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NONLINEAR SOLUTIONS OF LONG WAVELENGTH GRAVITATIONAL RADIATION

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In a significant improvement over homogeneous mini-superspace models, it is shown that the classical nonlinear evolution of inhomogeneous scalar fields and the metric is tractable when the wavelength of the fluctuations is larger than the Hubble radius. Neglecting second order spatial gradients, one can solve the energy constraint as well as the evolution equations by invoking a transformation to new canonical variables. The Hamilton-Jacobi equation is separable and complete solutions are given for gravitational radiation and multiple scalar fields interacting through an exponential potential. Although the time parameter is arbitrary, the natural choice is the determinant of the 3-metric. The momentum constraint may be simply expressed in terms of the new canonical variables, and several classes of solutions are given. The long wavelength analysis is essential for a proper formulation of stochastic inflation which enables one to model non-Gaussian primordial fluctuations for structure formation.

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I. INTRODUCTION

Currently, the most conservative theory for the generation of initial fluctuations for structure formation in the Universe is the inflation model. The Universe began with a large value of the cosmological constant which produced an inflating phase. Classical inhomogeneities were damped leaving only vacuum quantum fluctuations to serve as the source of inhomogeneities. The cosmological constant, modelled by a scalar field potential, then decayed, transforming its energy into radiation. However, observational evidence[1-5] is mounting which suggests that Gaussian primordial fluctuations with a scale-invariant spectrum as predicted by the simplest inflation models may, in fact, be incorrect. The correlation function of clusters and galaxies, pencil beam surveys and other redshift surveys seem to indicate discrepancies in the form of large scale power. The view that will be adopted here is that the inflation model is basically correct, but that previous models were too naive, and should be improved. In particular, it is *interesting to consider whether nonlinearities in the fields may produce significant* non-Gaussian fluctuations.

A disturbing property of inflation models is that scalar field quantum fluctuations eventually transform themselves to metric fluctuations requiring ultimately a theory of quantum gravity. In the standard approach to calculations in inflation models, one splits the fields into a homogeneous classical part, and quantum inhomogeneities are treated in linear theory.[6,7] These models are self-consistent and difficulties only arise when one considers nonlinearities. Since nonlinear problems are notoriously difficult to solve, I will split all fields into two parts, long and short wavelengths as compared to the Hubble radius during the inflation epoch. By extending previous work,[8] I will show that the nonlinear classical evolution of long wavelength gravitational and scalar fields is tractable. The initial conditions for the long wavelength problem are generated by short wavelength quantum fluctuations which expand beyond the Hubble radius. Since quantum gravity corrections are typically small, one can simply assume that the long wavelength quantum fluctuations have become classical in a process called stochastic inflation.[9-13]

When the wavelength of a mode exceeds the Hubble radius, it is an excellent approximation to neglect second order spatial gradients in the scalar field equations and Einstein's equations. The evolution equations and the energy constraint are identical to homogeneous and flat cosmological models. The only new ingredient is the momentum constraint which patches together different spatial points to make one Universe. An investigation of nonlinearity at short wavelengths is much more difficult, and will be considered in future work. Although nonlinear long wavelength evolution does not alter substantially the standard inflation predictions for a single scalar field,[13] non-Gaussian fluctuations can arise in our observable Universe through multiple interacting fields. One of the purposes of this paper is to develop the machinery describing such models.

In a previous paper by Salopek and Bond,[8] hereafter referred to as $SB1$, the long wavelength equations were solved making two simplifying assumptions. The shift function and the traceless part of the gravitational momenta were taken to be zero. Hence, the evolution of gravitational radiation was neglected. In this case, one could solve the momentum constraint. The Hubble parameter which describes the rate of expansion of the Universe is only a function of the scalar fields,

$$H(t,x) \equiv H\left(\phi_j(t,x)\right). \tag{1.1}$$

For example, the Hubble parameter does not depend explicitly on the time parameter, t. The momenta of the scalar fields are then given by partial differentiation of the Hubble function,

$$\frac{\dot{\phi}_j(t,x)}{N(t,x)} = -\frac{m_P^2}{4\pi}\frac{\partial H}{\partial \phi_j}, \tag{1.2}$$

where $N(t,x)$ is the lapse function. The energy constraint, then becomes the separated Hamilton-Jacobi equation. For a single scalar field, it is useful to choose ϕ as time because one could then generalize the variable ζ that was first introduced in linear theory by Bardeen, Steinhardt and Turner.[14] For multiple fields, the determinant of the 3-metric is the most useful. It is clear then that if one wishes to discuss the role of time in general relativity or even quantum cosmology, one should introduce inhomogeneities, because time is the observation surface that one views a 4-geometry. For example, homogeneous mini-superspace models are insufficient to address the choice of time hypersurface.

The full classical long wavelength problem will be solved using Hamilton-Jacobi theory. The 3-metric $\gamma_{ij}(t,x) \equiv \gamma^{1/3}(t,x)h_{ij}(t,x)$ will be expressed in terms of the conformal 3-metric h_{ij} with unit determinant. The terminology, gravitational radiation, will then refer to the conformal 3-metric without assuming any linear perturbation analysis.

In Sec. II, the equations for the long wavelength gravitational and scalar fields are enunciated. One neglects all second order spatial gradients in the action, or equivalently, in the equations of motion and the energy constraint. The resulting equations may be solved if one invokes a canonical transformation to new variables where the Hamiltonian density is identically zero. Since the Hamiltonian generates time evolution through Poisson brackets, the new coordinates are constants in time. The Hamilton-Jacobi equation is separable. Consequently, the determinant of the 3-metric is the natural time parameter. The only equation which remains is the momentum constraint, which may be conveniently written in terms of the new variables. The canonical transformation has disentangled the energy and momentum constraints.

