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PHASE TRANSITIONS WITH SUB-CRITICAL BUBBLES

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ABSTRACT

We study the dynamics of cosmological phase transitions initiated from a state of thermal equilibrium. If the effective potential satisfies certain general conditions, a homogeneous phase of false vacuum will form as the Universe expands, and the transition will proceed by well-known bubble nucleation processes. If such conditions do not hold, the Universe may instead be filled with a two-phase emulsion. The evolution of the transition will be determined by the free energy difference between the two phases and by the expansion rate of the Universe. Thermal fluctuations between the phases will determine the final distribution of regions of the Universe in each phase as they freeze-out. We develop a method to study the dynamics of such fluctuations, which we call sub-critical bubbles, and apply it to several situations of interest, including the symmetric and asymmetric double-well, and the Coleman-Weinberg scalar potentials. We show that in certain cases it is possible to avoid super-cooling, with the transition being *completed* by sub-critical fluctuations. Possible applications to the electroweak phase transition are briefly discussed.

1. Introduction

Since the discovery in the mid-seventies that gauge symmetries are restored at high temperatures, the study of phase transitions in the early Universe has been the object of much interest.[1] Within the context of the big-bang model, as the Universe expanded and cooled from its initially hot and dense state, symmetries were broken in succession until reaching the stage in which particle interactions are well described by the standard model group, $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$.

As in condensed-matter systems, these cosmological phase transitions may be first or higher order. Consider a model with a real scalar field ϕ that has both self-interactions and interactions with other fields. The dynamics of the scalar field will be determined by some finite temperature effective potential $V_T(\phi)$, with ϕ playing the role of the coarse-grained order parameter.[2] At high temperatures the system will be in its symmetric (disordered) phase, and as the temperature drops the system will experience symmetry breaking, as it settles into its ground state (ordered phase). If $V_T(\phi)$ exhibits a finite barrier between the two phases the transition will be first order. Otherwise, the transition is higher order.

An example of a first-order transition is shown in Fig. 1. At temperatures $T \gg T_c$, [curve (a)] where T_c is the critical temperature for the transition (in general of order the mass scale of the model), particles are effectively massless and the Lagrangian is symmetric. The system is assumed to be initially in thermal equilibrium with a heat bath at temperature T. For energy scales well below the Planck scale ($M_{Pl} \simeq 1.2 \times 10^{19}$ GeV), particle interactions will in general occur at a rate much faster than the Universe's expansion rate, justifying the assumption of thermal equilibrium. As the Universe expands and cools, a new minimum forms in the potential at a temperature T_1 [curve (b)]. The temperature continues to drop and at T_c the two minima become degenerate [curve (c)]. Notice the barrier separating the two minima. As T drops further, the new minimum becomes energetically favored, although the potential barrier "traps" the system in the symmetric phase, which becomes metastable. The Universe will be filled with a homogeneous phase of false vacuum. The metastable state decays by nucleation of bubbles of the broken symmetry phase; the bubbles expand and coalesce faster than they recede from each other due to the expansion of the Universe and the phase transition is eventually completed. Bubble nucleation occurs both at finite and zero temperatures. At finite temperatures, bubbles of the broken-symmetric phase can be thermally nucleated within the symmetric phase, triggering its decay. This classical process is exactly analogous to the nucleation of droplets in statistical physics,[2] and has been described in the context of field theory by Linde.[3] At low temperatures the decay process is dominated by quantum nucleation of bubbles, as shown by Coleman.[4] This mechanism is the field-theoretical generalization of barrier penetration in quantum mechanics. We will review both the finite and zero temperature false-vacuum decay processes in Section 2.

This scenario for the cosmological evolution of a first-order phase transition was very popular in the early eighties. It was invoked by Guth and Weinberg in the context of electroweak symmetry breaking with a light Higgs,[5] and also in the old inflationary model proposed by Guth to solve a series of problems of the standard big-bang model.[6] It was soon realized that in both cases the nucleation of bubbles, thermal or quantum, was a very rare process. The phase transition would not complete or would complete after extreme supercooling, both features not very desirable from either a cosmological or a particle physics point of view. Metastability should be avoided. The solution was to make the potential barrier either very small (hence the term "weakly first-order" transition) or disappear completely. In the electroweak case, the latter possibility was suggested by Witten, invoking quark condensation at temperatures of hundreds of MeV.[5] In the inflationary Universe case, new inflation was suggested[7] using the Coleman-Weinberg potential,[8] which has no barrier at zero temperatures and a very small one at finite temperatures. However, there is another way in which a homogeneous metastable state can be avoided. Namely, *it may not be formed* *at all* as the Universe cools below T_c. This possibility was first raised in the context of new inflation, although it can be applied in a wide range of situations, as we will see in this work.

The scenario for new inflation was criticized in the work of Mazenko, Unruh, and Wald (MUW).[9] They claimed that it was very unlikely that the field would remain localized around the origin as the Universe cooled to T_c, since thermal fluctuations would drive the field toward the other minima, overcoming the potential barrier separating them. Their arguments are of a general nature and are certainly valid when the field is in thermal equilibrium initially. In the particular case of inflation, thermal equilibrium as an initial condition is not guaranteed due to the very small couplings of the scalar field to itself and to the thermal bath.[10] However, *if* thermal equilibrium holds, one should take MUW's arguments seriously and investigate whether the effective potential indeed develops a metastable state. Recently one of us obtained two necessary conditions an effective potential should satisfy in order that a homogeneous metastable state forms:[11] i) the minimum at high temperatures $T \gg T_c$ should be located at the same side of the potential barrier as the false vacuum at zero temperatures, as is clearly the case in Fig. 1; ii) the probability that thermal fluctuations of the scalar field overcome the potential barrier must be strongly suppressed for $T_c < T < T_1$, so that the system gets effectively trapped in the metastable phase as the Universe cools to and below T_c. This last condition is generally translated into bounds on the typical couplings of the model.

Of course, these conditions are not always satisfied. It is very easy to think of models in which they are violated, as shown in Ref. 11. An extreme (and obvious) example is a higher-order phase transition that clearly violates condition i). It is then natural to ask how does the transition evolve when the requirements for metastability are not satisfied. The generally accepted qualitative picture goes as follows. Assuming for simplicity that $V_T(\phi)$ develops only two minima below a certain temperature T_1, (see Fig. 1) thermal fluctuations of the scalar field will in principle populate both minima, (*i.e.*, both "phases") as the temperature drops, with the relative probability of finding a certain region of the Universe in one phase being determined by the difference in free energy between the two phases. For the symmetric double-well potential (SDW) this probability is of course 0.5. The typical size of these fluctuations is determined by the correlation length of the scalar field, given by the temperature-dependent inverse mass scale of the model. These fluctuations become strongly suppressed at the so-called Ginzburg temperature, below which the thermal energy driving the fluctuations cannot overcome the potential barrier between the two minima.

The above qualitative remarks are mostly based on the works by Zel'dovich, Okun and Kobzarev,[12] Kibble,[13] and Vilenkin,[14] for the SDW potential, of interest due to the formation of domain walls.[15] For the asymmetric double-well potential (ADW), qualitative arguments concerning the nature of the transition were given in Refs. 16 and 11. It is our belief that there are many aspects of this problem that deserve further study from a more quantitative point of view. For example, can we build a coherent picture of these thermal fluctuations of the scalar field going over the barrier as field configurations that interpolate between the two minima at finite temperatures, similar to the "bubble" picture in the decay of metastable states? If so, can we compute their free energy, estimate the fluctuation rate per unit volume and compare it to the expansion rate of the Universe to determine the evolution of the phase transition? Or, in a nutshell, can we predict the final outcome of a phase transition simply by knowing the effective potential? This paper is an attempt to shed some light on these matters.

This work is organized as follows. In Section 2 we present a review on the decay of metastable states at zero and finite temperature. We start by discussing the one-dimensional case and generalize it to field theory. We conclude the review with an application of the results to the decay of metastable states in an expanding Universe. Readers familiar with this subject can skip this Section. In Section 3 we introduce the basic ideas of the paper. We motivate our choice of field configurations that represent the relevant thermal fluctuations of the scalar field and obtain the rate equation of fluctuations going both ways over the barrier in an expanding Universe. We apply our formalism to the SDW potential in Section 4 and to the ADW potential in Section 5. In both cases we obtain analitically and numerically the freeze-out temperature for the thermal fluctuations. In Section 4 we compare it to results obtained using qualitative arguments, while in Section 5 we estimate the fraction of volume of the Universe in each phase at freeze-out. In Section 6 we apply our general method to potentials which in principle would form a metastable state. We show that in some cases it is possible for thermal fluctuations to inhibit the formation of the metastable state, with the transition being completed without any supercooling. We apply these ideas to the Coleman-Weinberg potential. Finally, we conclude in Section 7 with a brief summary of our results and with an outlook to further work.

2. False-Vacuum Decay at Finite Temperature

In this Section we review some basic results on false-vacuum decay at zero and finite temperature and set up the formalism to be used in the next Sections. We start by examining metastability in a one-dimensional system and then generalize the results to field theory, emphasizing the passage from zero to finite temperature. After obtaining the decay rate per unit volume for a given temperature T, we apply the results to the problem of false-vacuum decay in an expanding Universe.

A. Metastability in One Dimension

Consider a system with a one-dimensional potential $V(x)$ which exhibits two non-degenerate minima, x_f and x_t, separated by a barrier of height V_h located at $x_{\max}$, as shown in Fig. 2. The potential is defined such that $V(x_f) = 0$ and $V''(x_f) \equiv \omega_f^2$, with the prime denoting derivative with respect to x. ω_f is the zero-point frequency of oscillations around x_f.

Imagine preparing the system such that initially there is a wave packet localized in the right well with center at x_f and with energy $E < V_h$. The system starts entirely localized in the metastable state. There are two mechanisms by which this state can decay to the ground state at x_t: At $T = 0$ or for $T \ll \omega_f$, the system would classically behave like a harmonic oscillator with frequency ω_f. However, it is rendered unstable by the quantum mechanical process of barrier penetration. At $\omega_f < T < V_h$, classical thermal fluctuations will be strong enough to induce a diffusion process over the barrier, destroying metastability. We will briefly obtain the rates for both processes, starting at $T = 0$.

In the semi-classical approximation ($\hbar \to 0$), we can use the WKB method to find the transmission amplitude across the barrier (x_{tp} is the classical turning point)

$$|T(E)| = A \exp\left[-2\int_{x_f}^{x_{tp}} dx\sqrt{2\left(V(x) - E\right)}\right] \quad , \tag{1}$$

where A is a normalization constant. The same result can be obtained using the language of path-integrals, which is more appropriate for applications to field theory. For a given unstable state $\Psi_f(t)$ with energy E_f, the decay rate Γ is given by the imaginary part of E_f,[17]

$$\Gamma = -\frac{2}{\hbar}\mathrm{Im}E_f \quad . \tag{2}$$

Thus, we must obtain the energy eigenvalue of the unstable state. To do this, one asks what is the probability amplitude for a transition between two position eigenstates $|x_1\rangle$ and $|x_2\rangle$, and then rewrites it in terms of a functional integral representation using the Feynman-Kac formula,[17]

$$I = \langle x_2|e^{-HT/\hbar}|x_1\rangle = N\int[dx]e^{-S_1(x)/\hbar} \quad , \tag{3.1}$$

with boundary conditions

$$x(T/2) = x_2 \quad \text{and} \quad x(-T/2) = x_1 \quad , \tag{3.2}$$

N a normalization factor and $S_1(x)$ the one-dimensional Euclidean action,

$$S_1(x) = \int_{-T/2}^{T/2} dt \left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)\right] \quad . \tag{4}$$

It can then be shown, taking $|x_1\rangle = |x_2\rangle = |x_f\rangle$ in the evaluation of $S_1(x)$, that the decay rate is

$$\Gamma = -2\lim_{T\to\infty}\frac{1}{T}\ln N\int[dx]e^{-S_1(x)/\hbar} \quad . \tag{5}$$

The calculation of the decay rate is reduced to the evaluation of the functional integral I.

To evaluate I, one uses the well-known semi-classical approximation, for which the dominant contribution to the transition amplitude comes from the stationary points of $S_1(x)$, that is, the solutions of the classical Euclidean equation of motion [equation of motion in the potential $-V(x)$] obtained from $\delta S_1(x) = 0$,

$$\ddot{x} = \frac{dV}{dx}, \quad \text{with } x(T/2) = x_2 \text{ and } x(-T/2) = x_1 \ , \tag{6}$$

where the overdot implies derivative with respect to Euclidean time τ. Denote these solutions by x_c. In the semi-classical limit, we must calculate small fluctuations around these solutions. Expanding $S_1(x)$ about x_c and keeping terms only up to second order in the fluctuations $\Delta x = x - x_c$,

$$I \simeq N e^{-S_1(x_c)/\hbar} \left[\det\left(-d^2/dt^2 + V''(x_c)\right)\right]^{-1/2} \quad . \tag{7}$$

In general there is more than one solution, x_c. Apart from the two trivial solutions for which x =constant, there is another solution which is of more interest. We can take $x_1 = x_2 = x_f$ and, in the limit $T \to +\infty$, imagine a solution in which the particle starts at $T \to -\infty$ at x_f, rolls down the hill $-V(x)$ reaching the turning point x_{tp} (see Fig. 2) at some time t_{tp} and then rolls back to rest at x_f at $T \to +\infty$. This solution is known as "the bounce". Let us denote it by $\bar{x}$. Using energy conservation, since $V(x_f) = 0$ and the motion starts with $\dot{x} = 0$, it is easy to obtain the Euclidean action corresponding to $\bar{x}$,

$$S_1(\bar{x}) = \lim_{T\to\infty} \int_{-T/2}^{T/2} dt \ \left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)\right] = -2\int_{x_f}^{x_{tp}} dx \ \sqrt{2V(x)} \ , \tag{8}$$

which reproduces the tunneling action obtained in Eq. 1 with the WKB approximation. The final answer is obtained by taking into account that the operator $d^2/dt^2 + V(\bar{x})$ has both a zero eigenvalue (from space translation invariance) and a negative eigenvalue (signaling the presence of a metastable state), and also by summing over configurations with an arbitrary number of bounces. The final decay rate is

$$\Gamma(\bar{x}) = \left(\frac{S_1(\bar{x})}{2\pi\hbar}\right)^{1/2} \left[\frac{\det(-d^2/dt^2 + \omega_f^2)}{\det'\left[-d^2/dt^2 + V''(\bar{x})\right]}\right]^{1/2} e^{-S_1(\bar{x})/\hbar} \simeq A\ \omega_f e^{-S_1(\bar{x})/\hbar} \quad , \tag{9}$$

where the prime in the denominator is a reminder to omit zero eigenvalues. For our purposes the simplified expression on the right hand side will be sufficient, with A being a constant of order unity.

