

## EUCLIDEAN ASYMPTOTIC EXPANSIONS OF GREEN FUNCTIONS OF QUANTUM FIELDS. (II) COMBINATORICS OF THE $As$ -OPERATION.

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### ABSTRACT

The results of [1] are used to obtain full asymptotic expansions of Feynman diagrams renormalized within the MS scheme in the regimes when some of the masses and external momenta are large with respect to the others. The large momenta are Euclidean, and the expanded diagrams are regarded as distributions with respect to them. The small masses may be equal to zero. The  $As$ -operation for integrals is defined and a simple combinatorial techniques is developed to study its exponentiation. The  $As$ -operation is used to obtain the corresponding expansions of arbitrary Green functions. Such expansions generalize and improve upon the well-known short-distance operator-product expansions, the decoupling theorem etc.; e.g. the low-energy effective Lagrangians are obtained to all orders of the inverse heavy mass. The obtained expansions possess the property of perfect factorization of large and small parameters, which is essential for meaningful applications to phenomenology. As an auxiliary tool, the inversion of the  $R$ -operation is constructed. The results are valid for arbitrary QFT models.

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# 1 Introduction

In the preceding paper [1] the problem of obtaining asymptotic expansions of multiloop Feynman diagrams was analyzed. From the point of view of phenomenology, it was stressed that such expansions must possess the property of “perfect factorization” of large and small dimensional parameters. From the point of view of mathematics, it was shown that the key technical problem (“the Master problem”) is that of obtaining asymptotic expansions—in powers and logarithms of the expansion parameter—of products of singular functions in the sense distributions. An explicit solution for the Master problem was obtained in the form of the so-called *As-operation for products of singular functions*, which acts on Euclidean momentum-space Feynman integrands and yields an expansion whose all terms are well-defined distributions (containing, in particular,  $\delta$ -functional contributions with coefficients depending non-analytically on the expansion parameter) so that the expansion allows termwise integrations with test functions.

The purpose of the present paper is to use the results of [1] and obtain explicit expansions of multiloop Feynman integrals in the form of *As-operation for integrals*.<sup>1</sup> It turns out that the combinatorial structure of the *As-operation* for integrals is very similar to that of the *R-operation*.<sup>2</sup> This similarity allows one to derive expansions of perturbative Green functions in a global form with the help of the same techniques as used in studying the exponentiation of the *R-operation*.

The present paper is a revised and simplified version of our publications [2], [3] and [4] which have been widely discussed in the literature [20]–[22].<sup>3</sup> The revision has not affected the results but only the order of presentation. The simplification is due to the fact that the starting point of [2] was the so-called EA-expansion for UV-renormalized integrals, which is a composition of *R-operation* and the *As-operation* in the sense of the present paper. Correspondingly, the *As-operation* as defined in [2] mixed up UV-renormalization and the expansion algorithm. In the present version we define the *As-operation* and study its exponentiation directly for the unrenormalized integrals, which results in further simplifications.

The attractiveness of the *As-operation* is due to the fact that it serves as an organizing center of the theory and fully exhibits the structure of the expansions, which is invaluable for applications where diagram-by-diagram analysis is necessary to enhance reliability of calculations. Its definition is also remarkably simple within dimensional regularization, which trivializes the study of exponentiation. It should be stressed, however, that the use of dimensional regularization is by no means essential for the expansion problem; a separate publication is devoted to a study of this point [14], [15], and especially [17]. Euclidean asymptotic expansions were studied in regularization independent manner within the framework of generalized MS schemes [15]. The main result of [17] is an expansion formula whose combinatorial structure is practically identical to that of the *As-operation* of the present paper. Therefore, all our combinatorial

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<sup>1</sup>The two *As-operations* are different representations of essentially the same operation, which justifies using the same name to denote them: the *As-operation* for integrals is an integrated version of the *As-operation* for products—cf. subsect. 5.1 below.

<sup>2</sup>Note that the *As-operation* for products also has close parallels with the *R-operation*—but with the *R-operation* in position representation.

<sup>3</sup>OPE in the MS scheme was also studied in [19].

results concerning exponentiation of the  $As$ -operation are immediately applicable to [17].

The plan of the present paper is as follows. It consists of five sections, each section is subdivided into subsections and contains a prologue where further information can be found. In the first three sections various aspects of combinatorial structure of the  $R$ -operation in the  $\overline{MS}$  scheme are studied. In sect. 2 the exponentiation formula for the  $R$ -operation is obtained using the functional techniques. The result is, of course, well-known. However, since we wish to use similar arguments in studying the  $As$ -operation, we feel it is useful to present an explicit derivation in such a way as to make it immediately applicable to the case of the  $As$ -operation studied in sects.4 and 5. In sect. 3 the inversion of the  $R$ -operation is derived, and in sect. 4 the renormalization of Green functions with multiple local operator insertions is considered.

In sects.4 and 5 we study the combinatorial structure of asymptotic expansions of Feynman diagrams. In sect. 5 we consider the special case of heavy mass expansions. The  $As$ -operation for integrals is introduced and its exponentiation is proved. Using the techniques developed in sects.1-3, the effective low-energy Lagrangian is presented in an explicitly convergent form to all orders in the inverse heavy mass. (It should be stressed that in this paper we only deal with purely combinatorial aspects of the theory. A fuller justification of a technical assumption (see subsect. 5.8 below) is presented in [17] (for an informal discussion see [16].) In sect. 6 the results of sect. 5 are extended to comprise the case of large external momenta, so that the familiar short-distance operator product expansion is reproduced and its generalizations are obtained.

Note that our final formulae immediately provide the very useful explicit expressions of OPE coefficient functions first announced in [11] (explicit examples of calculations going as far as 3-loop approximation can be found in [18]; for further references see also [23]).

The notations are on the whole consistent with those of [1].

# COMBINATORICS OF THE $R$ -OPERATION

## IN THE MS SCHEME

### 2 $R$ -operation in the MS scheme.

In this section we study the combinatorial structure of the  $R$ -operation applied to the perturbation series as a whole. The fact that the  $R$ -operation is equivalent to adding counterterms to the Lagrangian is of course well-known. But we are interested in exact formulae expressing this fact that would be convenient for practical calculations and valid for the MS scheme of UV renormalization. Therefore we reproduce a simple proof of the exponentiation  $R$ -operation in the MS scheme—see (2.25) below—as given in [2]. Our second aim is to present the derivation of the exponentiation formula (2.25) in a form sufficiently general to make it applicable to the case of the  $As$ -operation studied in Chapters 4 and 5. To make the paper more accessible to practitioners of applied QFT, our presentation is rather detailed.

We start in subsect. 2.1 with a description of the  $R$ -operation in the MS scheme. Our definitions are equivalent to the standard ones [5], and the differences are mainly notational. In subsect. 2.2 a convenient order of enumeration of UV subgraphs is fixed in preparation to studying the global structure of the  $R$ -operation. Subsect. 2.3 summarizes some results of the functional techniques that will be needed to us (for more details see e.g. [7]). In subsect. 2.4 a formal expression of the statement that the  $R$ -operation is equivalent to adding counterterms in the Lagrangian, the exponentiation formula (2.25), is derived; our reasoning is an extension of the arguments of [6]—for comments and comparison see subsect. 2.4). In subsect. 2.5 the results are extended to Green functions of fields. Complications due to composite operator insertions will be discussed in sect. 4.

#### 2.1 Basic definition of the $R$ -operation.

The effect of the  $R$ -operation in the MS-scheme on a single Feynman integral  $G$  can be expressed by the formula

$$\mathbf{R} \circ G = \sum_{\{g_\alpha\}} \left( \prod_{\alpha} \Delta_{UV \circ g_\alpha} \right) \cdot G / \left( \prod_{\alpha} g_\alpha \right). \quad (2.1)$$

Let us explain it.

One associates a Feynman graph, that will be referred to as “the graph  $G$ ”, with the integral  $G$ , and vice versa. The correspondence between graphs and integrals is established by Feynman rules.

Let us fix how  $\mu$ , the 't Hooft unit of mass, appears in the integrals. In momentum representation, having taken into account momentum conservation in all the vertices of the graph, one can obtain the integrand of the corresponding unrenormalized Feynman integral. Then for

each internal momentum integration one replaces the measure as

$$\frac{d^4 p}{(2\pi)^4} \rightarrow \mu^{2\epsilon} \frac{d^D p}{(2\pi)^D}, \quad 2\epsilon = 4 - D, \quad (2.2)$$

and assumes that integrations (and also the Dirac algebra etc.) are to be done within dimensional regularization. The role of  $\mu$  is to preserve the dimensionality of the integral as a whole under regularization.

An *UV subgraph*  $g$  of  $G$  consists of some vertices of the graph  $G$  and some of its lines attached by both ends to the vertices of  $g$ . If  $g$  consists of one vertex and no lines then we call it *elementary vertex*. If  $g$  has at least one loop and is connected and one-particle-irreducible then we call  $g$  a *proper UV subgraph*. (Note that if  $G$  is one-particle-irreducible then one of its proper UV subgraphs coincides with  $G$ .)

An *UV partition* of  $G$  is a set of its UV subgraphs  $\{g_\alpha\}$  such that each vertex of  $G$  belongs to one and only one  $g_\alpha$ . Note that one of the UV partitions consists of elementary vertices only, and another one consists of only one subgraph, namely,  $G$  as a whole. Summation in (2.1) runs over all UV partitions of  $G$ .

Now we have to define the operation  $\Delta_{UV} \circ g$  where  $g$  is an UV subgraph of  $G$ . By definition,  $\Delta_{UV} \circ g = g$  if  $g$  is an elementary vertex, and  $\Delta_{UV} \circ g = 0$  if  $g$  is not an UV subgraph. If  $g$  is a proper UV subgraph, then the action of  $\Delta_{UV}$  on  $g$  can be described as follows. The graph for  $(\Delta_{UV} \circ g) \cdot (G/g)$  can be obtained from the graph for  $G$  by shrinking  $g$  in  $G$  to a point. With the new vertex thus obtained one associates a factor—the UV counterterm—that is obtained from the integral corresponding to  $g$  via a special algorithm which need not be specified here (see, however, below). Using the Feynman rules thus extended one finally builds up the integral for  $(\Delta_{UV} \circ g) \cdot (G/g)$ . Note that the above rule for  $\mu$  is still operative, i.e. the number of the factors  $\mu^{2\epsilon}$  is equal to the number of loops of  $G$  with  $g$  shrunk to a point.

For the sake of completeness let us describe the recipe for evaluating the UV counterterm for a given proper UV subgraph  $g$ . If one rewrites the *R*-operation as

$$\mathbf{R} \circ g \equiv \mathbf{R}' \circ g + \Delta_{UV} \circ g, \quad (2.3)$$

then  $\Delta_{UV} \circ g$  is precisely the UV counterterm to be evaluated while  $\mathbf{R}'$  involves only counterterms for subgraphs of  $g$  which are assumed to be known already. Since we are working within dimensional regularization, all our integrals depend parametrically on  $D$ , the complex-valued dimension of space-time. Let  $\mathbf{K}$  be the operation that picks out the pole part at  $D = 4$  of any function on which it acts.  $\mathbf{R} \circ g$  should be finite at  $D = 4$ , therefore  $\mathbf{K} \circ \mathbf{R} \circ g = 0$ . On the other hand, the UV counterterms in the MS-scheme are pure poles. So, applying  $\mathbf{K}$  to (2.3), we get:

$$\Delta_{UV} \circ g = -\mathbf{K} \circ \mathbf{R}' \circ g. \quad (2.4)$$

Eqs. (2.1), (2.3) and (2.4) provide a convenient description of the *R*-operation in the MS-scheme. We would like to stress that all the manipulations are to be done in momentum representation, and that the rule for  $\mu$  remains operative. Also recall that (2.4) is a polynomial of masses and external momenta of  $g$  and is independent of  $\mu$ .

The most important points for us here are:

(i)  $\Delta_{UV} \circ g$ , whatever the subgraph  $g$  is, depends only on  $g$  but not on  $G$  as a whole.

(ii) Summation in (2.1) runs over all partitions of  $G$ .

(iii) An UV subgraph is defined as a subset of vertices and some (not necessarily all) of the lines connecting the vertices of this subset.

On the other hand, the fact that  $\Delta_{UV}$  is non-trivial only on *proper* UV subgraphs will be completely irrelevant.

## 2.2 Enumerating UV subgraphs.

Let us transform (2.1) to a more convenient form. Let  $v$  be a subset of vertices of  $G$  and

$$\Lambda_v \circ G \equiv \sum_{g \sim v} (\Delta_{UV} \circ g) \cdot (G/g), \quad (2.5)$$

where summation runs over all UV subgraphs  $g$  with the same set of vertices,  $v$ . Denoting the set of all vertices of  $G$  as  $V$ , we can rewrite (2.1) as:

$$\mathbf{R} \circ G = \sum_{\substack{v = \cup v_\alpha \\ v_\alpha \cap v_\beta = \emptyset, \text{ for } \alpha \neq \beta}} \left( \prod_\alpha \Lambda_{v_\alpha} \right) \circ G. \quad (2.6)$$

Eqs.(2.5)--(2.6) fix a convenient order of enumerating UV partitions of  $G$ .

## 2.3 $T$ -products and functional techniques.

Let  $\hat{\varphi}(x)$  be a free field operator;  $\varphi(x)$  will denote the corresponding classical field. Let  $F(\varphi)$  be a functional of the form

$$F(\varphi) = \sum_n \int F_n(x_1 \dots x_n) \varphi(x_1) \dots \varphi(x_n) dx_1 \dots dx_n. \quad (2.7)$$

Replacing the product of  $\varphi$ 's in (2.7) by the  $T$ -product of  $\hat{\varphi}$ 's one obtains an operator which we denote as  $TF(\hat{\varphi})$ . The Wick theorem allows one to reexpand  $TF(\hat{\varphi})$  in terms of the Wick normal products of  $\hat{\varphi}$  :

$$TF(\hat{\varphi}) = \sum_n \int F_n^N(x_1 \dots x_n) N[\hat{\varphi}(x_1) \dots \hat{\varphi}(x_n)] dx_1 \dots dx_n, \quad (2.8)$$

where  $F_n^N$  are related to  $F_n$  in a certain way (see below).

The generating functional for the coefficient functions is defined as:

$$F^N(\varphi) = \sum_n \int F_n^N(x_1 \dots x_n) \varphi(x_1) \dots \varphi(x_n) dx_1 \dots dx_n. \quad (2.9)$$

The relation between  $F(\varphi)$  and  $F^N(\varphi)$  is expressed, by definition, by

$$\mathbf{T}F(\hat{\varphi}) = N[F^N(\varphi)]_{\varphi=\hat{\varphi}}. \quad (2.10)$$

The well-known formal expression of the Wick theorem due to Khorikawa (see [7] and refs. therein) reads:

$$F^N(\varphi) = e^l F(\varphi), \quad (2.11)$$

where

$$l \equiv \frac{1}{2} \int dx dy \frac{\delta}{\varphi(x)} \Delta^c(x, y) \frac{\delta}{\varphi(y)} \equiv \frac{1}{2} \delta \Delta^c \delta, \quad (2.12)$$

and

$$\Delta^c(x, y) = \langle \mathbf{T} \hat{\varphi}(x) \hat{\varphi}(y) \rangle_0. \quad (2.13)$$

(Extension of all our formulae to the most general case of fermionic and complex fields is straightforward, therefore our analysis will be quite general.)

Our purpose is to study Green functions, but first it is convenient to consider the (off-shell)  $S$ -matrix. The unrenormalized  $S$ -matrix is expressed as:

$$S(\hat{\varphi}) = \mathbf{T} \exp[L(\hat{\varphi})], \quad (2.14)$$

where  $L(\hat{\varphi})$  denotes the interaction Lagrangian integrated over space-time and multiplied by  $i$  (the normal ordering of fields in  $L(\hat{\varphi})$  is *not* assumed).

The generating functional  $S^N$  is defined as ( cf. (2.10)):

$$\mathbf{T} \exp[L(\hat{\varphi})] = N[S^N(\varphi)]_{\varphi=\hat{\varphi}} \quad (2.15)$$

and from (2.11):

$$S^N = e^l \exp[L(\varphi)]. \quad (2.16)$$

Again we have to set the rules for  $\mu$ , in order to establish connection between our earlier definitions of Feynman integrals and the expressions generated using the functional technique. We assume that:

- (i) all momentum and space-time integrations,  $g$ -matrices etc. are  $D$ -dimensional;
- (ii) the dimensions of the fields  $\hat{\varphi}$  and the coupling constants are always the canonical 4-dimensional ones;
- (iii) each propagator bears an extra factor  $\mu^{2\epsilon}$ ; note that (2.12) is dimensionless since we assume that  $\delta/\delta\varphi(x) \cdot \varphi(y) = \delta(x-y)$  with  $D$ -dimensional  $\delta$ -function;
- (iv) the interaction Lagrangian  $L(\varphi)$  in (2.14) contains the factor  $\mu^{-2\epsilon}$  and is therefore dimensionless; note that condition (iii) is equivalent to (iv) extended to the quadratic part of the full Lagrangian ( cf. (2.35) below).

