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PRACTICAL APPLICATIONS OF SUPERSTRINGS: NEW ONE-LOOP RULES FOR GAUGE THEORIES

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ABSTRACT

We present new rules for computing one-loop gluon amplitudes in gauge theories. As an explicit example which illustrates their efficiency in comparison with conventional techniques, we present the details of the computation of one helicity amplitude for the one-loop $gg \to gg$ process.

1. Introduction

Hadron collider experiments over the coming decades, whether at the Tevatron, the SSC, or the LHC, promise the best means of understanding the main particle physics puzzles of our times, the forces underlying electroweak symmetry-breaking and generation of light fermion hierarchies and mixing angles. But in order to uncover new physics at these machines, one must have a thorough understanding of known physics, and this in turn will require computations in perturbative QCD to much higher order than have yet been carried out.

The traditional method of Feynman diagrams becomes increasingly difficult as the number of vertices and legs increases. Even for the one-loop four-point amplitude there are six Feynman diagram topologies for a total of twenty-seven pure gluon diagrams and twelve ghost diagrams. Since each non-abelian vertex is composed of six terms there would be on the order of 10^4 terms at the starting point of conventional Feynman diagram computation, making this computation difficult. Since gluons dominate at supercollider energies, the most important contributions are thus also the most difficult to compute.

In a recent paper we have outlined the use of a technology derived from four-dimensional heterotic strings[1] as an efficient technical tool for evaluating the pure glue contributions to the one-loop QCD amplitudes. Using this string-based technology, we have provided the first computation of the one-loop corrections to the $gg \to gg$ doubly-polarized cross-sections.[2] The sum over all helicities also provides the first complete check of the pure glue contributions to the unpolarized cross-section, first calculated

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by Ellis and Sexton.[3]

Because there is only a single string diagram at each loop order, the amplitude is organized into a compact expression which would be quite non-trivial to reproduce in a Feynman diagram language. One practical advantage of the string-based formalism is that the loop momentum is already integrated out, bypassing all algebra associated with the non-abelian gauge vertex as well as allowing for immediate simplifications when using the spinor helicity basis. The string also organizes the amplitude into smaller gauge invariant partial amplitudes, eliminating most of the large cancellations typical of Feynman diagram computations.

In these proceedings, we discuss a set of simplified rules for the one-loop n-gluon amplitude. The use of these rules requires no knowledge of string theory although their derivation[1,4,5] does rely on the technology of four-dimensional heterotic strings.[6,7,8] (The string models of interest here are not space-time supersymmetric, and their field theory limits are directly applicable to calculations in QCD.)

Since the amplitudes are infrared divergent we have developed string versions of ordinary dimensional regularizations, based on the work of Brink, Green and Schwarz.[9] Our discussion here will include dimensional regularization.

As a particular application we present the details of the computation of the $\mathcal{A}(1^-,2^+,3^+,4^+)$ helicitly amplitude. This amplitude is particularly simple to evaluate because no infrared or ultraviolet divergences are encountered. Our string-based method reduces the thirty-nine Feynman diagrams into two non-vanishing ϕ^3-type diagrams.

2. Spinor Helicity Basis

With four gluons there are 43 formally independent multilinear combinations of the polarization vectors with momenta (after making use of momentum conservation). They are not all independent in an amplitude, however, because they are related by the constraint of gauge invariance. The spinor helicity method[10] is an efficient technique for extracting the essential gauge-invariant content of an amplitude expressed in terms of polarization vectors and momenta. We will use the spinor helicity basis of Xu, Zhang and Chang.

In this, method the polarization vectors are written in terms of spinor quantities,

$$\varepsilon^+_\mu(p,q)=\frac{\langle q^-|\gamma_\mu|p^-\rangle}{\sqrt{2}\langle q^-|p^+\rangle}\,,\qquad\qquad \varepsilon^-_\mu(p,q)=\frac{\langle q^+|\gamma_\mu|p^+\rangle}{\sqrt{2}\langle p^+|q^-\rangle}\tag{1}$$

where p is the (on-shell) momentum of the gluon, q an arbitrary reference momentum such that $q^2=0$ and $p\cdot q\neq 0$ and the $|p^\pm\rangle$ are Weyl spinors. As usual, we will abbreviate $\langle k_1^-|k_2^+\rangle=\langle 1\,2\rangle$ and $\langle k_1^+|k_2^-\rangle=[1\,2]$. The spinor product is not merely an abstract object, but can be calculated *numerically* using the following pair of formulæ,

