2)$$

where

$$g_1(p, \kappa) = \frac{1}{p_1^2 + \kappa^2}, \quad (13.3)$$

$$g_2(p, \kappa) = \frac{1}{p_2^2 + \kappa^2},$$

$$g_3(p, \kappa) = \frac{1}{(p_1 - p_2)^2 + \kappa^2}.$$

The expansion of an isolated propagator in the sense of the distribution theory has been obtained and studied in sect. 7.

Each propagator in (13.2) generates, upon formal Taylor expansion in κ , singularities at the points where the corresponding momentum variable vanishes. Thus, the singular planes for each factor are:

$$\pi_1 = \{p \mid p_1 = 0\}, \quad (13.4)$$

$$\pi_2 = \{p \mid p_2 = 0\}, \quad (13.5)$$

$$\pi_3 = \{p \mid p_1 - p_2 = 0\}, \quad (13.6)$$

i.e. π_1 consists of all p such that $p_1 = 0$, etc.

The set of all IR-subgraphs of G comprises three subgraphs γ_i , $i = 1, 2, 3$, each consisting of one factor, and G itself. The two-factor subsets do not satisfy the completeness condition and therefore are not IR-subgraphs.

13.2 Fixing singularities corresponding to subgraphs.

Now we note that in the space of p *without* the point $p = 0$, the planes π_i do not intersect, so that a reasoning similar to that of subsects. 7.10 and 12.1 can be used. Indeed, consider a test function $\varphi_1(p)$ which is non-zero within the region \mathcal{O} shown in Fig.7b and therefore is zero in neighbourhoods of both π_2 and π_3 . Then

$$\int dp_1 dp_2 G(p, \kappa) \varphi_1(p) \equiv \int dp_1 g_1(p, \kappa) \tilde{\varphi}_1(p_1, \kappa), \quad (13.7)$$

where

$$\tilde{\varphi}_1(p_1, \kappa) = \int dp_2 G \setminus g_1(p, \kappa) \varphi_1(p) = \int dp_2 g_2(p, \kappa) g_3(p, \kappa) \varphi_1(p). \quad (13.8)$$

The expansion of the expression (13.7) can be obtained (cf. subsect. 11.3; also note that we present the formulae in dimensionally regularized form) by Taylor-expanding $\tilde{\varphi}_1(p_1, \kappa)$ (which does not give rise to any singularities in p_1) and using (7.38) to expand the propagator g_1 (cf. (11.10)). Denote:

$$[\mathbf{E}g_1](p, \kappa) = \sum_{\alpha} c_{\alpha}(\kappa) \delta_{\alpha}(p_1) \quad (13.9)$$

with $c_\alpha(\kappa)$ given by (7.39), where δ_α and \mathcal{P}_α have been introduced in (7.34), (7.35) (it is convenient to assume that $\delta_0 = \delta$ and $\mathcal{P} = 1$). Then one can rewrite the obtained result as follows:

$$G(p, \kappa) \underset{\kappa \rightarrow 0}{\simeq} (\mathbf{T}g_1(p, \kappa) g_2(p, \kappa) g_3(p, \kappa)) + (\mathbf{T}g_2(p, \kappa) g_3(p, \kappa)) [\mathbf{E}g_1](p, \kappa), \quad (13.10)$$

which is valid on the test functions φ_1 described above.

Analogous expansions can be obtained for the cases of test functions φ_2 and φ_3 that are in the same relation to the singular planes π_2 and π_3 , respectively, as φ_1 is to π_1 .

Consider a test function $\varphi_0(p)$ which is identically zero in some neighbourhood of the point $p = 0$. It can always be represented as

$$\varphi_0(p) = \varphi_1(p) + \varphi_2(p) + \varphi_3(p), \quad (13.11)$$

where all φ_i are as described above. Therefore, the three expansions that are valid on each φ_i can be glued together into an expansion that is valid on any φ_0 :

$$\begin{aligned} G(p, \kappa) &\underset{\kappa \rightarrow 0}{\simeq} (\mathbf{T}g_1(p, \kappa) g_2(p, \kappa) g_3(p, \kappa)) + (\mathbf{T}g_2(p, \kappa) g_3(p, \kappa)) [\mathbf{E}g_1](p, \kappa) \\ &+ (\mathbf{T}g_1(p, \kappa) g_3(p, \kappa)) [\mathbf{E}g_2](p, \kappa) + (\mathbf{T}g_1(p, \kappa) g_2(p, \kappa)) [\mathbf{E}g_3](p, \kappa) \\ &\equiv \mathbf{A}s'G(p, \kappa). \end{aligned} \quad (13.12)$$

This agrees with (12.1)–(12.2).