Given these rules, a connected Feynman integral will have the factor  $\mu^{2\epsilon(l-1)}$  where  $l$  is the number of its internal momentum integrations (loops)—which differs by  $\mu^{-2\epsilon}$  from what was postulated in subsect. 2.1. Such extra  $\mu^{-2\epsilon}$  to each connected component can be easily taken into account in what follows.

## 2.4 Exponentiation of the $R$ -operation.

The renormalized  $S$ -matrix is obtained by applying the  $R$ -operation to each Feynman integral contributing to the coefficient functions of  $S^N$ . Thus, the starting point of our analysis is the following expression:

$$S_R(\varphi) = \mathbf{R} \circ [\mathbf{T} \exp(L(\varphi))] = \sum_{N=0}^{\infty} \frac{1}{N!} \mathbf{R} \circ \mathbf{T} L^N(\varphi), \quad (2.17)$$

where we have introduced the convenient notation

$$\mathbf{T} \equiv e^l, \quad (2.18)$$

which will be used systematically in the context of the functional techniques.

Evaluating functional derivatives, one obtains a sum of terms, each one having a graphic representation. In the resulting expression one only has to replace products of the classical fields  $\varphi$  by the normal ( $N$ -) products of the free fields  $\hat{\varphi}$ , in order to obtain the operator of the  $S$ -matrix. (Note that since we wish to use the MS-scheme for UV renormalization, the  $S$ -matrix will not satisfy correct normalization conditions automatically, but this is of no importance because our study of the  $S$ -matrix is only an intermediate step in the study of Green functions.)

Note that in (2.17) each  $L(\varphi)$  corresponds to a vertex while each  $l$  (see (2.12)) generates one line of a Feynman graph. This allows one to conveniently perform all the enumerations of UV partitions inherent in the definition of the  $R$ -operation as given by eqs. (2.1)–(2.4).

First one has to enumerate all possible partitions of  $N$  vertices into groups of vertices of various sizes ( cf. (2.6)).  $N$  vertices can be split into  $n_1$  groups of 1 vertex,  $n_2$  groups of 2 vertices ...  $n_N$  groups of  $N$  vertices in

$$\left\{ \begin{matrix} N \\ n_1 \dots n_N \end{matrix} \right\} = N! \left[ \prod_{k=1}^N n_k! (k!)^{n_k} \right]^{-1}, \quad \sum_k k_k \cdot n_k = N, \quad (2.19)$$

ways. Therefore, applying (2.6) to (2.17) we get:

$$\mathbf{T} \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\substack{n_1 \dots n_N \\ \sum k n_k = N}} \left\{ \begin{matrix} N \\ n_1 \dots n_N \end{matrix} \right\} (\Lambda \circ L^{n_1}) \dots (\Lambda \circ L^{n_N}). \quad (2.20)$$

Now, for each group of vertices we have to expand  $\Lambda$  according to (2.5). We have to separate the lines connecting vertices inside each group  $\Lambda \circ L^k$  from the lines connecting vertices of different groups, and among the former to enumerate all subsets of lines that will form UV subgraphs on which  $\Delta_{UV}$  acts. This can be done by ascribing different labels to the  $n_1 + \dots + n_N$  groups of vertices and marking the field  $\varphi$  in each group by this label. Then we have:

$$\mathbf{T} \left[ \prod_{\alpha} \Lambda \circ L^{n_{\alpha}}(\varphi) \right] = \exp\left(\frac{1}{2} \delta \Delta^c \delta\right) \left[ \prod_{\alpha} \Lambda \circ L^{n_{\alpha}}(\varphi_{\alpha}) \right]_{\varphi_{\alpha} = \varphi} \quad (2.21)$$

$$\begin{aligned}
&= \left[ \exp \left( \sum_{\alpha < \beta} \delta_\alpha \Delta^c \delta_\beta + \sum_\alpha \frac{1}{2} \delta_\alpha \Delta^c \delta_\alpha \right) \prod_\alpha \Lambda^\circ L^{n_\alpha}(\varphi_\alpha) \right]_{\varphi_\alpha = \varphi} \\
&= \left[ \exp \left( \sum_{\alpha < \beta} \delta_\alpha \Delta^c \delta_\beta \right) \prod_\alpha \Lambda^\circ \left( \exp \left( \frac{1}{2} \delta_\alpha \Delta^c \delta_\alpha L^{n_\alpha}(\varphi_\alpha) \right) \right) \right]_{\varphi_\alpha = \varphi}
\end{aligned}$$

(here  $\delta_\alpha = \delta/\delta\varphi_\alpha$ ). In (2.21), each operator  $l_\alpha = \delta_\alpha \Delta^c \delta_\alpha$  generates one line attached to vertices of the  $\alpha$ -th group. There are  $\binom{M}{M'} = M!(M'!(M-M')!)^{-1}$  ways to choose  $M'$  lines from a set of  $M$  lines. Therefore,

$$\begin{aligned}
\Lambda^\circ(e^l F(\varphi)) &= \sum_{M=0}^{\infty} \frac{1}{M!} \Lambda^\circ(l^M F(\varphi)) \tag{2.22} \\
&= \sum_{M=0}^{\infty} \frac{1}{M!} \sum_{M'=0}^M \binom{M}{M'} l^{M-M'} \Delta_{UV}^f \circ (l^{M'} F(\varphi)) = e^l \Delta_{UV}^f \circ (e^l F(\varphi)),
\end{aligned}$$

where

$$\Delta_{UV}^f = \mu^{-2\epsilon} \Delta_{UV} \mu^{2\epsilon}. \tag{2.23}$$

(The origin of the seemingly bizarre  $\mu$ -factors in (2.23) is as follows. The  $R$ -operation and the operator  $\Delta_{UV}$  should be applied to “standard” dimensionally regularized Feynman integrals containing one factor  $\mu^{2\epsilon}$  per each loop. But the expression  $e^l F(\varphi)$  generates integrals with one such factor lacking per each connected component— cf. the end of subsect. 2.3. To remedy this, the powers of  $\mu$  are introduced into (2.23), where it has been taken into account that  $\Delta_{UV}$  is non-trivial only on integrals with one connected component. It should be noted that in the context of the functional techniques  $\Delta_{UV}$  is systematically replaced by  $\Delta_{UV}^f$ , so that the superscript  $f$  can and will be omitted.)

Using (2.22), we rewrite (2.21) as:

$$(2.21) = T \prod_\alpha \Delta_{UV} \circ (T L^{n_\alpha}(\varphi)). \tag{2.24}$$

Substituting this into (2.20), we finally obtain:

$$S_R(\varphi) = T \exp[L_R], \tag{2.25}$$

where

$$L_R = \Delta_{UV} \circ (T e^{L(\varphi)} - 1). \tag{2.26}$$

Note that  $\Delta_{UV} \circ (L(\varphi)) = L(\varphi)$  (recall that  $\Delta_{UV}$  is a unit operation on elementary vertices), therefore  $L_R$  can be represented as:

$$L_R = L(\varphi) + \text{“divergent UV counterterms”}, \tag{2.27}$$

where

$$\text{“divergent UV counterterms”} = \Delta_{UV} \circ [T e^{L(\varphi)} - L(\varphi) - 1] \tag{2.28}$$

(note that each term in the bracketed expression on the r.h.s. of (2.28) that gives a non-zero result after application of  $\Delta_{UV}$  has at least one loop). So, indeed, the effect of the  $R$ -operation

is equivalent to adding some counterterms to the interaction Lagrangian. Eq.(2.28) provides a convenient explicit expression for them.

For clarity's sake, let us explain the algorithm encoded in (2.26):

- (1) one writes down all the contributions to  $T \exp[L(\varphi)] = \exp(\frac{1}{2}\delta\Delta^c\delta) \exp[L(\varphi)]$ ;
- (2) then one discards all the terms except those that correspond to connected 1PI graphs;
- (3) each of the terms left has the form:

$$\mu^{-2\epsilon} \int \left( \prod_i dp_i \right) (2p)^D i\delta(\sum p_i) FI(p_1 \dots p_k) \tilde{\varphi}(p_1) \dots \tilde{\varphi}(p_k), \quad (2.29)$$

where

$$\tilde{\varphi}(p) = (2p)^{-D} \int e^{ipx} d^D x \varphi(x) \quad (2.30)$$

and  $\delta(\sum p_i)$  expresses momentum conservation while  $FI(p_1 \dots p_k)$  is a connected 1PI Feynman integral constructed according to the rules described after (2.1). To  $FI$ , one should apply  $\Delta_{UV}$  defined after (2.2), and  $\Delta_{UV}$  replaces  $FI$  in (2.29) by a polynomial of  $p_i$  with coefficients that are divergent in the limit  $D \rightarrow 4$ . That is,  $\Delta_{UV}$  transforms (2.29) into an integral over  $x$  of local products of  $\varphi(x)$  and its derivatives. (Note that in the case of the vacuum graphs ( $k = 0$ ) the  $\delta$ -function degenerates into an ill-defined factor  $\delta(0)$ , and a special axiom should fix a recipe for handling it. Such a recipe is completely non-interfering with what we are doing: the integral  $FI$  is correctly defined even in this case for both the  $R$ - and  $A$ s-operations to yield meaningful results when applied to it.)

The method for resolving combinatorics of the  $R$ -operation that we have used is an extension to the slightly more complicated case of the MS-scheme of a reasoning from subsect. IV.1.3 of [6]. In [6] the momentum subtraction scheme [8] was used (so that only one term on the r.h.s. of (2.4) was retained, namely, the one corresponding to the UV subgraph containing *all* the lines of the graph  $G$  which connect the vertices from the subset  $v$ ), and it was assumed that the Lagrangian is normally ordered. As a result, in [6] the counterterms in (2.28) were to be understood as normal products, while no such normal ordering is assumed in our case.

## 2.5 $R$ -operation on Green functions.

Let us now turn to Green functions. To obtain the generating functional  $G(J)$  for the Green functions one should add a source term to the Lagrangian:

$$L \rightarrow L + \varphi J, \quad (2.31)$$

and evaluate the vacuum average of the resulting  $S$ -matrix.

The term  $\varphi J$  generates a vertex that can be either isolated from, or connected by only one line with the rest of the graph. In either case the resulting graph is nullified by  $\Delta_{UV}$  unless it consists of a single vertex  $\varphi J$  or does not contain such vertices at all. Formally:

$$\Delta_{UV^\circ} (T e^{L+\varphi J}) = \Delta_{UV^\circ} (T e^L) + \Delta_{UV^\circ}(\varphi J) = \Delta_{UV^\circ} (T e^L) + \varphi J. \quad (2.32)$$

The last equation is due to the fact that  $\Delta_{UV}$  does not affect single vertices. Therefore, for the generating functional of the MS-renormalized Green functions one has:

$$G(J) \equiv \mathbf{R} \circ \langle T \exp[L(\varphi) + \varphi J] \rangle_0 = (T \exp[L_R + \varphi J])_{\varphi=0}. \quad (2.33)$$

It may be helpful to transform (2.33) to the form of a functional integral:

$$\begin{aligned} (T \exp[\chi(\varphi) + \varphi J])_{\varphi=0} &= \left( \exp\left(\frac{1}{2}\delta\Delta^c\delta\right) \exp\left[\chi\left(\frac{\delta}{\delta J}\right)\right] \exp[\varphi J] \right) \\ &= \exp\left[\chi\left(\frac{\delta}{\delta J}\right)\right] \exp\left(\frac{1}{2}J\Delta^c J\right) = \exp\left[\chi\left(\frac{\delta}{\delta J}\right)\right] \int d\varphi \exp[L_{\text{free}} + \varphi J] \\ &= \int d\varphi \exp[L_{\text{free}} + \chi(\varphi) + \varphi J], \end{aligned} \quad (2.34)$$

where

$$L_{\text{free}}(\varphi) = -\frac{1}{2}\varphi(\Delta^c)^{-1}\varphi \quad (2.35)$$

is the free classical action (multiplied by  $i\mu^{-2\epsilon}$ ) of the field  $\varphi$ . Denoting by  $L_{\text{tot}}$  the full classical action (multiplied by  $i\mu^{-2\epsilon}$ ) of the model:

$$L_{\text{tot}} = L_{\text{free}} + L, \quad (2.36)$$

and using (2.27), we have for (2.33):

$$G(J) = \int d\varphi \exp[L_{\text{tot}}(\varphi) + \text{“divergent UV counterterms”} + \varphi J]. \quad (2.37)$$

This equation together with (2.28) provides a convenient link between practical calculations of counterterms and the analysis of Green functions by the renormalization group method.

### 3 Inversion of the $R$ -operation and the $\xi$ -mapping.

Since the effect of the  $R$ -operation consists in adding to a graph  $G$  a linear combination of graphs with a lesser number of loops, it turns out possible to construct an inversion of the  $R$ -operation—in very much the same way as it is always possible to invert a triangle matrix with units on the main diagonal. We construct the inverted  $R$ -operation, and to this end the so-called formalism of  $\xi$ -mapping is developed. This formalism proves to be a useful conceptual tool for studying various aspects of renormalization owing to the universality of the  $\xi$ -mapping i.e. owing to the fact that it contains full information on the  $R$ -operation in a given subtraction scheme for arbitrary models with a given field content. The main application of the inverted  $R$ -operation will be the construction of Euclidean asymptotic expansion of Feynman diagrams in an explicitly convergent form.

After introducing some definitions in subsect. 3.1, in subsect. 3.2 the  $\xi$ -mapping is defined. In sects.2.3–2.5 its inversion, the mapping  $x^{-1}$  is proved to exist. A useful property of  $x^{-1}$  is pointed out in subsect. 3.6, and in subsect. 3.7 we briefly discuss how the structure of the renormalization group emerges from the point of view of the formalism of  $\xi$ -mappings. In subsect. 3.8 the inverted  $R$ -operation is defined and an explicit algorithm for calculating the corresponding counterterms is derived.

### 3.1 Local operators, local functionals, and Lagrangian functionals.

Let  $j_k(x)$  be a full linearly independent set of local products of  $\varphi(x)$  and its derivatives. In momentum representation, one has:

$$\begin{aligned} \tilde{j}_k(q) = \int d^D p_1 \dots d^D p_m (2p)^\mu \mu^{-2\epsilon} \delta \left( q + \sum_{k=1}^m p_k \right) \mathcal{P}_k \\ \times \tilde{\varphi}(p_1) \dots \tilde{\varphi}(p_m), \end{aligned} \quad (3.1)$$

with  $\tilde{\varphi}(p)$  defined in (2.30);  $\mathcal{P}_k$  is a polynomial of  $p_1 \dots p_m$ . For convenience we assume that  $\mathcal{P}_k$  are polynomials of the masses of the model and are allowed to contain factors depending only on  $q = -\sum p_k$ , or in terms of the coordinate representation  $j_k(x)$  are allowed to contain full derivatives in  $x$ .

We say that  $\mathcal{L}$  is a *local functional* of fields if

$$\mathcal{L} = \sum_n \int dq \mathcal{L}_n(q) \tilde{j}_n(q), \quad (3.2)$$

where  $\mathcal{L}_n(q)$  are coefficient functions that are independent of the fields.

We call  $\mathcal{L}$  defined in (3.2) a *Lagrangian functional* if

$$\mathcal{L}_n(q) = g_n \delta(q), \quad (3.3)$$

where  $g_n$  are coupling constants independent of  $q$ . For example,  $L$  and  $L_R$  in (2.14)–(2.16) and (2.26) are Lagrangian functionals.