$$
\begin{aligned}
\langle k_1\, k_2\rangle &= \sqrt{(k_1^t - k_1^z)\,(k_2^t + k_2^z)}\exp(i\,\mathrm{atan}(k_1^y/k_1^x)) - (1 \leftrightarrow 2) \\
&= \sqrt{\frac{k_2^t + k_2^z}{k_1^t + k_1^z}}\,(k_1^x + ik_1^y) - (1 \leftrightarrow 2) \\
[k_1\, k_2] &= \mathrm{sign}(k_1^t k_2^t)\,(\langle k_2\, k_1\rangle)^* .
\end{aligned}
\tag{2}
$$

Indeed, it is often advantageous in QCD calculations to calculate the amplitude numerically using complex computer arithmetic, and only then to square the answer, rather than squaring analytically and evaluating that formula numerically.

The advantages of the spinor helicity method become manifest with a judicious choice of reference momenta.[11] In particular for the amplitude $\mathcal{A}(1^-, 2^+, 3^+, 4^+)$, we choose reference momenta (k_4, k_1, k_1, k_1) for the legs (1,2,3,4) respectively, leading to the simplifications

$$
\begin{aligned}
\varepsilon_i \cdot \varepsilon_j = 0, \qquad & k_4 \cdot \varepsilon_1 = k_1 \cdot \varepsilon_2 = k_1 \cdot \varepsilon_3 = k_1 \cdot \varepsilon_4 = 0 \\
k_3 \cdot \varepsilon_1 = -k_2 \cdot \varepsilon_1\,, \qquad & k_4 \cdot \varepsilon_2 = -k_3 \cdot \varepsilon_2\,, \\
k_4 \cdot \varepsilon_3 = -k_2 \cdot \varepsilon_3 \qquad & k_3 \cdot \varepsilon_4 = -k_2 \cdot \varepsilon_4\,.
\end{aligned}
\tag{3}
$$

For practical calculations, it is simplest to use either a four-dimensional helicity scheme or the 't Hooft-Veltman dimensional regularization scheme in which all *observed* particles remain in four-dimensions, but the conventional scheme, where all states are continued to $4 - \epsilon$ dimensions, can also be used with the spinor helicity basis by introducing ϵ-helicities.[12]

3. General Structure of the Amplitude

In analogy to the color decomposition of the amplitude at tree level,[13] there is also a color decomposition of loop amplitudes.[4] In particular, the $SU(N)$ four-point gluon amplitude can be written in the following form,

$$
\begin{aligned}
\mathcal{A}_4^{\text{one-loop}} =& g^4 \sum_{\sigma \in S_4/Z_4} N_c\,\mathrm{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma}(4)}) A_{4;1}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \\
&+ \sum_{\sigma \in S_4/Z_2^3} \mathrm{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}})\,\mathrm{Tr}(T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) A_{4;3}(\sigma(1), \sigma(2); \sigma(3), \sigma(4))\,.
\end{aligned}
\tag{4}
$$

The notation 'S_4/Z_4' denotes the set of all permutations S_4 of four objects, omitting the purely cyclic transformations $(1 \to 2, 2 \to 3, 3 \to 4, 4 \to 1)$, etc. The notation '$S_4/Z_2^3$' refers again to the set of permutations of four objects but with permutations considered equivalent (and only one representative picked) if they exchange labels within a single trace or exchange the two traces: $S_4/Z_2^3 = \{(1\,2\,3\,4), (1\,3\,2\,4), (1\,4\,2\,3)\}$.

In general, the one-loop amplitude consists of a sum over all inequivalent one and two trace terms, with the single-trace terms carrying an explicit factor of the number of colors. Three and higher trace terms do not appear in the one-loop amplitude, and the *partial amplitudes* $A_{n;j}$ are independent of the specific $SU(N)$ gauge group.