In Sec. III, I consider solutions where the dynamic effects of gravitational radiation are not important. This situation was also investigated in SB1, but their approach cannot be applied directly to gravitational radiation. The more powerful method of canonical transformations is required. Exact complete solutions are obtained for m massless scalar fields evolving under a cosmological constant. This example is solved in detail because it will guide the solution for gravitational radiation. Exact complete solutions are obtained for multiple scalar fields interacting though an exponential potential. The momentum constraint may be integrated exactly, and the results are compared with earlier work.

Sec. IV actually treats the evolution of gravitational radiation. The 6-dimensional separated Hamilton-Jacobi equation which governs the canonical transformation may be solved explicitly if only a cosmological constant is present. This simple case illustrates the solution to the more general situation where scalar fields are also present. The 6 gravitational degrees of freedom may essentially be reduced to a single massless scalar field. Complete canonical transformations are given for the case of gravitational radiation interacting with a scalar field which has an exponential potential. Sec. V contains a summary of results as well as conclusions.

II. HAMILTON-JACOBI THEORY FOR LONG WAVELENGTH INHOMOGENEOUS UNIVERSES

The action principle for the gravitational field and n scalar fields is,[15]

$$
\begin{aligned}
\mathcal{I} &= \int d^4x\sqrt{-g}\Big\{ \frac{m_{\mathcal{P}}^2}{16\pi} R - \tfrac{1}{2} g^{\mu\nu}\partial_\mu\phi_k\partial_\nu\phi_k - V(\phi_k) \Big\}, \\
&= \int d^4x N\sqrt{\gamma}\Big\{ \frac{m_{\mathcal{P}}^2}{16\pi}\big[{}^{(3)}R + K_{ij}K^{ij} - K^2 \big] \\
&\qquad + \tfrac{1}{2}\big[(\dot\phi_k - N^i\phi_{k|i})^2/N^2 - \phi_{k|i}\phi_k{}^{|i} \big] - V(\phi_k) \Big\}.
\end{aligned} \tag{2.1}
$$

The basic variables are the scalar fields, ϕ_k, the lapse and shift functions, N and N^i, and the 3-metric, γ_{ij}. The extrinsic curvature 3-tensor K_{ij} is a generalization of the Hubble parameter that appears in isotropic cosmologies:

$$
g_{00} = -N^2 + \gamma^{ij}N_iN_j, \quad g_{0i} = g_{i0} = N_i, \quad g_{ij} = \gamma_{ij}, \tag{2.2a}
$$

$$
K_{ij} = \big(N_{i|j} + N_{j|i} - \frac{\partial\gamma_{ij}}{\partial t}\big)/(2N). \tag{2.2b}
$$

It proves more useful to consider a Hamiltonian formulation. One defines the momenta densities, π^{ϕ_k}, $[\pi^\gamma]^{ij}$ for scalar and gravitational fields, respectively,

$$
\pi^{\phi_k} = \sqrt{\gamma}(\dot\phi_k - N^i\phi_{,i}) \;, \quad [\pi^\gamma]^{ij} = \frac{m_{\mathcal{P}}^2}{16\pi}\sqrt{\gamma}(\gamma^{ij}K - K^{ij}) \;, \tag{2.3}
$$

and the corresponding action is

$$
\mathcal{I} = \int d^4x\big(\pi^{\phi_k}\dot\phi_k + [\pi^\gamma]^{ij}\dot\gamma_{ij} - N\mathcal{H} - N^i\mathcal{H}_i\big) \tag{2.4}
$$

where the energy density, $\mathcal{H}$, and the momentum density, $\mathcal{H}_i$ are given by,

$$
0 = \mathcal{H} = \frac{16\pi}{m_{\mathcal{P}}^2}\gamma^{-1/2}\Big(\gamma_{jk}\gamma_{li}[\pi^\gamma]^{ij}[\pi^\gamma]^{kl} - \frac{1}{2}(\pi^\gamma)^2\Big) + \frac{1}{2}\gamma^{-1/2}{\pi^{\phi_k}}^2 + \gamma^{1/2}V(\phi_k) \tag{2.5a}
$$

$$
\Big\{ -\frac{m_{\mathcal{P}}^2}{16\pi}\gamma^{1/2}({}^3R) + \frac{1}{2}\gamma^{1/2}\gamma^{ij}\phi_{k,i}\phi_{k,j} \Big\}, \quad \text{(neglected terms)}
$$

$$
0 = \mathcal{H}_i = -2\big(\gamma_{il}[\pi^\gamma]^{lk}\big)_{,k} + [\pi^\gamma]^{lk}\gamma_{lk,i} + \pi^{\phi_k}\phi_{k,i} \;; \tag{2.5b}
$$

here $\pi^\gamma \equiv [\pi^\gamma]^{ij}\gamma_{ij}$ is the trace of $[\pi^\gamma]^{ij}$. The long wavelength equation are obtained by dropping all second order spatial gradients. In (2.5a), the terms in braces which include the 3-curvature, ${}^3R$, and the scalar field spatial derivatives, $\gamma^{ij}\phi_{k,i}\phi_{k,j}/2$, are neglected.