Consider the expansion (13.12) to order $o(\kappa^2)$:

$$\begin{aligned} \mathbf{A}s'_2 G(p, \kappa) &= \frac{1}{p_1^2 p_2^2 (p_1 - p_2)^2} \\ &- \kappa^2 \left[\frac{1}{p_1^4 p_2^2 (p_1 - p_2)^2} + \frac{1}{p_1^2 p_2^4 (p_1 - p_2)^2} + \frac{1}{p_1^2 p_2^2 (p_1 - p_2)^4} \right] \\ &+ c_0(\kappa) \left[\delta(p_1) \frac{1}{p_2^2 (p_1 - p_2)^2} + \frac{1}{p_1^2} \delta(p_2) \frac{1}{(p_1 - p_2)^2} + \frac{1}{p_1^2 p_2^2} \delta(p_1 - p_2) \right]. \end{aligned} \quad (13.13)$$

Recall that

$$c_0(\kappa) = \int d^D q (q^2 + \kappa^2)^{-1} = \kappa^{2(D-4)} c_0(1). \quad (13.14)$$

One sees that the $O(\kappa^2)$ terms possess a logarithmic singularity at $p = 0$. Therefore, the r.h.s. of (13.13) is well-defined on test functions $\varphi(p)$ such that $\bar{\varphi}(0) = 0$. Moreover, on such test functions the asymptotic character of the expansion is preserved which means (to order $o(\kappa^2)$) that:

$$\int d^{2D} p \bar{\varphi}(p) [G(p, \kappa) - \mathbf{A}s'_2 G(p, \kappa)] = o(\kappa^2). \quad (13.15)$$

This should be compared with (7.3). An explicit proof of (13.15) proceeds along the same lines as that of (7.3). A new element here is that one should consider cone regions in the integration space, as follows. (There is a limited resemblance between such cone regions and the Hepp sectors used in the theory of α -representation. However, the Hepp sectors correspond to a complete resolution of the recursive structure of singularities, while in our case only one-level descent—form the complete graph to its largest subgraphs—is performed.)

13.3 Approximation properties of As' .

Let $\theta_1(p)$ be such that $\theta_1(p) \equiv 1$ around each non-zero point of π_1 , $\theta_1(p) \equiv 0$ around non-zero points of π_2 and π_3 , and $\theta_1(\lambda p) = \theta_1(p)$ for each λ , $p \neq 0$. Consider the following contribution to (13.15):

$$\int d^{2D}p \theta_1(p) \bar{\varphi}(p) [G(p, \kappa) - \mathbf{As}'_2 G(p, \kappa)]. \quad (13.16)$$

One splits the integration region into two parts: $|p_1| > \kappa$ and $|p_1| < \kappa$ (cf. Fig.7c and subsect. 7.2). In the region $|p_1| < \kappa$, one rescales $p \rightarrow \kappa p$ then takes into account the scaling properties of G (which are not violated by the As -operation) and the fact that $\bar{\varphi}(p) \sim p$ near $p = 0$, and finds that

$$\int_{|p_1| < \kappa} d^{2D}p \theta_1(p) \bar{\varphi}(p) [G(p, \kappa) - \mathbf{As}'_2 G(p, \kappa)] = O(\kappa^3). \quad (13.17)$$

In the region $|p_1| > \kappa$, it is convenient to reorder the integrand of (13.16) as follows:

$$\begin{aligned} & \int_{|p_1| > \kappa} d^{2D}p \theta_1(p) \bar{\varphi}(p) \left\{ G(p, \kappa) - \frac{1}{p_1^2 p_2^2 (p_1 - p_2)^2} \right. \\ & \left. + \kappa^2 \left[\frac{1}{p_1^4 p_2^2 (p_1 - p_2)^2} + \frac{1}{p_1^2 p_2^4 (p_1 - p_2)^2} + \frac{1}{p_1^2 p_2^2 (p_1 - p_2)^4} \right] \right\} \\ & = \int_{|p_1| > \kappa} d^{2D}p \theta_1(p) \bar{\varphi}(p) \times \left\{ \left[\frac{1}{p_1^2 + \kappa^2} - \frac{1}{p_1^2} + \kappa^2 \frac{1}{p_1^4} \right] \frac{1}{p_2^2 + \kappa^2} \frac{1}{(p_1 - p_2)^2 + \kappa^2} \right. \\ & \left. + \left[\frac{1}{p_1^2} \right] \left[\frac{1}{p_2^2 + \kappa^2} \frac{1}{(p_1 - p_2)^2 + \kappa^2} - \frac{1}{p_2^2 (p_1 - p_2)^2} + \kappa^2 \left(\frac{1}{p_2^4 (p_1 - p_2)^2} + \frac{1}{p_2^2 (p_1 - p_2)^4} \right) \right] \right. \\ & \left. + \left[-\kappa^2 \frac{1}{p_1^4} \right] \left[\frac{1}{p_2^2 + \kappa^2} \frac{1}{(p_1 - p_2)^2 + \kappa^2} - \frac{1}{p_2^2 (p_1 - p_2)^2} \right] \right\}. \end{aligned} \quad (13.18)$$

Consider e.g. the first term contributing to the r.h.s.:

$$\begin{aligned} & \int_{|p_1| > \kappa} d^{2D}p \theta_1(p) \bar{\varphi}(p) \\ & \times \left[\frac{1}{p_1^2 + \kappa^2} - \frac{1}{p_1^2} + \kappa^2 \frac{1}{p_1^4} - c_0(\kappa) \delta(p_1) \right] \frac{1}{p_2^2 + \kappa^2} \frac{1}{(p_1 - p_2)^2 + \kappa^2} \\ & = \int_{|p_1| > \kappa} d^D p_1 \bar{\varphi}(p_1, \kappa) \left[\frac{1}{p_1^2 + \kappa^2} - \frac{1}{p_1^2} + \kappa^2 \frac{1}{p_1^4} \right], \end{aligned} \quad (13.19)$$

where

$$\bar{\varphi}(p_1, \kappa) = \int d^D p_2 \frac{\theta_1 \bar{\varphi}}{(p_2 + \kappa^2)((p_1 - p_2)^2 + \kappa^2)}. \quad (13.20)$$

The square-bracketed term on the r.h.s. of (13.19) is bounded by $\text{const} \times \kappa^4 / |p_1|^6$. To estimate $\bar{\varphi}$, one should take into account that the integration over p_2 in (13.20) runs over the region

$|p_2| < k|p_1|$, and that $\bar{\varphi}$ can be represented as $p_1\tilde{\varphi}_1 + p_2\tilde{\varphi}_2$, where both $\tilde{\varphi}_i$ are smooth functions [63]. Rescaling the integration variable as $p_2 \rightarrow |p_1| \cdot p_2$, one gets:

$$\bar{\varphi}(p_1, \kappa) = |p_1|^{D-4+1} \int_{|p_2|<k} d^D p_2 \frac{\theta_1(\hat{p}_1, p_2) \bar{\varphi}_1(p_1, |p_1| \cdot p_2)}{(p_2^2 + \kappa^2/|p_1|^2)((\hat{p}_1 - p_2)^2 + \kappa^2/|p_1|^2)}. \quad (13.21)$$

Since $\bar{\varphi}_1(p_1, |p_1| \cdot p_2)$ can be bounded by $\text{const} \times |p_1|$, one sees that $\bar{\varphi}(p_1, \kappa)$ is bounded by $\text{const} \times |p_1|$. (It is not difficult to understand that since the expansion (13.12) should be asymptotic only at $D = 4$, one may check estimates like (13.15) only at $D = 4$.) One arrives at the conclusion that the expression (13.19) is $o(\kappa^2)$, as expected. The remaining terms in (13.18) are considered in the same way, and, finally, one obtains (13.15).

13.4 The last counterterm.

The result of the preceding subsection (the estimate (13.15)) allows one to repeat the reasoning of subsect. 12.3.