At temperatures $\omega_f < T \lesssim V_h$ we expect that thermal fluctuations will dominate over quantum fluctuations. The dominant process responsible for the decay of the metastable state will now be thermal diffusion over the barrier as opposed to quantum tunneling. The rate for a thermally activated process can in general be written as

$$\Gamma(T) = A\,\omega e^{-E_a/T} \quad , \tag{10}$$

where ω is a fundamental fluctuation rate and E_a is the typical activation energy. For the potential of Fig. 2, one would naively write

$$\Gamma(T) = A\omega_f e^{-V_h/T} \tag{11}$$

for diffusion processes over the barrier. Indeed, if we consider a Boltzmann distribution of particles incident on the barrier from the right with momentum $-p$, the rate for making transitions over the barrier can be identified with the flux of probability over the barrier,[18]

$$\Gamma(T) = \frac{\int dp\; dx(-p)e^{-[p^2/2+V(x)]/T}\delta(x - x_{\rm max})\Theta(-p)}{\int dp\; dx e^{-[p^2/2+V(x)]/T}} \quad , \tag{12}$$

where $\delta(x - x_{\rm max})$ gives the location of the barrier and $\Theta(-p)$ is inserted so that particles are only incident from the right. After evaluating the momentum integral, the integral in x in the denominator can be approximated by a Gaussian integral (around x_f) and the result is,

$$\Gamma(T) \simeq \frac{\omega_f(T)}{2\pi} e^{-V_h/T} \quad . \tag{13}$$

At finite T, the decay rate is controlled by the ratio between the potential barrier height V_h and the available thermal energy T. This basic result carries over to field theory, although the barrier height will be equivalent to the free energy required to "excite" a field configuration that interpolates between the two minima through the potential's saddle point at $V(x_{\rm max})$.

B. Metastability in Field Theory

It is possible to generalize the previous results to field theory. We start by discussing zero temperature vacuum decay[4] and then include finite temperature effects.[3] Consider a scalar field in 3+1 dimensions with a Lagrangian density

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \quad , \tag{14}$$

where $V(\phi)$ is a potential with similar profile to that of Fig. 2, with a false vacuum at $\phi = \phi_f$, $V(\phi_f) = 0$, and $V''(\phi_f) = m_f^2$, with the prime denoting differentiation with respect to ϕ. The true vacuum is denoted by ϕ_t. According to Coleman, and Callan and Coleman,[4] the decay rate per unit volume is

$$\Gamma = \left(\frac{S_4(\bar{\phi})}{2\pi}\right)^2 \left[\frac{\det(-\Box_E + \mathrm{m}_f^2)}{\det'\left[-\Box_E + V''(\bar{\phi})\right]}\right]^{1/2} e^{-S_4(\bar{\phi})/\hbar} \simeq A\, m_f^4 e^{-S_4(\bar{\phi})/\hbar} \quad , \tag{15}$$

where $\Box_E = \partial^2/\partial\tau^2 + \nabla^2$, A is a constant of order unity, and

$$S_4(\bar{\phi}) = \int d\tau d^3x \left[\frac{1}{2}\left(\frac{\partial\bar{\phi}}{\partial\tau}\right)^2 + \frac{1}{2}\left(\nabla\bar{\phi}\right)^2 + V(\bar{\phi})\right] \tag{16}$$

is the Euclidean action in four dimensions of $\bar{\phi}$, the non-trivial solution of

$$\Box_E \phi = V'(\phi) \quad , \tag{17}$$

with boundary conditions $\lim_{\tau\to\pm\infty} \phi(\tau, \mathbf{x}) = \phi_f$ and $\lim_{|\mathbf{x}|\to+\infty} \phi(\tau, \mathbf{x}) = \phi_f$. For simplicity we only consider the last expression on Eq. 15 for the decay rate.

The solutions $\bar{\phi}$ are a straight forward generalization of the quantum-mechanical bounce introduced before. The decay rate is obtained by using a semi-classical approximation so that the dominant contribution to the functional integral comes from field configurations with lowest action. As shown by Coleman, Glaser, and Martin,[19] such configurations exhibit $O(4)$-symmetry, allowing the problem to be reduced to one degree of freedom, in the radial direction $r^2 = |\mathbf{x}|^2 + \tau^2$ in Euclidean space. In this case the Euclidean action is simply,

$$S_4(\phi) = 2\pi^2 \int r^3 dr \left[\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 + V(\phi)\right] \quad , \tag{18}$$

and the bounce is the solution to the equation of motion

$$\frac{d^2\phi}{dr^2} + \frac{3}{r}\frac{d\phi}{dr} = \frac{dV}{d\phi} \tag{19}$$

with boundary conditions $\lim_{r\to+\infty} \phi = \phi_f$ and $\frac{d\bar{\phi}}{dr}|_{r=0} = 0$. If we think of ϕ as position and r as time, it is easy to see that the bounce is the solution to the equation of a particle moving in the potential $-V(\phi)$, subject to a friction force which is inversely proportional to time. The particle is released from rest at $t = 0$ at some initial position ϕ_i and comes to rest at ϕ_f at $t \to \infty$, as shown in Fig. 3.

In describing the properties of the bounce, it is convenient to introduce the false-vacuum energy density $\varepsilon \equiv V(\phi_f) - V(\phi_t)$. From Fig. 3, it is clear that if ε is small compared to the potential barrier, $\varepsilon \ll V_h$, ϕ_i will be very close to ϕ_t. In this case, ϕ will start rolling *very* slowly down the hill, slowly enough that by the time it actually rolls down, the friction term is negligible. This limit is the well-known thin-wall limit. The bounce describes a bubble of radius R in Euclidean space with a wall of thickness $\Delta \simeq m^{-1}$ through which ϕ quickly evolves from its interior value $\phi \sim \phi_t$ to its exterior value $\phi \sim \phi_f$. Thus the decay of the homogeneous metastable state is triggered by the quantum nucleation and subsequent growth of a bubble of true vacuum, very much like the nucleation of droplets in statistical physics. That the bubble can grow is a consequence of a delicate balance between the gain in volume energy from being in the true phase, $|E_V| = \frac{4\pi}{3}\varepsilon R^3$ and the deficit in surface energy, $E_S = 4\pi S_1(\bar{\phi})R^2$, where $S_1(\bar{\phi}) = \int dr[\frac{1}{2}(d\bar{\phi}/dr)^2 + V(\bar{\phi})]$ is the surface density of the bubble. This can be seen explicitly by extremizing the action

$$S_4(R) \lesssim -\frac{\pi^2}{2}\varepsilon R^4 + 2\pi^2 S_1 R^3 \tag{20}$$

to find the critical radius $R_c = 3S_1/\varepsilon$, for which $|E_V|/E_S = 1$.

If the potential is not nearly degenerate, one must rely on numerical integration of the equation of motion to find the bounce. We will show a few examples shortly. First, it is useful to make a few comments in this case. Back to Fig. 3, it is clear that as ε increases ϕ_i quickly moves away from ϕ_t; the friction force, which is a burden in the nearly degenerate case, becomes an asset in slowing the downhill motion so that the final position at ϕ_f is reached as $r \to \infty$. The motion now starts at the inclined portion of the hill with reasonable acceleration, with the obvious consequence that the bubble radius decreases considerably. In fact, in the only exactly soluble model (at least to our knowledge) beyond the thin-wall approximation, with a potential $V(\phi) = \frac{m^2}{2}\phi^2[1 - \ln(\phi^2/\phi_0^2)]$, the bounce solution is $\phi(r) = \phi_0 \exp(-m^2r^2/2)$, and thus has a "radius" $R \sim m^{-1}$ of the same order as the wall's thickness.[20]

As an application of the previous discussion, consider the potential

$$V(\phi) = \frac{\lambda_1}{8}\left(\phi^2 - \phi_0^2\right)^2 + \frac{\lambda_2}{3}\phi_0\left(\phi - \phi_0\right)^3 \quad . \tag{21}$$

This potential has the same profile as that of Fig. 2. $V(\phi)$ has a false vacuum at $\langle\phi_f\rangle = \phi_0$ and a true vacuum at $\langle\phi_t\rangle = -\frac{\phi_0}{2}[(1+2a)+(1+12a+4a^2)^{1/2}]$, where $a \equiv \lambda_2/\lambda_1$ measures the relative asymmetry between the two minima. Defining the dimensionless variables $X = \phi/\phi_0$ and $\rho = \sqrt{\lambda_1}\phi_0 r$, the Euclidean action becomes

$$S_4(X) = \frac{2\pi^2}{\lambda_1}\int \rho^3\, d\rho\left[\frac{1}{2}\left(\frac{dX}{d\rho}\right)^2 + V(X)\right] \quad , \tag{22}$$

with $V(X) = V(\phi)/\lambda_1\phi_0^4 = \frac{1}{8}(X^2-1)^2 + \frac{a}{3}(X-1)^3$. Bounce solutions for different values of the asymmetry parameter a are shown in Fig. 4, while in Table 1 we show both the value of the bounce action $\bar{S}_4 \equiv \frac{\lambda_1}{2\pi^2}S_4(\bar{\phi})$ and its "radius", which we define as

$$R \equiv \frac{(\sqrt{\lambda_1}\phi_0)^{-1}}{\bar{S}_4}\int \rho^4\, d\rho\left[\frac{1}{2}\left(\frac{dX}{d\rho}\right)^2 + V(X)\right] \quad . \tag{23}$$

It is easy to see that as the asymmetry increases the bounce solution approaches $\phi(r) \to \exp(-m^2r^2/2)$; the critical bubbles become coreless, as we anticipated in the previous qualitative discussion.[20]

So far we have been dealing with the zero temperature case. In order to describe bubble nucleation at finite temperatures, we make the formal substitution $\tau \to \hbar\tau/T$ in Eq. 16, impose the periodic (anti-periodic for fermions) boundary condition $\phi(0,\mathrm{x}) = \phi(\hbar/T,\mathrm{x})$, and integrate in the "time" direction only in the interval $0 \leq \tau \leq 1$. The Euclidean action is now,

$$\frac{S_4(\phi)}{\hbar} = T^{-1} \int_0^1 d\tau \int d^3x \left[\frac{T^2}{2\hbar^2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + V_T(\phi) \right] \quad , \tag{24}$$

where $V_T(\phi)$ includes finite temperature corrections to $V(\phi)$. For a model with one scalar field it is given by[21]

$$V_T(\phi) = V(\phi) + \frac{T^4}{2\pi^2} \int_0^\infty x^2 \, dx \ln \left\{ 1 - \exp \left[- \left(x^2 + m^2(\phi)/T^2 \right)^{1/2} \right] \right\} \quad , \tag{25}$$

where $m^2(\phi) \equiv d^2V/d\phi^2$.

Back to the expression for the Euclidean action, note that the kinetic term increases quadratically with the temperature $\sim \frac{T^2}{2\hbar^2} (\partial\phi/\partial\tau)^2$. As we are interested in configurations that minimize the classical Euclidean action, for high temperatures it is clear that *static* (*i.e.* τ-independent) configurations will dominate the functional integral. The theory is effectively reduced to three dimensions and the Euclidean action becomes

$$\frac{S_4(\phi)}{\hbar} \to S_3(\phi, T)/T = T^{-1} \int d^3x \left[\frac{1}{2} (\nabla \phi)^2 + V_T(\phi) \right] \quad . \tag{26}$$

The rate per unit volume for thermal nucleation of critical bubbles is[3]

$$\Gamma(T) = T \left(\frac{S_3(\bar{\phi}, T)}{2\pi T} \right)^{3/2} \left[\frac{\det[-\nabla^2 + V_T''(\phi_f)]}{\det'[-\nabla^2 + V_T''(\bar{\phi})]} \right]^{1/2} e^{-S_3(\bar{\phi}, T)/T} \simeq A \, m_f^4(T) e^{-S_3(\bar{\phi}, T)/T} \quad , \tag{27}$$

where A is a constant, $m_f^2(T) = d^2V_T/d\phi^2|_{\phi_f}$, and $S_3(\bar{\phi}, T)$ is the Euclidean action of the non-trivial solution of $\nabla^2\phi = V_T'(\phi)$ with boundary condition $\lim_{|\mathbf{x}| \to \infty} \phi = \phi_f$. As discussed before, the decay rate is dominated by maximally symmetric solutions, now with $O(3)$-symmetry. The finite temperature bounce is then a solution of

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = V_T'(\phi) \quad , \tag{28}$$

with boundary conditions $\lim_{r \to \infty} \phi = \phi_f$ and $\frac{d\phi}{dr}|_{r=0} = 0$.

The last expression on the right hand side of Eq. 27 is the approximation to the decay rate that we will use in this work. Not surprisingly, the expression for the decay rate is equivalent to the general formula for a thermally activated process, Eq. 10. Note that the fundamental frequency of oscillations around ϕ_f is itself a function of the temperature, reflecting the fact that the potential changes with T. $S_3(\bar{\phi}, T)$ is the "activation energy" for the thermal process. It is equivalent to the free energy of the bubble of true vacuum nucleated in a homogeneous phase of false vacuum and represents the free energy barrier the system must overcome for the thermal transition to occur.[3] It is clear from Eq. 27 that thermal fluctuations are a purely classical effect; Planck's constant has disappeared from the decay rate.

All previous comments describing the properties of the bounce solutions apply here, with the main difference that the bounce must now be calculated for each value of T. It is again possible to obtain $S_3(\bar{\phi}, T)$ in the thin-wall limit, for which the nucleation rate is, $\Gamma(T) \sim \exp(-16\pi S_1^3/3\varepsilon^2 T)$, a result well-known from nucleation theory in statistical physics.[3] At finite temperatures, the thin-wall limit is useful to describe thermal nucleation just below the critical temperature, as can be seen from Fig. 1. As $T \to T_c$, $S_3(\bar{\phi}, T) \to \infty$, since the double-well limit is achieved. The same happens as $T \to 0$ since thermal fluctuations become strongly suppressed at low temperatures. Typically, $S_3(\bar{\phi}, T)$ reaches a minimum at some temperature $T_{\max}$ for which the rate is maximum, increasing steadily for lower temperatures until it is overcome by quantum fluctuations at some temperature T_0. This qualitative behavior of the Euclidean action as a function of T is shown in Fig. 5. We can now apply these results to the early Universe.