### 3.2 The $\xi$ -mapping.

One can see that the proof of eqs. (2.17) and (2.25) is valid not only for Lagrangian, but also for general local functionals  $\mathcal{L}$ :

$$\mathbf{R} \circ \mathbf{T} \exp \mathcal{L} = \mathbf{T} \exp \xi[\mathcal{L}], \quad (3.4)$$

where

$$\xi[\mathcal{L}] \equiv \Delta_{UV} \circ (\mathbf{T} \exp \mathcal{L} - 1).$$

It is not difficult to see that  $\xi[\mathcal{L}]$  is also a local functional:

$$\xi[\mathcal{L}] = \sum_n \int dq \xi_n[\mathcal{L}; q] \tilde{j}_n(q). \quad (3.5)$$

In momentum representation its coefficient functions are formal series of the form:

$$\xi_n[\mathcal{L}; q] = \sum_{N=0}^{\infty} \sum_{n_1} \dots \sum_{n_N} \int dq_1 \dots \int dq_N \delta \left( q + \sum_{i=1}^N q_i \right) \quad (3.6)$$

$$\times \mathcal{P}_{n;n_1 \dots n_N}^{(N)}(q_1 \dots q_N) \mathcal{L}_{n_1}(q_1) \dots \mathcal{L}_{n_N}(q_N),$$

where  $\mathcal{P}$  are polynomials in  $q_i$ . The UV poles are contained in the coefficients of the polynomials  $\mathcal{P}$ . In the coordinate representation,  $\xi_n[\mathcal{L}; x]$  are sums of local products of  $\mathcal{L}_i(x)$  and their derivatives.

The representation of  $\xi[\mathcal{L}]$  in the form of (3.5) and (3.6) is in general not unique because a factor depending only on  $q$  as a whole can be included either into  $\xi_n[\mathcal{L}; q]$  or into  $\tilde{j}_n(q)$ . But it can be made unique if one requires that  $\mathcal{P}$  from (3.6) should not contain such factors. We assume that this condition is always fulfilled.

If  $\mathcal{L}$  is a Lagrangian functional ( cf. (3.3)) then such is  $\xi[\mathcal{L}]$  as well, as can be seen from (3.6).

So far we have done very little beyond rewriting the results of sect. 2 in a new form. But this exercise is by no means trivial because it exhibits the fact that the  $\xi$ -mapping is universal in the sense that it depends only on the field content of the model but not on the interaction. We will see that the  $\xi$ -mapping is a convenient tool for studying UV renormalization of interaction Lagrangians as well as composite operators.

### 3.3 Inversion of the $\xi$ -mapping.

We are going to prove that there exists a mapping  $\xi^{-1}[\mathcal{L}]$  of the form of (3.5)–(3.6) with  $\mathcal{P}$  replaced by some other polynomials  $\bar{\mathcal{P}}$  satisfying the same restrictions. More precisely,

$$\xi[\xi^{-1}[\mathcal{L}]] = \mathcal{L} \quad \text{and} \quad \xi^{-1}[\xi[\mathcal{L}]] = \mathcal{L} \quad (3.7)$$

for any local functional  $\mathcal{L}$ . The equations (3.7) allow one to write down the following inversion of (2.3):

$$\mathbf{R} \circ \mathbf{T} \exp \xi^{-1}[\mathcal{L}] = \mathbf{T} \exp \mathcal{L}. \quad (3.8)$$

This equation is one of the key results of the present paper.

### 3.4 The structure of the $\xi$ -mapping.

Let us study the structure of  $\xi[\mathcal{L}]$ . (Throughout this section we denote  $\Delta \equiv \Delta_{UV}$ .)

The terms that are linear in  $\mathcal{L}$  can be extracted from  $\xi[\mathcal{L}]$  as follows:

$$\xi^{-1}[\mathcal{L}] \equiv \Delta \circ (\mathbf{T} e^{\mathcal{L}} - 1) = \Delta \circ (\mathbf{T} \mathcal{L}) + \Delta \circ \mathbf{T} (e^{\mathcal{L}} - \mathcal{L} - 1) \quad (3.9)$$

Then

$$\Delta \circ (\mathbf{T} \mathcal{L}) = \Delta \circ \mathbf{T} \left[ \sum_m \int dq \mathcal{L}_m(q) \tilde{j}_m(q) \right] = \sum_m \int dq \mathcal{L}_m(q) \Delta \circ (\mathbf{T} \tilde{j}_m(q)) \quad (3.10)$$

and

$$\Delta \circ (\mathbf{T} \tilde{j}_m(q)) = \sum_n z_{m,n} \tilde{j}_n(q). \quad (3.11)$$

Indeed,  $\tilde{j}_m(q)$  is the Fourier transform of a local products of fields so that the operation of  $T$ -product generates all possible tadpole graphs from  $\tilde{j}_m(q)$ .  $\Delta$  replaces the loop integrals corresponding to such diagrams by counterterms, and the result is a linear combination of local operators, which is what (3.11) states. In any subtraction scheme

$$z_{n,n} = 1. \quad (3.12)$$

In the MS scheme,  $z_{m,n}$  for  $n \neq m$  are poles in  $D - 4$  with numeric coefficients that are independent of the dimensional parameters. (Recall that we include masses into the local operators  $j$ .)

Let us show that the matrix  $z_{m,n}$  has an inverse. To this end we choose a special basis  $\tilde{j}_n$  such that each  $\tilde{j}_n$  is a monomial of fields and masses. Moreover, it is easy to see that the basis can be chosen to be ordered so that if the  $n$ -th operator is built of a lesser number of fields than the  $n'$ -th one (provided the dimensionalities in mass units and the numbers of full derivatives are the same) or if the dimensionality of the  $n$ -th operator is less than that of the  $n'$ -th one, then  $n < n'$ . In such a basis,  $m < n$  implies that  $z_{m,n} = 0$ . Therefore,  $z_{m,n}$  is a block-triangle matrix with units on the diagonal. It follows immediately that  $z^{-1}$  exists and has the same block-triangle structure as  $z$ , and the matrix elements of  $z^{-1}$  are polynomials of non-zero elements of  $z$ .

### 3.5 Existence of $\xi^{-1}$ .

Now we can solve the equation

$$\bar{\mathcal{L}} = \xi[\mathcal{L}] \quad (3.13)$$

with respect to  $\mathcal{L}$ .

Rewrite (3.7) in the form of an equation for the sources:

$$\bar{\mathcal{L}}_n(q) = \xi_n[\mathcal{L}; q], \quad (3.14)$$

using the above information on the structure of  $\xi$ , extract the terms that are linear in  $\mathcal{L}$  on the r.h.s. of (3.14):

$$\bar{\mathcal{L}}_n(q) = \sum_m \mathcal{L}_m(q) z_{m,n} + 2\xi_n[\mathcal{L}; q]. \quad (3.15)$$

Multiplying (3.15) by  $z^{-1}$ , we can rewrite it as:

$$\mathcal{L}_n(q) = \sum_m \bar{\mathcal{L}}_m(q) z_{m,n}^{-1} - \sum_m 2\xi_m[\mathcal{L}; q] z_{m,n}^{-1}. \quad (3.16)$$

Now the expression of  $\mathcal{L}_n$  in terms of  $\bar{\mathcal{L}}_n$  can be obtained by iterating (3.16), taking as a starting value the first term on the r.h.s. of (3.16).  $k$  iterations allow one to obtain an expression of  $\mathcal{L}_n$  in terms of  $\bar{\mathcal{L}}_n$  to order  $k$  in  $\bar{\mathcal{L}}_n$ . Denoting the obtained solution of (3.16) as

$$\mathcal{L} = \xi^{-1}[\bar{\mathcal{L}}], \quad (3.17)$$

one can easily see that the mapping  $\xi^{-1}$  has the properties described in subsect. 3.3.

### 3.6 A useful property of $\xi$ and $\xi^{-1}$ .

If  $J$  is a source for the field  $\varphi$  then (2.32) can be rewritten as:

$$\xi[\mathcal{L} + \varphi J] = \xi[\mathcal{L}] + \varphi J. \quad (3.18)$$

It is not difficult to see that

$$\xi^{-1}[\mathcal{L} + \varphi J] = \xi^{-1}[\mathcal{L}] + \varphi J. \quad (3.19)$$

### 3.7 $\xi$ -mapping and renormalization group.

It might be interesting to note that the above proof did not use any specific properties of the MS scheme, though  $\xi$  and  $\xi^{-1}$  do depend on the details of the  $R$ -operation via  $\mathcal{P}$  and  $\tilde{\mathcal{L}}$  in (3.6). It follows that the above formalism offers an alternative way of studying renormalization group problems.

For instance, let  $\mathbf{R}_a$  and  $\mathbf{R}_b$  be two  $R$ -operations differing by the choice of subtraction operators. Then:

$$\mathbf{R}_a \circ \Gamma e^{\mathcal{L}} = \mathbf{T} e^{\xi_a[\mathcal{L}]} = \mathbf{R}_b \circ \mathbf{T} e^{\xi_b^{-1}[\xi_a[\mathcal{L}]]} \equiv \mathbf{R}_b \circ \mathbf{T} e^{\xi_{ab}^{-1}[\mathcal{L}]}. \quad (3.20)$$

One sees that  $\xi$  has the form of (3.5)–(3.6) with suitable polynomials  $\mathcal{P}$  which—as can be easily understood—stay finite when the regularization is removed. Thus the structure of the renormalization group emerges very naturally<sup>4</sup>.

The Zimmermann identities for the Green functions with local operator insertions<sup>5</sup> can be easily obtained by performing suitable variations of (3.20) with respect to the classical sources  $\mathcal{L}_n$  entering  $\mathcal{L}$  in (3.2) ( cf. sect. 4 below).

### 3.8 Practical calculations of $\mathbf{R}^{-1}$ .

Let us derive a diagram-by-diagram recipe for calculating the mapping  $\xi^{-1}$ . To this end it is convenient to introduce the inversion of the  $R$ -operation, i.e. an operation  $\mathbf{R}^{-1}$  such that its structure is the same as that of the  $R$ -operation (see (2.1)) but the counterterms are evaluated according to a different rule in order to ensure that  $\mathbf{R}^{-1} \circ \mathbf{R} = 1$  and  $\mathbf{R} \circ \mathbf{R}^{-1} = 1$ .

By definition, for the inverse  $R$ -operation one would have:

$$\mathbf{R}^{-1} \circ \mathbf{T} e^{\mathcal{L}} = \mathbf{T} \exp \xi^{-1}[\mathcal{L}]. \quad (3.21)$$

There should exist the operator  $\Delta^{-1}$  related to  $\mathbf{R}^{-1}$  in the same way as  $\Delta$  to  $\mathbf{R}$  (see (2.1)); recall that throughout this section  $\Delta \equiv \Delta_{UV}$ ). Then:

$$\xi^{-1}[\mathcal{L}] \equiv \Delta^{-1} \circ (\mathbf{T} e^{\mathcal{L}} - 1). \quad (3.22)$$

<sup>4</sup>It may be interesting to compare the above rather compact reasoning with the more cumbersome derivation of the renormalization group transformation given in [24] in the BPHZ framework.

<sup>5</sup>It should be noted that we use the term “Zimmermann identities” in a slightly different sense than e.g. in [24]: we use it to denote the relations between the MS-renormalized and bare (unrenormalized) operators, without  $\epsilon$ -dependent factors like those considered in [24] for the purposes of studying anomalies.

From the second equation in (3.7) the following condition on  $\Delta^{-1}$  emerges:

$$\mathcal{L} = \Delta^{-1} \circ [\mathbf{T} \exp \Delta \circ (\mathbf{T} e^{\mathcal{L}} - 1) - 1]. \quad (3.23)$$

Now (3.23) can be rewritten as:

$$\begin{aligned} & \Delta^{-1} \circ (\mathbf{T} e^{\mathcal{L}} - 1 - \mathcal{L}) + (\Delta^{-1} \circ \mathcal{L} - \mathcal{L}) \\ &= -\Delta^{-1} \circ [\mathbf{T} \exp \Delta \circ (\mathbf{T} e^{\mathcal{L}} - 1) - \mathbf{T} e^{\mathcal{L}}], \end{aligned} \quad (3.24)$$

where we have assumed linearity of  $\Delta^{-1}$ . Note that the argument of  $\Delta^{-1}$  on the r.h.s. of (3.24) is equal to  $(\mathbf{R} - 1) \circ (\mathbf{T} e^{\mathcal{L}})$ .

Assume that

$$\Delta^{-1} \circ [\mathcal{L}] = \mathcal{L}, \quad (3.25)$$

which means that  $\Delta^{-1}$  is a unity operation on an elementary vertex. Then (3.24) can be rewritten as:

$$\Delta^{-1} \circ (\mathbf{T} e^{\mathcal{L}} - 1 - \mathcal{L}) = \Delta^{-1} \circ [(\mathbf{R} - 1) \circ \mathbf{T} e^{\mathcal{L}}]. \quad (3.26)$$

Therefore, for (3.23) to hold, it is sufficient that the operator  $\Delta^{-1}$  has the following properties: linearity; the property (3.25); equality to 0 on unconnected and non-1PI diagrams; its action on 1PI diagrams should be defined by the following recursion:

$$\Delta^{-1} \circ [G] = -\Delta^{-1} \circ [(\mathbf{R} - 1) \circ G]. \quad (3.27)$$

The r.h.s. should be understood as follows.  $(\mathbf{R} - 1)$  replaces  $G$  by a sum of integrals obtained from  $G$  by replacing one or more non-trivial UV subgraphs by the corresponding UV counterterm (see the definition of the  $R$ -operation in subsect. 2.1). Therefore,  $\Delta^{-1}$  on the r.h.s. acts on a linear combination of Feynman integrals with divergent coefficients, all the integrals having one loop less as compared to  $G$ , while the divergent coefficients remain unaffected by  $\Delta^{-1}$  because of its linearity. On one loop integrals:

$$\Delta^{-1} \circ [g] \equiv -\Delta^{-1} \circ \Delta \circ [g] = -\Delta \circ [g], \quad (3.28)$$

and the recursion stops correctly.

Eq.(3.27) provides a most convenient recipe for explicit calculations. For example, in the scalar  $g\varphi^4$  model one has:

$$\begin{aligned} \Delta^{-1} \left[ \text{fish} \right] &= -\Delta \left[ \text{fish} \right], \\ \Delta^{-1} \left[ \text{bubble} \right] &= 3 \Delta \left[ \text{fish} \right] \cdot \Delta \left[ \overline{\text{bubble}} \right] \\ &\quad - \Delta \left[ \text{bubble} \right], \end{aligned} \quad (3.29)$$

and so on.

## 4 Renormalization of multilocal operator insertions.

It is not difficult to generalize the above results to arbitrary Green functions of composite operators. The main point here is that it is simply sufficient to consider  $T$ -exponents of arbitrary local functionals  $\mathcal{L}$  as defined in (3.2) instead of Lagrangian functionals, and then derive expressions for specific composite (multi-) local operator insertions by performing suitable variations. In subsect. 4.1 we present some explicit formulae for the case when one wishes to express renormalized Green functions in terms of unrenormalized ones, and in subsect. 4.2 we consider the opposite case. Note that the case of one local operator insertion within the momentum subtraction scheme was first considered by Zimmermann [9]<sup>6</sup>.

### 4.1 Generalized Zimmermann identities.

Consider a Green function of arbitrary local operators (without loss of generality we take the local operators to belong to the basis introduced in subsect. 3.1):

$$\mathbf{R}_\circ \langle \mathbf{T} \tilde{j}_1(q_1) \dots \tilde{j}_n(q_n) e^L \rangle_0, \quad (4.1)$$

where  $\tilde{j}_k$  are Fourier transformed local products of  $\varphi(x)$  and its derivatives as defined in (3.1). Using the formalism introduced in the preceding section, one gets:

$$\begin{aligned} \text{eq.(4.1)} &= \left( \frac{\delta}{\delta \mathcal{L}_{n_N}(p_N)} \dots \frac{\delta}{\delta \mathcal{L}_{n_1}(p_1)} \mathbf{R}_\circ \mathbf{T} e^{\mathcal{L}} \right)_{\mathcal{L}=L, \varphi=0} \\ &= \mathbf{T} \left( \frac{\delta}{\delta \mathcal{L}_{n_N}(p_N)} \dots \frac{\delta}{\delta \mathcal{L}_{n_1}(p_1)} e^{\xi[\mathcal{L}]} \right)_{\mathcal{L}=L, \varphi=0} \\ &= \mathbf{T} \left\{ \left[ e^{-\xi[\mathcal{L}]} \frac{\delta}{\delta \mathcal{L}_{n_N}(p_N)} \dots \frac{\delta}{\delta \mathcal{L}_{n_1}(p_1)} e^{\xi[\mathcal{L}]} \right]_{\mathcal{L}=L} e^{\xi[L]} \right\}_{\varphi=0}. \end{aligned} \quad (4.2)$$

The expression in the square brackets on the r.h.s. of (4.2) is a sum of products of terms like

$$\left[ \frac{\delta}{\delta \mathcal{L}_{i_n}(p_n)} \dots \frac{\delta}{\delta \mathcal{L}_{i_1}(p_1)} \xi[\mathcal{L}] \right] = \Delta_\circ \left[ \mathbf{T} \tilde{j}_{i_1}(p) \dots \tilde{j}_{i_n}(p_n) \exp L \right]. \quad (4.3)$$

It is not difficult to see that (4.3) is a local functional which can be expanded in  $\tilde{j}$  as:

$$\text{eq.(4.3)} = \sum_m \tilde{j}_m(p_1 + \dots + p_n) \tilde{Z}_{i_1 \dots i_n; m}^{(n)}(p_1 \dots p_n), \quad (4.4)$$

where  $\tilde{Z}$  are polynomials in  $p$  and in the parameters of  $L$ .  $Z$  is independent of  $p$  if the dimensionality of  $\tilde{j}$  in units of mass is greater than that of the product  $\tilde{j}_{i_1} \dots \tilde{j}_{i_n}$ . Moreover, since we include full derivatives into  $j$ ,  $Z$  can not contain factors depending only on  $p_1 + \dots + p_n$ ; in particular,  $\tilde{Z}_n^{(1)}(p)$  are independent of  $p$ .