The resulting system is mathematically self consistent because the Poisson brackets of the various *constraints* return the constraints:

$$
\{\mathcal{H}(x), \mathcal{H}(x')\} = 0, \tag{2.6a}
$$

$$\{\mathcal{H}(x), \mathcal{H}_i(x')\} = -\mathcal{H}(x')\frac{\partial}{\partial x'^i}\delta^3(x - x'), \tag{2.6b}$$

$$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_j(x)\frac{\partial}{\partial x^i}\delta^3(x - x') + \mathcal{H}_i(x')\frac{\partial}{\partial x^j}\delta^3(x - x'). \tag{2.6c}$$

If one neglects the scalar fields, this system is equivalent to the large gravitational coupling $G = m_{\mathcal{P}}^{-2} \to \infty$ limit considered by many researchers.[16–18] (However, in inflation models, scalar fields are essential to model primordial inhomogeneities for structure formation.) Several authors works have attempted to formulate a quantum limit, where unfortunately the momentum constraint was usually ignored (Pilati,[16] Teitelboim,[17] SB1). In SB1, the classical system was solved, including the momentum constraint, if the evolution of gravitational radiation was neglected. This special case is the most relevant for stochastic inflation and will be considered from a different point of view in Sec. III. In this paper, the classical system, including gravitational radiation will be solved completely in several situations.

Unlike the case for pure general relativity, the first Poisson bracket vanishes because the Hamiltonian density is a function of ultra local values of the fields; *i.e.* it does not depend on spatial gradients. If the shift function vanishes, then Eq.(2.6a) implies that the energy constraint is preserved in evolution,

$$\frac{\partial \mathcal{H}(x')}{\partial t} = \{\mathcal{H}(x'), \int d^3x N(x)\mathcal{H}(x)\} = 0.$$

Since this true independent of the the momentum constraint, it is therefore possible to solve the energy constraint and the evolution equations in isolation. Analogously, the second Poisson bracket states that the momentum constraint is preserved in evolution only if the Hamiltonian vanishes.

By taking variations of the action (2.4) with respect to the canonical variables, one finds the evolution equations to be: (1) the definitions of the scalar and gravitational field momenta,

$$(\dot{\phi}_k - N^i\phi_{k,i})/N = \gamma^{-1/2}\pi^{\phi_k}, \tag{2.7a}$$

$$(\dot{\gamma}_{ij} - N_{i|j} - N_{j|i})/N = \frac{16\pi}{m_{\mathcal{P}}^2}\gamma^{-1/2}\left(2[\pi^\gamma]_{ij} - \gamma_{ij}\pi^\gamma\right), \tag{2.7b}$$

and (2) the evolution equations for the momenta:

$$\left(\dot{\pi}^{\phi_k} - \left(N^i\pi^{\phi_k}\right)_{,i}\right)/N = -\gamma^{1/2}\frac{\partial V}{\partial \phi_k}, \tag{2.7c}$$

$$\left(\frac{\partial[\pi^\gamma]^i_j}{\partial t} - \left(N^m[\pi^\gamma]^i_j\right)_{,m} + N^i_{,m}[\pi^\gamma]^m_j - N^m_{,j}[\pi^\gamma]^i_m\right)/N =$$

$$\gamma^{1/2}\,\delta^i_j\,\{\frac{16\pi}{m_{\mathcal{P}}^2}\left([\pi^\gamma]^{lm}[\pi^\gamma]_{lm} - \frac{1}{2}\pi^{\gamma 2}\right) + \frac{1}{2}\pi^{\phi_k 2}\}. \tag{2.7d}$$

The constraints (2.5a,b) follow from the variation of the lapse and shift functions.

In SB1, it was assumed that the traceless part of the gravitational momentum density vanished so that $[\pi^{\gamma}]^{ij} = \pi^{\gamma}\gamma^{ij}/3$. In this case, the momentum constraint (2.5b) reduces dramatically,

$$\frac{2}{3}(\gamma^{-1/2}\pi^{\gamma})_{,i} = \gamma^{-1/2}\pi^{\phi_k}\phi_{k,i}. \tag{2.8}$$

In fact, it can be solved exactly, implying that $\gamma^{-1/2}\pi^{\gamma}$ is not an independent variable, but rather is an arbitrary function of ϕ_k and possibly time, whereas $\gamma^{-1/2}\pi^{\phi_k}$ is given by partial derivatives of that arbitrary function:

$$\gamma^{-1/2}\pi^{\gamma} \equiv -\frac{3m_{\mathcal{P}}^2}{8\pi}H(\phi_k, t), \quad \gamma^{-1/2}\pi^{\phi_k} = -\frac{m_{\mathcal{P}}^2}{4\pi}\frac{\partial H(\phi_k, t)}{\partial \phi_k}. \tag{2.9a}$$

Physically, $H(\phi_k, t)$ is the Hubble parameter

$$H \equiv \left(\frac{\partial}{\partial t}\ln(\sqrt{\gamma}) - N^i_{|i}\right)/(3N)$$

which measures the rate of change of the log of the scale factor, $\ln(a) \equiv \ln(\gamma)/6$. One can show using that the evolution equation for π^{γ} that there is no explicit time dependence in the Hubble parameter, $H \equiv H(\phi_k)$. (Of course, since ϕ_k depends on time, H depends implicitly on time.) Substituting these expressions for the momenta into energy constraint (2.5a) leads to a separated Hamilton-Jacobi equation,

$$H^2 = \frac{m_{\mathcal{P}}^2}{12\pi}\left(\frac{\partial H}{\partial \phi_k}\right)^2 + \frac{8\pi}{3m_{\mathcal{P}}^2}V(\phi_k), \quad \text{(No Evolution in Gravitational Radiation)}, \tag{2.9b}$$

Hence, in SB1, one first integrated the momentum constraint, and then solved the energy constraint as well as the evolution equations. Unfortunately, by assuming the traceless part of the gravitational momentum tensor vanishes, there is no evolution in the gravitational radiation. When one relaxes this assumption, one finds that the momentum constraint may not be solved using the simple method that led to (2.9a). However, the basic ingredient, a Hamilton-Jacobi-like equation has mysteriously appeared, although we have not explicitly employed a canonical transformation. In fact, I will use canonical transformations to incorporate the effects of gravitational radiation.