One finally obtains:

$$G(p, \kappa) \underset{\kappa \rightarrow 0}{\simeq} \mathbf{As}_2 G(p, \kappa) + o(\kappa^2), \quad (13.22)$$

where

$$\mathbf{As}_2 G(p, \kappa) = \mathbf{As}'_2 G(p, \kappa) + c_{G,0}(\kappa) \delta(p) \quad (13.23)$$

($\delta(p) \equiv \delta(p_1) \delta(p_2)$) and

$$c_{G,0}(\kappa) = \int d^{2D} p G(p, \kappa) \equiv \int d^D p_1 d^D p_2 G(p_1, p_2, \kappa). \quad (13.24)$$

Examples of higher-order counterterms are:

$$c_{G,2,0}(\kappa) \partial^2 \delta(p_1) \delta(p_2), \quad (13.25)$$

where

$$c_{G,2,0}(\kappa) = \int d^D p_1 d^D p_2 p_1^2 G(p_1, p_2, \kappa), \quad (13.26)$$

and

$$c_{G,0,2}(\kappa) \partial_\mu \delta(p_1) \partial^\mu \delta(p_2), \quad (13.27)$$

where

$$c_{G,0,2}(\kappa) = \int d^D p_1 d^D p_2 (p_1 \cdot p_2) G(p_1, p_2, \kappa), \quad (13.28)$$

etc. Concerning diagrammatic interpretation of (13.24), (13.26) and (13.28), see Fig.7d.

This completes our discussion of the example.

The essential point to be stressed here is that the proof involves nothing but the power counting. The same remains true in the most general case as well, which justifies the conclusions of subsect. 12.2. For a more formal treatment of this point see [20].

Note that in practice there is no real need to performed the detailed analysis as above. It is simply sufficient to enumerate singularities of the formal expansion, determine (by power

counting) what counterterms are needed, and to find counterterms by formal integration of the ansatz for the As -operation with suitable polynomials to project out the coefficients.

Acknowledgments.

I would like to express my gratitude to V.A.Matveev, A.V.Radyushkin, A.N.Tavkhelidze and A.A.Vladimirov for the interest in, and support of this work since its early stages. I am indebted to G.B.Pivovarov, V.V.Vlasov and A.N.Kuznetsov for collaboration on putting the theory of asymptotic expansions of distributions on a rigorous foundation, and to A.A.Pivovarov for a discussion of the sum rules method. I would also like to thank N.Atakishiev, C.Burgess, G.Efimov, M.Fukugita, E.Remiddi, D.Robaschik, D.Shirkov, A.Slavnov, D.Slavnov, and S.Zlatev for encouragement and stimulating discussions.

The main part of this paper was written during my stay at Jilin University (Changchun, China), and it is my pleasure to thank Professor Wu Shishu and the staff of the Physics Department for their warm hospitality. I am grateful to A.P.Contogouris for hospitality at the McGill University (Montreal, Canada), and to W.A.Bardeen and R.K.Ellis for their kind hospitality at the Theory division of Fermilab where the manuscript was completed.

Unfortunately, Sergei Gorishny, who contributed so much to the theory of Euclidean asymptotic expansions, will never read these lines. The only thing I can do to appreciate his generous help and advice is to dedicate this publication to his memory.

Finally, my thanks are due to John Collins for a careful reading of the manuscript and stimulating discussions concerning extension of the described methods to Minkowskian asymptotic regimes.

References

- [1] D.A.Slavnov, *Teor. Mat. Fiz.* 17 (1973) 169.
- [2] E.B.Manoukian, "Renormalization," Academic Press, New York, 1983.
- [3] M.C.Bergère, C.de Calan, and A.P.C.Malbouisson, *Commun. Math. Phys.* 62 (1978) 137.
- [4] E.B.Manoukian: *J. Phys. G7*, (1981) 1159.
- [5] K.Pohlmeyer: *J. Math.Phys.* 23 (1982) 2511.
- [6] P.Kosinski and P.Maslanka: *Lett. Math. Phys.* 9 (1985) 157.
- [7] K.G.Chetyrkin, S.G.Gorishny, and F.V.Tkachov: *Phys.Lett.* 119B (1982) 407.
- [8] K.Wilson: *Phys.Rev.* 179 (1969) 1499.
- [9] W.Zimmermann, in: "Lectures on elementary particles and quantum field theory," (S.Deser, M.Grisaru and S.Pendleton, Eds.), v.1, MIT Press, Cambridge (Mass.), 1970; *Ann.Phys.(N.Y.)* 77 (1973) 570.
- [10] W.A.Bardeen, A.J.Buras, D.W.Duke and T.Muta,: *Phys. Rev. D*18 (1978) 3998.
- [11] F.V.Tkachov: *Phys.Lett.* 125B (1983) 85.
- [12] F.V.Tkachov, in: "Proc. Int. Seminar Quarks-82," Inst. Nucl. Research, Moscow, 1983.
- [13] F.V.Tkachov: *Phys.Lett.* 124B (1983), 212.
- [14] S.G.Gorishny, S.A.Larin and F.V.Tkachov: *Phys.Lett.* 124B (1983) 217.
- [15] F.V.Tkachov: preprint INR P-332, Moscow, 1984; preprint INR P-358, Moscow, 1984; "Euclidean asymptotic expansions of Green functions of quantum fields. (I) Expansions of products of singular functions", preprint McGill/91-06.
- [16] G.B.Pivovarov and F.V.Tkachov: preprint INR P-370, Moscow, 1984; preprint INR P-459, Moscow, 1986; G.B.Pivovarov and F.V.Tkachov, in: "Quarks-86. Proc. of the Seminar." VNU Science Press, Utrecht, 1987.
- [17] G.B.Pivovarov and F.V.Tkachov: *Teor. Mat. Fiz.* 77 (1988) 51.
- [18] G.B.Pivovarov and F.V.Tkachov: "Euclidean asymptotic expansions of Green functions of quantum fields. (II) Combinatorics of the As-operation", preprint FERMILAB-PUB-91/345-T.
- [19] A.N.Kuznetsov and F.V.Tkachov: "Euclidean asymptotic expansions of Green functions of quantum fields. (III) MS scheme as subtraction of asymptotics in momentum space." (in preparation).