These partial amplitudes satisfy the $U(1)$ decoupling equation[4]

$$A_{4;3}(1,2,3,4) = \sum_{\sigma \in S_4/Z_4} A_{4;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \ . \tag{5}$$

The idea is to evaluate each independent partial amplitude for a fixed order of its arguments, but for all desired helicity configurations. Since the partial amplitudes are not themselves Bose symmetric (the Bose symmetry appears only in the sums over permutations), the pattern of helicities is important. Thus $A_{4;1}(2^+,1^+,3^-,4^+)$ can be determined from $A_{4;1}(1^-,2^+,3^+,4^+)$ by cyclic permutation and renaming of arguments, whereas $A_{4;1}(1^-,2^-,3^+,4^+)$ and $A_{4;1}(1^-,2^+,3^-,4^+)$ are independent partial amplitudes.

To evaluate the partial amplitudes, one starts with the kinematic terms[1,5]

$$\begin{aligned}\mathcal{K}(\{k_i,\varepsilon_i\}) = \int \prod_i^n dx_i \int \left(\prod_{i=1}^n d\theta_{i3}\, d\theta_{i4}\right) \prod_{i<j}^n \exp\Big(k_i\cdot k_j G_B(x_{ij})\Big)\\ \times \prod_{i\neq j}^n \exp\Bigg\{\frac{1}{2}\Big[-\theta_{i3}\theta_{j3}k_i\cdot k_j\, G_F(x_{ij}) + 2i\theta_{i3}\theta_{j4}k_i\cdot\varepsilon_j\, G_F(x_{ij}) - 2i\theta_{i3}\theta_{i4}k_j\cdot\varepsilon_i\, \dot G_B(x_{ij})\\ + \theta_{i4}\theta_{j4}\varepsilon_i\cdot\varepsilon_j\, G_F(x_{ij}) + \theta_{i3}\theta_{i4}\theta_{j3}\theta_{j4}\varepsilon_i\cdot\varepsilon_j\, \ddot G_B(x_{ij})\Big]\Bigg\}\end{aligned} \tag{6}$$

which contains just the right-mover (superstring) part of the heterotic string amplitude. (The string tension has been omitted, or equivalently all momenta have absorbed the square root of the string tension, since it drops out in the gauge-theory limit.) The $\theta_{i,m}$ are Grassmann parameters; θ_{i4} simply ensures that the amplitude is multi-linear in the ε_i as it should be. In string theory the functions G_F and $\dot G_B$ are Green functions on the torus, but for our purposes these may be thought of as functions which keep track of Feynman parameters. The notation $\dot G_B(x_{ij})$ means $\partial_{x_i} G_B(x_{ij})$, while the notation $\ddot G_B(x_{ij})$ means $\partial^2_{x_i} G_B(x_{ij})$. The arguments of the Green functions $x_{ij} = x_i - x_j$ are related to ordinary Feynman parameters by $x_i = \sum_{m=1}^j y_m$. (The x_i are also trivially related to the imaginary parts of the usual closed string parameters.) The functions $\dot G_B$ and G_F are odd functions of their arguments, while $\ddot G_B$ is an even function of its argument. Note that this formula expresses the (string) partial amplitudes directly in terms of dot products of the *external* momenta and polarization vectors; no off-shell momenta or polarization vectors appear anywhere.

One can therefore use the spinor-helicity basis immediately; to do so, pick a set of reference momenta for the gluons, and substitute the appropriate expressions for the dot products. One then performs the Grassmann parameter integrals. Next, one removes all $\ddot{G}_B$s from the amplitude via integration by parts. The rules for performing these integrations by parts are straightforward. One simply continues to integrate by parts (ignoring surface terms) until all the $\ddot{G}_B$ have been eliminated; this is always possible as was shown in appendix II of reference [4]. As a simple example of the integration-by-parts process, consider the term in the four-point amplitude

$$\begin{aligned}&\ddot{G}_B(x_{12})\dot{G}_B(x_{23})\dot{G}_B(x_{34})\\&\quad\to -\Big[k_1\cdot k_2\dot{G}_B(x_{12})+k_1\cdot k_3\dot{G}_B(x_{13})+k_1\cdot k_4\dot{G}_B(x_{14})\Big]\dot{G}_B(x_{12})\dot{G}_B(x_{23})\dot{G}_B(x_{34})\end{aligned}\tag{7}$$

where we have integrated by parts with respect to x_1. The additional Green functions in the brackets have been brought down from the $\exp(k_i\cdot k_jG_B(x_{ij}))$ factor in the kinematic contributions (6).