C. Metastability in the Early Universe

Let us assume that we are studying a model with a potential that behaves like Fig. 1. Within the big-bang model, it is reasonable to assume that early in the evolution of the Universe the temperature was high enough ($T \gg T_c$) so that the field ϕ was "localized" at

$\phi = 0$. (More on the notion of localized later.) As the Universe expands and cools, the temperature will eventually reach T_c and, if the conditions for metastability described in Ref. 11 are satisfied, the field will remain localized at $\phi \simeq 0$ below T_c. In this case the Universe will be filled with a homogeneous phase of false vacuum and radiation, and its evolution will be described by Einstein's equation [assuming a flat Robertson-Walker metric, $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$]

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\left[\frac{\pi^2}{30} g_* T^4 + V_0\right] \quad , \tag{29}$$

where the first term in the square brackets is the energy density in radiation ($\equiv \rho_{\rm rad}$) with g_* degrees of freedom at T and V_0 is the false-vacuum energy density. In order for the expansion to be radiation dominated, the temperature should satisfy,

$$T > \left(\frac{30V_0}{\pi^2 g_*}\right)^{1/4} \quad . \tag{30}$$

Otherwise the Universe will enter a de Sitter phase as in inflationary models. Recall that the thermal rate achieves a maximum at a temperature $T_{\rm max}$, as shown in Fig. 5. Thus, unless thermal nucleation is efficient at $T \sim T_{\rm max}$, the Universe will supercool in the false-vacuum phase and the decay will occur only by tunneling effects. [This, of course, assumes that no new physical effects come into play at lower temperatures, modifying the effective potential. (Ref. 5.)] In order to estimate if thermal nucleation is an efficient mechanism for false-vacuum decay in an expanding Universe, we must compare it with the expansion rate of the Universe per unit volume, $\Gamma_U = H^4$. From Eqs. 27 and 29 we can write,

$$\frac{\Gamma(T)}{\Gamma_U} \simeq 2.3 \times 10^{71} \left(\frac{\tilde{m}_f(T)}{\tilde{T}^2}\right)^4 \left(\frac{\rm GeV}{\phi_0}\right)^4 e^{-\tilde{S}_3(\tilde{\phi},T)/\tilde{T}} \quad , \tag{31}$$

where quantities with a tilde are scaled by ϕ_0 and $g_* = 110$ was used. From this expression, it is clear that only if the ratio between the two rates is of order unity or larger will thermal nucleation be effective. For an order of magnitude estimate, setting $\phi_0 = 1$GeV, $\tilde{m} \sim \tilde{T}$, we obtain,

$$\frac{\Gamma(T)}{\Gamma_U} \gtrsim 1 \quad \Rightarrow \quad \frac{\tilde{S}_3(\tilde{\phi},T)}{\tilde{T}} \lesssim 160 \quad . \tag{32}$$

As often remarked in the literature, such small bounce actions are uncommon although not impossible. (See Table 1. For $T = 0$ the condition reads $S_4(\bar{\phi}) \lesssim 160$.) If such bubbles are nucleated, they will expand with a speed $v \sim \varepsilon/\rho_{\rm rad}$ and may coalesce, completing the phase transition.[3] A detailed study of the kinetics of first-order phase transitions in a radiation dominated Universe is still lacking, although we will not be tackling these questions here. (See however the work of Guth and Weinberg on coalescence in a de Sitter Universe.[22]) This concludes our review on metastability. The methods we presented are applicable when the potential is such that a homogeneous metastable state forms as the Universe cools below the critical temperature for the transition. As we remarked in the introduction, even if the potential does behave like Fig. 1, the existence of a metastable state is not at all guaranteed. Thermal fluctuations of the scalar field may "leak" over the barrier to the other accessible minimum, destroying metastability. Those fluctuations are of course of a different nature from the bubbles of the nucleation processes, since they are bubbles of false vacuum being nucleated in true vacuum. They are typically small bubbles of ephemeral existence, since they are unstable against collapse due both to unfavorable surface and volume energy. Nevertheless, at high enough temperatures there may be enough thermal energy available to excite such field configurations, the typical scalar field fluctuations between the two minima of the potential. In the next Section we will motivate the study of such configurations, explaining their role in the dynamics of phase transitions and calculate their free energy.

3. Free Energy of Thermal Fluctuations: General Formalism

In the last Section we reviewed the formalism to study the decay of a homogeneous false-vacuum state at finite temperature. Here we will develop a method to study the dynamics of the transition when the general requirements for metastability do no hold: As T drops close to T_c thermal fluctuations may populate *both* vacuum states and the Universe would instead be filled by a two-phase "emulsion" characterized by the different average values of the order parameter around each vacuum. The evolution of the phase transition will depend crucially on the relative free energy difference between the two vacuum states, and on the ratio between the thermal fluctuation rates between the two states and the expansion rate of the Universe.

A. Sub-Critical Bubbles

In the usual picture of false-vacuum decay at finite temperatures, bubbles of true vacuum are thermally nucleated in the homogeneous false-vacuum phase and will quickly expand, converting false vacuum into true. In order for the bubbles to expand rather than contract, it is necessary that the gain in volume energy from the bubble interior being in the lower free energy phase overcomes the unfavorable surface tension of the bubble. This will happen if the bubble radius is larger than the critical radius R_c for which the two effects balance out. As we saw before, the radius of the critical bubble is very sensitive to the energy difference between the two minima; the closer to degeneracy the larger the bubble radius, with $R_c \to \infty$ in the limit of the SDW. For nearly degenerate potentials, $R_c \gg \xi(T)$, where $\xi(T) = m^{-1}(T)$ is the temperature dependent correlation length of the field ϕ, given by the inverse temperature dependent mass of excitations around the minimum. For distances larger than $\xi(T)$ correlations in the field are exponentially small.[13] Thus, for small asymmetry between the two minima of $V_T(\phi)$, critical size bubbles correspond to large fluctuations in the free energy. (Recall that the free energy of a critical bubble is given by the $O(3)$-

symmetric bounce action defined in Eq. 26.) Their nucleation will be a rare process, and metastable states in this case can be quite long lived compared to the typical time scales in the system. As the asymmetry between the two minima increases, the critical radius of the bubble decreases and so does its free energy. For large enough asymmetry, the radius becomes of the same order as the correlation length of the field, $R_c \rightarrow \xi(T)$. These "bubbles" are still critical in the sense that it is favorable for them to grow, although the usual picture of a bubble becomes somewhat blurred, since the bubble radius becomes comparable to the bubble wall's thickness.[20] Such bubbles were discussed in Section 2 and their profile is displayed in Fig. 4 for the potential of Eq. 21 at $T = 0$. As can be seen from Eq. 16 in the $T = 0$ case and Eq. 27 in the finite T case, the decay rate depends exponentially on the bubble's Euclidean action. From Table 1, and in a cosmological context from Eq. 32, it can be seen that even for large asymmetries the decay of metastable states is a rare process.

However, sub-critical sized fluctuations will have lower free energy and will occur more frequently. As we are interested in the dynamics of the transition in the hot early Universe, we will focus on thermal fluctuations. They will certainly play the dominant role in determining the outcome of the transition at temperatures close to the critical temperature. What then can be said, in general terms, about sub-critical thermal fluctuations? The higher the temperature, *i.e.* the higher the available thermal energy, the more the field fluctuates about its equilibrium value. For example, in Fig. 1 at $T \gg T_c$ the volume averaged value of the field is $\phi \sim 0$, although locally the field can make large excursions about this value. Also, at temperatures above T_c the correlation length $\xi(T) \sim m(T)^{-1}$ typically decreases like T^{-1}, so that the volume of an average fluctuation decreases with T. That the effective potential seems to restrict the value of the field closer to the minimum with increasing temperature reflects the fact that fluctuations in the field will average out on smaller and smaller scales.[9] In practice this means that if we were to measure the value of the field in a volume of the order of $\xi^3(T)$, the probability of finding $\phi \sim 0$ would be quite small at high temperatures. As the temperature drops, the fluctuation rate decreases and the fluctuation volume increases. The localization around $\phi = 0$ becomes more and more effective, although not necessarily efficient. If we continue to follow the evolution of $V_T(\phi)$ as pictured in Fig. 1, we see that at $T = T_1$ a new locally stable point appears at some value of ϕ that we write as $\langle\phi_t\rangle_T$, with the subscript t serving as a reminder that at $T < T_c$ this minimum becomes the global (true) minimum and the angular brackets with the subscript T as a reminder that the value of $\langle\phi_t\rangle_T$ changes with temperature. In the usual picture of metastability, even though there is a new stable point in the potential, thermal fluctuations from $\phi = 0$ to $\phi = \langle\phi_t\rangle_T$ are strongly suppressed so that as the temperature drops to T_c the field will be homogeneously localized at $\phi = 0$. Thus, the system goes out of equilibrium since in equilibrium the relative probability of being in each minima is given by the Boltzmann factor. (More about this shortly.) As we mentioned in the Introduction, this "localization" assumption should be examined in more detail. It should be possible to obtain a criterion in order to establish quantitatively for a given model when such assumption holds.[11] As a first step in this direction, we start by discussing what are the expected qualitative properties of such fluctuations.

Let us assume that for $T_c < T < T_1$ there will be fluctuations from $\phi = 0$ to $\phi = \langle\phi_t\rangle_T$ and back. One can picture these fluctuations as an attempt by the system to reach equilibrium; as long as there is thermal energy available to induce such transitions, the system will try to equilibrate while it is adiabatically cooled by the expansion of the Universe. Of course these fluctuations will be quite different from the familiar critical bubbles from false-vacuum decay. Critical bubbles are energetically favored since their interior is in the state with lower free energy. Hence they grow after being nucleated. These fluctuations are not energetically favored and are only possible due to the available thermal energy causing the fluctuations in the scalar field. However, once they are possible, there will be a non-zero probability in a given volume of finding the scalar field at $\phi = \langle\phi_t\rangle_T$. The smaller the volume we look at, the higher the probability of having this volume in the new unfavored vacuum.

Conversely, if we recall that fluctuations in ϕ are correlated up to $\xi(T)$, the maximum volume for which there is an appreciable probability of finding ϕ in the new vacuum is $V_c(T) \simeq \xi^3(T)$. In other words, the typical maximum fluctuation volume will be determined by $V_c(T)$. We will soon argue that these are also the statistically relevant fluctuations. Larger fluctuations are possible but exponentially suppressed. (See the review by J.S. Langer in Ref. 2.)

One may in principle think about these thermal fluctuations as being sub-critical "bubbles" of all possible radii, $R \lesssim \xi(T)$. The reason why we use the word bubble for these field fluctuations is because they should still have preferably $O(3)$-symmetry, since other fluctuations with less symmetry would cost more free energy. (They are there but are sub-dominant. We are not including here nearest neighbor effects that can enhance the fluctuations. In a future treatment these contributions should be included.) Of course, there is an absolute lower bound on the radius of the fluctuations below which our arguments break down. It is related with the ultra-violet cutoff which is implicit when adopting a coarse-grained description of the system. Loosely speaking, the spatial derivatives of the field should change smoothly within a spatial scale l. [In a lattice description of the field dynamics, the lattice spacing plays the role of the ultra-violet cutoff, and the consistency condition reads (lattice spacing) $\ll l$.] A physical requirement for l is that it should be the smallest length on which a phase can be defined.[2] A natural choice is the correlation length $\xi(T)$. Accordingly, we make the standard assumption that the statistically relevant fluctuations are the ones of correlation volume. The reason for this is simple. For a system in thermal equilibrium, once there are regions of the Universe in which the field is at $\langle\phi_t\rangle_T$, there will also be fluctuations back to $\phi = 0$. Within a horizon volume, we may picture the Universe as a 3-dimensional lattice with cells of volume $\sim \xi^3(T)$ in which we may find the scalar field at one or the other vacuum with a relative probability given by the Boltzmann factor,[16,15]

$$\frac{W_t}{W_0} \simeq \exp(-\Delta F/T) \quad , \tag{33}$$

where W_t (W_0) is the probability of finding the field at $\langle\phi_t\rangle_T$ ($\phi = 0$) and ΔF is the free energy difference between the two minima. (We are neglecting the difference in the oscillation frequencies around each minima here. $\xi(T)$ is in principle different at each vacuum.) As the temperature drops, the two phases will compete for dominance, with thermal fluctuations between them occurring fast compared to the expansion rate of the Universe. Thus, relevant fluctuations can convert one correlation size region in one vacuum into a correlation size region in another vacuum. Smaller fluctuations within a correlation region are ineffective at altering the *average* relative distribution of cells in each vacuum (in a stochastic description of the dynamics, they are the white noise coming from higher momentum modes and usually disappear once an averaging over a correlation volume is performed), while larger fluctuations will be sub-dominant. This justifies our taking the dominant fluctuations to be of correlation volume. A very schematic illustration of the fluctuation dynamics is shown in Fig. 6.

We should try to clarify what is meant by "dominant" fluctuations. We are interested in computing fluctuations in the scalar field that interpolate between the two vacuum states in $V_T(\phi)$. These fluctuations can go both ways; a region of correlation volume with $\phi = 0$ is converted into a region of correlation volume with $\phi = \langle\phi_t\rangle_T$ and vice-versa. That is, false-vacuum regions thermally fluctuate into true vacuum regions and true vacuum regions thermally fluctuate into false-vacuum regions. The question then is how to compute the free energy of these fluctuations and their rate. In the usual false-vacuum decay picture, the computation for the decay rate reduces to estimating the functional integral for the transition amplitude between the two states. This is done, as explained in Section 2, by taking a semi-classical approximation to the integral, using the fact that the action is minimized by the bounce solutions to the Euclidean equations of motion. Here, the sub-critical bubbles are not solutions to the equations of motion and we must find criteria that will help us select what are the field configurations that give the dominant contribution to the transition amplitude. Hence our choice of field configurations with $O(3)$-symmetry and with volume $V_c(T) = \xi^3(T)$. Further, we take the point of view that the general formula for a thermally activated process, Eq. 10, is applicable for fluctuations going in both directions.[23] The calculation of the fluctuation rate reduces to the calculation of E_a, the activation energy for the transition, which is simply the free energy of the fluctuations that interpolate between the two vacua.

B. Free Energy of Fluctuations: Anatomy of Sub-Critical Bubbles

Consider a model with a real scalar field with a potential $V_T(\phi)$ which below the temperature T_1 exhibits two locally stable points at $\langle\phi_f\rangle_T$ and $\langle\phi_t\rangle_T$, like in Fig. 1. (In Fig. 1 $\langle\phi_f\rangle_T \sim 0$ but this may not be the case in general, as in the example of Section 5.) The probability of a fluctuation of the scalar field around its equilibrium values is

$$W(\phi) \equiv W(\langle\phi_{f(t)}\rangle_T \rightarrow \phi) \sim \exp\left[-S_3^{f(t)}(\phi,T)/T\right] \quad , \tag{34}$$

where $S_3^{f(t)}(\phi,T)$ represents the change in free energy (or the work done by the system) due to the fluctuations $\Delta\phi = |\langle\phi_{f(t)}\rangle_T - \phi|$ around the minima, defined in Eq. 26. In this work we will assume that the 1-loop finite temperature effective potential is a good approximation to the homogeneous part of the free energy; the minimum of $V_T(\phi)$ determines the homogeneous state of thermal equilibrium of the field ϕ at T. For a discussion of the validity of this approximation we refer the reader to Ref. 9.

For small fluctuations, it is possible to make a Gaussian approximation to the free energy and estimate $W(\phi)$.[11] However, we are interested in fluctuations that for $T < T_1$ overcome the barrier and thus go beyond the validity of the Gaussian approximation. One possible way of estimating $S_3(\phi,t)$ is by using trial functions for the field configurations that interpolate between the two extrema. As discussed before, there are three assumptions we adopt to describe such fluctuations; first, they are $O(3)$-symmetric, so that we can reduce the volume integral for $S_3(\phi,T)$ to a one dimensional integral in the radial coordinate r. Non-spherical fluctuations are certainly present but are assumed to be sub-dominant due to a higher "cost" in free energy. Second, they interpolate between the two minima, fixing their beginning and end points. Third, they must do this within one correlation volume of the field $V_c(T) =$

$\frac{4\pi}{3}\xi^3(T)$. Based on these assumptions we can ask what is the typical anatomy of such fluctuations. To answer this question we invoke the discussion on Section 2 concerning "coreless" bubbles, *i.e.*, critical bubbles that are solutions of the Euclidean equations of motion but that, due to the large non-degeneracy in the potential, have radii only slightly larger than the correlation length of the field. Such bubbles are shown in Fig. 4 for the potential of Eq. 21. Accordingly, we take for the sub-critical bubbles,

$$\phi_{f(t)}(r) = \left(\langle\phi_{f(t)}\rangle_T - \langle\phi_{t(f)}\rangle_T\right) e^{-r^2/\ell^2_{f(t)}(T)} + \langle\phi_{t(f)}\rangle_T \quad , \tag{35}$$

where $\phi_{f(t)}(r)$ denotes a fluctuation of false (true) vacuum inside a true (false) vacuum region. This trial function will represent a typical bubble with size $\ell_{f(t)}(T)$, which will later be taken to be the correlation length in the corresponding vacuum.