- [20] F.V.Tkachov and V.V.Vlasov: "Multiloop Feynman diagrams and distribution theory. (I), (II)", preprints McGill/91-03 and 04.
- [21] A.N.Kuznetsov and F.V.Tkachov: "MS-Scheme without dimensional regularization.": Talk at "Quarks-88. Int. Seminar", Tbilisi, USSR, 1988; "Multiloop Feynman diagrams and distribution theory. (III) UV renormalization as subtraction of asymptotics in momentum space", preprint NIKHEF/90-17, Amsterdam.
- [22] A.N.Kuznetsov, F.V.Tkachov and V.V.Vlasov: "Multiloop Feynman diagrams and distribution theory. (IV) Asymptotic expansions of renormalized diagrams", preprint FERMILAB-PUB-91/346-T.
- [23] L.Schwartz: "Théorie des Distributions," Hermann, Paris, 1966.
- [24] N.N.Bogoliubov and D.V.Shirkov: "Introduction to the Theory of Quantized Fields," Nauka, Moscow, 1984.
- [25] G.'t Hooft: Nucl.Phys. B61 (1973), 455.
- [26] J.C.Collins: "Renormalization," Cambridge Univ. Press, Cambridge, 1984.
- [27] J.C.Collins: "Sudakov form factors", in: "Perturbative QCD" (A.H.Mueller, ed.), World Scientific, Singapore, 1989.
- [28] S.R.Surguladze and F.V.Tkachov: Nucl.Phys. B331 (1990) 35; S.A.Larin, F.V.Tkachov, and J.A.M.Vermaseren: Phys.Rev.Lett. 66 (1991) 862; S.A.Larin and J.A.M.Vermaseren: "The $O(\alpha_s^3)$ corrections to the Bjorken sum rule for polarized electroproduction and to the Gross-Llewellyn Smith sum rule", preprint NIKHEF-H/90-23, Amsterdam, 1991.
- [29] K.G.Chetyrkin and F.V.Tkachov: Phys.Lett. 114B (1982) 340.
- [30] K.G.Chetyrkin, S.G.Gorishny, S.A.Larin and F.V.Tkachov: Phys.Lett. 132B (1983) 351.
- [31] S.G.Gorishny, S.A.Larin and F.V.Tkachov: Phys.Lett. 101A (1984) 120.
- [32] G.'t Hooft and M.Veltman: Nucl.Phys. B44 (1972) 189.
- [33] P.Breitenlohner and D.Maison: Commun. Math. Phys.: 52 (1977) 11.
- [34] P.M.Blekher: Teor.Mat.Fiz. 50 (1982), 370.
- [35] L.Rosen and J.D.Wright: Comm. Math. Phys. 134 (1990) 433.
- [36] F.V.Tkachov (in preparation).
- [37] M.A.Shifman, A.I.Vainshtein and V.I.Zakharov: Nucl.Phys. B147 (1979), 385, 448.
- [38] F.V.Tkachov, in: Proc. Int. Seminar QUARKS-90, World Scientific, Singapore, 1991; "Mathematical methods of perturbative QCD: a new paradigm", preprint McGill/05-91.
- [39] S.Gorishny: Nucl.Phys. B319 (1989) 633.