There is a simple check on the integration by parts algebra: the kinematic factor must vanish identically under the substitution $G_F(x)\to -\dot{G}_B(x)$. This property follows from the world sheet supersymmetry of the string.[14,15]

After all integration by parts we can simply drop the exponentiated Green functions from the kinematic contributions (6) since the rules below will include their contributions. The exponentiated Green functions in fact lead to a standard Feynman denominator, while the remaining kinematic pieces will yield a numerator polynomial in the Feynman parameters.

4. New One-Loop Rules

The new rules organize the various contributions to the n-gluon amplitude into ϕ^3-type diagrams. At first sight, this may seem puzzling because non-abelian gauge theories contain a four-point contact interaction. However, with our rules the contact interaction arises from a cancellation of momentum invariants in the numerator against poles. Our rules directly give the amplitude in terms of a Feynman parameter integral, but with the loop momentum integrated out. Here we present the rules for partial amplitudes associated with a single trace; the double trace terms [which are not needed in the computation of the four-point amplitude because of their symmetries and the tree-level decoupling equation] will be presented elswhere.[5] We now give the diagrammatic rules for $A_{n;1}(1,2,\cdots n)$ which is the coefficient of the color trace $\mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})$.

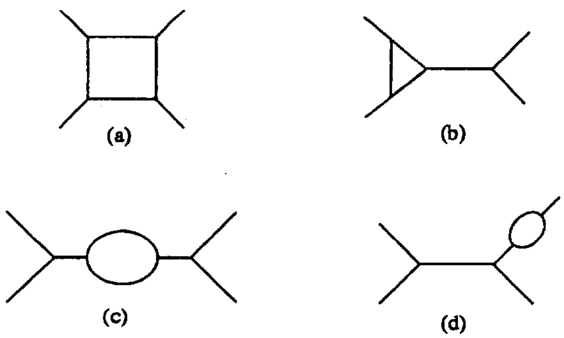

Figure 1. The kinds of ϕ^3 diagrams associated with the one-loop four-point amplitude.

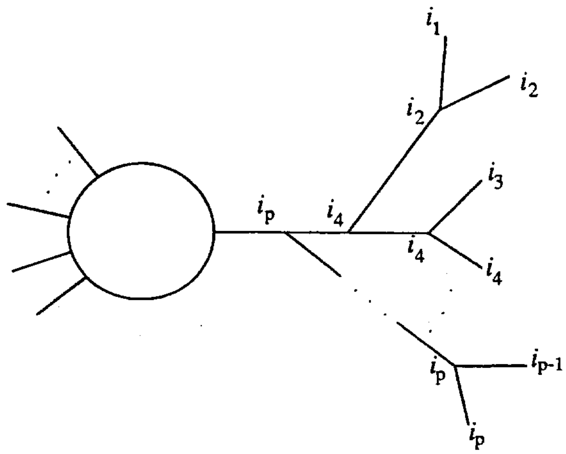

Figure 2. Labelling the trees attached to the loop: the clockwise ordering of the tree legs follows the ordering in the color trace associated with the partial amplitude.

General Diagrammatic Rules for Single Trace Partial Amplitudes:

1) Draw all n-point ϕ^3 diagrams in a planar fashion. (Exclude tadpole diagrams.) For example, the types of ϕ^3 diagrams associated with the four-point partial amplitude $A_{4;1}(1,2,3,4)$ are given in figure 1.

2) External legs are labeled clockwise in the order in which they appear in the trace corresponding to the partial amplitude. In the trees attached to the loop, one works inwards, labeling each inner line attached to the vertex with the last label (that is, most clockwise) of the labels on the outer branches attached to that vertex, as depicted in figure 2. The line attaching the tree to the loop is then labeled by the last tree leg. The contributions for a given diagram topology are given by summing over all labeled diagrams which satisfy these conditions. For example, the set of all allowed labeled diagrams of the type in figure 1b is given in figure 3.