Before one proceeds further, a simple analogy proves instructive. If one considers the set of all points in Euclidean 3-space a unit distance from the origin, $x^2+y^2+z^2 = 1$, then there are only 2 degrees of freedom and it is convenient to introduce spherical coordinates, θ and ϕ, with $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$, so that the constraint is satisfied. In the same way, the energy constraint (2.5a) is telling us that one should choose new canonical variables so that the Hamiltonian vanishes strongly. Since the Hamiltonian generates time evolution through the Poisson bracket relations, the new canonical variables are constant in time, although they may be spatially dependent. The only equation that remains to be solved is the momentum constraint which can be conveniently expressed as a function of the new variables (Sec. IV).

As the initial step, I review the theory of canonical transformations for gravity,[15] which is quite similar to the classical treatment.[19] One defines new fields, denoted by a tilde, $\tilde{\phi}_k(t,x)$, $\pi^{\tilde{\phi}_k}(t,x)$, $\tilde{\gamma}_{ij}(t,x)$, $[\pi^{\tilde{\gamma}}]^{ij}(t,x)$, so that Hamilton's equations are preserved.

This implies that the new action has the same form as the original except that it may have a total time derivative added to it:

$$\mathcal{I} = \int d^4x \left(\pi^{\tilde{\phi}_k}\dot{\tilde{\phi}}_k + [\pi^{\tilde{\gamma}}]^{ij}\dot{\tilde{\gamma}}_{ij} - N\tilde{\mathcal{H}} - N^i\tilde{\mathcal{H}}_i\right) + \int dt\,\dot{\mathcal{S}}, \tag{2.10}$$

where $\mathcal{S}$ is a three-functional which depends on ϕ_k, $\tilde{\phi}_k$, γ_{ij} and $\tilde{\gamma}_{ij}$. It will be assumed that $\mathcal{S}$ does not depend explicitly on time. In this sense, the theory of canonical transformations for gravitational fields differs from standard theory.

Applying the chain rule,

$$\dot{\mathcal{S}} = \int d^3x \left[\frac{\delta \mathcal{S}}{\delta\phi_k(x)}\dot{\phi}_k(t,x) + \frac{\delta \mathcal{S}}{\delta\tilde{\phi}_k(x)}\dot{\tilde{\phi}}_k(t,x) + \frac{\delta \mathcal{S}}{\delta\gamma_{ij}(x)}\dot{\gamma}_{ij}(t,x) + \frac{\delta \mathcal{S}}{\delta\tilde{\gamma}_{ij}(x)}\dot{\tilde{\gamma}}_{ij}(t,x)\right], \tag{2.11}$$

and comparing (2.10) with (2.11), one derives the canonical transformation linking the various variables,

$$\mathcal{H}(x) = \tilde{\mathcal{H}}(x), \quad \mathcal{H}_i(x) = \tilde{\mathcal{H}}_i(x), \tag{2.12a}$$

$$\pi^{\phi_k}(x) = \frac{\delta \mathcal{S}}{\delta\phi_k(x)}, \quad [\pi^{\gamma}]^{ij} = \frac{\delta \mathcal{S}}{\delta\gamma_{ij}(x)}, \quad \pi^{\tilde{\phi}_k}(x) = -\frac{\delta \mathcal{S}}{\delta\tilde{\phi}_k(x)}, \quad [\pi^{\tilde{\gamma}}]^{ij} = -\frac{\delta \mathcal{S}}{\delta\tilde{\gamma}_{ij}(x)}. \tag{2.12b}$$

The new variables will be chosen so that the new Hamiltonian density vanishes functionally, $\tilde{\mathcal{H}}\left(\tilde{\phi}_k(x), \pi^{\tilde{\phi}_k}(x), \tilde{\gamma}_{ij}(x), [\pi^{\tilde{\gamma}}]^{ij}(x)\right) \equiv 0$, (in the language of Dirac,[20] it vanishes strongly), leading to the Hamilton-Jacobi equation,

$$\begin{aligned}
&\frac{16\pi}{m_{\mathcal{P}}^2}\gamma^{-1/2}\frac{\delta \mathcal{S}}{\delta\gamma_{ij}(x)}\frac{\delta \mathcal{S}}{\delta\gamma_{kl}(x)}\left[\gamma_{jk}(x)\gamma_{il}(x) - \frac{1}{2}\gamma_{ij}(x)\gamma_{kl}(x)\right] \\
&+ \frac{\gamma^{-1/2}(x)}{2}\left(\frac{\delta \mathcal{S}}{\delta\phi_k(x)}\right)^2 + \gamma^{1/2}(x)V(\phi_k(x)) \equiv 0.
\end{aligned} \tag{2.13}$$