<sup>6</sup>see also the remark in the footnote in the last paragraph of subsect. 3.7.

Examples will help to understand the above. Denoting (4.3) as  $\Delta(j_1 \dots j_n)$ , one has ( $L_R$  was defined in (2.26)):

$$\mathbf{R}_\bullet \langle T j_1 e^L \rangle_0 = (T \Delta(j_1) \exp[L_R])_{\varphi=0}, \quad (4.5)$$

$$\mathbf{R}_\bullet \langle T j_1 j_2 e^L \rangle_0 = (T [\Delta(j_1) \Delta(j_2) + \Delta(j_1 j_2)] \exp[L_R])_{\varphi=0}, \quad (4.6)$$

$$\mathbf{R}_\bullet \langle T j_1 j_2 j_3 e^L \rangle_0 = (T [\Delta(j_1) \Delta(j_2) \Delta(j_3) + \Delta(j_1 j_2) \check{\Delta}(j_3) \quad (4.7)$$

$$+ \check{\Delta}(j_1 j_3) \check{\Delta}(j_2) + \check{\Delta}(j_2 j_3) \check{\Delta}(j_1) + \check{\Delta}(j_1 j_2 j_3)] \exp[L_R])_{\varphi=0}.$$

Since  $\check{\Delta}(\varphi) = \varphi$  and  $\check{\Delta}(\varphi_{j_1 \dots j_k}) = 0$  (this is because it is impossible to form a 1PI graph with one “ $\varphi$ -insertion”), one can see that for  $j = \varphi$  the above formulae degenerate into

$$\mathbf{R}_\bullet \langle T \tilde{\varphi}(q_1) \dots \tilde{\varphi}(q_n) e^L \rangle_0 = (T \tilde{\varphi}(q_1) \dots \tilde{\varphi}(q_n) e^{L_{\mathbf{R}}(\varphi)})_{\varphi=0}, \quad (4.8)$$

which agrees with (2.33), as expected.

Using (4.4), one can represent (4.5) as:

$$\mathbf{R}_\bullet (T \tilde{j}_1(p) e^L) = \sum_m \tilde{Z}_{1;m}^{(1)} + (T \tilde{j}_m(p) \exp[L_R]) \quad (4.9)$$

(identities of this kind were first obtained by Zimmermann [9]),

$$\mathbf{R}_\bullet (T \tilde{j}_1(p_1) \tilde{j}_2(p_2) e^L) = \sum_m \tilde{Z}_{1,2;m}^{(2)}(p_1, p_2) (T \tilde{j}_m(p_1 + p_2) \exp[L_R]) \quad (4.10)$$

$$+ \sum_m \tilde{Z}_{1;m}^{(1)} \sum_n \tilde{Z}_{1;n}^{(1)} (T \tilde{j}_m(p_1) \tilde{j}_n(p_2) \exp[L_R]),$$

etc. Note that all the sums in (4.9) and (4.10) are finite. Identities like (4.10) and their generalizations can be called (generalized) Zimmermann identities for multilocal operator insertions.

## 4.2 Inverted Zimmermann identities.

The equations derived in the preceding subsection allow one to express renormalized Green functions in terms of unrenormalized ones. Using the inversion of the  $R$ -operation obtained in sect. 3, one can easily do the opposite, i.e. reexpress unrenormalized quantities in terms of renormalized ones. Thus, instead of (4.1)–(4.2) one has:

$$\begin{aligned} & \langle T \tilde{j}_1(q_1) \dots \tilde{j}_n(q_n) e^L \rangle_0 \\ &= \left( \frac{\delta}{\delta \mathcal{L}_{n_N}(p_N)} \dots \frac{\delta}{\delta \mathcal{L}_{n_1}(p_1)} T e^{\mathcal{L}} \right)_{\mathcal{L}=L, \varphi=0} \end{aligned} \quad (4.11)$$

$$= \mathbf{R} \circ \mathbf{T} \left\{ \left[ e^{-\xi[L]} \frac{\delta}{\delta \mathcal{L}_{n_N}(p_N)} \cdots \frac{\delta}{\delta \mathcal{L}_{n_1}(p_1)} e^{\xi[L]} \right]_{\mathcal{L}=L} e^{\xi[L]} \right\}_{\varphi=0},$$

and instead of (4.9) one has:

$$\mathbf{T} \tilde{j}_1(p) e^L = \sum_m \tilde{Z}_{1;m}^{(1)} \mathbf{R} \circ (\mathbf{T} \tilde{j}_m(p) \exp \xi^{-1}[L]) \quad (4.12)$$

(note that  $\tilde{Z}^{(1)} = [\tilde{Z}^{(1)}]^{-1}$ ),

$$\begin{aligned} \mathbf{T} \tilde{j}_1(p_1) \tilde{j}_2(p_2) e^L &= \sum_m \tilde{Z}_{1,2;m}^{(2)} \mathbf{R} \circ (\mathbf{T} \tilde{j}_m(p_1 + p_2) \exp \xi^{-1}[L]) \\ &+ \sum_m \tilde{Z}_{1;m}^{(1)} \sum_n \tilde{Z}_{1;n}^{(1)} \mathbf{R} \circ (\mathbf{T} \tilde{j}_m(p_1) \tilde{j}_n(p_2) \exp \xi^{-1}[L]), \end{aligned} \quad (4.13)$$

(one can check that  $\tilde{Z}_{1,2;m}^{(2)} = -\sum_l \tilde{Z}_{1,2;l}^{(2)} \tilde{Z}_{l;m}^{(1)}$ ). And so on.

In a similar manner one can relate expressions renormalized in different renormalization schemes, cf. subsect. 3.7.

## AS-OPERATION FOR MULTILOOP INTEGRALS.

### 5 Heavy mass expansions.

We now turn to asymptotic expansions of Feynman diagrams. In this section we consider the case when the set of the heavy parameters contains only masses, which corresponds to studying low-energy effective Lagrangians to all orders in the inverse heavy mass. The reasoning in the more general case is very much the same, so that in the next section we will be able to concentrate only on specific complications due to large external momenta.

In subsect. 5.1 the *As*-operation for unrenormalized (or UV-convergent) integrals is defined as an integrated version of the *As*-operation for products of singular functions introduced in [1], in subsect. 5.2 the combinatorial structure of the corresponding subgraphs is studied. In subsect. 5.3 the  $\delta$ -functions corresponding to the counterterms for the IR-subgraphs are integrated out and in subsect. 5.4 the final form of the *As*-operation for 1PI integrals is presented. In subsect. 5.5 the obtained expression for the *As*-operation is extended to non-1PI integrals, in subsect. 5.6 its exponentiation is proved, and in subsect. 5.7 it is transformed to an explicitly convergent form using the inversion of the *R*-operation. In sects. 4.8 and 4.9 the results are extended to UV-renormalized diagrams and models, respectively. Explicit expressions for low-energy effective Lagrangians are presented to all orders of the inverse heavy mass. Subsect. 5.10 contains a simple example.

#### 5.1 *As*-operation for integrals: definition.

Let us introduce notations and formulate the problem that we are going to consider here. Our notations on the whole will be consistent with subsect. 3.1 of [1].

Let  $\Gamma$  be an  $l$ -loop 1PI Feynman diagram, and let  $p = (p_1 \dots p_l)$  denote its integration momenta. We assume in this section that  $\Gamma$  depends on the light and heavy masses  $m$  and  $M$ , and that all the external momenta of  $\Gamma$  are light (and denoted as  $k$ ). All  $M$  are non-zero and independent of the expansion parameter denoted as  $\kappa$ , while  $m, k = O(\kappa)$  and some masses  $m$  may be equal to zero. Unlike the abstract analysis of [1], here it will be convenient to show the dependence on  $m$ ,  $k$  and  $M$  explicitly.

Thus, the unrenormalized integrand is denoted as  $\Gamma(p, m, k, M)$ . We assume that  $\Gamma$  is UV-convergent, i.e.  $\Gamma(p, \dots)$  is absolutely integrable over  $p$  in infinite limits. Denote the integrated diagram as:

$$\Gamma(k, m, M) \stackrel{\text{def}}{=} \int dp \Gamma(p, k, m, M). \quad (5.1)$$

In [1] we introduced the *As*-operation for products of singular functions like  $\Gamma(p, k, m, M)$  which yields asymptotic expansions of such products with respect to the light parameters in the sense of the distribution theory. It is natural to expect (for more details see subsect. 5.8; a fuller justification is given in [17]) that if (5.1) is UV-convergent, then the asymptotic expansion of

$\Gamma(k, m, M)$  can be obtained by expanding its integrand  $\Gamma(p, k, m, M)$  using the *As*-operation of [1] and then performing termwise integration of the resulting series over  $p$ .

Let us introduce an operation that acts on *integrated* Feynman diagrams in accordance with the above recipe. This new operation will be referred to as *the As-operation for integrals*. To distinguish the two types of *As*-operations, one could use e.g. the notations  $\mathbf{As}_{\text{prod}}$  and  $\mathbf{As}_{\text{int}}$ . But as it is normally clear whether the *As*-operation acts on a product of singular functions or on an integral, we will use the same notation  $\mathbf{As}$  without subscripts in both cases. Formally, one has:

$$\mathbf{As}\circ\Gamma(k, m, M) \stackrel{\text{def}}{=} \int dp \mathbf{As}\circ\Gamma(p, k, m, M). \quad (5.2)$$

Let us exhibit the structure of the *As*-operation on the r.h.s. of (5.2). The general expression for it was given in [1]. Specifying it to the case considered here, one obtains:

$$\begin{aligned} \mathbf{As}\circ\Gamma(p, k, m, M) & \quad (5.3) \\ &= \mathbf{T}_{m,k}\circ\Gamma(p, k, m, M) + \sum_{\gamma} [\mathbf{E}\circ\gamma(p, k, m)] \times [\mathbf{T}_{m,k}\circ\Gamma\setminus\gamma(p, k, m, M)], \end{aligned}$$

where the notations used are as follows.

The operation  $\mathbf{T}_{m,k}$  performs the Taylor expansion in powers of the light parameters. Summation runs over all IR-subgraphs of  $\Gamma$ . Specifying to the present case the definitions of subsects.9.2–3 of [1], one sees that an IR-subgraph  $\gamma$  can be described as follows: (i) the set of lines of  $\gamma$  is a full subset of the light lines of  $\Gamma$ , i.e. if one puts to zero the momenta flowing through all the lines of  $\gamma$  together with the light external momenta  $k$ , then no light line which does not belong to  $\gamma$  should have the momentum flowing through it nullified owing to momentum conservation (cf. [1]); (ii) any vertex of  $\Gamma$  whose incident lines all belong to  $\gamma$ , is included into  $\gamma$ . The product of the lines and vertices of  $\gamma$  is independent of  $M$  and is denoted as  $\gamma(p, k, m)$ .

The product  $\Gamma\setminus\gamma$  in (5.3) comprises all the factors from  $\Gamma$  that do not belong to  $\gamma$ . Graphically,  $\Gamma\setminus\gamma$  is obtained from  $\Gamma$  by deleting all the lines and vertices of  $\gamma$ .

The operation  $\mathbf{E}$  (expressions for it were derived in [1]) applied to an IR subgraph returns a linear combination of counterterms that are proportional to  $\delta$ -functions of  $p$ . Our first aim here is to integrate out those  $\delta$ -functions explicitly.

## 5.2 Structure of $\Gamma\setminus\gamma$ .

It is not difficult to verify that in the case under consideration (no heavy external momenta) the IR-subgraphs are uniquely characterized by the properties of their complements. Indeed, denote the connected components of  $\Gamma\setminus\gamma$  as  $h$ . These subgraphs (which will be referred to as *heavy knots*) have the following properties:

(i) since  $\gamma$  consists of only light lines, each  $h_i$  has only light external lines while all the heavy lines of  $\Gamma$  are hidden within all  $h_i$ ;

(ii) since  $\gamma$  is full, none of  $h_i$  can have a light line whose momentum vanishes due to momentum conservation when one puts to zero all the external momenta of  $h_i$ . This is an analytical formulation. Graphically, it implies that each  $h_i$  is “1PI with respect to light lines” which means that it cannot be divided into two disconnected graphs (which may consist of a single vertex) by cutting any one of its light lines.

Note that a heavy knot  $h_i$  may consist of a single heavy line (including both vertices to which it is attached).

One has:

$$\Gamma \setminus \gamma = \prod_i h_i. \quad (5.4)$$

In the final form of the  $As$ -operation for integrals, the heavy knots will play the same role as UV-subgraphs in the expression (2.1) for the  $R$ -operation.

Example. Consider the diagram  $\Gamma$  as shown in Fig.1a. Fat lines correspond to propagators of heavy particles, the rest to light particles. In Figs.1b–1d various IR subgraphs are shown with dashed lines. Heavy knots in each case are clearly visible as connected components built of solid—normal and fat—lines.

### 5.3 Integrating $\delta$ -functions.

Before substituting explicit expressions for  $\mathbf{E} \circ \gamma$  into (5.3), the dependence of each term in (5.3) on  $p$  should be studied. One can see that  $\gamma$  does not depend on those components of  $p$  that correspond to the loops of  $\Gamma \setminus \gamma$ . Let us denote the collection of such momenta as  $p_{\Gamma \setminus \gamma, \text{int}}$ . Furthermore, each loop of  $\Gamma \setminus \gamma$  belongs to only one heavy knot  $h_i$  from the decomposition (5.4), so that:

$$p_{\Gamma \setminus \gamma, \text{int}} = \{p_{i, \text{int}}\}_i. \quad (5.5)$$

The remaining momentum components of  $p$  form exactly the proper variables of  $\gamma$  introduced in [1] and denoted as  $p_\gamma$ , so that

$$p = (p_\gamma, p_{\Gamma \setminus \gamma, \text{int}}). \quad (5.6)$$

Note that  $\Gamma \setminus \gamma$  is independent of those components of  $p_\gamma$  that correspond to the loops of  $\gamma$ . Denote as  $p_{\Gamma \setminus \gamma, \text{ext}}$  those components of  $\gamma$  on which  $\Gamma \setminus \gamma$  does depend.