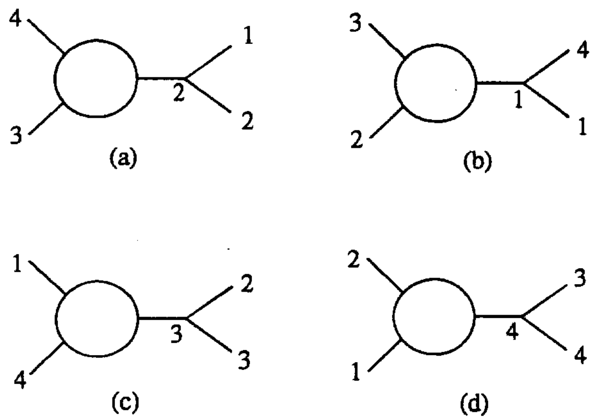

Figure 3. All allowed labellings of figure 1(b).

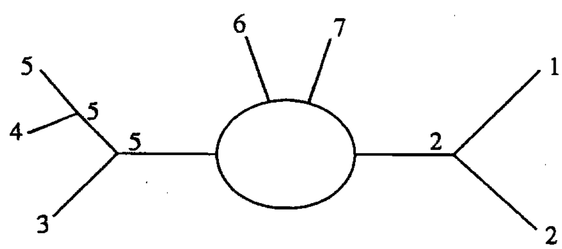

Figure 4. A diagram with two independent trees attached to the loop.

3) Each diagram consists of loop and tree contributions which are analyzed separately. Independent trees can be treated independently. An example of this situation is depicted in figure 4 where legs $1,2$ and $3,4,5$ are part of two independent trees.
4) The tree contributions $\mathcal{T}$ yield the usual ϕ^3 poles (with external legs truncated) in the momentum invariants multiplied by a factor of either 0,1 or -1 depending on the Green function structure as specified by the tree rules below.
5) The loop yields a parametric integral

$$\mathcal{L} = \frac{(4\pi)^{\epsilon/2}}{(16\pi^2)}\Gamma(N_L - 2 + \epsilon/2)\int_0^1 dx_{i_{N_L-1}} \int_0^{x_{i_{N_L-1}}} dx_{i_{N_L-2}} \cdots \int_0^{x_{i_3}} dx_{i_2} \int_0^{x_{i_2}} dx_{i_1} \times \frac{P_L(x_i, \varepsilon_i, k_i, \epsilon)}{\left(\sum_{l<m}^{N_L} k_{i_l} \cdot k_{i_m} x_{i_l i_m}(1 + x_{i_l i_m})\right)^{N_L - 2 + \epsilon/2}} \tag{8}$$

where N_L is the number of legs attached to the loop and the i_m are ordered labels for the legs attached to the loop as depicted in figure 5. In this expression $x_{N_L} = 1$. The k_{i_m} are the momenta of the lines attached to the loop. P_L is a polynomial in the polarization vectors, momenta, and integration parameters, specified by the loop rules below. Were $P_L = 1$ this integral would be *precisely* the loop for a massless ϕ^3 field theory in $4 - \epsilon$ dimensions. Note that the polynomials P_L for diagrams with different orderings are ***not*** necessarily trivially related (because of the possibility of adding total derivative terms to the string integrand). The x_i are simply related to conventional Feynman parameters as mentioned previously.

The partial amplitude associated with a given trace structure is then obtained by putting the above pieces together for a given ϕ^3 diagram and summing over all diagrams

$$g^n A_{n;1} = i(-\sqrt{2} g \mu^{\epsilon/2})^n \sum_{\text{diagrams}} \mathcal{T}\,\mathcal{L}\,. \tag{9}$$

The overall normalization takes into account our choice of conventions $\mathrm{Tr}(T^a T^b) = \delta^{ab}$.

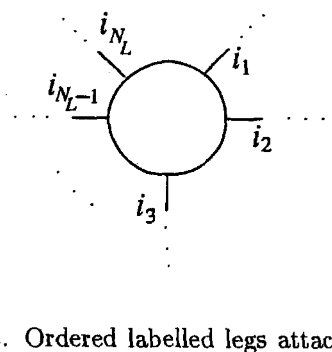

Figure 5. Ordered labelled legs attached directly to the loop. The labels are inherited from the last label on the tree if the leg is attached to a tree.

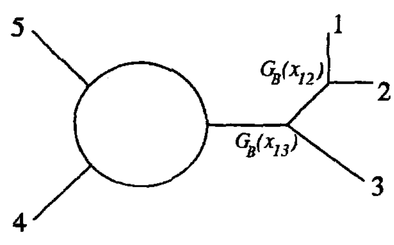

Figure 6. An example of a tree attached to the loop, with the contributing Green functions labelled.