It will be assumed that the momentum constraint vanishes weakly but not strongly; *i.e.* it constrains the new canonical variables, but its Poisson bracket with other fields will not in general vanish. A redundancy theorem advanced by Moncrief and Teitelboim[21] claimed that if $\mathcal{S}$ satisfied the energy constraint, it automatically satisfied the momentum constraint. This theorem cannot be applied for long wavelength fields because the Poisson bracket (2.6a) vanished, whereas in pure Einstein gravity it returned the momentum constraint.[8]

Since $\tilde{\mathcal{H}} \equiv 0$, the new fields are in fact constants in time if the shift function vanishes. For example, the time evolution of $\tilde{\phi}_k(x)$ is generated by the Poisson bracket relation with the new Hamiltonian, $\dot{\tilde{\phi}}_k(x) = \{\tilde{\phi}_k(x), \tilde{H}_{am}\}$, where $\tilde{H}_{am} = \int d^3x (N\tilde{\mathcal{H}}(x) + N^i\tilde{\mathcal{H}}_i(x))$ vanishes. (See Sec. III.A for the case where the shift does not vanish).

Of course, the great simplification that arises for long wavelength universes is that there is no causal contact between different spatial points. Each spatial point evolves as an independent homogeneous universe. Mathematically, it is possible to write the generating functional S as a sum over all independent points. One attempts the ansatz,

$$S = -\frac{m_P^2}{4\pi}\int d^3x\sqrt{\gamma}\,H\Big(\phi_k(x), h_{ij}(x); \tilde{\phi}_k(x), \tilde{h}_{ij}(x)\Big), \tag{2.14a}$$

where $h_{ij} \equiv \gamma^{-1/3}\gamma_{ij}$ and $\tilde{h}_{ij} \equiv \tilde{\gamma}^{-1/3}\tilde{\gamma}_{ij}$ are conformal 3-metrics, each having unit determinant. The function, H, which will be referred to as the Hubble function, has no explicit spatial dependence except through the fields themselves and the additional parameters, $\tilde{\phi}_k$, $\tilde{h}_{ij}$:

$$H \equiv H(\phi_k, h_{ij}; \tilde{\phi}_k, \tilde{h}_{ij}).$$

Applying the result that

$$\frac{\partial H}{\partial \gamma_{ij}} = \gamma^{-1/3}\Big[\frac{\partial H}{\partial h_{ij}} - \frac{1}{3}\frac{\partial H}{\partial h_{kl}}h_{kl}h^{ij}\Big]$$

one finds that the Hubble function satisfies,

$$H^2 = \frac{8}{3}\Big[h_{jk}h_{li}\frac{\partial H}{\partial h_{ij}}\frac{\partial H}{\partial h_{kl}} - \frac{1}{3}\Big(h_{ij}\frac{\partial H}{\partial h_{ij}}\Big)^2\Big] + \frac{m_P^2}{12\pi}\Big(\frac{\partial H}{\partial \phi_k}\Big)^2 + \frac{8\pi}{3m_P^2}V(\phi_k), \tag{2.14b}$$

which will be referred to as the separated Hamilton-Jacobi equation (SHJE). The derivatives appearing in (2.14b) are determined by assuming that all the h_{ij} are independent; only after differentiation does one set $det(h_{ij}) = 1$. The momenta are given by functional derivatives of Hamilton's principal functional S:

$$[\pi^{\gamma}]^{ij} = -\frac{m_P^2}{4\pi}\sqrt{\gamma}\Big[\frac{1}{2}H\gamma^{ij} + \gamma^{-1/3}\Big(\frac{\partial H}{\partial h_{ij}} - \frac{1}{3}\frac{\partial H}{\partial h_{kl}}h_{kl}h^{ij}\Big)\Big], \tag{2.14c}$$

$$[\pi^{\tilde{\gamma}}]^{ij} = \frac{m_P^2}{4\pi}\sqrt{\gamma}\tilde{\gamma}^{-1/3}\Big(\frac{\partial H}{\partial \tilde{h}_{ij}} - \frac{1}{3}\frac{\partial H}{\partial \tilde{h}_{kl}}\tilde{h}_{kl}\tilde{h}^{ij}\Big), \tag{2.14d}$$

$$\pi^{\phi_k} = -\frac{m_P^2}{4\pi}\sqrt{\gamma}\frac{\partial H}{\partial \phi_k}, \tag{2.14e}$$

$$\pi^{\tilde{\phi}_k} = \frac{m_P^2}{4\pi}\sqrt{\gamma}\frac{\partial H}{\partial \tilde{\phi}_k}. \tag{2.14f}$$

It is useful to note that the trace of the gravitational momentum tensor is proportional to the Hubble parameter,

$$\pi^{\gamma} \equiv \gamma_{ij}[\pi^{\gamma}]^{ij} = -\frac{3m_P^2}{8\pi}\sqrt{\gamma}H. \tag{2.15}$$

The SHJE contains no reference to spatial variables nor to the time coordinate; it is actually solved in *field space* where ϕ_k and h_{ij} are the intrinsic variables. By assuming that the Hubble function depends on γ_{ij} only through the combination, $h_{ij} \equiv \gamma^{-1/3}\gamma_{ij}$, one has effectively decomposed the gravitational momentum tensor into a trace contribution (the first term on the right hand side of (2.14c)) and a traceless part which describes evolving gravitational radiation. Similarly, by assuming that the

H depends on $\tilde{\gamma}_{ij}$ only through $\tilde{h}_{ij}$, one finds that the new gravitational momentum tensor is traceless, $[\pi^{\tilde{\gamma}}]^{ij}\tilde{h}_{ij} = 0$.