Now, substituting (5.3) into (5.2) and using the explicit expressions for  $\mathbf{E} \circ \gamma$  derived in [1], we can rewrite (5.2) as:

$$\begin{aligned} \mathbf{As} \circ \Gamma(k, m, M) &= \mathbf{T}_{m, k} \circ \Gamma(k, m, M) \\ &+ \sum_\gamma \sum_\alpha \left\{ \left[ \int dp'_\gamma \mathcal{P}_{\gamma, \alpha}(p'_\gamma) \gamma(p'_\gamma, k, m) \right] \int dp_\gamma \int dp_{\Gamma \setminus \gamma, \text{int}} \delta_{\gamma, \alpha}(p_\gamma) \right. \\ &\quad \left. \times \mathbf{T}_{m, k} \circ \left[ \Gamma \setminus \gamma(p_{\Gamma \setminus \gamma, \text{ext}}, p_{\Gamma \setminus \gamma, \text{int}}, k, m, M) \right] \right\}, \end{aligned} \quad (5.7)$$

The polynomials  $\mathcal{P}_\alpha$  and the  $\delta$ -functions  $\delta_\alpha$  in (5.7) form full dual sets (cf. [1]) so that the sum over  $\alpha$  represents the Taylor expansion:

$$\sum_\alpha \mathcal{P}_{\gamma, \alpha}(p'_\gamma) \int dp_\gamma \delta_{\gamma, \alpha}(p_\gamma) \cdot f(p_\gamma) \equiv \mathbf{T}_{p'_\gamma} \circ f(p'_\gamma). \quad (5.8)$$

(It does not matter that the expression that plays the role of  $f$  in (5.7) depends not on all  $p_\gamma$  but only on some of the components of  $p_\gamma$ , namely, on  $p_{\Gamma \setminus \gamma, \text{ext}}$ .) Using (5.8) and renaming  $p'_{\gamma, \text{ext}} \rightarrow p_{\gamma, \text{ext}}$ , we get:

$$\begin{aligned} \mathbf{As} \circ \Gamma(k, m, M) &= \mathbf{T}_{m, k} \circ \Gamma(k, m, M) \\ &+ \sum_{\gamma} \left\{ \int dp_{\gamma}(p_{\gamma}, k, m) \times \mathbf{T}_{m, k} \circ \mathbf{T}_{p_{\Gamma \setminus \gamma, \text{ext}}} \circ [\Gamma \setminus \gamma(p_{\Gamma \setminus \gamma, \text{ext}}, p_{\Gamma \setminus \gamma, \text{int}}, k, m, M)] \right\}. \end{aligned} \quad (5.9)$$

Now recall the representation of  $\Gamma \setminus \gamma$  in terms of heavy knots (5.4), (5.5). It is easy to see that the last line in (5.9) can be rewritten as

$$\prod_i \tau \circ h_i(p_{i, \text{int}}, p_{i, \text{ext}}, k, m, M), \quad (5.10)$$

where  $p_{i, \text{ext}}$  comprises those components of  $p_{\Gamma \setminus \gamma, \text{ext}}$  on which the heavy knot  $h_i$  depends. The operation  $\tau$  in (5.10) Taylor-expands in  $m$  and in all the momenta that are external with respect to the subgraph  $h_i$ , i.e. both  $p_{i, \text{ext}}$  and the corresponding components of  $k$ . It should be stressed that each factor in (5.10) is independent of the rest of the original diagram  $\Gamma$ . Therefore, we arrive at the following expression of the  $As$ -operation for integrated Feynman diagrams:

$$\mathbf{As} \circ \Gamma = \tau \circ \Gamma + \sum_{\{h_i\}} \left( \prod_i \tau \circ h_i \right) \cdot [\Gamma / \prod_i h_i]. \quad (5.11)$$

The meaning of this formula is as follows. One enumerates all sets of pairwise non-intersecting heavy knots  $h_i$  and contracts each heavy knot  $h_i$  to a point, replacing it by a formal series in its external momenta which is obtained by Taylor-expanding the corresponding Feynman integral.

To avoid confusion, it should be understood that the IR-counterterms  $\mathbf{E} \circ \gamma$  in (5.3) have transformed into the expression in the square brackets in (5.11), while the "counterterms"  $\tau \circ h_i$  in (5.11) have originated from the expression  $\Gamma \setminus \gamma$  in (5.3). In other words, "counterterms" and "non-counterterms" have changed places in the transition from (5.3) to (5.11).

## 5.4 $As$ -operation for integrals: the final form.

Eq.(5.11) can be rewritten in the following form which is analogous to the representation (2.1) of the  $R$ -operation:

$$\mathbf{As} \circ \Gamma = \sum_{\{h_i\}} (\Delta_{\text{as}} \circ h_i) \cdot [\Gamma / \prod_i h_i]_{M=\infty}. \quad (5.12)$$

The summation here runs over all sets of pairwise non-intersecting subgraphs  $h_i$  of  $\Gamma$  such that each vertex of  $\Gamma$  belongs to one of  $h_i$ . (A subgraph  $h$  is defined as a subset of vertices of  $\Gamma$  together with some lines of  $\Gamma$  which are connected by both ends to the vertices included into  $h$ . Subgraphs are non-intersecting if they have no common vertices and, consequently, no common lines.) One of such sets consists of a single subgraph  $h = \Gamma$  which corresponds to the first term  $\tau \circ \Gamma$  on the r.h.s. of (5.11). The operation  $\Delta_{\text{as}}$  coincides with  $\tau$  on heavy knots, is a unity operation on single vertices without loops, and returns zero otherwise. Setting  $M = \infty$  is a formal expression of the fact that one should discard from the r.h.s. those terms in which some heavy lines remained outside all  $h_i$  so that (5.11) could be restored from (5.12).

There is no need to introduce special rules for heavy knots containing tadpoles made of light lines (cf. Fig.2), our formalism takes them into account correctly. For practical purposes it is sufficient to note that such heavy knots are nullified by the operation  $\tau$  due to the well-known property of the dimensional regularization to nullify momentum integrals without dimensional parameters.

The combinatorial resemblance of (5.12) and the expression (2.1) for the  $R$ -operation is nearly complete except for the minor difference described above. However, this difference can be easily taken into account to make the reasoning of sects.1–3 to be applicable to the  $As$ -operation. But first expansions of diagrams  $\Gamma$  other than 1PI should be considered.

### 5.5 $As$ -operation for non-1PI diagrams.

We have derived explicit expressions for the operation  $As$  on 1PI integrals. Remarkably, one can expand arbitrary—not necessarily 1PI—integrals using the same expression (5.12).

For the operation  $As$  to be applicable to any graph  $\Gamma$ , its structure should satisfy certain *a priori* requirements. Let us first enumerate them.

If  $\Gamma$  is disconnected, i.e. there are two subgraphs  $\Gamma_1$  and  $\Gamma_2$  such that  $\Gamma = \Gamma_1 \cdot \Gamma_2$ , then the expansion of  $\Gamma$  is a product of the expansions of  $\Gamma_1$  and  $\Gamma_2$ , i.e.

$$\mathbf{As}^\circ(\Gamma_1 \cdot \Gamma_2) = (\mathbf{As}^\circ\Gamma_1) \cdot (\mathbf{As}^\circ\Gamma_2). \quad (5.13)$$

If  $\Gamma$  falls into two parts connected by only one light line (the corresponding propagator is denoted as  $D_l$ ), i.e.  $\Gamma = \Gamma_1 \cdot D_l \cdot \Gamma_2$ , then

$$\mathbf{As}^\circ(\Gamma_1 \cdot D_l \cdot \Gamma_2) = (\mathbf{As}^\circ\Gamma_1) \cdot D_l \cdot (\mathbf{As}^\circ\Gamma_2). \quad (5.14)$$

This is due to the fact that both the mass of this line and the momentum flowing through it (which is a combination of the light external momenta of  $\Gamma$ ) are expansion parameters so that  $D_l$  is proportional to a negative power of  $\kappa$  and need not be expanded.

If  $\Gamma$  consists of two subgraphs connected by one heavy line (the corresponding propagator is denoted as  $D_h$ ), i.e.  $\Gamma = \Gamma_1 \cdot D_h \cdot \Gamma_2$ , then

$$\mathbf{As}^\circ(\Gamma_1 \cdot D_h \cdot \Gamma_2) = (\mathbf{As}^\circ\Gamma_1) \cdot (\mathbf{T}_k^\circ D_h) \cdot (\mathbf{As}^\circ\Gamma_2). \quad (5.15)$$

(Recall that  $\mathbf{T}_k$  Taylor-expands in  $k$ , and that the momentum flowing through  $D_h$  is a combination of the light external momenta of  $\Gamma$ .)

Now, note that the definition (5.12) does not use 1PI-ness of  $\Gamma$ . Therefore, let us define the operation  $As$  by (5.12) on any graph  $\Gamma$ . Let us prove that the operation thus defined satisfies the above *a priori* criteria (5.13)–(5.15).

Eq.(5.13) is true because each  $h_i$  from (5.12) lies strictly within either  $\Gamma_1$  or  $\Gamma_2$ , so that each partition of  $\Gamma$  consists of a partition of  $\Gamma_1$  and a partition of  $\Gamma_2$ , and a sum over all partitions of  $\Gamma$  is equivalent to a sum over partitions of  $\Gamma_1$  times a sum over partitions of  $\Gamma_2$ .

The same argument proves (5.14) because  $D_i$  cannot belong to any of  $h_i$  without making it 1PI with respect to light lines, which implies that each  $h_i$  with non-zero  $\Delta_{as}$  lies strictly within either  $\Gamma_1$  or  $\Gamma_2$ .

To prove (5.15) we note that  $D_h$  must always belong to one of the  $h_i$ . Denote this heavy knot as  $h$ . Then  $h = h' \cdot D_h \cdot h''$  (where  $h'$  and  $h''$  each consist of at least one vertex) and

$$\Delta_{as} \circ (h' \cdot D_h \cdot h'') = (\Delta_{as} \circ h') \cdot (\mathbf{T}_{k \circ} D_h) \cdot (\Delta_{as} \circ h''). \quad (5.16)$$

Then one applies the same argument as above.

An induction with respect to the number of 1PI components of  $\Gamma$  enables one to conclude that (5.12) remains correct on all Feynman diagrams.

## 5.6 Exponentiation of $\Delta_{as}$ .

Now we are in a position to apply the combinatorial results of sects.1–3 to (5.12).

Let  $\varphi$  and  $\Phi$  be the fields of the light and heavy particles that have the masses  $m$  and  $M$ , respectively. Let  $L = L(\varphi, \Phi)$  be an interaction Lagrangian. Consider the generating functional of Green functions of the light fields  $\langle \mathbf{T} \exp[L(\varphi, \Phi) + \varphi J] \rangle_0$ . It is assumed that all the external momenta of the Green functions are light, i.e.  $O(\kappa)$ . We ignore the UV divergences for a while, because we are now interested only in the formal global structure of the operation  $\mathbf{As}$ ; one can simply assume that there are no UV divergences at all in the model described by  $L$ .

As was noted above, the combinatorial structures of the  $As$ - and  $R$ -operation s are very similar. The similarity goes far enough to allow one to write down the following analogue of (2.31):

$$\mathbf{As} \circ \langle \mathbf{T} \exp[L(\varphi, \Phi) + \varphi J] \rangle_0 = \langle \mathbf{T} \exp[L_{\text{eff,bare}}(\varphi) + \varphi J] \rangle_0, \quad (5.17)$$

where

$$L_{\text{eff,bare}}(\varphi) = \Delta_{as} \circ \left[ \mathbf{T} e^{L(\varphi, \Phi)} - 1 \right]_{\Phi=0}. \quad (5.18)$$

Setting  $\Phi$  to zero in (5.18) is formally equivalent to setting  $M = \infty$  in (5.12) and is an expression of the fact that each heavy line in (5.11) must belong to one of the heavy knots.

The algorithm of (5.18) is as follows:

- (i) all the diagrams contributing to  $\mathbf{T} \exp L(\varphi, \Phi)$  are evaluated;
- (ii)  $\Delta_{as}$  nullifies all the terms except those which correspond to connected diagrams that are 1PI with respect to light lines;
- (iii) on single vertices,  $\Delta_{as}$  acts as a unit operation, i.e.  $\Delta_{as} \circ [L(\varphi, \Phi)]_{\Phi=0} = [L(\varphi, \Phi)]_{\Phi=0}$ ;
- (iv) the remaining terms are exactly the heavy knots and can be represented as (2.28), where  $FI$  also depends on  $m$  and  $M$ .  $\Delta_{as}$  performs the formal Taylor expansion of  $FI$  with respect to  $m$  and all the external momenta of such terms, i.e.  $p_i$  in (2.28).

Note that the recipe for  $\Delta_{\text{as}}$  allows a remarkably “fool-proof” reformulation (which may be useful in writing computer programs): it is sufficient to apply the  $\tau$ -operation to *all* connected terms generated by the  $T$ -exponent. Then the terms that should be discarded will either develop singularities  $1/0$ , or be nullified owing to the property of the dimensional regularization to nullify integrals without dimensional parameters like

$$\int d^D p p^{-2\alpha} = 0.$$

It is no less remarkable that this “fool-proof” reformulation stays valid in the most general case comprising heavy external momenta with arbitrary linear restrictions.

### 5.7 Explicitly convergent form for the heavy mass expansion (5.17).

As was noted in [1], our method of expansion leads to expressions containing spurious divergences in different terms which should cancel in the final result. Let us reorganize our asymptotic expansions in such a way as to reexpress them in an explicitly convergent form. In (5.18),  $L_{\text{eff,bare}}(\varphi)$  is a Lagrangian functional (in terms of subsect. 3.1) of the field  $\varphi$  whose coefficients (“coupling constants”) are functions of the heavy masses. Irrespective of the specific values of these functions, the diagrams on the r.h.s. of (5.17) may contain UV divergences. Using (3.11), one can explicitly extract the  $R$ -operation on the r.h.s. of (5.17) and rewrite it as:

$$\mathbf{As} \circ \langle T \exp[L(\varphi, \Phi) + \varphi J] \rangle_0 = \mathbf{R} \circ \langle T \exp[L_{\text{eff}}(\varphi) + \varphi J] \rangle_0, \quad (5.19)$$

where

$$L_{\text{eff}}(\varphi) = \xi^{-1}[L_{\text{eff,bare}}(\varphi)]. \quad (5.20)$$

Since the mapping  $\xi^{-1}$  transforms a Lagrangian functional again into a Lagrangian functional,  $L_{\text{eff}}(\varphi)$  can be represented as

$$L_{\text{eff}}(\varphi) = \sum_n \int dx g_{\text{eff},n}(g, M, \mu, \epsilon) j_n(x), \quad (5.21)$$

where  $j_n(x)$  are defined in subsect. 3.1 and  $g$  denotes the set of coupling constants of the original interaction Lagrangian  $L$ ; the “effective couplings”  $g_{\text{eff},n}$  have the form:

$$g_{\text{eff},n}(g, M, \mu, \epsilon) = \frac{1}{M^{d_n}} \sum_{k=0}^{\infty} g^k \sum_{l \geq 0} \left(\frac{\mu}{M}\right)^{l(2\epsilon)} g_{\text{eff},n,l}(\epsilon), \quad (5.22)$$

where the summation over powers of  $g$  reminds one that we are working within perturbative framework;  $g_{\text{eff},n,l}(\epsilon)$  have a pole singularity at  $\epsilon \sim 0$  of a finite order depending on  $n$  and  $l$ , while the integer exponents  $d_n$  (which can be negative) are determined on dimensional grounds.

Indeed, the coefficients  $g_{\text{eff},n}$  are polynomial expressions built of UV counterterms (each of which is just a polynomial in  $1/\epsilon$ ) and heavy knots which are Taylor-expanded in powers of their external momenta and  $m$ . The resulting loop integrals depend only on  $M$ , and on dimensional grounds this dependence has the form  $M^{d'-2\epsilon l'} \times c(\epsilon)$  where  $d'$  is an integer and  $l'$

the corresponding loop number, while  $c(\epsilon)$  has a pole singularity  $1/\epsilon''$  (the latter information is strictly speaking not necessary). The powers of momenta and the light masses  $m$  resulting from the above Taylor expansion are included into  $j$  (recall the definition in subsect. 3.1). Recalling that the UV renormalization parameter  $\mu$  enters all expressions via integer powers of  $\mu^{2\epsilon}$ , one arrives at (5.21) and (5.22). Note that the range of summation over  $l$  is finite for each  $n$  and  $k$ .

Now one can easily convince oneself that the parameters of the effective Lagrangian are finite at  $\epsilon = 0$ . Indeed, the initial expression (the l.h.s. of (5.19)) is finite at  $\epsilon = 0$  by construction in each order of the perturbation theory (in the absence of UV divergences in the initial model—recall the assumption at the beginning of this section). When transformed into the form of the r.h.s. of (5.19), it is, of course, still finite but has now the form of UV-renormalized (and therefore, finite) loop integrals times products of  $g_{\text{eff},n}(g, M, \mu, \epsilon)$ , in each order of the perturbation theory. Taking into account linear independence of  $j_n$ , one concludes that all  $g_{\text{eff},n}(g, M, \mu, \epsilon)$  are finite at  $\epsilon = 0$ .

## 5.8 *As*-operation on UV-divergent diagrams.

Eq.(5.2) was used to define the *As*-operation on UV convergent diagrams. However, that definition can be formally extended to the UV-divergent case (then, of course, the UV divergences should be regularized; but the dimensional regularization which is used at the intermediate steps of the *As*-operation for products as derived in [1], regularizes UV divergences as well; therefore, no new regulators are really needed). The combinatorial results (5.17)–(5.20) remain valid though now the effective couplings  $g_{\text{eff},n}(g, M, \mu, \epsilon)$  are not finite at  $\epsilon \sim 0$  because they inherit the UV divergences of the unrenormalized Green functions on the l.h.s. of (5.17).

Let  $\mathbf{R}\circ\Gamma(k, m, M)$  be a renormalized graph (the dependence on the renormalization parameter  $\mu$  is implicit). The *R*-operation was defined by (2.1), which can be represented as follows:

$$\mathbf{R}\circ\Gamma(k, m, M) = \sum_i \mathcal{Z}_i(1/\epsilon, m, M)\Gamma_i(k, m, M), \quad (5.23)$$

where  $i$  enumerates all the individual terms in the sum in (2.1),  $\mathcal{Z}_i$  are products of divergent coefficients of the corresponding UV counterterms and are polynomials of all their arguments,  $\Gamma_i$  are the (unrenormalized) diagrams obtained from  $\Gamma$  by shrinking the corresponding UV subgraphs to points.