- [40] C.H.Llewellyn Smith and J.P.de Vries: Nucl.Phys. B296 (1988) 991.
- [41] V.A.Smirnov: Commun. Math. Phys. 134 (1990) 109.
- [42] K.G.Chetyrkin: preprint MPI-Ph/PTh 13/91.
- [43] M.C.Bergere and F.David: Comm. Math. Phys. 81 (1981) 1.
- [44] G.Bonneau: Nucl.Phys. B167 (1982) 261.
- [45] L.J.Reinders, H.Rubinstein and S.Yazaki: Phys.Rep. 127 (1985), 1.
- [46] A.V.Radyushkin: Soviet J. Part. Nuclei 14 (1983), 58.
- [47] S.Wandzura: Nucl.Phys. B122 (1977), 412.
- [48] B.L.Ioffe, L.N.Lipatov and V.A.Khoze: "Deep inelastic processes". Energoatom, Moscow, 1983 (in Russian).
- [49] A.N.Tikhonov et al.: "Regularizing algorithms and apriori information," Mir, Moscow, 1983 (in Russian).
- [50] M.R.Pennington, G.G.Ross and R.G.Roberts: Phys.Lett. 120B (1983), 204.
- [51] P.S.Isaev: "Quantum electrodynamics at high energies," Energoatom, Moscow, 1984 (in Russian).
- [52] B.L.Ioffe and A.V.Smilga: Phys.Lett. 114B (1982), 353; V.A.Nesterenko and A.V.Radyushkin: Phys.Lett. 115B (1982), 410.
- [53] I.I.Balitsky and A.Yung: Phys.Lett. 129B (1983), 328; B.L.Ioffe and A.V.Smilga, Pis'ma v Zh.E.T.F.: 37 (1983), 250.
- [54] F.V.Tkachov: Phys.Lett. 100B (1981), 65; Teor.Mat.Fiz. 56 (1983), 350; K.G.Chetyrkin and F.V.Tkachov: Nucl.Phys. B192 (1981), 159; S.G.Gorishny, S.A.Larin, L.R.Surguladze and F.V.Tkachov: Comp. Phys. Commun. 55 (1989) 381; L.R.Surguladze and F.V.Tkachov: Comp. Phys. Commun. 55 (1989) 205.
- [55] K.Symanzik: Comm.Math.Phys. 37 (1973), 7; T.Appelquist and J.Carrazzone: Phys.Rev. D11 (1975), 298; J.Ambjørn: Commun. Math. Phys. 66 (1979) .
- [56] G.Altarelli and L.Maiani: Phys.Lett. 52B (1974), 351; M.Gaillard and B.W.Lee: Phys.Rev.Lett. 33 (1974), 108.
- [57] B.Grinstein: Phys.Lett. 134B (1984), 111.
- [58] V.A.Beylin and A.V.Radyushkin: Yad.Phys.(Sov.J.Nucl.Phys.) 45 (1987), n.2.
- [59] I.I.Balitsky: Phys.Lett. 114B (1982), 53.
- [60] A.A.Pivovarov, N.N.Tavkhelidze and V.F.Tokarev: Phys.Lett. 132B (1983), 402.

- [61] F.W.J.Olver: "Introduction to asymptotic methods and special functions", Academic Press, New York, 1974.
- [62] V.G.Kanovei: "Axiom of Choice and Axiom of Determination", Nauka, Moscow, 1984 (in Russian).
- [63] L.Schwartz: "Analyse," vol.1, Hermann, Paris, 1968.
- [64] S.Libby and G.Sterman: Phys.Rev. D18 (1978) 3252, 4737.
- [65] S.G.Gorishny and S.A.Larin: Nucl.Phys. B283 (1987), 452.
- [66] G.B.Pivovarov: Teor. Mat. Phys. 80 (1989) 702.
- [67] J.C.Collins, D.E.Soper and G.Sterman: "Factorization of hard processes in QCD", in: "Perturbative QCD" (A.H.Mueller, ed.) (World Scientific, Singapore, 1989).
- [68] R.K.Ellis, M.A.Furman, H.E.Haber and I.Hinchliffe: Nucl.Phys. B173 (1980) 397.

Figure captions.

Fig.1. Kinematics of three-point functions.