The determinant of the 3-metric does not enter in (2.14b); this gravitational degree of freedom has been separated from the others using the ansatz (2.14a). For this reason, γ is the natural time variable for classical long wavelength fields. For example, one may invert the canonical transformation (2.14c-f) to obtain ϕ_j and h_{ij} as a function of γ and the new canonical variables, which are constants if the shift vanishes; hence γ is the obvious time choice. However, when one includes quantum diffusion from short wavelength fluctuations, then $T = \ln(H\gamma^{1/6})$ is more useful.[13]

In summary, eqs. (2.14) and the momentum constraint (2.5b) are the fundamental equations of this paper. The evolution equations (2.7) will be of secondary importance. The goal then is find solutions of (2.14b) which depend on arbitrary parameters, $\tilde{\phi}_k$, and $\tilde{\gamma}_{ij}$, which are the new canonical variables. The momentum constraint will then be solved in the last step, as illustrated explicitly in Sec. III and Sec. IV.

III. LONG WAVELENGTH SCALAR FIELD SOLUTIONS NEGLECTING EVOLUTION OF GRAVITATIONAL RADIATION

If one neglects the evolution of gravitational radiation, then one can solve[8] the long wavelength problem of scalar fields interacting through a potential by finding solutions, $H \equiv H(\phi_k)$ of the separated Hamilton-Jacobi equation (2.9b). The momentum constraint is automatically satisfied. Unfortunately, this technique does not apply to the case where the Hubble function depends on the conformal 3-metric h_{ij}. This is the situation where the evolution of gravitational radiation is important. In this section, I will examine the problem of solving n scalar fields without gravitational radiation from a different vantage point: one looks for $n-$parameter solutions of the SHJE equation. These solutions will be called complete because they generate a canonical transformation which describes the general evolution of n scalar fields. This method is sufficiently powerful to encompass gravitational radiation as will be shown in Sec. IV.

In Sec. A, the general theory of scalar fields interacting with an arbitrary potential is developed. In Sec. B, examples of complete solutions of the SHJE are given for m massless scalar fields. In general, it is shown how the m massless fields may be reduced to a single massless degree of freedom. In Sec. C, it is shown, how all the solutions of the SHJE for a given potential may be obtained from a complete solution. In Sec. D and Sec. E, a complete solution is given for two scalar fields, one massless and the other interacting through an exponential potential.

A. General Theory of Scalar Fields with an Arbitrary Potential

If the Hubble function does not depend on the conformal 3-metric, h_{ij}, then the SHJE for n scalar fields and gravity simplifies to eq.(2.9b), which was applied extensively in SB1. The gravitational momentum tensor is then proportional to the 3-metric,

$$[\pi^{\gamma}]^{ij} = -\frac{m_P^2}{8\pi}\sqrt{\gamma}\, H\, \gamma^{ij}.$$

It will now be assumed that the Hubble function depends on n independent parameters, $\tilde{\phi}_k, k = 1, ..., n$,

$$H \equiv H(\phi_k, \tilde{\phi}_k). \tag{3.1}$$

Such solutions are typically not unique. According to (2.14), the new parameters will be interpreted as new canonical variables, whose conjugate momenta, $\pi^{\tilde{\phi}_k}$ are given by (2.14f). After substitution of eq.(2.14e), one finds that the momentum constraint (2.5b) reduces to,

$$\mathcal{H}_i(x) = \frac{m_P^2}{4\pi}\gamma^{1/2}\left(H_{,i} - \frac{\partial H}{\partial \phi_k}\phi_{k,i}\right).$$

In contrast to SB1, I will not assume that any of the new variables are homogeneous. Expanding the spatial gradient of the Hubble parameter,

$$H_{,i} = \frac{\partial H}{\partial \phi_k}\phi_{k,i} + \frac{\partial H}{\partial \tilde{\phi}_k}\tilde{\phi}_{k,i}, \tag{3.2}$$

one finds that the momentum constraint may be written simply in terms of the new variables,

$$0 = \tilde{\mathcal{H}}_i(x) = \pi^{\tilde{\phi}_k}\tilde{\phi}_{k,i}. \tag{3.3}$$

The new Hamiltonian then contains only a contribution from the momentum constraint,

$$\tilde{H}_{am} = \int d^3x N^i \pi^{\tilde{\phi}_k}\tilde{\phi}_{k,i}. \tag{3.4}$$

The time derivative of the new variables are then given by Poisson brackets with the new Hamiltonian:

$$\dot{\tilde{\phi}}_k - N^i\tilde{\phi}_{k,i} = 0, \quad \dot{\pi}^{\tilde{\phi}_k} - \left(N^i\pi^{\tilde{\phi}_k}\right)_{,i} = 0. \tag{3.5}$$

Thus in general, the new canonical variables are not constants in time. However, eq.(3.5) states that they are indeed constants along trajectories *normal* to the time hypersurface. For example, if the spatial coordinates are chosen so that the shift function vanishes, then it is true that $\tilde{\phi}_k(x)$ and $\pi^{\tilde{\phi}_k}(x)$ are constants for a fixed spatial coordinate. Thus, it is not necessary to assume that the shift vanishes, although it simplifies the analysis. From now on in this section, N^i will be set to zero.