Even though the free energy of these fluctuations will depend on the particular model we consider, there are a few general properties worth exploring before we go on to apply these ideas to different potentials. Using the dimensionless variables $X = \phi/\phi_0$ and $\rho = r\phi_0$ the free energy can be written as

$$S_3(\phi, T) = 4\pi\phi_0 \int \rho^2 d\rho \left[\frac{1}{2}\left(\frac{dX}{d\rho}\right)^2 + V_T(X)\right] \quad . \tag{36}$$

Examining the integral and taking into account the trial function used, it is clear that the gradient term grows linearly with $\ell_{f(t)}(T)$ and the potential term will grow as $\ell^3_{f(t)}(T)$. We can write, in general,

$$S_3^{f(t)} \lesssim \frac{3\pi}{4}\left(\frac{\pi}{2}\right)^{1/2} \tilde{\ell}_{f(t)}(T) \left(\langle\phi_{f(t)}\rangle_T - \langle\phi_{t(f)}\rangle_T\right)^2 \left[1 + A_{f(t)}(T)\tilde{\ell}^2_{f(t)}(T)\right] \quad , \tag{37}$$

where $\tilde{\ell}(T) \equiv \ell(T)m(T)$ and $A_{f(t)}(T)$ is a temperature-dependent coefficient which must be evaluated for a given potential $V_T(\phi)$ for each of the fluctuations. $S_3^{f(t)}$ is the free energy for nucleating a region of false (true) vacuum of volume $V = 4\pi\ell^3_{f(t)}/3$ inside a true (false) vacuum region. In general, $A_{f(t)}(T) > 0$ and the free energy is a monotonically increasing function of the "radius" $\ell(T)$.

C. Fluctuations in an Expanding Universe: Rate Equation

In the last subsection we obtained the general expression for the free energy of the fluctuations of the scalar field that go both ways over the potential barrier. Now we should examine the dynamics of the thermal fluctuations in an expanding Universe. From the previous discussion, we can write the rates for the fluctuations per unit time per unit volume as

$$\Gamma_{f\to t} \equiv \Gamma_f \simeq m_f^4(T)e^{-S_3^t(\phi,T)/T} \tag{38a}$$

for a true vacuum sub-critical bubble nucleated within a false-vacuum region, and

$$\Gamma_{t\to f} \equiv \Gamma_t \simeq m_t^4(T)e^{-S_3^f(\phi,T)/T} \tag{38b}$$

for a false-vacuum sub-critical bubble nucleated within a true vacuum region. Thus, $\Gamma_{f(t)}$ denotes the rate at which correlation volume false (true) vacuum regions disappear due to the nucleation of similar volume regions of true (false) vacuum in their interior. In the limit of the SDW, the two rates are the same. We will discuss this case in Section 4.

We are interested in following the evolution of the fluctuating regions as the Universe expands. Starting at a homogeneous (but locally strongly fluctuating) phase of ϕ at its high-temperature minimum, we ask how will the system behave as T drops below T_1 and a new minimum appears. If enough thermal energy is available, the system will try to equilibrate with fluctuations going both ways until a Boltzmann distribution is achieved. Eventually, due to the adiabatic cooling of the Universe, fluctuations are progressively more suppressed and may freeze out, fixing the relative volume in each phase. In what follows we present a method to describe the dynamics of thermal fluctuations between the two minima using the sub-critical bubbles. Our intention is not to obtain precise quantitative results (which is probably impossible without large scale computer simulations), but a coherent picture of the fluctuation dynamics in an expanding Universe.

As we mentioned before, the Universe can be pictured as a "chess-board" with regions of both vacua rapidly interconverting with the rates given above. At a given temperature T, there will be many correlation volumes inside a horizon volume. For a radiation dominated Universe with expansion rate given by Eq. 29, the number of correlation volumes within a horizon volume V_H is

$$V_H/V_c \simeq \left(\frac{m(T)M_{Pl}}{aT^2}\right)^3 \quad , \tag{39}$$

where $a \equiv (4\pi^3 g_*/45)^{1/2}$. Consider a volume $V \gg V_c$ but smaller than V_H containing many regions of false and true vacua. It is convenient to introduce N_f (N_t), the average number of correlation volume false (true) vacuum regions in the volume V. Accordingly, the total number of correlation volume regions in the volume V is $N = N_f + N_t$. The fluctuation rates are obtained by simply multiplying the rates in Eq. 38 by V_c. Even though V_c will change with temperature, the change is quite small in the range of temperatures we are interested in. We also take $V_c = \xi_f^3(T)$, neglecting the difference between the correlation lengths in each vacua. We can then write the Master equations for the rate of change of the average number of correlation regions of each vacuum in V as

$$\frac{dN_f}{dt} = V_c\left[-N_f\Gamma_f \;+\; N_t\Gamma_t\right] \tag{40a}$$

and

$$\frac{dN_t}{dt} = V_c\left[-N_t\Gamma_t \;+\; N_f\Gamma_f\right] \quad . \tag{40b}$$

Note that we did not include in the above equations another potential mechanism by which false-vacuum regions can be converted into true vacuum regions. Namely, since the false-vacuum sub-critical bubbles are energetically unfavored, they could be converted into true vacuum regions not only by thermal fluctuations but simply by shrinking away. Although in principle one might think that the time scale for shrinking is faster than for thermal fluctuations over barriers, there are quite a few factors that would slow down the shrinking. Imagine we are dealing with a theory with only a real scalar field. In this case the dynamics should be described by following the evolution of the scalar field coupled to a thermal bath. This coupling acts as an effective friction force (and also noise) which will influence the kinetics of the bubble walls.[24] In a more realistic situation, one may imagine that massless particles will be trapped in the false vacuum, as in ordinary Higgs mechanisms, providing an effective kinetic pressure that will slow down or even halt the shrinking of the walls rendering the false-vacuum regions stable. This effect was used before in connection with the primordial formation of non-topological solitons,[16] although it is applicable to more general situations as well. Accordingly, we will adopt the point of view that the dominant time scale is associated with the thermal production of sub-critical bubbles. Numerical simulations of the dynamics of such configurations are presently under way.

If we introduce the ratio $Y \equiv N_f/N_t$, such that in equilibrium, as $T \to 0$, $Y \to 0$ (*i.e.*, the system would reach its ground state if it remained in equilibrium until $T = 0$), the rate equation for Y is

$$\frac{dY}{dt} = -V_c\Gamma_f\,(1+Y)\,(Y - Y^{\rm eq}) \quad , \tag{41}$$

where $Y^{\rm eq}$ is the equilibrium ratio of regions in each vacua, given by Eqs. 33 and 38,

$$Y^{\rm eq} \simeq \exp\left\{-\left[S_3^f(\phi,T) - S_3^t(\phi,T)\right]/T\right\} \quad . \tag{42}$$

As we are interested in following the distribution of fluctuating regions as the temperature drops, it is convenient to introduce the dimensionless "time" variable $x \equiv \phi_0/T$ and reexpress Eq. 41 in terms of x. The rate equation for Y becomes, finally,

$$\frac{dY}{dx} = -\frac{x}{x_{Pl}}\frac{V_c\Gamma_f}{\phi_0}\,(1+Y)\,(Y - Y^{\rm eq}) \quad , \tag{43}$$

where $x_{Pl} \equiv 1.66 g_*^{1/2}\phi_0/M_{Pl}$. This general formula can be applied to study the dynamics of thermal fluctuations for any potential that exhibits two minima below a certain temperature. In Section 5 we will apply it to study the dynamics of thermal fluctuations for the the ADW potential of Eq. 21 and in Section 6 we will apply it to the Coleman-Weinberg potential. Before that we examine the simpler case of the SDW in the next Section.

4. Thermal Fluctuations and Freeze-Out in a Double-Well Potential

In this Section we apply the ideas developed in Section 3 to the important case of the symmetric double-well potential. We obtain the free energy of correlation volume fluctuations that interpolate between the two vacua and follow their evolution in an expanding Universe. As the potential is exactly degenerate, the two fluctuation rates will be the same and we need only to compare it to the expansion rate of the Universe, without having to integrate the rate equation.

Consider the potential of Eq. 21 with the asymmetry parameter λ_2 set to zero. The free energy $S_3(\phi, T)$ is given in Eq. 36, and the finite temperature potential $V_T(\phi)$ is defined in Eq. 25 with $m^2(\phi) = \lambda_1(3\phi^2 - \phi_0^2)/2$. At $T = 0$ the potential has two minima at $\langle\phi_c\rangle = \pm\phi_0$. For non-zero T the minima will change as shown in Fig. 7 such that, at T_c, $d^2V_T/d\phi^2|_{\phi=0} = 0$. Using a high temperature expansion for the integral in Eq. 25 valid for $T \gg m(\phi)$, we find that $T_c = 2\phi_0$. This result agrees remarkably well with a numerical evaluation of T_c.

We are interested in studying the dynamics of thermal fluctuations as T drops below T_c. For $T \lesssim T_c$ new stable points appear at $\langle\phi\rangle = \pm\langle\phi\rangle_T$ as can be seen in Fig. 7. These points are separated by a potential barrier of height $V_T(\phi = 0) - V_T(\phi = \langle\phi\rangle_T)$. However, for temperatures close to T_c there will be plenty of thermal energy to induce fluctuations over the barrier. This behavior is well-known from condensed-matter systems. In particular, the critical behavior of a scalar field with the potential above is the same as that of an Ising ferromagnet, for which it is known that domains of spins up and spins down permeate the lattice with fluctuations establishing a regime of detailed balance, depending on how the system is cooled.[25] To the best of our knowledge, there has been no attempt to study the dynamics of these fluctuations from a quantitative point of view, at least in a cosmological context.[15] Given that the correlation length of the scalar field is $\xi(T) = m^{-1}(T)$, where $m^2(T) = d^2V_T/d\phi^2|_{\langle\phi\rangle_T}$, the typical volume of the correlated domains will be $V_c(T) = \xi^3(T)$.

As discussed earlier, the thermal fluctuation rate per unit volume is taken to be

$$\Gamma(T) = m^4(T) e^{-S_3(\phi,T)/T} \quad . \tag{44}$$

We are interested in fluctuations that interpolate between the two minima. These fluctuations will be statistically relevant if their rate is faster than the expansion rate of the Universe. Eventually, as the Universe expands and cools these fluctuations will be progressively more suppressed until they freeze-out at a temperature T_{fo}. The freeze-out temperature has been qualitatively estimated before by Kibble,[13,11] neglecting the Universe's expansion. In this case, the freeze-out temperature is known as the Ginzburg temperature, T_G. Later we will compare the two approaches. In order to obtain $\Gamma(T)$ we must calculate $S_3(\phi, T)$, the free energy of the fluctuations. As we discussed before, we impose boundary conditions on $\phi(r)$ such that it describes a coreless bubble with its center in one vacuum at $r = 0$ approaching the other vacuum as $r \to \infty$, with a "size" $\ell \simeq \xi(T)$. Of course, fluctuations will go both ways with the same rate since the potential is exactly degenerate. It suffices to estimate $S_3(\phi, T)$ for one of the configurations. To obtain the free energy, we use the trial function described in Eq. 35, which can be written as

$$\phi(r) = \langle\phi\rangle_T \left[2e^{-r^2/\ell^2(T)} - 1\right] \quad . \tag{45}$$

This trial function describes a sub-critical bubble of 'positive' vacuum nucleated in the 'negative' vacuum. If $\ell(T) \simeq \xi(T)$, this bubble actually converts a whole correlation volume 'negative' region into a correlation volume 'positive' region. Due to the T-dependent integral in $V_T(\phi)$, we cannot estimate $S_3(\phi, t)$ analytically for all T. Before we present the results of the numerical integration for all T, we will obtain the free energy taking $T = 0$. As long as T_{fo} is small enough compared to T_c this approximation should give good results. We will see that this is indeed the case.

For $T = 0$, it is convenient to define the dimensionless distances $\rho = \sqrt{\lambda_1}\phi_0 r$, $\tilde{\ell} = \sqrt{\lambda_1}\phi_0 \ell$. Using Eq. 45 with $\langle\phi\rangle_T = \phi_0$, the free energy is, in terms of the dimensionless radius $\tilde{\ell}$

$$S_3(\tilde{\ell},0) \lesssim \frac{\pi}{2}\left(\frac{\pi}{\lambda_1}\right)^{1/2} \phi_0 \tilde{\ell}\left[\frac{3\sqrt{2}}{2} + \tilde{\ell}^2\left(\frac{9+18\sqrt{2}-16\sqrt{3}}{18}\right)\right] \quad . \tag{46}$$

Note that $S_3(\tilde{\ell},0)/T \gg 1$ must be satisfied at all temperatures. This condition imposes a natural lower bound on the "size" of the fluctuations. We must check if it is satisfied for the sub-critical bubbles we are interested in. The correlation length at $T=0$ is $\xi^{-1}(0) = \sqrt{\lambda_1}\phi_0$. Thus, a correlation length bubble would have $\tilde{\ell}=1$, and the free energy is

$$S_3(1,0) \simeq \frac{12.9}{\sqrt{\lambda_1}}\phi_0 \quad . \tag{47}$$

So, the condition $S_3(\tilde{\ell},0)/T \gg 1$ is easily satisfied for all $T \lesssim T_c$.

We can now calculate the freeze-out temperature for the fluctuations by comparing the thermal fluctuation rate per unit volume with the expansion rate of the Universe per unit volume. As discussed in Section 2, the freeze-out temperature is defined as the temperature at which the two rates are the same. Using Eqs. 31 and 47, we can write the freeze-out temperature $\tilde{T}_{fo} \equiv T_{fo}/\phi_0$, as

$$\tilde{T}_{fo} \simeq \frac{3.23}{\sqrt{\lambda_1}\left[41.1 + \ln\sqrt{\lambda_1} - \ln\left(\phi_0/\mathrm{GeV}\right) - 2\ln\left(\tilde{T}_{fo}\right)\right]} \quad , \tag{48}$$

where $g_* = 110$ was used and the mass scale ϕ_0 is expressed in units of a GeV. In Fig. 8 we show how T_{fo} changes with both the mass scale and the coupling λ_1. The continuous lines use the $T=0$ approximation of Eq. 48, while the dots are the results obtained numerically using the full temperature dependent potential. In the latter case $\Gamma(T)$ is evaluated at each temperature and compared to Γ_U. Notice that, for a given mass scale, as the coupling gets weaker T_{fo} increases. As T_{fo} approaches the critical temperature $T_c = 2\phi_0$, the approximation of considering only the zero temperature potential becomes unreliable. However, it is clear that for small couplings there will be a maximum mass scale above which $T_{fo} > T_c$; for small enough couplings the thermal fluctuations will never be large enough to overcome the barrier separating the two minima.

As the cosmological evolution of the double-well potential is related to the formation of domain walls, we can consider T_{fo} as the effective temperature at which walls are formed.[13,16] (It will thus influence the initial density in walls. The same arguments can be used for other topological defects as well.) It is interesting to compare our results for T_{fo} with those obtained by Kibble.[13] He estimated T_{fo} as the temperature below which fluctuations from the ordered phase characterized by $\phi = \langle\phi\rangle_T$ back to the symmetric phase $\phi = 0$ are strongly suppressed. Accordingly, he wrote

$$T_{fo} \simeq [V_T(0) - V_T(\langle\phi\rangle_T)]\, V_c(T) \quad , \tag{49}$$

where $V_c(T)$ is the correlation volume of the fluctuations. Using a high temperature expansion for $V_T(\phi)$, and taking $V_c(T) = 4\pi\xi^3(T)/3$, one obtains,

$$\tilde{T}_{fo} = \left(\frac{36}{\pi^2}\lambda_1 \; + \; \frac{1}{4}\right)^{-1/2} \quad . \tag{50}$$

For $\lambda_1 = 1.0,\ 0.1$, and 0.01, one obtains respectively, $\tilde{T}_{fo} \simeq 0.51,\ 1.28$, and 1.87. Since the expansion rate of the Universe is neglected in this approach, the results are independent of the mass scale and only depend on the coupling. However, given its simplicity, this approach is useful for order of magnitude estimates of the freeze-out temperatures, if care is taken for sufficiently high mass scales and small couplings.