Kinematic Factor Tree Rules:

1) In order for there to be a non-vanishing tree contribution it must be possible to associate to every vertex in a tree *exactly* one $\dot{G}_B$ or G_F whose argument is a difference of x_{i_u,i_l}, with one label from each branch entering into the vertex from an outermore part of the diagram. If more than one, or no Green function can be associated with a tree vertex there is no contribution. The Green functions which can be matched with the vertices are removed from the term under consideration, while in the remaining factors all indices which appear in the tree are replaced with the last label (following the clockwise ordering) of the tree.

2) Each Green function which has been associated with a vertex yields either a plus sign or minus sign. As we go clockwise around the labeled tree, depicted in figure 2, if i_l appears before i_m a $\dot{G}_B(x_{i_m i_l})$ or $G_F(x_{i_l i_m})$ contributes a plus sign while a $\dot{G}_B(x_{i_l i_m})$ or $G_F(x_{i_m i_l})$ contributes a minus sign.

As a simple example of the tree rules consider the ϕ^3 diagram depicted in figure 6. In order to get a tree contribution for the configuration depicted in figure 6 we must have a $G(x_{12})$ and either one of $G(x_{13})$ or $G(x_{23})$ where G is either a $\dot{G}_B$ or a G_F. In particular, for the partial amplitude associated with the color trace structure $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5})$, the tree contribution for the combination of Green functions $\dot{G}_B(x_{12})\dot{G}_B(x_{13})\dot{G}_B(x_{14})G_F^2(x_{25})$ is

$$\mathcal{T} = \frac{1}{2k_1\cdot k_2}\,\frac{1}{2(k_1\cdot k_2 + k_1\cdot k_3 + k_2\cdot k_3)}\,. \tag{10}$$

The Green functions which remain for the loop rules are $\dot{G}_B(x_{34})G_F^2(x_{25})$. The overall sign in the tree contribution is plus because the two Green functions $\dot{G}_B(x_{12})$ and $\dot{G}_B(x_{13})$ each give a minus sign.

As another example, for the same color trace and ϕ^3 diagram the combination of Green functions $\dot{G}_B(x_{12})G_F(x_{13})\dot{G}_B(x_{23})G_F^2(x_{25})$ vanishes because $G_F(x_{13})$ and $\dot{G}_B(x_{23})$ can both be associated to the same vertex.

Since all tree indices in Green functions which remain after the Green functions which are associated with the tree vertices have been removed are to be set equal to the last tree label, cancellation can arise between different terms. This type of cancellation occurs in the sample calculation below.

Kinematic Factor Loop Rules:

It is first useful to introduce the concept of a 'cycle' of G_F's which is a group of G_Fs which can be written in the form

$$G_F(x_{i_1 i_2})G_F(x_{i_2 i_3})\ldots G_F(x_{i_{m-1}\, i_m})G_F(x_{i_m i_1})\,. \tag{11}$$

Since G_F is antisymmetric, replacing any $G_F(x_{ij})$ by $-G_F(x_{ji})$ still forms a cycle. The G_F in fact always form cycles, but these do not necessarily correspond to the ordering of the external legs in a diagram. We may now classify the simplification rules according to the cycles of G_Fs. For a loop where the attached legs are labeled as in figure 5, the following simplification rules hold:

No G_Fs: Such terms receive a factor

$$2\mathcal{R} + N_s \tag{12}$$

where N_s is the number of real adjoint scalars (which is of course zero in the standard model) and the regularization factor $\mathcal{R} = 1$ for the four-dimensional helicity scheme[5] and $\mathcal{R} = (1 - \epsilon/2)$ for either the 't Hooft-Veltman[16] or conventional scheme, used for example by Ellis and Sexton.[3]

Two G_Fs:

$$G_F(x_{i_2 i_1})G_F(x_{i_1 i_2}) \longrightarrow -2\,. \tag{13}$$

Three or more G_Fs in a single cycle:

$$G_F(x_{i_2 i_1})G_F(x_{i_3 i_2})\cdots G_F(x_{i_1 i_m}) \longrightarrow -1 \qquad (m > 2) \tag{14}$$

where the case where the arguments of the G_F's differ by a sign can be trivially related by the antisymmetry of the Green functions $G_F(x_{ij}) = -G_F(x_{ji})$.