The momentum constraint (3.3) may be integrated by dividing out by one of the new momentum variables, say $\pi^{\tilde{\phi}_1}$, giving

$$\tilde{\phi}_{1,i} = -\sum_{k=2}^{n}\frac{\pi^{\tilde{\phi}_k}}{\pi^{\tilde{\phi}_1}}\tilde{\phi}_{k,i} \tag{3.6}$$

whose solution implies that $\tilde{\phi}_1$ is an arbitrary function of the remaining fields, $\tilde{\phi}_k$, $k = 2, ..., n$:

$$\tilde{\phi}_1 = f(\tilde{\phi}_2, \tilde{\phi}_3, ..., \tilde{\phi}_n), \quad \text{with} \quad \tilde{\phi}_k \equiv \tilde{\phi}_k(x), \quad k = 2, ..., n. \tag{3.7a}$$

As a result, one must identify,

$$\pi^{\tilde{\phi}_k}/\pi^{\tilde{\phi}_1} = -\frac{\partial f}{\partial \tilde{\phi}_k} \quad k = 2, ..., n, \quad \text{where} \quad \pi^{\tilde{\phi}_1} \equiv \pi^{\tilde{\phi}_1}(x). \tag{3.7b}$$

$\pi^{\tilde{\phi}_1}$ and $\tilde{\phi}_k$, k=2,...,n are spatially dependent.

For a single scalar field, the integral of the momentum constraint simplifies. $\tilde{\phi}$ can be taken to be spatially homogeneous whereas $\pi^{\tilde{\phi}} \equiv \pi^{\tilde{\phi}}(x)$ can have an arbitrary spatial dependence. In this case, uniform Hubble surfaces are the same as uniform ϕ (comoving) surfaces. The constant parameter ζ first introduced by Bardeen, Steinhardt and Turner[14] in linear perturbation theory to describe metric fluctuations in inflation models is defined to be the variation of $\ln\sqrt{\gamma}$ on a uniform H slice,[8]

$$\zeta(x) \equiv \ln\left[\sqrt{\gamma}(H,x)/(\sqrt{\gamma}(H,x_0)\right] = \ln\left[\pi^{\tilde{\phi}}(x)/\pi^{\tilde{\phi}}(x_0)\right], \quad \text{(Single Scalar Field)} \tag{3.8}$$

(However, note that this quantity is three times their original definition: $\zeta = 3\zeta_{BST}$.) Here, x_0 is some fixed fiducial point. Eq.(3.8) follows from the definition of the new canonical momentum (2.14f). It is important to note that ζ is constant in time even if the Universe is not inflating because the present formalism is valid whenever the wavelength of the fluctuations is larger than the Hubble radius. (An example of a non-inflating universe is given in Sec.D.2). For multiple fields, ζ as defined above is not constant for all times. However, if one field dominates at late times, then asymptotically ζ approaches a constant. For multiple fields, one should in general characterize the system using the constants $\tilde{\phi}_k(x)$, $\pi^{\tilde{\phi}_k}(x)$ which are constrained by (3.7). They characterize both adiabatic and isothermal fluctuations.

In summary, eqs.(3.7a,b) represent a general solution of the momentum constraint: the spatial dependence of $\pi^{\tilde{\phi}_1}$ and $\tilde{\phi}_k$, $k = 2, n$ is arbitrary whereas the remainder of the scalar field variables, $\tilde{\phi}_1$ and $\pi^{\tilde{\phi}_k}$, $k = 2, ..., n$, are constrained in terms of the arbitrary function, f. Using the canonical transformation (2.14e,f), one can determine the evolution of $\phi_k(t,x)$ and $\pi^{\phi_k}(t,x)$ as a function of γ and the constants $\tilde{\phi}_k(x)$ and $\pi^{\tilde{\phi}_k}(x)$ (assuming the shift vanishes). γ then becomes the natural time parameter.

B. Complete Solutions of the SHJE for Massless Fields with a Cosmological Constant

The separated Hamilton-Jacobi equation of m massless scalar fields evolving under the influence of a cosmological constant (neglecting gravitational radiation) is easily solved and provides one with a simple class of canonical transformations. This example also illuminates the more general situation where gravitational radiation also evolves.

1. Solving the SHJE for Massless Fields

Neglecting gravitational radiation, $\partial H/\partial h_{ij} = 0$, the SHJE describing m massless scalar fields evolving through a constant potential, $V(\phi_k) = V_0$ is

$$H^2 = \frac{m_{\mathcal{P}}^2}{12\pi}\sum_{k=1}^{m}\left(\frac{\partial H}{\partial \phi_k}\right)^2 + \frac{8\pi V_0}{3m_{\mathcal{P}}^2}. \tag{3.9}$$

For a single scalar field, the equation may be rearranged,

$$d\phi = \frac{m_{\mathcal{P}}}{\sqrt{12\pi}}\frac{dH}{\sqrt{H^2 - H_0^2}}, \tag{3.10}$$

and it is easily integrated:

$$H(\phi,\tilde{\phi}) = H_0 \cosh\left[\sqrt{12\pi}(\phi-\tilde{\phi})/m_{\mathcal{P}}\right] \quad \text{where} \quad H_0^2 = 8\pi V_0/(3m_{\mathcal{P}}^2). \tag{3.11a}$$

The solution is a function of an arbitrary parameter $\tilde{\phi}$. The relationship between (ϕ, π^{ϕ}) and the new canonical variables $(\tilde{\phi}, \pi^{\tilde{\phi}})$ are given through eqs.(2.14e,f), leading to

$$\phi = \tilde{\phi} - \frac{m_{\mathcal{P}}}{\sqrt{12\pi}} \sinh^{-1}\left[\sqrt{\frac{4\pi}{3}}\gamma^{-1/2}\pi^{\tilde{\phi}}/(H_0 m_{\mathcal{P}})\right], \quad \pi = \pi^{\tilde{\phi}} \tag{3.11b}$$

For a fixed spatial coordinate, the new variables are constants, and eq. (3.11b) describes how the scalar field evolves in time which can be taken to be $\gamma^{1/2}$. $\tilde{\phi}$ is the final value of the scalar field as $\gamma \to \infty$.