5. Thermal Fluctuations and Freeze-Out in Asymmetric Double-Well Potentials

In this Section we apply our method to study the dynamics of thermal fluctuations for asymmetric double-well potentials. We start by discussing the finite temperature behavior of the effective potential and go on to calculate the free energy of the sub-critical bubbles of false and true vacua. We then solve the rate equation both analytically and numerically and obtain the freeze-out temperature and the relative fraction of the volume of the Universe in each phase for different values of the couplings in the potential.

A. Finite Temperature Behavior of Effective Potential

We will adopt the potential of Eq. 21 as our model of the ADW. In the next Section we will discuss the effects of couplings of the scalar field to other fields. This potential has no reflection symmetry even at tree-level. The cubic term, included to lift the degeneracy, breaks the reflection symmetry explicitly. So, when we analyse the thermal evolution of $V_T(\phi)$, we will not speak of a symmetry breaking mechanism but of a possible phase transition in which regions of the Universe may be filled by the two phases represented by the two minima of $V_T(\phi)$. The finite temperature corrections to a potential for a real scalar field were defined in Eq. 25 where now, $m^2(\phi) = \frac{\lambda_1}{2}(3\phi^2 - \phi_0^2) + 2\lambda_2\phi_0(\phi - \phi_0)$. For $T \gg m(\phi)$, $V_T(\phi)$ can be approximated by

$$V_T(\phi) \simeq V(\phi) - \frac{\pi^2}{30}T^4 + \frac{T^2}{24}m^2(\phi) \quad , \tag{51}$$

where $V(\phi)$, the $T = 0$ potential, is given in Eq. 21. The finite temperature behavior of $V_T(\phi)$ is shown in Fig. 9. We *define* T_c as the temperature at which a new stable minimum appears [curve (c)]. A similar potential was discussed in Ref. 11. Note that the high temperature minimum is located to the left of the barrier at $T = 0$, violating condition i) for metastability (see Introduction and Ref. 11). Indeed, from the approximate expansion of Eq. 51, we obtain the location of the minimum at high temperatures as

$$(\phi_t)_T \simeq -\frac{2\lambda_2}{3\lambda_1}\phi_0 \quad . \tag{52}$$

The asymmetry parameter λ_2 causes the minimum to be displaced from the origin, toward the $T = 0$ true vacuum. For $\lambda_2 = 0$ we recover the SDW result. Thus, as the Universe cools below T_c thermal fluctuations *may* overcome the potential barrier and populate the false vacuum at $\langle\phi_f\rangle_T$. In this case, the Universe will be filled with a two phase emulsion, with fluctuations rapidly converting one phase into another, similarly to the SDW case studied before but now with different fluctuation rates between the two vacua. Otherwise, the field will smoothly evolve to the true minimum at $T = 0$ and no false vacuum will form. By using the rate equation we will be able to predict, given the parameters of the model, what will the outcome of the transition be. Basically, if $T_{f_0} \simeq T_c$ fluctuations over the barrier will be strongly suppressed and no transition occurs. This fact can have many interesting consequences for early Universe physics. In particular, models with an asymmetry may never form domain walls, since the field will always be in one phase.

B. Free Energy of Fluctuations

Let us assume that as T drops below T_c thermal fluctuations populate both vacua, so that there will be a non-zero probability of finding the field in the new vacuum at $\langle\phi_f\rangle_T$. (The reader need not worry about a potential loophole in this assumption. If the false-vacuum regions are never populated we will simply find that there is never equilibrium, i.e., the rate for thermal hopping over the barrier will never be faster than the expansion rate of the Universe.) Around each stable minimum the fluctuations on the field are correlated up to $\xi(T)$. For non-degenerate potentials the correlation length will be different at each minimum. For simplicity, we will take both correlation lengths to be given by $m_f^{-1}(T)$, the mass at the false vacuum. Typically, $m_f(T) < m_t(T)$. Improvements on this approach can be saved for future work. As we explained before, we consider fluctuations that interpolate between the two vacua. In Eqs. 35 to 37, we obtained the general expressions for the free energies of both configurations of interest. We now simply apply these formulas to the particular case of $V_T(\phi)$ given by Eq. 21 supplemented by the temperature corrections of Eq. 25.

As we remarked before for the SDW, since we do not have an analytical expression of $V_T(\phi)$ for all T, we cannot evaluate $S_3^{f(t)}(\phi, T)$ in general. We will present the results obtained both by a full numerical evaluation of the free energies and by considering only the $T = 0$ potential. As we showed before for the SDW, this approximation should be very useful in the cases where T_{f_0} is sufficiently smaller than T_c. Consider the potential of Eq. 21. For $T = 0$ it has two minima at $\langle\phi_f\rangle = \phi_0$ and at $\langle\phi_t\rangle = -\frac{\phi_0}{2}\left[(1+2a) + (1+12a+4a^2)^{1/2}\right]$, with $a \equiv \lambda_2/\lambda_1$. Let us start with $S_3^t(\phi, 0)$. It is the free energy of a sub-critical bubble of true vacuum inside a false-vacuum region. Its center is at $\langle\phi_t\rangle$ and, as $r \to \infty$, it approaches ϕ_0. Using the trial function of Eq. 35, and the expression for the free energy of Eq. 36 with the potential of Eq. 21 we obtain,

$$S_3^t(\ell, 0) \lesssim \pi\left(\frac{\pi}{\lambda_1}\right)^{1/2}(1-X_t)^2\,\phi_0\tilde{\ell}\left[\frac{3\sqrt{2}}{8} + \tilde{\ell}^2\left(\frac{(1-X_t)^2}{64} + \frac{\sqrt{2}}{8} - \frac{\sqrt{3}(1-X_t)}{9}\left(\frac{1}{2}+\frac{a}{3}\right)\right)\right], \tag{53}$$

where as before $\tilde{\ell} \equiv \ell\sqrt{\lambda_1}\phi_0$, and we will take $\ell = m_f^{-1} = (\sqrt{\lambda_1}\phi_0)^{-1}$, $X_t \equiv \langle\phi_t\rangle/\phi_0$. The finite temperature evaluation of S_3^t takes into account the variation of the potential and its minima with T. It is interesting to note that for large enough asymmetry (in this example $a \gtrsim 0.2$) the sub-critical bubble ansatz used above approximates very well the profile (but not the value at the center) of the critical bubble obtained by false-vacuum decay techniques. (See Fig. 4.)

The free energy of a false-vacuum bubble inside a true vacuum region is obtained precisely in the same way. Of course, the end points of the configuration are reversed and we must add the false-vacuum energy $\Lambda \equiv -\frac{1}{8}(X_t^2-1)^2 - \frac{a}{3}(X_t-1)^3$ in order to obtain a finite energy for the bubble. After some algebra, we obtain for the $T = 0$ potential,

$$\begin{aligned} S_3^f(\ell, 0) \lesssim \pi\left(\frac{\pi}{\lambda_1}\right)^{1/2}(1-X_t)^2\,\phi_0\tilde{\ell}\Big[&\frac{3\sqrt{2}}{8} + \tilde{\ell}^2\Big(\frac{(1-X_t)^2}{64} + \frac{\sqrt{3}(1-X_t)}{9}\left(\frac{X_t}{2}+\frac{a}{3}\right) \\ &+ \frac{\sqrt{2}}{4}\left(\frac{1}{4}(3X_t^2-1) - a(1-X_t)\right) + a(1-X_t) - \frac{1}{2}X_t(X_t+1)\Big)\Big]. \end{aligned} \tag{54}$$

The reader can verify that the expressions for S_3^f and S_3^t reduce to the expression for the SDW, Eq. 46, in the limit $a = 0$.

Once we have the free energies for both fluctuations, we can solve the rate equation obtained in Section 3. As for the SDW case, we obtain results for both the analytical approximation of taking $T = 0$ and for the full numerical integration of the rate equation, Eq. 43.

C. Rate Equation and Freeze-Out

We start by obtaining an analytical approximation to the freeze-out temperature. This approximation is based on the assumption that the system will quickly reach equilibrium as T drops below T_c and will then track the equilibrium fraction $Y^{\rm eq}$ until T_{fo}. The approximation fails as $T_{fo} \to T_c$. In this case we claim that no equilibrium is possible, as is well-known in weakly-coupled systems.

Introduce $\bar{Y} \equiv Y - Y^{\rm eq}$, the departure from equilibrium. If the departure from equilibrium is small, that is, if the system quickly equilibrates below T_c, we can set $d\bar{Y}/dx \simeq 0$ and write Eq. 43 as

$$\frac{dY^{\rm eq}}{dx} \simeq -\frac{x}{x_{Pl}}\frac{V_c\Gamma_f}{\phi_0}\left(1+\bar{Y}+Y^{\rm eq}\right)\bar{Y} \quad . \tag{55}$$

Defining the freeze-out temperature as the temperature at which the departure from equilibrium is of order the equilibrium fraction, that is, at T_{fo}, $\delta \equiv \bar{Y}/Y^{\rm eq} \sim 1$, the rate equation becomes at T_{fo},

$$\frac{dY^{\rm eq}}{dx} \simeq -\frac{x_{fo}}{x_{Pl}}\frac{V_c\Gamma_f}{\phi_0}\delta Y^{\rm eq}\left[1+Y^{\rm eq}(1+\delta)\right] \quad , \tag{56}$$

with all quantities evaluated at T_{fo}. In order to solve for T_{fo} we need an expression for $Y^{\rm eq} = \exp\left[-\left(S_3^f - S_3^t\right)/T\right]$. As in the SDW case, we do not have a simple expression for the free energy due to the complicated temperature dependence of $V_T(\phi)$. In order to continue with an analytical approximation we again take the $T = 0$ limit for the potential. This approximation should be reasonable if T_{fo} is sufficiently smaller than T_c. Taking $\delta = 1$

we obtain, at freeze-out

$$Y^{\rm eq} \simeq \frac{1}{2}\left[\frac{\Delta S_3(0) x_{Pl}}{x_{fo} V_c \Gamma_f} - 1\right] \quad , \tag{57}$$

where $\Delta S_3(0) \equiv S_3^f(\phi, 0) - S_3^t(\phi, 0)$. Using Eqs. 53 and 54 and that at T_{fo}, $Y^{\rm eq} = \exp(-\Delta S_3(0)/T_{fo})$ we can obtain T_{fo} for each value of the asymmetry parameter a. The results are given in Fig. 10 for $\lambda_1 = 1.0$ and several values of a ($\equiv \lambda_2$ in this case). Notice that as the asymmetry increases freeze-out occurs earlier since the corresponding actions increase as well. We show the critical temperature for each value of a studied. T_c decreases as a increases making it progressively more difficult for the system to equilibrate. This trend is confirmed by the numerical evolution of the rate equation. The results for the numerical calculation are shown in Fig. 10 for $a = 0.15$ and different mass scales. The agreement with the analytical calculation is very good. For smaller values of the quartic coupling, the system could still be able to reach equilibrium, although the freeze-out temperatures will approach the critical temperature considerably faster (see results for the SDW) decreasing the fraction of volume in the false vacuum.

Finally, in Fig. 11 we present the results for the numerical evolution of the rate equation for $a = 0.15$. Note that the relative fraction Y tracks the equilibrium fraction until the freeze-out temperature, below which it remains practically constant. By varying the mass scale ϕ_0 we can see that the fraction of volume in the false vacuum at freeze-out decreases with ϕ_0. The lower the mass scale the slower the expansion rate of the Universe, giving more 'time' for the false vacuum regions to leak back to the true vacuum.

A question of interest is then what happens with the false-vacuum regions after freeze-out. Depending on the relative probability of being in the false vacuum they may or may not percolate.[16,26] The percolation probability is roughly $p_c \sim 0.3$. So, if $Y(T_{fo}) \gtrsim 0.43$ the Universe will be permeated by the two phases with a domain wall separating them which will eventually start moving toward the false vacuum, wiping it away. If $Y(T_{fo}) \lesssim 0.43$ the false vacuum does not percolate and the Universe will be in the true vacuum with isolated islands of false vacuum. Of course, in the simple model studied here there is nothing to stop the collapse of these regions and they will shrink away. However, in more realistic models the scalar field interacts with other fields and the whole picture can be quite different. An example already mentioned is the formation of non-topological solitons, where the collapse is halted by massless particles. To the determine the abundance of solitons, knowledge of $Y(T_{fo})$ is crucial. From Figs. 10 and 11 it can be seen that percolation is strongly suppressed both by the asymmetry in the potential and by the mass scale at which the transition takes place.

In very general terms, understanding the dynamics of the transition depends on our ability to track the fraction of volume of the Universe in each phase. In this Section we have shown how this is done for a self-interacting real scalar field. In order to study the effects of the interactions with other fields in situations in which there is symmetry breaking, in the next Section we study the Coleman-Weinberg potential. We show that the dynamics of the transitions can be quite different from the cases analysed so far.

6. Escaping Metastability

In this Section we will apply the method developed in Section 3 to the Coleman-Weinberg potential.[8] We will show that below a certain mass scale, it is possible for the transition to be completed by sub-critical bubbles without any supercooling. For the sake of definiteness (and not of conviction) we will use the minimal SU(5) model and will only include interactions with the gauge fields. This example exhibits symmetry breaking through radiative corrections and was very popular in the early eighties in connection with the new inflationary models.[7] (Although here we assume thermal equilibrium.) It should serve as a simple enough prototype in order to show how a metastable state may never be formed in the evolution of a phase transition even though one would naively assume so.

To one loop order, the finite temperature effective scalar potential is

$$V_T(\phi) = B\phi^4\left[\ln\left(\phi^2/\phi_0^2\right) - \frac{1}{2}\right] + \frac{18}{\pi^2}T^4\int_0^\infty dx\, x^2 \ln\left\{1 - \exp\left[-\left(x^2 + 25g^2\phi^2/8T^2\right)^{1/2}\right]\right\} \tag{58}$$

where $B \equiv \frac{5625}{1024\pi^2}g^4$, g is the gauge coupling constant, and the adjoint Higgs field, Φ, is written as $\phi(1,1,1,-\frac{3}{2},-\frac{3}{2})$. ϕ_0 fixes the mass scale of symmetry breaking. In Fig. 12 we show the potential above for different temperatures. Note that at high temperatures the minimum is at $\phi = 0$, and as the temperature drops to $T = T_1$ a new minimum is formed which becomes the true minimum, $(\phi_t)_T$, for $T < T_c$. At $T = 0$ the barrier disappears and $(\phi_t)_T = \phi_0$. The question we wish to examine is the formation (or not) of the metastable state at $\phi = 0$ as the temperature drops below T_c. Can sub-critical bubbles populate the new minimum as $T \leq T_1$ such that as T_c is approached there will be an appreciable fraction of the volume of the Universe already at the true minimum? Will it be above percolation threshold allowing the transition to be completed?