Fig.2. (a) Kinematics of light-by-light scattering. It differs from the general 4-point function (b) by the restriction $q_1 + q_3 = 0$.

Fig.3. Three light-mass propagators connected to a common vertex form a complete IR-subgraph.

Fig.4. The property of being an IR-subgraph depends on restrictions imposed on heavy external momenta: in (a) the pair of vertical lines constitutes an IR subgraph, while in (b) it does not.

Fig.5. This graph is its own IR-subgraph if the contact terms localized at $Q = 0$ are to be taken into account; it is not, if Q is just fixed at a non-zero value.

Fig.6. A simple example of the division of the proper variables of IR-subgraphs into “internal” components corresponding to loops and the “external” components.

Fig.7. (a) A graphical representation of the product (13.2). (b) The test function φ_1 is non-zero within the region enclosed by the dashed line. (c) Geometry of integrations in (13.19) and (13.20). (d) A graphical interpretation of the expressions (13.24), (13.26) and (13.28).

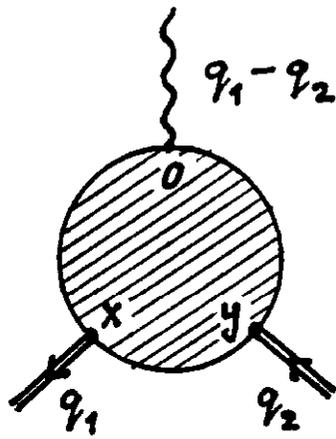
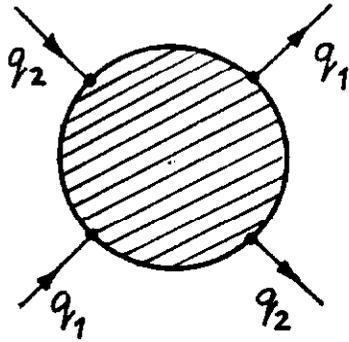
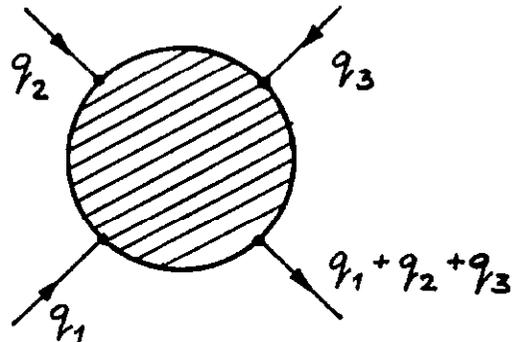


Fig. 1



(a)



(b)

Fig. 2.