Naively, to expand (5.23) one would expand each term on the r.h.s.: the factors  $\mathcal{Z}_i$  are polynomials in  $m$  and are already “expanded”, and  $\Gamma_i$  can be expanded by a straightforward application of the definition (5.2). Indeed, all the intricacies of the *As*-operation are essentially aimed at extracting the terms that are non-analytical in light parameters, and the very fact of the polynomial dependence of UV counterterms on masses and momenta makes one expect that UV renormalization in the MS scheme and the *As*-expansion are, in a sense, “orthogonal” (cf. the analysis of [17]).

Motivated by the above, we define the *As*-operation on renormalized graphs by applying

the  $As$ -operation (5.2) termwise to  $\Gamma_i$  on the r.h.s. of (5.23) as follows:

$$\mathbf{As} \circ \mathbf{R} \circ \Gamma(k, m, M) \stackrel{\text{def}}{=} \sum_i \mathcal{Z}_i(1/\epsilon, m, M) \mathbf{As} \circ \Gamma_i(k, m, M). \quad (5.24)$$

We assert that the expansion thus obtained is a correct asymptotic expansion for  $\mathbf{R} \circ \Gamma(k, m, M)$ . This is not obvious, but for that matter neither is obvious the UV finiteness of the r.h.s. of (5.23) or even (2.1). In the next paper of the present series [16] we will represent the  $R$ -operation in such a form that its UV finiteness will be made apparent [15] and the proof of correctness of the above definition of the  $As$ -operation will be reduced to a rather straightforward technical exercise.

For a more practically oriented reader it may be sufficient to learn the following. On the one hand, the final expressions for the coefficient functions of short-distance expansions (see subsect. 6.3) imply highly non-trivial cancellations of UV and IR divergences between different terms which cannot be understood without the underlying theory. Such cancellations were checked by explicit three-loop calculations in [11] and in several other calculations (see e.g. [18]). On the other hand, from the point of view of our methods there is no essential difference between the short-distance operator-product expansion and the most general Euclidean regimes with heavy masses, and all the non-trivial patterns of combination of UV and IR divergences manifest themselves at the three-loop level. One should also take into account that we are dealing with relatively primitive integrals of rational functions where all the effects are governed by the power counting so that there is no reason to expect any pathologies. Therefore, the explicit calculations of [11] and [18] provide a very strong evidence in favour of the above assumption while [17] provides formal arguments. In what follows, we will concentrate on the combinatorial structure of the definition (5.24).

(Note that the aggregate operation  $\mathbf{As} \circ \mathbf{R}$  defined in (5.24) was denoted as  $\mathbf{As}$  and studied as a whole in our original publication [2], and the starting point there was the so-called EA-expansion. Such a way of reasoning resulted in superfluous complications which are avoided in the present exposition.)

## 5.9 $As$ -operation in models with UV divergences.

If the model described by the interaction Lagrangian  $L(\varphi, \Phi)$  possesses UV divergences, then the UV-renormalized generating functional of Green functions can be represented as (cf. (2.33)):

$$\mathbf{R} \circ \langle \text{T exp}[L(\varphi, \Phi) + \varphi J] \rangle_0 = \langle \text{T exp}[L_R(\varphi, \Phi) + \varphi J] \rangle_0. \quad (5.25)$$

with  $L_R(\varphi, \Phi)$  defined analogously to (2.33).

Since the  $As$ -operation is defined to act formally on each term of the renormalized diagrams, one can apply it to both sides of (5.25). Then one can simply use the results obtained for the case of the models without UV divergences (see (5.17)–(5.20)). Our final formulae are as follows:

$$\mathbf{R} \circ \langle \text{T exp}[L(\varphi, \Phi) + \varphi J] \rangle_0 \underset{M \rightarrow \infty}{\simeq} \mathbf{As} \circ \mathbf{R} \circ \langle \text{T exp}[L(\varphi, \Phi) + \varphi J] \rangle_0 \quad (5.26)$$

$$\equiv \mathbf{R} \circ \mathbf{R}^{-1} \circ \mathbf{A}_s \circ \mathbf{R} \circ \langle \mathbf{T} \exp[L(\varphi, \Phi) + \varphi J] \rangle_0 \equiv \mathbf{R} \circ \langle \mathbf{T} \exp[L_{\text{eff}}(\varphi) + \varphi J] \rangle_0,$$

where

$$L_{\text{eff}}(\varphi) = \Delta_{\text{UV}}^{-1} \circ [\mathbf{T} e^{L_{\text{eff,bare}}(\varphi)} - 1], \quad (5.27)$$

$$L_{\text{eff,bare}}(\varphi) = \Delta_{\text{as}} \circ [\mathbf{T} e^{L_R(\varphi, \Phi)} - 1]_{\Phi=0}, \quad (5.28)$$

$$L_R(\varphi, \Phi) = \Delta_{\text{UV}} \circ [\mathbf{T} e^{L(\varphi, \Phi)} - 1]. \quad (5.29)$$

The algorithms of the three  $\Delta$ -operations have been described in detail in sects.1.1, 2.8, and 4.4.

Eq.(5.26) means that the effects of virtual presence of heavy particles on the effective low energy theory of light particles can be described by an effective Lagrangian  $L_{\text{eff}}$  (explicitly given by (5.27)–(5.29)) to all orders in  $1/M$ . Our derivation of this result is valid within the MS-like renormalization schemes and in all models including those with massless particles like QCD.

Note that the  $R$ -operations in the original model and in the effective model need not be the same; this can be equivalently described by saying that the renormalization parameters in the two models may be different. In such a case the parameters of  $L_{\text{eff}}$  will depend on *both* renormalization parameters.

It should also be recalled that the  $A_s$ -operation derived in [1] is essentially unique because it leads to series in powers and logarithms of the expansion parameters. This uniqueness property (up to the choice of renormalization schemes) is inherited by (5.26)–(5.29), whatever the guise in which such expansions might appear when derived by alternative methods. Our form of presentation is dictated by computational convenience.

If one deals with gauge models, then a most convenient way of studying gauge properties of  $L_{\text{eff}}$  is to employ the techniques described in [10].

## 5.10 Example.

To get a feeling of how the above formulae work, consider QED with a light electron  $\psi$  and a heavy muon  $\Psi$ . The well-known interaction Lagrangian is denoted as  $L(\psi, \Psi, A)$  where  $A$  is the photon field. Restricting ourselves to the two-photon sector in the 1-loop approximation, we get (fat lines correspond to the muon):

$$\begin{aligned} L_R(\psi, \Psi, A) &= \Delta_{\text{UV}} \circ [\mathbf{T} e^{L_{\text{QED}}(\psi, \Psi, A)} - 1] \\ &= \Delta_{\text{UV}} \circ \left[ \text{Diagram 1} \right] + \Delta_{\text{UV}} \circ \left[ \text{Diagram 2} \right] + \dots \\ &= \delta Z i \int dx \left[ -\frac{1}{4} F_{\mu\nu}^2(x) \right] + \dots, \\ L_{\text{bare,eff}}(\psi, A) &= \Delta_{\text{as}} \circ [\mathbf{T} e^{L_R(\psi, A)} - 1]_{\psi=0} \\ &= L_R(\psi, A) + \Delta_{\text{as}} \circ \left[ \text{Diagram 3} \right] + \dots, \end{aligned} \quad (5.30)$$

and, finally,

$$\begin{aligned} L_{\text{eff}}(\psi, A) &= \Delta_{\text{UV}}^{-1} \left[ \text{T} e^{L_{\text{eff,bare}}(\psi, A)} - 1 \right] \\ &= L_{\text{eff,bare}}(\psi, A) + \Delta_{\text{UV}}^{-1} \left[ \text{Diagram} \right] + \dots \end{aligned} \quad (5.31)$$

Since

$$\Delta_{\text{UV}}^{-1} \left[ \text{Diagram} \right] = -\Delta_{\text{UV}} \left[ \text{Diagram} \right], \quad (5.32)$$

we obtain:

$$\begin{aligned} L_{\text{eff}}(\psi, A) &= \Delta_{\text{as}} \left[ \text{Diagram} \right] + \Delta_{\text{UV}} \left[ \text{Diagram} \right] + \dots \\ &= \delta z(M^2, \mu^2) i \int dx \left[ -\frac{1}{4} F_{\mu\nu}^2(x) \right] + \dots, \end{aligned} \quad (5.33)$$

where the dots denote other field structures, and

$$\delta z(M^2, \mu^2) = \frac{e^2}{16\pi^2} \left[ c_1 + c_2 \log(M^2/\mu^2) \right] + O(1/M^2). \quad (5.34)$$

Note that cancellation of the UV divergences in the final result is obvious even without explicit calculations.

## 6 Generalized operator-product expansions.

In this section we turn to the case when the set of heavy parameters includes external momenta. Recall that we regard the diagrams and Green functions to be expanded as distributions with respect to the heavy external momenta [1].

We start by considering in subsects. 6.1–6.3 the case corresponding to the usual short-distance operator-product expansion, i.e. when there are no heavy masses and no linear restrictions on the heavy external momenta except the overall momentum conservation. In subsect. 6.1 notations are introduced and the  $As$ -operation for integrals is derived for the case under study, in subsect. 6.2 the global structure of the  $As$ -operation as applied to Green functions is studied, and in subsect. 6.3 the final results are presented. Modifications due to heavy masses are studied in subsect. 6.4.

In subsect. 6.5 we impose the so-called “natural” linear restrictions on the heavy momenta, and in subsect. 6.6 the corresponding version of the  $As$ -operation and the expansions for Green functions are presented. In subsect. 6.7 the structure of the general formulae is explained with a simple example.

In subsect. 6.8 we briefly discuss the contact terms in the obtained expansions. The section is concluded in subsect. 6.9 with a discussion of the most general linear restrictions imposed on heavy momenta; it is pointed out that the contact terms in such a case may contain the so-called paralocal operators.

## 6.1 As-operation and heavy momenta.

Let us assume that all the particles of the model are light, but some of the external momenta of the Green function to be expanded are heavy.

More precisely, let  $j$  numerate the heavy external lines of the diagram  $\Gamma$ , then  $Q_j$  are the corresponding ingoing external momenta. There are also some light external momenta collectively denoted as  $k$ , which are of order  $O(\kappa)$  together with the masses  $m$  (recall that  $\kappa$  is our standard notation of the expansion parameter [1]). The fact that  $Q$  are heavy is formally expressed as

$$Q = O(1) \quad \text{as } \kappa \rightarrow 0. \quad (6.1)$$

One cannot assume all  $Q_j$  to be independent of  $\kappa$  because of the momentum conservation which reads:

$$\sum Q = -\sum k = O(\kappa). \quad (6.2)$$

Therefore, let us introduce the ‘‘heavy’’ components of  $Q$  that are independent of  $\kappa$ :

$$\bar{Q}_j \equiv Q_j|_{\kappa=0}. \quad (6.3)$$

$Q$  can be represented as:

$$Q_j = \bar{Q}_j + q_j, \quad (6.4)$$

where  $q_j = O(\kappa)$  are linear combinations of  $k$ .

The momentum conservation should hold separately for heavy and light components, so that:

$$\sum_j \bar{Q}_j = 0. \quad (6.5)$$

We have assumed that there are no other restrictions on  $\bar{Q}$  except (6.5). Considering the diagram  $\Gamma$  as a distribution with respect to the heavy momenta means that we expand expressions of the form

$$\Gamma(k, m, F) \equiv \int d\bar{Q}' F(\bar{Q}) \int dp \Gamma(p, \bar{Q} + q, k, m), \quad (6.6)$$

where  $F$  is a smooth test function independent of  $\kappa$ , and the integration runs over the manifold described by (6.5).

It is convenient to represent  $F$  as a vertex attached to the vertices corresponding to  $Q$  by the lines that are heavy by definition—see Fig.3.

To expand  $\Gamma(k, m, F)$ , one repeats the reasoning of subsects.4.1-4.4 and arrives at the following equation instead of (5.12):

$$\mathbf{As} \circ \Gamma = \sum_h (\Delta_{\mathbf{as}} \circ h) \cdot [\Gamma/h]. \quad (6.7)$$

Now there is only one heavy knot shrunk to the point in each term in the sum. Its description—owing to the above agreement that the lines connecting the  $F$ -vertex with  $\Gamma$  are heavy—coincides with that given in subsect. 5.2: it must be 1PI with respect to light lines. But it may be easier to follow the ‘‘fool-proof’’ recipe given at the end of subsect. 5.6, which remains valid here.

A reasoning similar to that of subsect. 5.5 shows that the above formula is also valid in the case of a non-1PI graph  $\Gamma$ ; in particular, it is valid for disconnected graphs.

## 6.2 *As*-operation on Green functions.

Let us now turn to Green functions. Let  $H_j(x)$  be local products of light fields  $\varphi(x)$  and their derivatives. Consider the following generating functional of Green functions:

$$\langle T \left\{ \prod_j \tilde{H}_j(Q_j) e^{\mathcal{L}} \right\} \rangle_0, \quad (6.8)$$

where the tilde marks Fourier transforms and  $\mathcal{L}$  is a local functional (see the definition in subsect. 3.1). To obtain specific correlators from (6.8), it is sufficient to perform suitable variations with respect to the coefficient functions of  $\mathcal{L}$  and replace  $\mathcal{L}$  by the Lagrangian  $L$ . The momenta corresponding to any additional operator insertions are light, i.e.  $O(\kappa)$ , by definition.

Without loss of generality, the test function is introduced as follows:

$$G(F, \mathcal{L}) \equiv \langle T \{ F * \mathcal{H} e^{\mathcal{L}} \} \rangle_0, \quad (6.9)$$

where

$$F * \mathcal{H} = \int d\bar{Q}' F(\bar{Q}) \prod_j \tilde{H}_j(\bar{Q}_j + q_j), \quad (6.10)$$

where the momenta  $\bar{Q}_j$  and  $q_j$  are the same as defined in subsect. 6.1

The sum of all  $q_j$  should in general not be taken to be zero in (6.9) in order to get rid of the disconnected diagrams contributing to (6.9) which normally are of no interest in phenomenological applications. If, however, the kinematics of the problem require that  $\sum q_j = 0$  for the connected component, one can take the corresponding limit termwise in the final expansion. Taking such limits commutes with the expansion procedure unless there are connected diagrams like the one shown in Fig.4 where the wavy lines correspond to massless particles and may give rise to an infrared divergence. Whether or not the problem under study allows such terms can be checked by a straightforward inspection. However, it is still possible to regularize such contributions e.g. by introducing a non-zero mass (which, of course, should be considered as light in the expansion procedure); then the  $\bar{Q}$ -dependent part of the final expansion as obtained by our methods will be insensitive to the light mass structure of the model (see below).

Now, using (6.7) and reasoning as in subsect. 5.6, we obtain the following analogue of (5.17):

$$\text{As} \circ G(F, \mathcal{L}) = \langle T \left\{ \Delta_{\text{as}} \circ [TF * \mathcal{H} e^{\mathcal{L}}] e^{\mathcal{L}} \right\} \rangle_0. \quad (6.11)$$

The algorithm for evaluating  $\Delta_{\text{as}}[TF * \mathcal{H} \exp \mathcal{L}]$  is exactly the same as described in subsect. 5.6 (recall that according to the graphical conventions introduced after (6.6), the lines connecting the  $F$ -vertex with the proper Feynman diagram are heavy by definition; note also that the  $\delta$ -function which expresses momentum conservation in the relevant diagrams contains a sum of only light momenta due to (6.5), so that a formal expansion in light parameters will not affect it; if one wishes, one could introduce a formal integration in some of the  $q_j$  to get rid of it

completely). One sees that  $\Delta_{\text{as}}[TF * \mathcal{H} \exp \mathcal{L}]$  is a local functional in the sense of subsect. 3.1, so that (cf. subsect. 6.8):

$$\Delta_{\text{as}} [TF * \mathcal{H} e^{\mathcal{L}}] = \sum_n C_{\mathcal{H},n}^{\text{bare}}(F, \mathcal{L}) \tilde{j}_n(q), \quad (6.12)$$

where  $q = \sum q_j$  and

$$C_{\mathcal{H},n}^{\text{bare}}(F, \mathcal{L}) = \int d\bar{Q}' F(\bar{Q}) C_{\mathcal{H},n}^{\text{bare}}(\bar{Q}, \mathcal{L}). \quad (6.13)$$

We have introduced the subscript “bare” to indicate that the  $C$ ’s contain divergences, so that a procedure analogous to what was described in subsect. 5.7 is needed, in order to transform (6.11) to an explicitly convergent form.