Introducing the dimensionless variables $\tilde{T} = T/g\phi_0$, $X = \phi/\phi_0$, and $\rho = g^2\phi_0 r$, the free energy (Eq. 36) can be written as

$$S_3(\phi, T) = \frac{4\pi\phi_0}{g^2} \int \rho^2 d\rho \left[\left(\frac{dX}{d\rho} \right)^2 + V_{\tilde{T}}(X) \right] \quad . \tag{59}$$

The free-energy scales with g^{-2}, a well-known result. For weak couplings transitions over barriers are suppressed and equilibrium between the two phases more difficult. Using the dimensionless temperature $\tilde{T}$ it is possible to show numerically (a high temperature expansion, although widely used, is very unreliable here) that $T_1 \simeq 0.66g\phi_0$, while the critical temperature is $T_c \simeq 0.55g\phi_0$.

Unfortunately in the present case it is not possible to calculate the free energies of the sub-critical bubbles analytically due to the complexity of the potential. Also, we cannot use the $T = 0$ approximation to obtain the freeze out temperature since we are interested in temperatures around T_c. The work has to be done numerically. Accordingly, we evaluated with the potential of Eq. 58, for $T \leq T_1$, the free energies for the sub-critical bubbles going both ways over the barrier using Eqs. 35 and 37. With the free energies $S_3^{f(t)}$, we evolved the rate equation, Eq. 43, and obtained the fraction of volume of the Universe in each phase ($\phi = 0$ or $\phi = \langle\phi_t\rangle_T$) at freeze-out, T_{fo}.

The results are shown in Fig. 13 for different mass scales and for $g = 1$. (We will not worry here about the running of g since our intention is to illustrate our method and not to examine this model in detail.) For high mass scales, $\phi_0 \lesssim 10^{16}$ GeV we can see that no equilibrium is ever achieved and our numerical integration is unreliable. The fraction Y should track, at least initially, the equilibrium fraction Y_{eq}. For those mass scales, metastability becomes a problem of initial conditions. As the mass scale drops, equilibrium can be achieved and we notice that for $\phi_0 = 10^{15}$ GeV freeze-out occurs for $T < T_c$ with a relative fraction $Y(T_{fo}) \simeq 0.4$, right around the percolation fraction, as we explained in the last Section. Thus, the Universe will consist of very large true and false-vacuum regions, with the false-vacuum regions quickly being wiped out due to their higher free energy. (The actual kinetics may be quite complicated due to the mass gap of particles between the symmetric and broken-symmetric phases.) The phase transition can be completed without a metastable state being formed at $\phi = 0$. As the mass scale drops even further, the true vacuum regions very quickly dominate the volume of the Universe. For $\phi_0 = 10^{14}$ GeV at T_{fo} only about 1% of the volume of the Universe remains in the false vacuum. From our results and within the approximations of our method, it is quite clear that at least in this and similar models, the formation of metastable states must rely on weak couplings and/or very early time scales for the transition.

7. Conclusions and Outlook

In this paper we presented a method that can be used to follow the evolution of phase transitions in an expanding Universe. We considered systems with interactions described by a potential which exhibits two stable minima below temperatures roughly of the order of the mass scale ϕ_0. At temperatures well above the critical temperature of the transition the system is in thermal equilibrium with a thermal bath with the equilibrium state determined by the extremum of the homogeneous part of the free energy (i.e. the temperature corrected potential). As the Universe expands and the temperature drops, a new minimum appears in the free energy and the system goes out of equilibrium (Fig. 1). Driven by the available thermal energy of the bath, the system will try to reach equilibrium by means of transitions between its two possible states. Our method consists in estimating the free energy of these transitions, which we dubbed sub-critical bubbles, and in solving a rate equation which determines the dynamics of this equilibration mechanism as the Universe expands and cools.

In order to evaluate the free energy of the field configurations that interpolate between the two minima we made three assumptions. The statistically relevant fluctuations exhibit $O(3)$ symmetry, are roughly of a correlation volume, and interpolate between the two minima of the homogeneous free energy. These fluctuations are thus different from the familiar critical bubbles of false-vacuum decay mechanisms. We are not interested in studying the decay of a homogeneous false vacuum, a well-known topic, but instead in studying, among other things, the conditions that lead to the formation of such states.

We applied our method to three situations of interest. We started with the symmetric double-well (Fig. 7) for which the free energy of the transitions going both ways over the barrier is the same. In this case it is sufficient to compare the thermal fluctuation rate to the expansion rate of the Universe. Next we applied the method to the asymmetric double-well, for which the free energy of the configurations interpolating between the two minima is, of course, different (Fig. 9). In both cases we performed both analytical and numerical calculations of the freeze-out temperature, below which the fluctuations are strongly suppressed. We studied the dependence of the freeze-out temperature both on mass scales and on couplings of the models. For the asymmetric double-well, we evaluated the fraction of the volume of the Universe in each phase at freeze-out. We show that for high energy scales and/or for weak couplings equilibration never obtains.

Finally, we applied our method to the Coleman-Weinberg potential in the context of the standard SU(5) model. This is a very interesting case since it is believed to exhibit a metastable state at the symmetric phase ($\phi = 0$, see Fig. 12) as the temperature drops below the critical temperature. By using our method, it is possible to study the dynamics of the thermal fluctuations and to quantitatively estimate , within our approximations, what is the fraction of the volume of the Universe in each phase at freeze-out. Our results indicate that for mass scales below or around 10^{15} GeV a metastable state never forms and the transition is completed by the percolating sub-critical bubbles at freeze-out.

Understanding the dynamics of primordial phase transitions is clearly of topical importance if we want to develop a coherent picture of the physical processes that took place in the early Universe. An example of great interest is the possibility of producing the baryon asymmetry in the electroweak phase transition.[27] One of the conditions for a successful scenario is precisely that the system goes out of equilibrium around the electroweak scale. Two recent scenarios naturally invoked a first-order transition to satisfy this condition.[28] Our method should be a useful tool in order to test if these or any other models proposed truly develop a metastable state at the desired energy scale. In the same way that the finite temperature sphaleron scenario gained support from numerical computations on the lattice,[29] our method should also be tested numerically. Simulations of the dynamics of sub-critical bubbles are currently under way.[30]

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Table Caption

The bounce action $\bar{S}_4 \equiv \frac{\lambda_1}{2\pi^2} S_4(\bar{\phi})$ and radius (defined in Eq. 23) for different values of the asymmetry parameter a.

Figure Captions

Figure 1: Temperature evolution of effective potential for a model that may exhibit a metastable state. In curve (a) $T \gg T_c$; In curve (b) a new locally stable point appears at $T = T_1$; In curve (c) $T = T_c$ and the two minima are degenerate; In curve (d) $T = 0$.

Figure 2: Asymmetric double-well potential

Figure 3: "Upside-down potential" that appears in Euclidean equation of motion. ϕ_i is the value of the scalar field in the center of the bubble.

Figure 4: Bounce solutions for potential of Eq. 21 for different values of the asymmetry parameter a. Note that for large enough asymmetry the bubble picture becomes invalid.

Figure 5: Qualitative behavior of free energy of critical bubbles with temperature. For $T \lesssim T_0$ quantum nucleation dominates over thermal nucleation of critical bubbles.

Figure 6: Nucleation of correlation volume sub-critical bubbles inside correlation volume regions of a given phase is pictured as the dominant mechanism by which the two phases interconvert. Smaller size bubbles are not effective in altering the average distribution of cells in each phase.

Figure 7: Temperature behavior of double-well potential. The critical temperature is defined as the temperature at which the origin becomes an inflexion point. For the present model, $T_c \simeq 2\phi_0$.

Figure 8: Freeze-out temperatures for different mass scales for the double-well potential. Note that for small enough couplings and/or large enough masses T_{fo} approaches T_c. In this case, no equilibrium is possible and the vacuum structure is fixed at T_c. Numerical results are indicated by dots.

Figure 9: Temperature behavior for the asymmetric double-well potential of Eq. 21. The parameters are $\lambda_1 = 1.0$ and $\lambda_2 = 0.1$. Note how the high-temperature minimum [curve (a)] is biased toward the zero temperature true vacuum [curve (d)]. At $T = T_c$ a new locally stable minimum appears. [Curve (c).]

Figure 10: The freeze-out temperature as a function of the mass scale and different asymmetry parameter a. Numerical results are indicated by dots.

Figure 11: Results from the integration of the rate equation are shown for several mass scales. The volume fraction tracks the equilibrium fraction until freeze-out occurs. The asymmetry parameter is $a = 0.15$. For small mass scales, the fraction of false-vacuum regions in the Universe is negligible.

Figure 12: Temperature dependence of the Coleman-Weinberg potential for the standard $SU(5)$ model. For $T < T_c$ (dotted line), the symmetric phase can become metastable.

Figure 13: The volume fraction as function of inverse temperature for the Coleman-Weinberg potential. For mass scales below 10^{15} GeV, most of the volume of the Universe is in the true vacuum and the transition is completed without the formation of a metastable state at the symmetric phase.