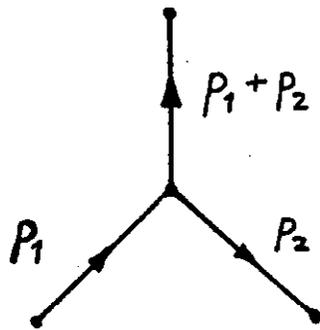


Fig. 3

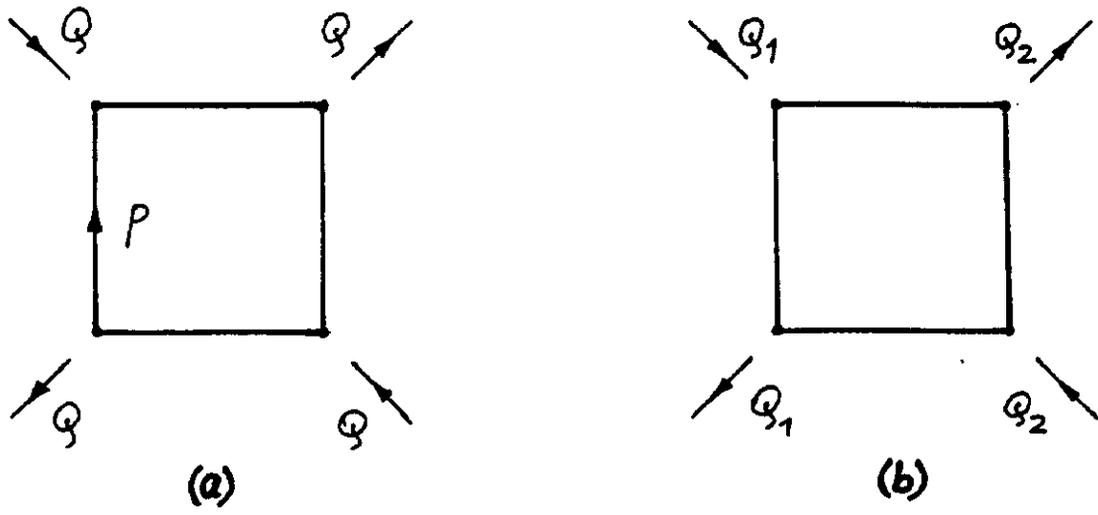


Fig. 4

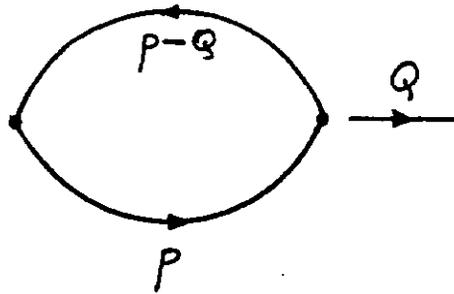


Fig. 5

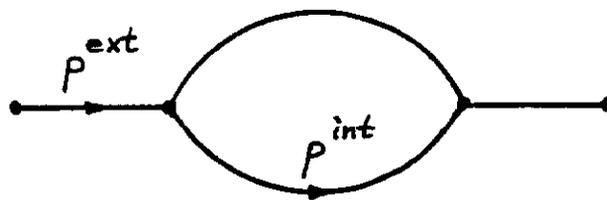
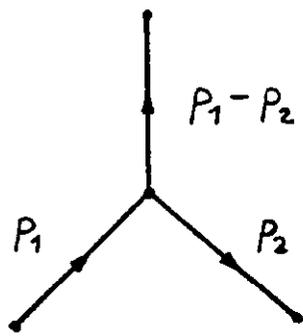
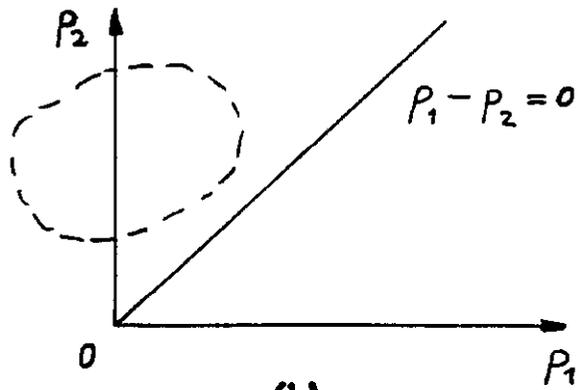


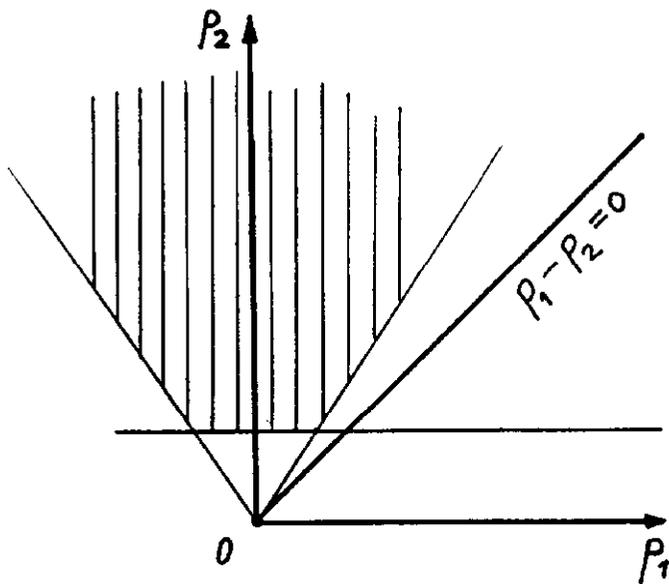
Fig. 6



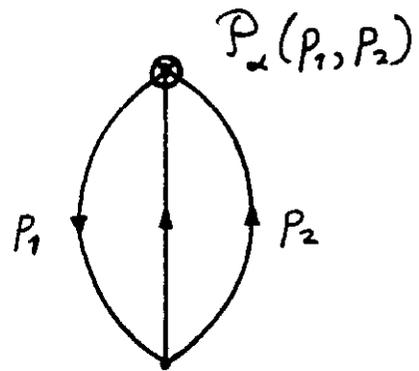
(a)



(b)



(c)



(d)

Fig.7.