Any other combinations of G_Fs will lead to a vanishing result; G_Fs which do not follow the cyclic ordering in figure 5 or are arranged in two or more cycles do not contribute. Any $\dot{G}_B$s which multiply the above expressions are simply replaced by the polynomial

$$\dot{G}_B(x_{ij}) \longrightarrow \frac{1}{2}\left(-\mathrm{sign}(x_{ij}) + 2x_{ij}\right)\,. \tag{15}$$

For massless adjoint fermion contributions to the n-gluon amplitude once again one starts from the kinematic terms (6) and follows the same procedures discussed above. However, instead of making the replacements (12), (13) and (14) one multiplies by an overall factor of $-4N_f$, where N_f is the number of Dirac fermions, makes the $\dot{G}_B$ substitutions (15) and the G_F substitutions

$$G_F(x_{ij}) \longrightarrow \frac{1}{2}\mathrm{sign}(x_{ij}) \tag{16}$$

without regard to the cyclic ordering.

The case of fundamental fermions requires a simple modification of the color trace structure. The inclusions of masses for the internal fermions or scalars is straightforward, but the case of external fermions is more complicated. We will discuss these issues elsewhere.

5. Sample Calculation

We now apply the rules to a calculation of the pure glue contributions to the $\mathcal{A}(-+++)$ four-gluon helicity amplitude. From a conventional Feynman diagram point of view, this computation would be quite non-trivial as there would be a total of thirty-nine diagrams to evaluate. Our technique reduces the computation to a total of two non-vanishing ϕ^3-like diagrams. Since this particular helicity amplitude contains neither ultraviolet nor infrared singularities (nor are singularities encountered in intermediate steps), the parameter integrals are particularly easy to evaluate.

As discussed in the last section, the first step is to insert the spinor helicity simplifications (3) into the kinematic factors (6). After integrating out the Grassmann parameters the kinematic terms simplify into a single term

$$
\begin{aligned}
K =& \frac{s^2 t}{4} \frac{[2\,4]^2}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]} \Big(\dot{G}_B(x_{21}) - \dot{G}_B(x_{31})\Big) \\
& \times \Big(\dot{G}_B(x_{42}) G_F^2(x_{43}) - \dot{G}_B(x_{32}) G_F^2(x_{43}) - 2G_F(x_{32}) G_F(x_{42}) G_F(x_{43}) \\
& - \dot{G}_B(x_{42}) \dot{G}_B^2(x_{43}) + \dot{G}_B(x_{32}) \dot{G}_B^2(x_{43}) - G_F^2(x_{42}) \dot{G}_B(x_{43}) \\
& + \dot{G}_B^2(x_{42}) \dot{G}_B(x_{43}) - 2\dot{G}_B(x_{32}) \dot{G}_B(x_{42}) \dot{G}_B(x_{43}) - G_F^2(x_{32}) \dot{G}_B(x_{43}) \\
& + \dot{G}_B^2(x_{32}) \dot{G}_B(x_{43}) - \dot{G}_B(x_{32}) G_F^2(x_{42}) + \dot{G}_B(x_{32}) \dot{G}_B^2(x_{42}) \\
& + G_F^2(x_{32}) \dot{G}_B(x_{42}) - \dot{G}_B^2(x_{32}) \dot{G}_B(x_{42})\Big) \,.
\end{aligned} \tag{17}
$$

Since our choice of spinor helicity basis has set all $\varepsilon_i \cdot \varepsilon_j = 0$, no $\ddot{G}_B$s appear in the amplitude and there is no need to integrate by parts. Although this expression may seem complicated at first sight, one should compare this to the $\sim 10^4$ terms of a conventional Feynman diagram computation.

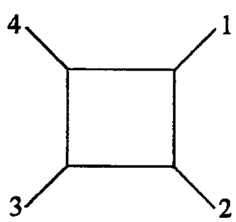

Figure 7. The only box diagram with the required ordering of labelled legs for $A_{4;1}(1,2,3,4)$.