To accomplish this, one proceeds as follows:

$$\begin{aligned} \text{As} \circ G(F, \mathcal{L}) &= \sum_n C_{\mathcal{H},n}^{\text{bare}}(F, \mathcal{L}) \langle T \{ \tilde{j}_n(q) e^{\mathcal{L}} \} \rangle_0 \\ &= \sum_n C_{\mathcal{H},n}^{\text{bare}}(F, \mathcal{L}) \frac{\delta}{\delta \mathcal{L}_n(q)} \langle T e^{\mathcal{L}} \rangle_0 \\ &= \sum_n C_{\mathcal{H},n}^{\text{bare}}(F, \mathcal{L}) \frac{\delta}{\delta \mathcal{L}_n(q)} \mathbf{R} \circ \langle T e^{\xi^{-1}[\mathcal{L}]} \rangle_0 \\ &= \sum_m \left\{ \sum_n C_{\mathcal{H},n}^{\text{bare}}(F, \mathcal{L}) \mathcal{Z}_{n,m} \right\} \mathbf{R} \circ \langle T \{ \tilde{j}_m(q) e^{\xi^{-1}[\mathcal{L}]} \} \rangle_0, \end{aligned} \quad (6.14)$$

where we have used the fact that  $\xi^{-1}[\mathcal{L}]$  defined in sect. 3 is a local functional.

To complete the reasoning, one should take into account that the initial expression on which the  $\text{As}$ -operation acts may contain UV divergences and be renormalized via the  $R$ -operation, as follows:

$$G_{\mathbf{R}}(F, \mathcal{L}) \equiv \mathbf{R} \circ \langle T \{ F * \mathcal{H} e^{\mathcal{L}} \} \rangle_0. \quad (6.15)$$

The arguments justifying correctness of a straightforward application of  $\text{As}$  to renormalized expressions are the same as in subsect. 5.8.

### 6.3 Short-distance expansions.

Now let us exhibit the general structure of the expressions for the case when  $\mathcal{L} = L + \varphi J$  where  $L$  is a Lagrangian functional.

Using the results of sect. 4 (cf. (4.9)), one represents (6.15) as:

$$G_{\mathbf{R}}(F, L + \varphi J) = \sum_a \langle T \{ F * \mathcal{Z}_{\mathcal{H},a} * \mathcal{J}_a e^{[L_R + \varphi J]} \} \rangle_0, \quad (6.16)$$

where  $\mathcal{J}_a$  are multilocal products of the currents  $j_n$  defined in subsect. 3.1,  $\mathcal{Z}$  is a “matrix” of divergent coefficients, and the  $*$ ’s denote some contractions over the heavy momenta.  $L_{\mathbf{R}}$  is defined in (2.26). Applying  $\text{As}$  to both sides and using (6.14), one gets:

$$\text{As} \circ G_{\mathbf{R}}(F, L + \varphi J) = \int d\bar{Q}' F(\bar{Q}) \sum_m c_m(\bar{Q}, g, \mu) \mathbf{R} \circ \langle T \{ \tilde{j}_m(q) e^{[L + \varphi J]} \} \rangle_0, \quad (6.17)$$

where

$$\int d\bar{Q}' F(\bar{Q}) c_m(\bar{Q}, g, \mu) \equiv \sum_a \sum_n C_{\mathcal{J}_{a,n}}^{\text{bare}}(F * \mathcal{Z}_{\mathcal{H},a}, L_R) \mathcal{Z}_{n,m} \quad (6.18)$$

are linear functionals of the test function  $F$  that are finite in the limit  $\epsilon \rightarrow 0$  (which is proved similarly to the reasoning at the end of subsect. 5.7);  $g$  are the coupling constants of the Lagrangian  $L$ , and  $\mu$  is the renormalization parameter. The expansion (6.17) has the form of a familiar operator product expansion at short distances.

It should be stressed that (6.18)—despite its cumbersome appearance—describes a fully constructive algorithm for getting explicit expressions for the coefficient functions  $C_m(\bar{Q}, g, \mu)$ . This algorithm is equivalent to the one described in [11], where an assumption on existence and properties of the operator-product expansion in the MS scheme was made. Namely, in [11] it was assumed that the coefficient functions of such OPE are analytical in masses. One can easily see that the expressions (6.17)–(6.18) prove that assumption: indeed, the coefficient functions as defined above have turned out to be independent of  $m$ , but the local operators  $j$  have been allowed to contain non-negative integer powers of  $m$ ; if the local operators are built of only fields, then the coefficient functions would become analytical in masses. Non-trivial examples of two- and three-loop calculations of coefficient functions using this algorithm together with explicit formulae can be found in [11] and [18].

Finally, note that if one wishes to consider Green functions of composite operators, then suitable variations in  $\mathcal{L}$  should be introduced into (6.14), (6.15) etc., giving rise to additional terms in the final result. Thus, if one considers the expansion of a correlator of the form

$$\langle T \{ \tilde{j}_n(Q) \tilde{j}_m(-Q + q) \tilde{j}_l(k) e^L \} \rangle_0, \quad (6.19)$$

then the final expansion will contain the correlators

$$\langle T \{ \tilde{j}_a(q) \tilde{j}_b(k) e^L \} \rangle_0 \quad \text{and} \quad \langle T \{ \tilde{j}_a(q + k) e^L \} \rangle_0. \quad (6.20)$$

The latter expression is a typical “vacuum condensate”, i.e. a vacuum expectation value of a local operator (cf. [12]). It is not equal to zero, even within perturbation theory, because the normal ordering of local products of fields is not used in the MS scheme.

## 6.4 Effects of heavy particles on operator expansion.

If there are heavy masses in the model, then the above results will get modified in the following way. The expression (5.12) for the  $As$ -operation will remain valid provided one includes the heavy knots corresponding to the  $F$ -vertex that were described after (6.7). Note that the “fool-proof” recipe of subsect. 5.2 is still valid here. Instead of (6.11) one will have:

$$\mathbf{As}^\circ G(F, \mathcal{L}) = \langle T \{ \Delta_{\text{as}^\circ} [TF * \mathcal{H}e^{\mathcal{L}}] \exp \Delta_{\text{as}^\circ} [T e^{\mathcal{L}} - 1] \} \rangle_0. \quad (6.21)$$

And the final result (6.17) will take the form

$$\mathbf{As}^\circ G_R(F, L(\varphi, \Phi) + \varphi J) \quad (6.22)$$

$$= \int d\bar{Q}' F(\bar{Q}) \sum_m c_m(\bar{Q}, M, g, \mu, q) \times \mathbf{R}^\circ \langle T \{ j_m(\sum q) \exp [L_{\text{eff}}(\varphi) + \varphi J] \} \rangle_0,$$

where  $L_{\text{eff}}(\varphi)$  is defined in (5.20), while  $c_m$  are analytical in  $q$ . Note that the operators  $j_m$  are here built of the light fields only, while the currents entering into the multilocal operator  $\mathcal{H}$  in the initial expression are allowed to contain heavy fields as well.

## 6.5 Natural restrictions on heavy momenta.

Let us turn to the case when there are linear restrictions imposed on the heavy external momenta other than the overall momentum conservation. Of immediate phenomenological importance are the so-called *natural restrictions* [3], [1]. In terms of position space, one can describe them as follows: all the “heavy” operator insertions are arranged into several groups, the distances within each group tend to zero while the distances between groups stay finite (cf. Fig.5). More precise definitions in terms of momentum representation are presented below.

Consider (6.8) and let the “heavy” operator insertions  $H_j$  (which are allowed to be built of both heavy and light fields) be divided into several non-intersecting groups (numerated by  $\lambda$ ) in such a way that the heavy momentum conservation holds separately within each group:

$$\sum_{j \in \lambda} Q_j = O(\kappa), \quad \text{for each } \lambda. \quad (6.23)$$

In terms of the independent momenta  $\bar{Q}$  :

$$\sum_{j \in \lambda} \bar{Q}_j = 0, \quad \text{for each } \lambda. \quad (6.24)$$

For each group it is natural to introduce a separate test function  $F_\lambda(\bar{Q}_\lambda)$  (where  $\bar{Q}_\lambda$  denotes the set of all  $\bar{Q}_j$  for  $j \in \lambda$ ), and the corresponding integration  $d\bar{Q}'_\lambda$  over the manifold described by (6.24). Now eqs.(6.9) and (6.10) are replaced by:

$$G(F_\lambda, \mathcal{L}) \equiv \langle T \left\{ \prod_\lambda F_\lambda * \mathcal{H}_\lambda e^{\mathcal{L}(\varphi, \Phi)} \right\} \rangle_0, \quad (6.25)$$

where

$$F_\lambda * \mathcal{H}_\lambda = \int d\bar{Q}'_\lambda F_\lambda(\bar{Q}_\lambda) \prod_{j \in \lambda} \tilde{H}_j(\bar{Q}_j + q_j). \quad (6.26)$$

To use graphical representation, one introduces an  $F$ -vertex of the same type as described after (6.6) for each of the test functions  $F_\lambda$ .

## 6.6 Generalized operator expansions.

The  $As$ -operation on a diagram  $\Gamma$  contributing to (6.25) will have the form

$$\mathbf{As} \circ \Gamma = \sum_{\{h_j\}} \left( \prod_i \Delta_{\text{as}} \circ h_i \right) \cdot \left[ \Gamma / \prod_i h_i \right]_{M=\infty, F=0} \quad (6.27)$$

(cf. (5.12)), where the heavy knots  $h$  are 1PI with respect to light lines and must contain at least one  $F$ -vertex or heavy line. Setting  $M = \infty$  and  $F = 0$  is an expression of the fact that all the “heavy” elements ( $F$ -vertices and heavy lines) must be contained within the  $h$ 's.

Applying  $\mathbf{As}$  to (6.25), one obtains the following expression instead of (6.21):

$$\begin{aligned} & \mathbf{As} \circ G(F_\lambda, \mathcal{L}(\varphi, \Phi)) \\ &= \sum_{\{\Lambda\}} \langle \mathbf{T} \left\{ \prod_{\Lambda \in \{\Lambda\}} \Delta_{\mathbf{as}} \circ \left[ \mathbf{T} \prod_{\lambda \in \Lambda} F_\lambda * \mathcal{H}_\lambda e^{\mathcal{L}(\varphi, \Phi)} \right] \times \exp \Delta_{\mathbf{as}} \circ [\mathbf{T} e^{\mathcal{L}(\varphi, \Phi)} - 1] \right\} \rangle_0, \end{aligned} \quad (6.28)$$

where  $\{\Lambda\}$  denotes a splitting of the set of all  $\lambda$  into non-intersecting subsets numerated by  $\Lambda$ , so that

$$\prod_{\Lambda \in \{\Lambda\}} \prod_{\lambda \in \Lambda} F_\lambda * \mathcal{H}_\lambda = \prod_{\lambda} F_\lambda * \mathcal{H}_\lambda, \quad (6.29)$$

and the first summation on the r.h.s. of (6.28) runs over all such splittings  $\{\Lambda\}$ .

The action of  $\Delta_{\mathbf{as}}$  can be most easily described using the “fool-proof” recipe presented in subsect. 5.2:  $\Delta_{\mathbf{as}}$  nullifies unconnected diagrams as well as all those connected ones which result in meaningless expressions like  $1/0$  when formally Taylor-expanded in light parameters. The remaining diagrams are exactly 1PI with respect to light lines and get formally Taylor-expanded in the light parameters.

Each  $\Delta_{\mathbf{as}}$  results in a local functional (cf. (6.12)), so that (6.28) can be represented as

$$\begin{aligned} & \mathbf{As} \circ G(\{F_\lambda\}, \mathcal{L}(\varphi, \Phi)) \\ &= \sum_{\{\Lambda\}} \langle \mathbf{T} \left\{ \prod_{\Lambda \in \{\Lambda\}} \left[ \sum_{\alpha_\Lambda} C_{\Lambda, \alpha_\Lambda, \text{bare}}(\{F_\lambda\}_{\lambda \in \Lambda}, M, \mathcal{L}, \{q_j\}_{j \in \Lambda}, \epsilon) \tilde{j}_{\alpha_\Lambda} \left( \sum_{j \in \Lambda} q_j \right) \right] \right. \\ & \quad \left. \times \exp \mathcal{L}_{\text{eff, bare}}(\varphi) \right\} \rangle_0, \end{aligned} \quad (6.30)$$

where each  $C$  is a coefficient function which is independent of fields and is at the same time a linear functional with respect to each  $F_\lambda$ . The dependence on  $q_j$  is analytical.

To obtain an expansion for the renormalized Green functions in an explicitly convergent form one should—as has been done in all the special cases considered above—“sandwich” the  $\mathbf{As}$ -operation between  $\mathbf{R}$  and  $\mathbf{R}^{-1}$  (cf. (5.26)). Further reasoning is similar to that of subsects. 4.7 and 5.2–3: one should first get rid of the  $R$ -operation using the generalized Zimmermann identities of subsect. 4.1, then apply the  $\mathbf{As}$ -operation, and, finally, use the identities of subsect. 4.2 to extract the  $R$ -operation and thus transform the expansion to an explicitly convergent form. In this way one arrives at the following final result:

$$\begin{aligned} & \mathbf{As} \circ \mathbf{R} \circ G(\{F_\lambda\}, L(\varphi, \Phi) + \varphi J) \\ &= \sum_{\{\Lambda\}} \mathbf{R} \circ \langle \mathbf{T} \left\{ \prod_{\Lambda \in \{\Lambda\}} \left[ \sum_{\alpha_\Lambda} c_{\Lambda, \alpha_\Lambda}(\{F_\lambda\}_{\lambda \in \Lambda}, M, g, \{q_j\}_{j \in \Lambda}) \tilde{j}_{\alpha_\Lambda} \left( \sum_{j \in \Lambda} q_j \right) \right] \right. \\ & \quad \left. \times \exp [L_{\text{eff}}(\varphi) + \varphi J] \right\} \rangle_0, \end{aligned} \quad (6.31)$$

which should be compared with (6.17);  $L_{\text{eff}}$  is given by (5.20) and has the form of (5.21).

### 6.7 Example.

To clarify the above general expansion, consider the case of two groups of two currents each (the corresponding phenomenological problem is the deep inelastic scattering of two deeply virtual photons [13]). There are two heavy momenta  $\bar{Q}_1$  and  $\bar{Q}_2$ , and two test functions,  $F_{AB}(\bar{Q}_1)$  and  $F_{CD}(\bar{Q}_2)$ . The expression to be expanded is:

$$\int \int d\bar{Q}_1 d\bar{Q}_2 F_{AB}(\bar{Q}_1) F_{CD}(\bar{Q}_2) \quad (6.32)$$

$$\times \mathbf{R}^\circ < \mathbf{T} \left\{ \tilde{j}_A(\bar{Q}_1) \tilde{j}_B(-\bar{Q}_1 + q_1) \tilde{j}_C(\bar{Q}_2) \tilde{j}_D(-\bar{Q}_2 + q_2) e^{L(\varphi, \Phi) + \varphi J} \right\} >_0,$$

where  $q$  are introduced to get rid of the non-connected components. Its expansion is as follows. There are two sets  $\{\Lambda\}$  over which the summation in (6.31) runs:  $\{(A, B), (C, D)\}$  and  $\{A, B, C, D\}$ . The first of them corresponds to that term in the expansion where the two pairs of currents are “shrunk” into (linear combinations of) local operators separately; the second corresponds to a single sum of local operators. The resulting expansion reads (cf. Fig.6):

$$\int \int d\bar{Q}_1 d\bar{Q}_2 F_{AB}(\bar{Q}_1) F_{CD}(\bar{Q}_2) \quad (6.33)$$

$$\times \left( \sum_a \sum_b c_{AB,a}(\bar{Q}_1, M, g, \mu) c_{CD,b}(\bar{Q}_2, M, g, \mu) \right.$$

$$\times \mathbf{R}^\circ < \mathbf{T} \left\{ \tilde{j}_a(q_1) \tilde{j}_b(q_2) \exp[L_{\text{eff}}(\varphi) + \varphi J] \right\} >_0$$

$$\left. + \sum_a c_{ABCD,a}(\bar{Q}_1, \bar{Q}_2, M, g, \mu) \mathbf{R}^\circ < \mathbf{T} \left\{ \tilde{j}_a(q_1 + q_2) \exp[L_{\text{eff}}(\varphi) + \varphi J] \right\} >_0 \right).$$

### 6.8 Contact terms in operator expansions.

The expansions that we have derived should be valid for arbitrary test functions  $F(\bar{Q})$ . For example, if there are two independent heavy momenta,  $\bar{Q}_1$  and  $\bar{Q}_2$ , then the expansion will in general contain terms proportional to  $\delta(\bar{Q}_1)$ ,  $\delta(\bar{Q}_2)$ ,  $\delta(\bar{Q}_1 + \bar{Q}_2)$ ,  $\delta(\bar{Q}_1)\delta(\bar{Q}_2)$  etc. (for a discussion of the role of such terms see [1]). Let us explain how such terms are generated by the  $As$ -operation.