TABLE 1

a	$\bar{S}_4$	R
0.07	131.4	9.1
0.15	85.5	6.8
0.3	22.8	4.1

FIGURE 2

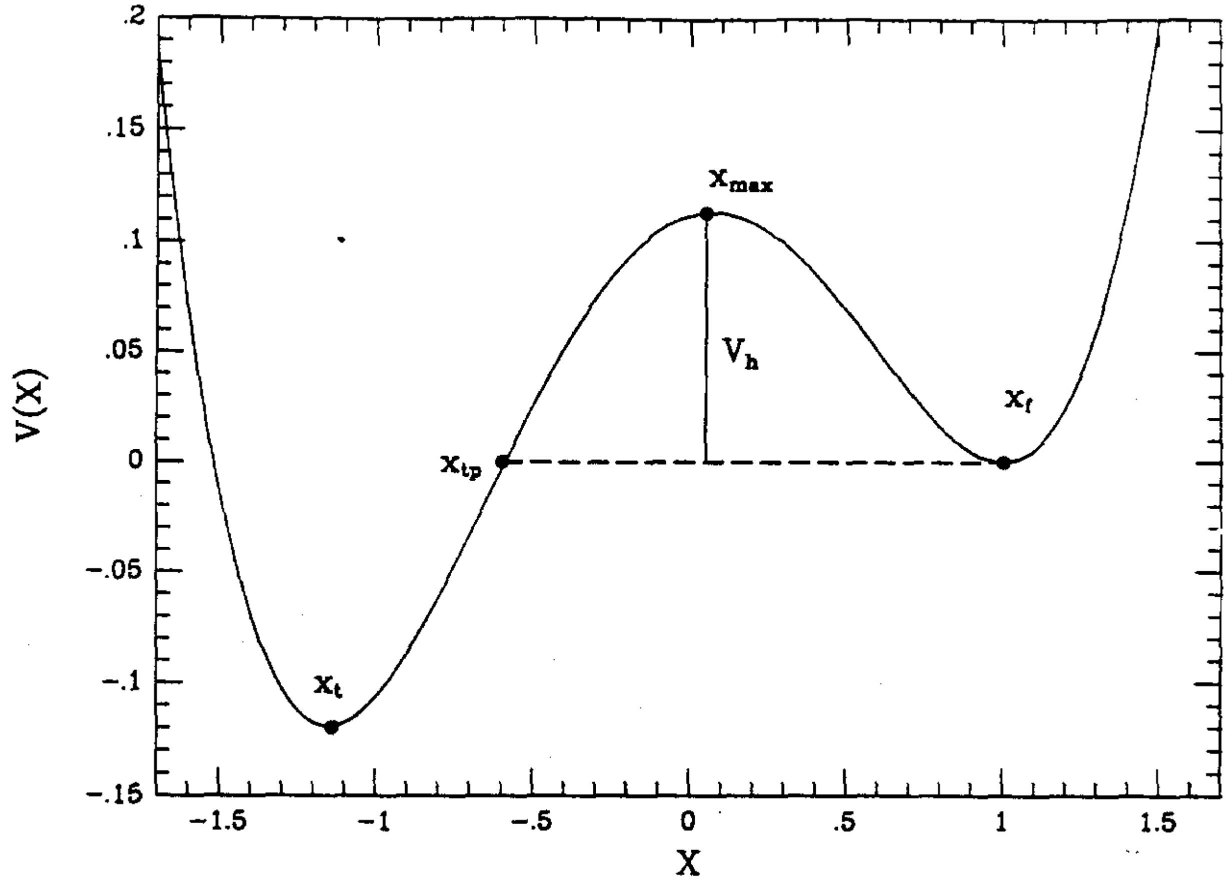

FIGURE 1

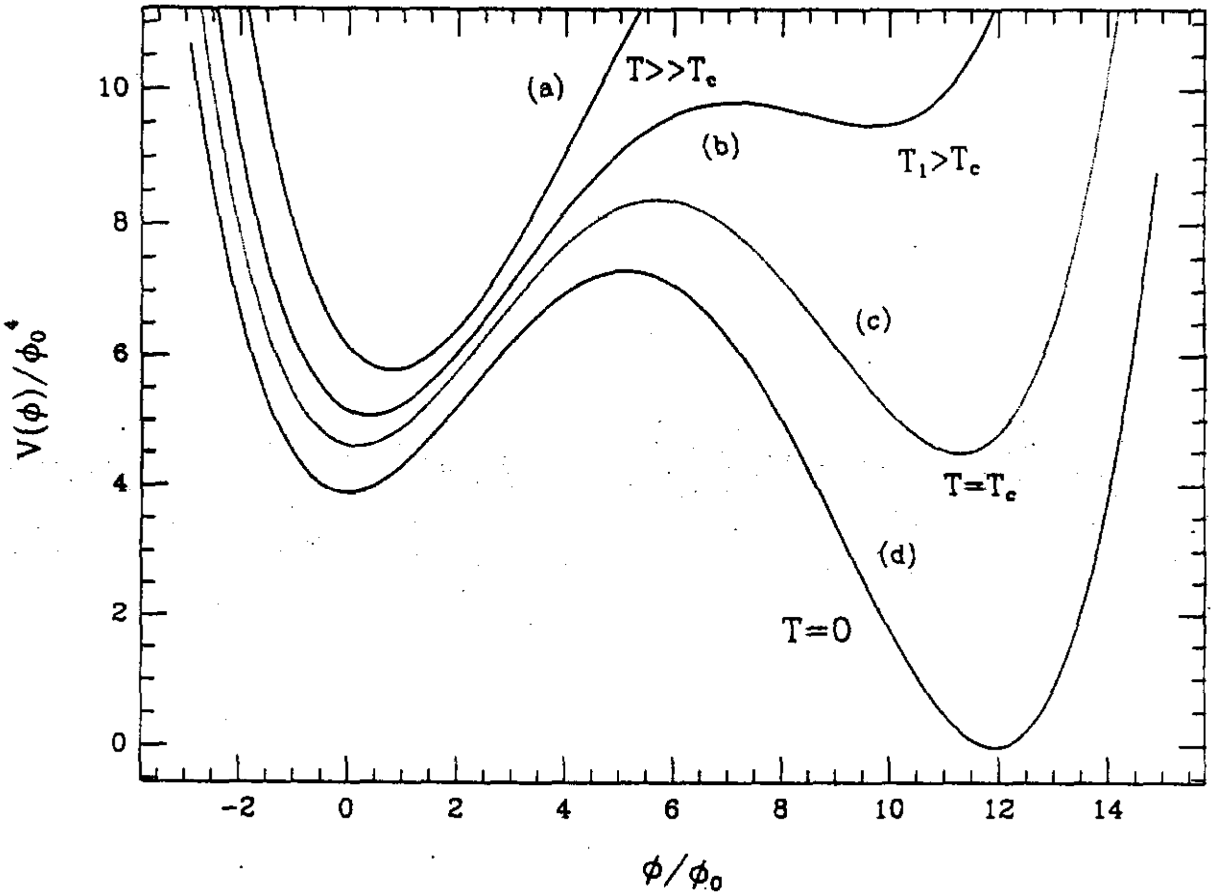

FIGURE 4

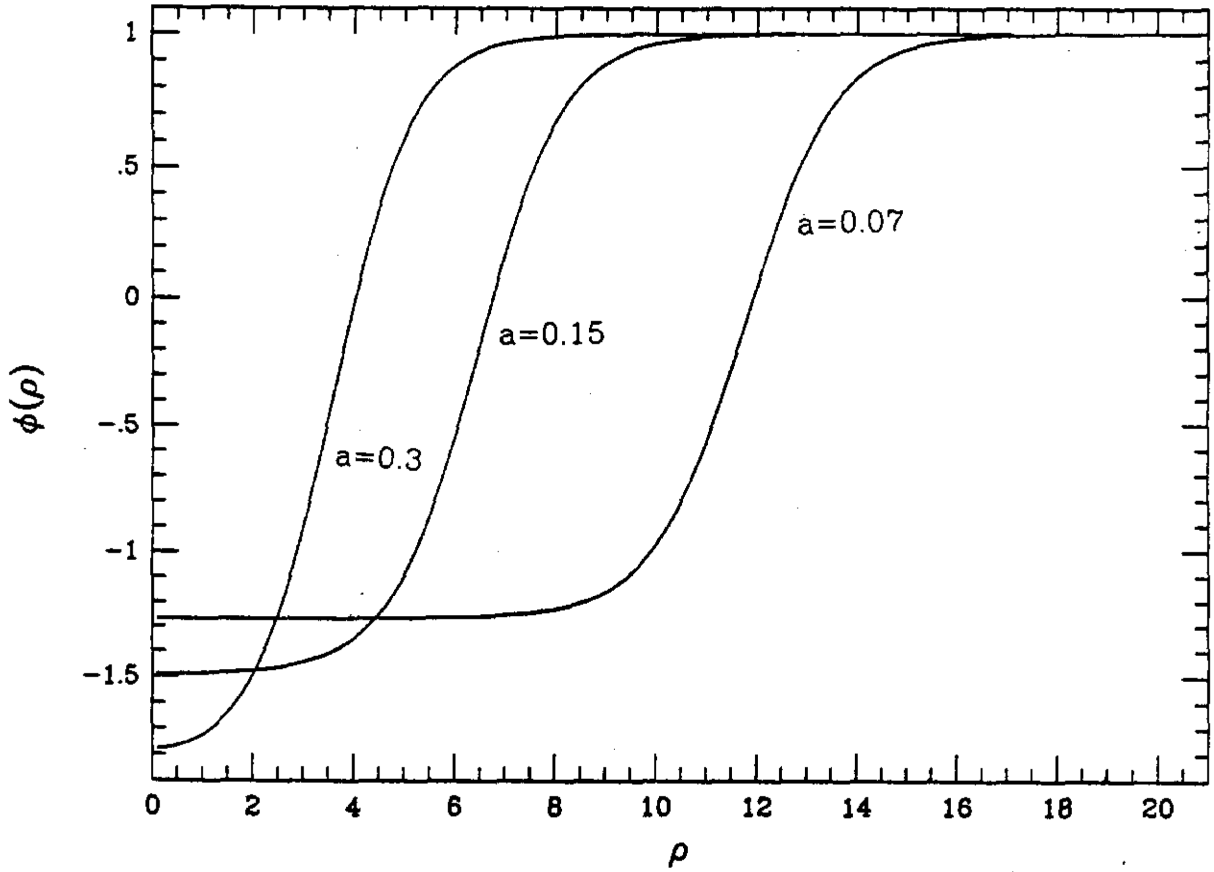

FIGURE 3

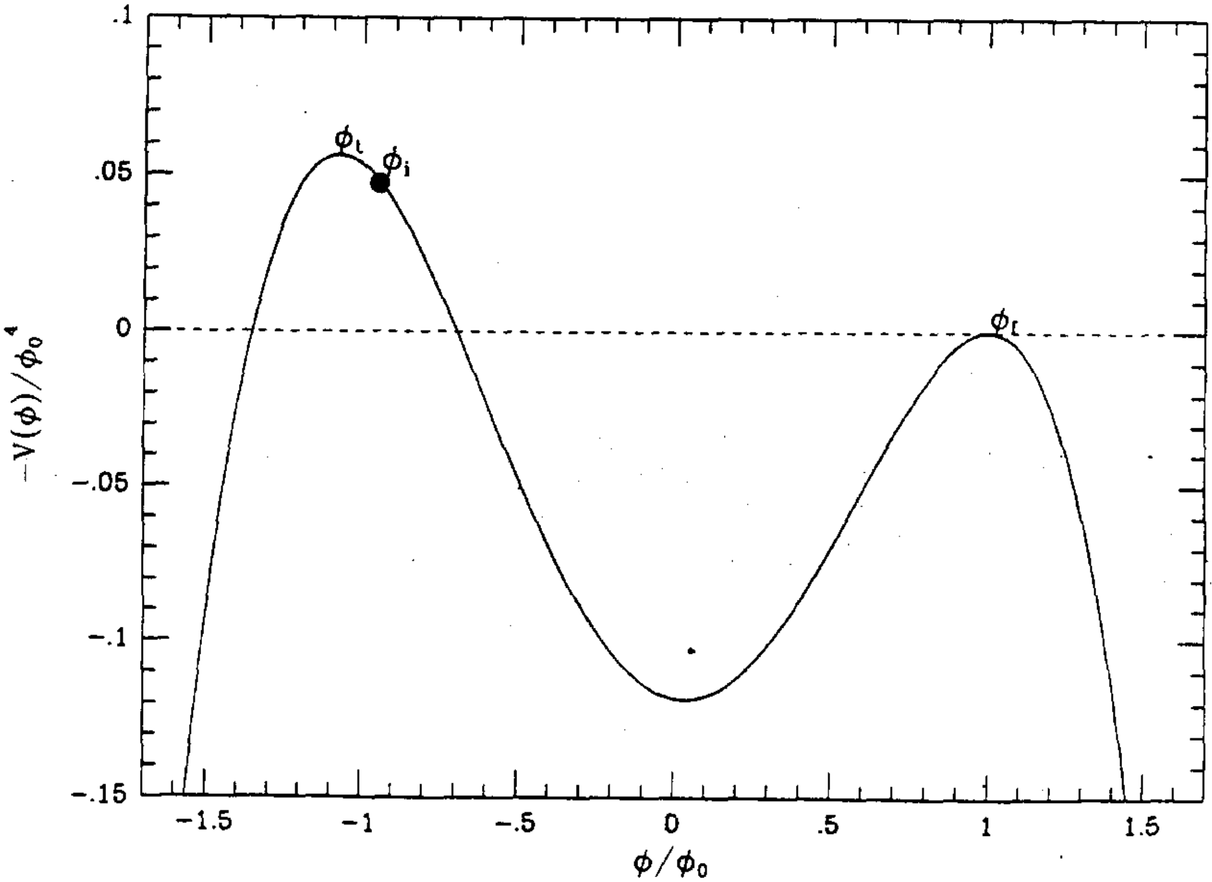

FIGURE 7

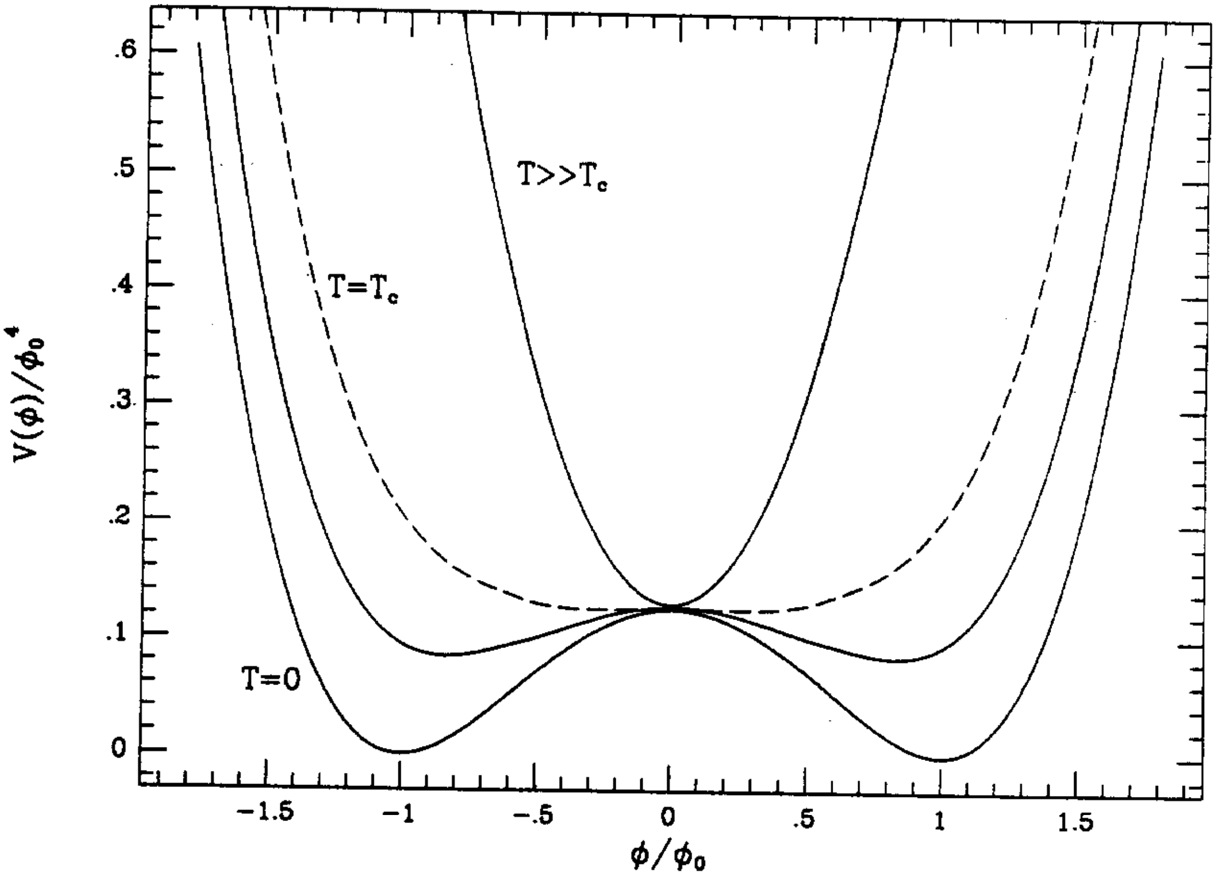

FIGURE 6