The types of diagrams for the partial amplitude $A_{4;1}(1^-,2^+,3^+,4^+)$ are given in figure 1. First consider the only labeled box diagram which is depicted in figure 7. Since there are no trees sewn onto the loop we may apply the kinematic factor loop rules directly on equation (17) to obtain the loop polynomial

$$
P_L = \frac{s^2 t}{2} \frac{[2\,4]^2}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]}\, x_2 (1 - x_3)(x_3 - x_2)^2 \tag{18}
$$

after a bit of algebra to compress the result. The kinematic variables $s = 2k_1 \cdot k_2$ and $t = 2k_1 \cdot k_4$ are the usual Mandelstam variables.

Including the integral over parameter space, the overall normalization, and the scalar denominator yields

$$\begin{aligned} &\frac{i}{2}\,\frac{1}{4\pi^2}s^2 t\frac{[2\,4]^2}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]}\int_0^1 dx_3\int_0^{x_3} dx_2\int_0^{x_2} dx_1 \\ &\qquad\qquad \times\frac{x_2(1-x_3)(x_3-x_2)^2}{(sx_1x_2+tx_2x_3+ux_1x_3+t(x_1-x_2))^2} \\ &= \frac{i}{48\pi^2}s\frac{[2\,4]^2}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]}\,. \end{aligned} \tag{19}$$

Since the expression is completely finite we have set the dimensional regularization parameter $\epsilon = 0$.

We now compute the contributions associated with diagrams of the type in figure 1b, as depicted in figure 3. However, all the diagrams vanish except for diagram 3a. Diagram 3b vanishes because there is no $\dot{G}_B(x_{14})$ or $G_F(x_{14})$ in the kinematic term (17). The others vanish because of a trivial cancellation amongst terms in the kinematic factor (17) after implementing the kinematic factor tree rule that Green functions which are not associated with any tree vertices have all tree labels replaced by the one on the last tree leg.

For the non-vanishing diagram 3a, the tree contribution

$$\mathcal{T} = \frac{1}{2k_1\cdot k_2} = \frac{1}{s} \tag{20}$$

where we have obtained a sign of plus from $G_B(x_{21})$ which is associated with the tree vertex. The loop polynomial is

$$P_L = \frac{s^2 t}{2}\frac{[2\,4]^2}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]}x_2(1-x_3)(x_2-x_3)\,. \tag{21}$$

Since the loop scalar denominator is $(sx_2(x_2-x_3))^{-1}$, the complete contribution from diagram 3a is

$$\begin{aligned} &\frac{i}{8\pi^2}\,t\frac{[2\,4]^2}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]}\int_0^1 dx_3\int_0^{x_3} dx_2\,(1-x_3) \\ &= \frac{i}{48\pi^2}\,t\frac{[2\,4]^2}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]} \end{aligned} \tag{22}$$

where we have again set $\epsilon = 0$ because the integral is completely finite.

Finally, we observe that, with our choice of reference momenta, diagrams of the type in figure 1c also vanish, while diagram 1d never contributes to on-shell amplitudes; in general with dimensional regularization there is a complete cancellation of the infrared and ultraviolet contributions leading to a vanishing result.[17,5] For this particular helicity amplitude there are anyways neither ultraviolet nor infrared divergences so the last diagram vanishes identically.

Combining the results from the two diagrams then leads to the final result for the amplitude

$$A_{4;1}(1^-,2^+,3^+,4^+) = \frac{i}{48\pi^2}\frac{(s+t)[2\,4]^2}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]}\,. \tag{23}$$

From the $U(1)$ decoupling equation (5) we also have

$$A_{4;3}(1^-,2^+,3^+,4^+) = \frac{i}{8\pi^2}\frac{st[2\,4]^2}{(s+t)\,[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]}\,. \tag{24}$$

These partial amplitudes can be combined into the complete amplitude using equation (4).

6. Conclusions

We have presented new rules for evaluating one-loop n-gluon amplitudes including the contributions of adjoint scalars and fermions and have presented a sample computation. The corresponding computation using Feynman diagrams would involve 39 diagrams with $\sim 10^4$ terms; with the new rules only two simpler ϕ^3-like diagrams need to be evaluated. Although these rules were derived from four-dimensional heterotic string theory, they can be used in practical calculations without any knowledge of string theory.

It should be possible to extend these rules to multi-loop level as well as to the case of external fermions. We expect that our rules will lead to a deeper understanding of gauge theories in general and QCD in particular.

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