Within the context of subsect. 6.2 (no heavy masses etc.), consider the case corresponding to the product of two currents in (6.8):

$$F * \mathcal{H} = \int d\bar{Q} F(\bar{Q}) H(\bar{Q} + q) H(-\bar{Q}). \quad (6.34)$$

The non-trivial terms are generated from the following expression (cf. (6.12)):

$$\Delta_{\text{as}^\circ} \left[ \mathbf{T} F * \mathcal{H} e^L \right] = \sum_n C_{\mathcal{H},n}^{\text{bare}}(F, L) \tilde{j}_n(q), \quad (6.35)$$

where  $L$  is the interaction Lagrangian.

There are two classes of terms generated by  $TF * \mathcal{H}e^L$  that give non-zero result after application of  $\Delta_{\text{as}}$ . The first class contains the terms whose graphs remain connected after the  $F$ -vertex is deleted. Such terms can be represented as:

$$\mu^{-2\epsilon} \int d^D \bar{Q} F(\bar{Q}) f(\bar{Q}, p_1, \dots, p_n, m, \mu) \times \delta(q + \sum p_j) \prod_j (\tilde{\varphi}(p_j) d^D p_j), \quad (6.36)$$

where the Feynman amplitude  $f$  depends on the masses  $m$  of the theory, contains one factor  $\mu^{2\epsilon}$  per each loop, and the  $\delta$ -function that expresses the momentum conservation is shown explicitly. The action of  $\Delta_{\text{as}}$  on (6.36) consists in Taylor-expanding  $f$  in masses and  $p_j$ . The result has the form:

$$\begin{aligned} \Delta_{\text{as}} \circ (6.36) &= \int d^D \bar{Q} F(\bar{Q}) \sum_a f_a(\bar{Q}, \mu) \\ &\times \left[ \mu^{-2\epsilon} \int \mathcal{P}_a(m, p_1, \dots, p_n) \delta(q + \sum p_j) \prod_j (\tilde{\varphi}(p_j) d^D p_j) \right]. \end{aligned} \quad (6.37)$$

The square-bracketed term in (6.37) corresponds to  $\tilde{j}$  in (6.12), and the dependence of  $f_a(\bar{Q}, \mu)$  on  $\bar{Q}$  and  $\mu$  is as follows:

$$f_a(\bar{Q}, \mu) = \bar{f}_a(\bar{Q}, \left(\frac{m^2}{Q^2}\right)^\epsilon). \quad (6.38)$$

In the  $l$ -loop approximation,  $\bar{f}_a$  is a polynomial of order  $l$  in its second parameter, while its dependence on the first parameter is determined by power counting and covariance properties.

The second class contains the heavy knots that consist of two parts connected only via the  $F$ -vertex (cf. Fig.7). If there are no heavy masses, then there is only one such subgraph that is not nullified by the operation  $\tau$ —it consists of the two vertices corresponding to the “heavy” operators  $H$  (see (6.34)) connected to the  $F$ -vertex. The corresponding contribution to (6.35) has the following form (for definiteness we assume that  $H(x) = \varphi(x)^2$ ):

$$\begin{aligned} \Delta_{\text{as}} \circ & \left[ \int dQ F(Q) \delta(q + Q + p_1 + p_2) \delta(-Q + p'_1 + p'_2) \right. \\ & \times \tilde{\varphi}(p_1) \tilde{\varphi}(p_2) \tilde{\varphi}(p'_1) \tilde{\varphi}(p'_2) dp'_1 \dots dp'_2 \left. \right] \\ & = \tau [dQ F(Q) \delta(-Q + p'_1 + p'_2)] \\ & \times \delta(q + p_1 + p_2 + p'_1 + p'_2) \tilde{\varphi}(p_1) \tilde{\varphi}(p_2) \tilde{\varphi}(p'_1) \tilde{\varphi}(p'_2) dp_1 \dots dp'_2, \end{aligned} \quad (6.39)$$

where the two  $\delta$ -functions in the first line are inherited from the two operators  $H$  (cf. (3.1)), and we have extracted a  $\delta$ -function in the last line that depends only on the light momenta. We have deliberately not performed the integration over  $Q$  on the r.h.s. because  $\tau$  should Taylor-expand the square-bracketed expression in  $p'_i$  and this can be done by applying  $\tau$  directly to the  $\delta$ -function. The result has the form

$$\sum_a \int dQ [F(Q) \delta^{(\alpha)}(Q)] \times j_\alpha(q), \quad (6.40)$$

and one sees that the “bare” coefficient functions in (6.12) receive  $\delta$ -functional contributions. Such contributions—“dressed” by the appropriate divergent renormalization factors (cf.

(6.18))—are finally inherited by the coefficient functions of the short-distance expansion (6.17). It may be not quite obvious that the coefficient functions thus constructed will be integrable around  $Q \sim 0$ . However, cancellations between different terms can be traced if one performs the expansion procedure explicitly starting from the  $As$ -operation for products of singular functions prior to performing the integrations of  $\delta$ -functions described in subsect. 5.3. Alternatively, an interested reader can verify validity of our recipes by straightforward calculations in a simplest situation, e.g. within the model  $\varphi^3$  in two dimensions for the currents  $H(x) = \varphi^2(x)$  in one loop approximation.

## 6.9 General linear restrictions on heavy momenta and paralocal operators.

Consider the following operator product instead of (6.10):

$$F * \mathcal{H} = \int d\bar{Q}' F(\bar{Q}) \left[ \delta(q + \sum_j \bar{Q}_j) \prod_\lambda \delta(q_\sigma + \sum_j c_{\lambda,j} \bar{Q}_j) \right] \left[ \prod_j \tilde{H}_j(\bar{Q}_j + q_j) \right], \quad (6.41)$$

where the  $\delta$ -functions are introduced to impose linear restrictions on the heavy momenta. With a special choice of  $c_{\lambda,j}$  one reproduces the restrictions (6.24). Examples of more general restrictions are presented in Fig.8. We are not aware of any phenomenological applications where such restrictions might emerge naturally. Therefore, we offer only a few comments concerning the most general case (6.41).

First, it should be stressed that the expansion procedure as described by (6.11) remains fully correct and well-defined provided the “fool-proof” recipe for  $\Delta_{as}$  from subsect. 5.6 is used. No simple graphical description for heavy knots exists, though.

Second, the representation (6.12) should be replaced by a more general one: the sum on the r.h.s. may now contain operators that can not be interpreted as local operators and differ from (3.1) by additional  $\delta$ -functions besides the one expressing momentum conservation. Such operator monomials can be called *paralocal operators* [3]. A simple example of a paralocal operator in position-space representation may be as follows:

$$P(z) = \int \prod_\lambda d\xi_\lambda \prod_j \varphi(z + \sum_\lambda \xi_\lambda c_{\lambda,j}), \quad (6.42)$$

but more complicated patterns are possible: note in this respect that a natural position-space representation for the multilocal operators which are a special case of paralocal ones, is

$$J_1(z_1) \dots J_n(z_n), \quad (6.43)$$

where all  $J$  are local operators.

The renormalization properties of paralocal operators can be studied most easily using the techniques of [15] (also [16]) and are similar to those of (multi-) local ones: a paralocal operator renormalizes via paralocal ones. Zimmermann identities (including the inverted ones) can also be generalized to this case. Therefore, expansions in explicitly convergent form for the case of

general non-natural restrictions on heavy momenta can be obtained in a manner completely similar to the case of natural restrictions.

### Conclusions.

We have developed a simple combinatorial technique for studying global exponentiation properties of the  $As$ -operation and obtained Euclidean asymptotic expansions for  $MS$ -renormalized Green functions of arbitrary local operators in arbitrary models. The expansions are true infinite asymptotic series that run in powers and logs of the expansion parameter. The obtained expansions exhibit perfect factorization of heavy and light dimensional parameters (masses and external momenta) which means e.g. that the coefficient functions of operator expansions are analytical in light masses and momenta. It should be stressed that the expansions are valid in models with massless particles like QCD, and are most convenient for practical calculations. Uniqueness of such expansions (cf. [1]) greatly facilitates study of their properties in gauge models.

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## Figure captions.

Fig.1. A diagram (a) and its heavy knots (a)–(d). All the external momenta are light. The fat lines correspond to propagators of heavy particles. Various IR subgraphs are shown with dashed lines in (b)–(d).

Fig.2. (a) An example of a diagram with a "light" tadpole subgraph (the notations are the same as in Fig.1). The tadpole corresponds to a non-expandable global factor. The  $As$ -operation should not affect such factors. (b) and (c) are two IR-subgraphs. The contribution corresponding to the IR-subgraph shown in (b) is automatically set to zero by the operation  $\tau$  due to the properties of dimensional regularization, so that the tadpole can be factored out in the result produced by the  $As$ -operation, as expected.

Fig.3. A graphical representation of the test function corresponding to the heavy external momenta. The lines connecting the  $F$ -vertex with the rest of the diagram are heavy by definition.

Fig.4. If the wavy lines correspond to massless scalar particles then setting  $q = 0$  would result in an IR divergence in  $D = 4$ .

Fig.5. An example of kinematics of heavy momenta with linear restrictions. (a) differs from (b) by the restriction  $Q_3 = -Q_2$ . (c) shows the corresponding position space picture:  $\xi, \zeta \rightarrow 0$ .

Fig.6. A diagrammatic illustration of (6.33). The two terms on the r.h.s. correspond to two ways of cutting off the heavy knots corresponding to the heavy external momenta.

Fig.7. Graphical representation of the heavy knots contributing to contact terms. Each blob corresponds to a subgraph that is 1PI with respect to light lines. If there are no heavy masses then each blob is just 1PI and is nullified by  $\tau$  unless it consists of a single vertex.

Fig.8. Examples of "non-natural" restrictions on heavy momenta. (a) differs from Fig.6 by an additional restriction  $Q_1 = Q_2$ .

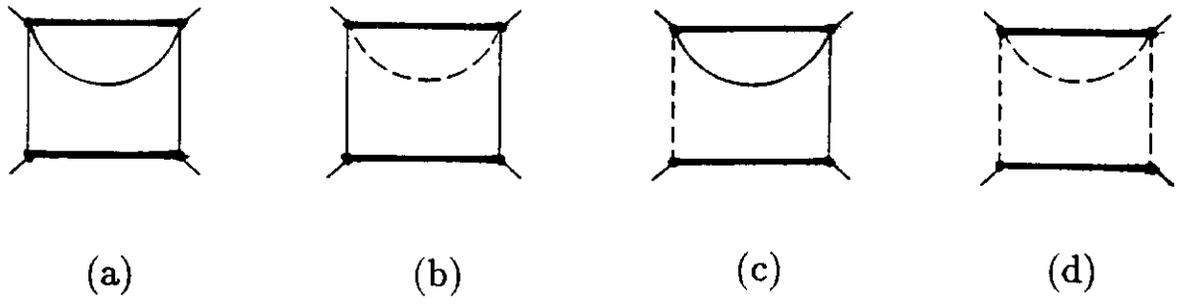


Fig. 1

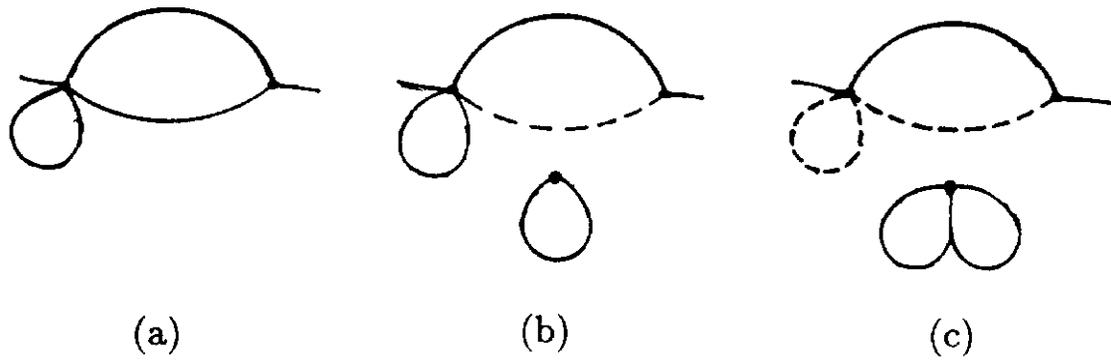


Fig. 2

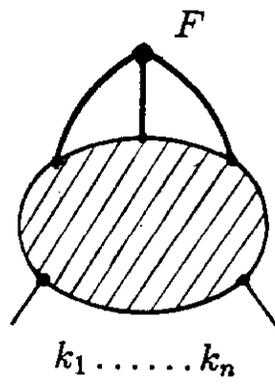


Fig. 3

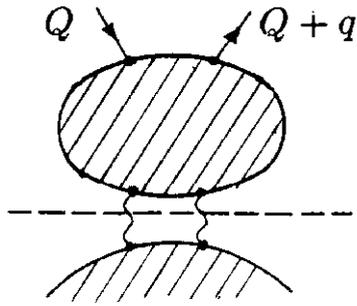
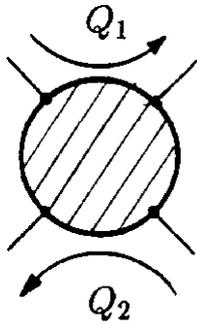
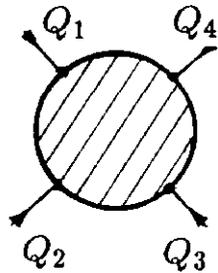


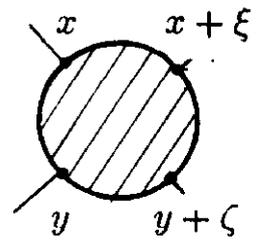
Fig. 4



(a)



(b)



(c)

Fig. 5

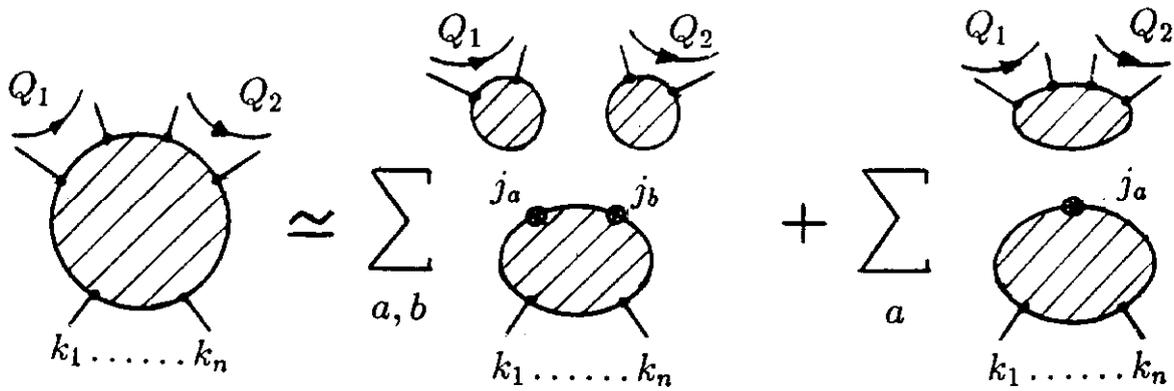


Fig. 6

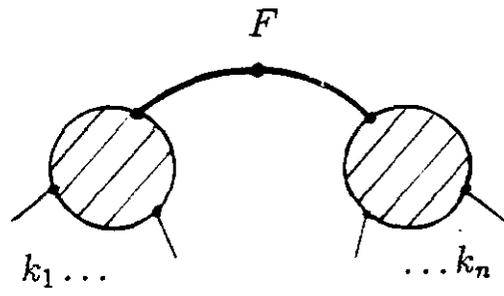
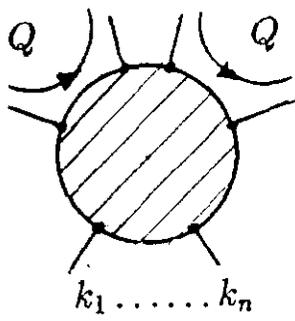
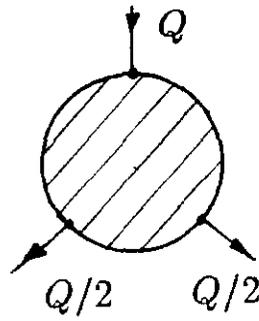


Fig. 7



(a)



(b)

Fig. 8