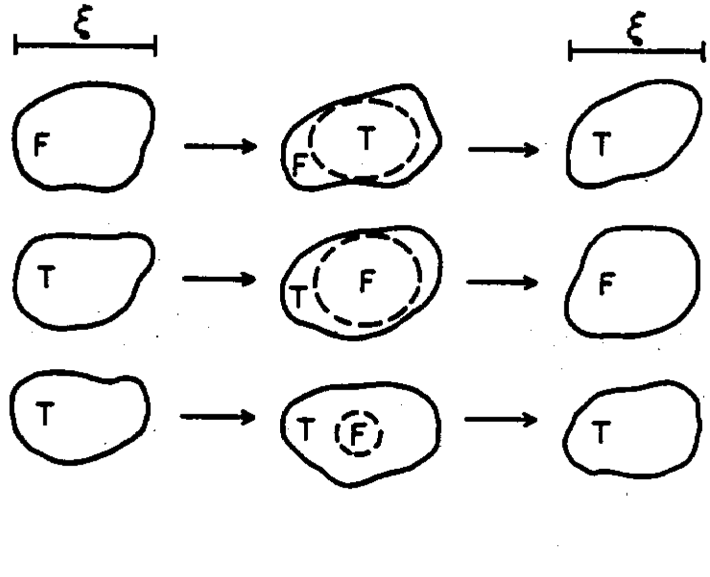

FIGURE 5

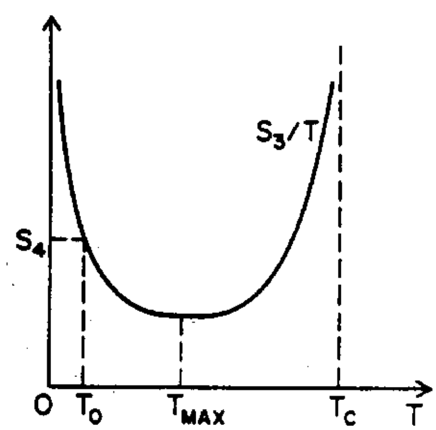

FIGURE 9

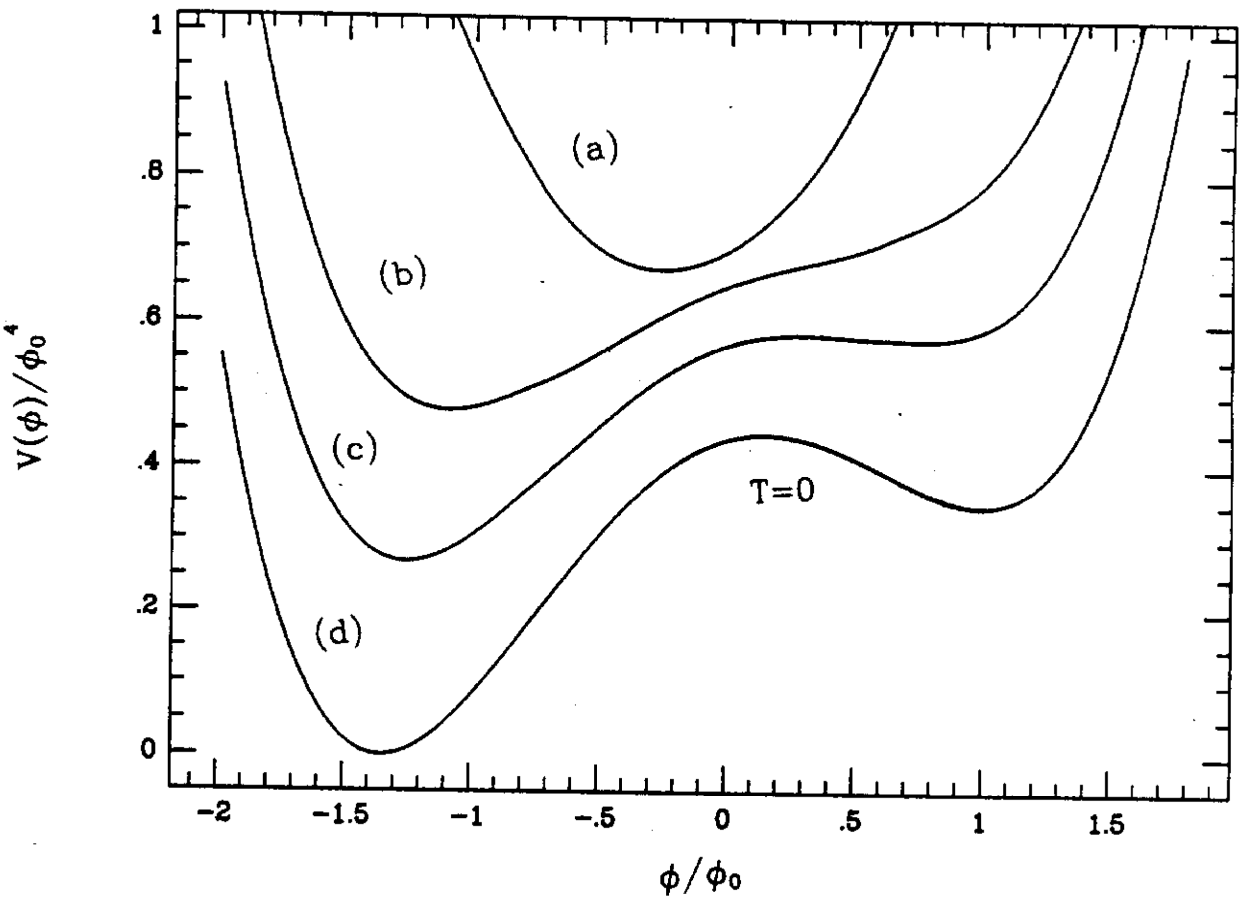

FIGURE 8

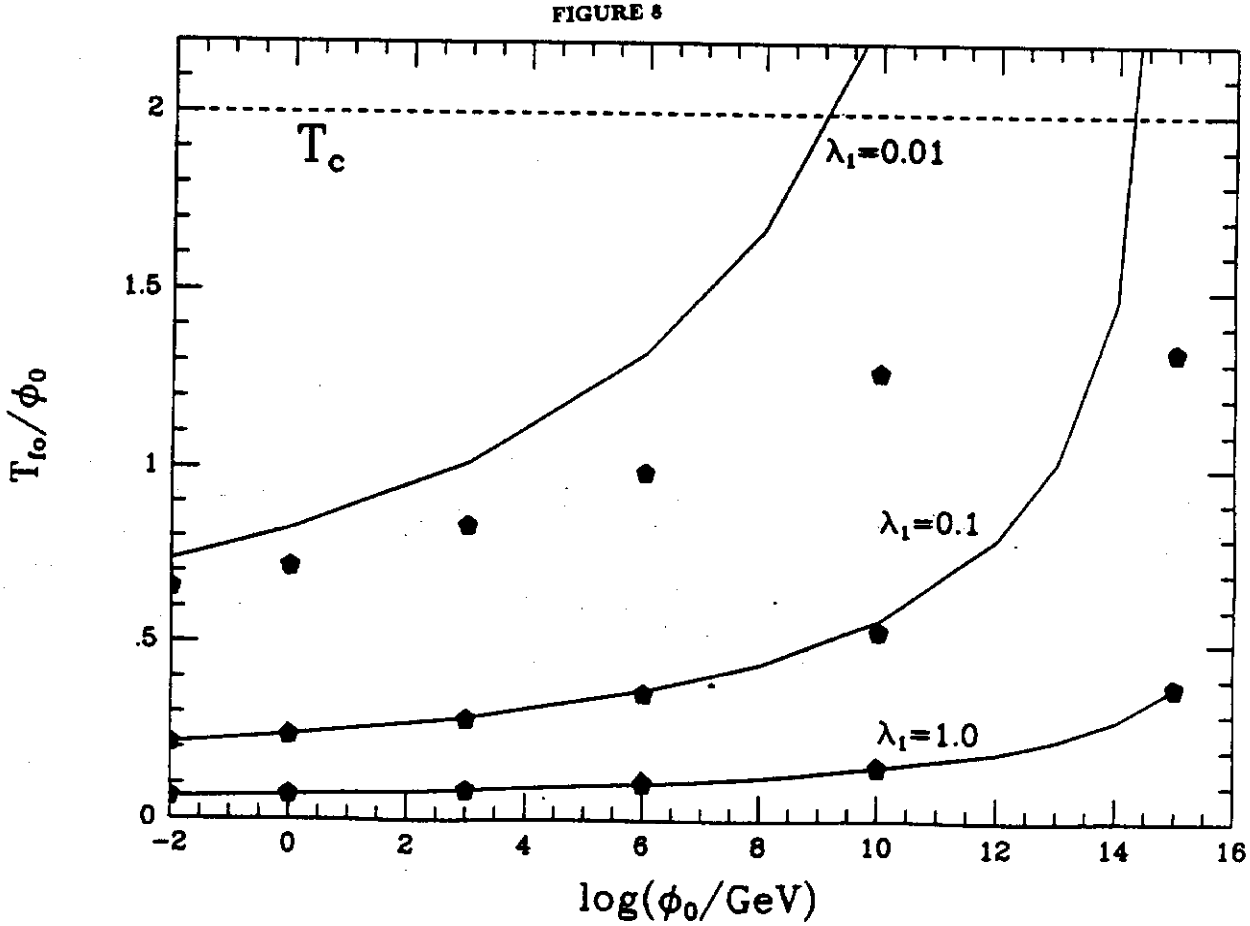

FIGURE 11

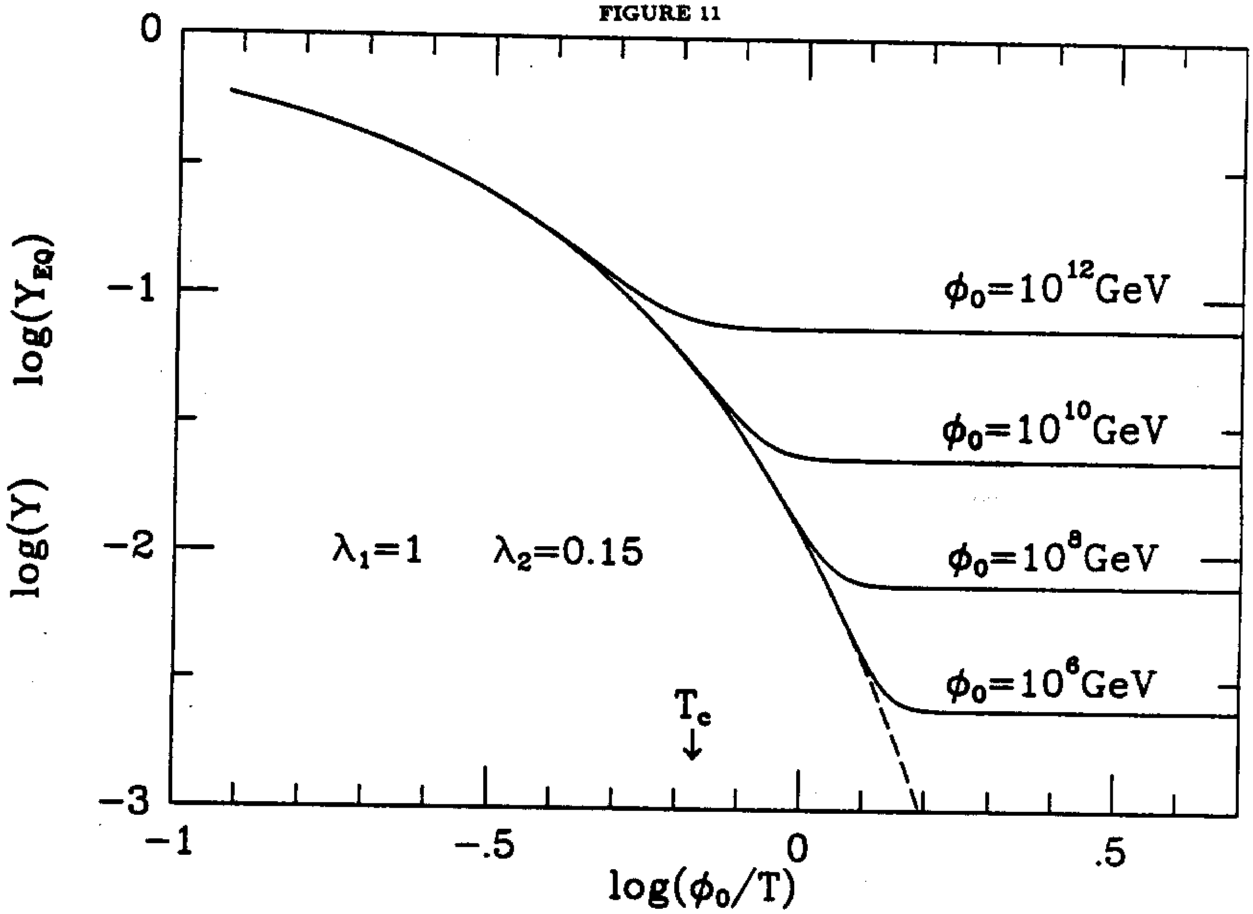

FIGURE 10

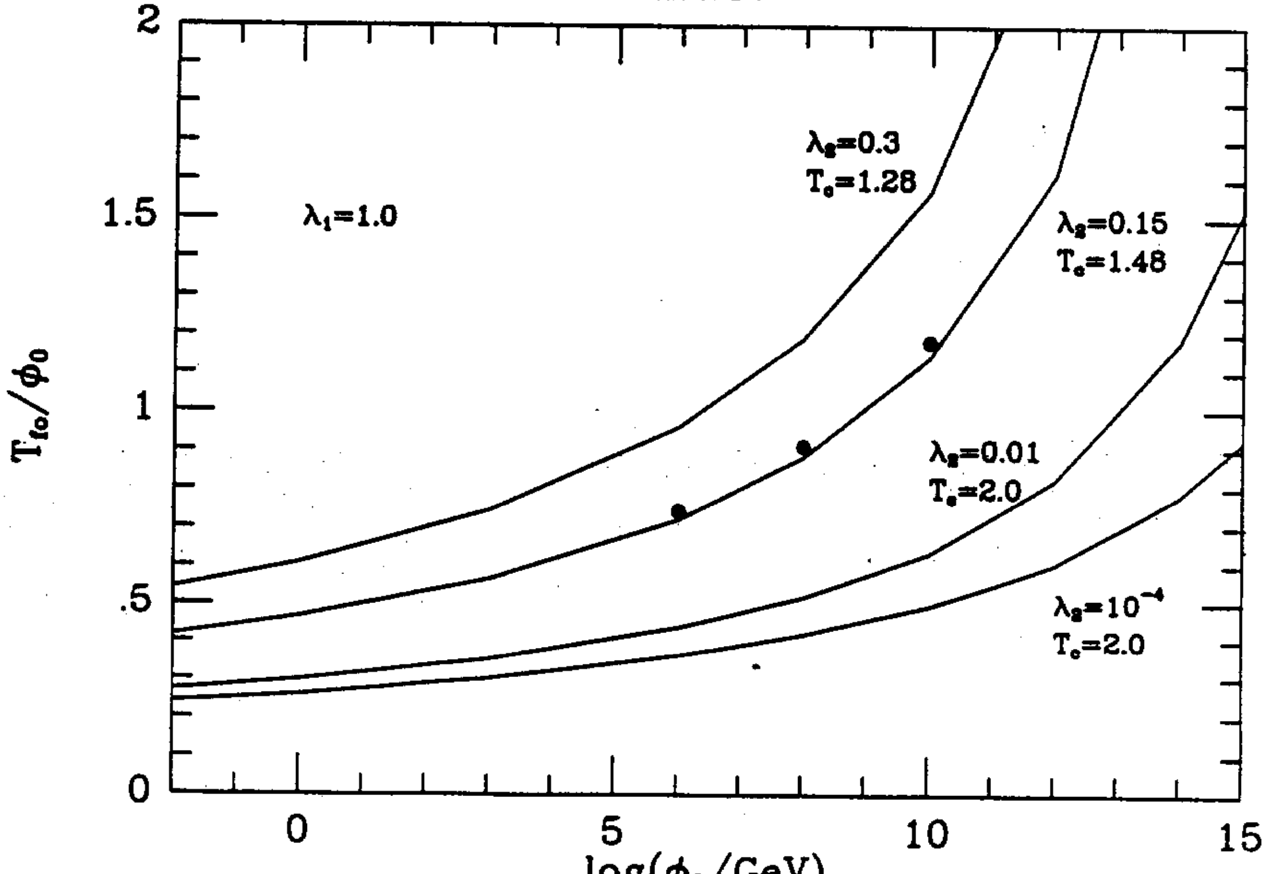

FIGURE 12

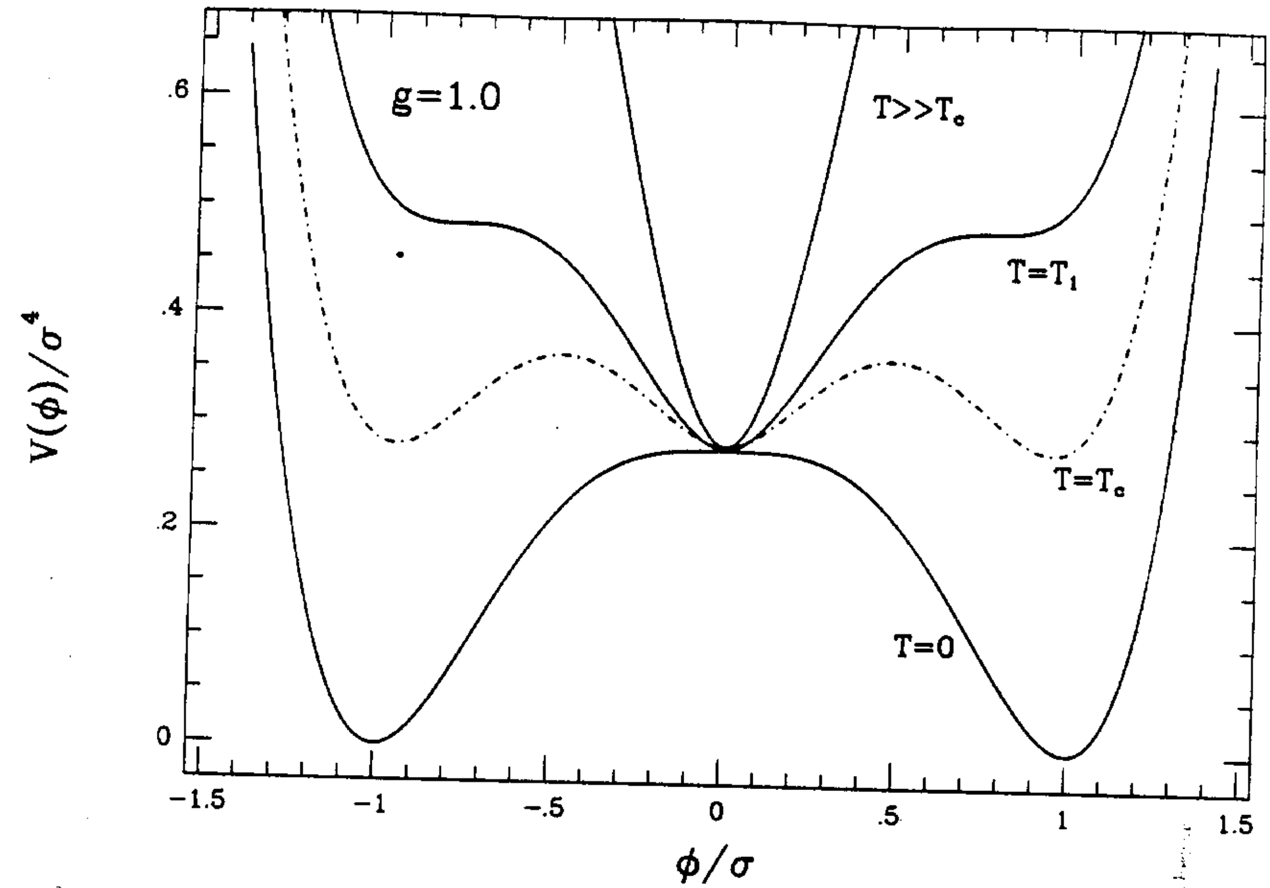

FIGURE 13