According to Sec. A, the momentum constraint can be satisfied if $\tilde{\phi}$ is homogeneous while $\pi^{\tilde{\phi}}$ has an arbitrary spatial dependence. It is certainly unusual that at late times, ϕ assumes a uniform value. This is a peculiar feature of assuming only a single massless scalar field. In this case, the momentum constraint is actually a singular equation. Another possible solution of (3.3) is $\pi^{\tilde{\phi}} = 0$ and $\tilde{\phi} \equiv \tilde{\phi}(x)$ having an arbitrary spatial dependence but no time evolution. This solution cannot be obtained smoothly from the other class of solutions if one employs a single scalar field. When one includes other fields, then the spatial dependence of ϕ_1 at late times is certainly allowed (eq. (3.7)).

For m scalar fields, one may easily guess a solution which depends on m arbitrary parameters, $\tilde{\phi}_k$. One simply takes the solution of the one dimensional problem and replaces $\phi - \tilde{\phi}$ by $[\sum_{k=1}^{m}(\phi_k - \tilde{\phi}_k)^2]^{1/2}$ which will be denoted by $|\vec{\phi} - \vec{\tilde{\phi}}|$:

$$H(\phi_k, \tilde{\phi}_k) = H_0 \cosh\left[\sqrt{12\pi}|\vec{\phi} - \vec{\tilde{\phi}}|/m_{\mathcal{P}}\right]. \tag{3.12a}$$

The validity of the solution may be justified by explicit substitution in the SHJE. The solution is a direct consequence of the rotationally symmetric form of the SHJE equation. The old variables are related to the new ones through,

$$\phi_k = \tilde{\phi}_k - \frac{m_{\mathcal{P}}}{\sqrt{12\pi}} \frac{\pi^{\tilde{\phi}_k}}{|\pi^{\vec{\tilde{\phi}}}|} \sinh^{-1}\left[\sqrt{\frac{4\pi}{3}}\gamma^{-1/2}|\pi^{\vec{\tilde{\phi}}}|/(m_{\mathcal{P}} H_0)\right], \quad \text{with} \quad \pi^{\phi_k} = \pi^{\tilde{\phi}_k}. \tag{3.12b}$$

This equation describes the evolution of $\phi_k(t,x)$ and $\pi^{\phi_k}(t,x)$ in time at a fixed spatial coordinate. Surfaces of constant H are concentric circles about the point $\vec{\tilde{\phi}}$ as shown in Fig. 1. The physical trajectories are straight lines which are orthogonal to the circles.

In order to verify (3.12b), one begins with the canonical transformation (2.14e,f). Because of the symmetry between ϕ_k and $\tilde{\phi}_k$, the canonical transformation eq.(2.14e,f) implies that $\pi^{\tilde{\phi}_k} = \pi^{\phi_k}$, and that

$$\pi^{\tilde{\phi}_k} = \frac{m_{\mathcal{P}}^2}{4\pi}\gamma^{1/2}\frac{\partial H}{\partial \tilde{\phi}_k} = -\sqrt{\frac{3}{4\pi}} H_0 m_{\mathcal{P}}\, \gamma^{1/2} \sinh\left[\sqrt{12\pi}|\vec{\phi} - \vec{\tilde{\phi}}|/m_{\mathcal{P}}\right] \; (\phi_k - \tilde{\phi}_k)/|\vec{\phi} - \vec{\tilde{\phi}}|. \tag{3.13}$$

one squares both sides and then sums over the index k, giving

$$|\pi^{\bar{\vec{\phi}}}| = \sqrt{\frac{3}{4\pi}} H_0 m_{\mathcal{P}} \gamma^{1/2} \sinh\left[\sqrt{12\pi}|\vec{\phi} - \bar{\vec{\phi}}|/m_{\mathcal{P}}\right], \tag{3.14}$$

where $|\pi^{\bar{\vec{\phi}}}|$ denotes $(\sum_k {\pi^{\bar{\phi}_k}}^2)^{1/2}$. Dividing eq.(3.13) by (3.12), one finds that $\pi^{\bar{\phi}_k}$ and ϕ_k share the same direction,

$$\frac{\pi^{\bar{\phi}_k}}{|\pi^{\bar{\vec{\phi}}}|} = -\frac{(\phi_k - \bar{\phi}_k)}{|\vec{\phi} - \bar{\vec{\phi}}|}.$$

Solving for ϕ_k and then applying (3.14) leads to the stated relationship between the new and old variables, (3.12b).

In this instance, it was trivial to guess the solution (3.12a) of the SHJE for multiple fields given the one dimensional solution (3.11a). However, it will prove useful to develop more systematic methods which will guide our analysis through non-trivial problems.

2. Integration of SHJE through Evolution Equations

Consider a solution of the SHJE equation which describes trajectories emanating from a single point in field space, $\phi_k = \bar{\phi}_k$. Given some initial direction for the momentum of the scalar field Π^{ϕ_k}, one may integrate the equations of motion,

$$\pi^\gamma \equiv -\frac{3m_{\mathcal{P}}^2}{8\pi}\gamma^{1/2} H = -\left(\frac{3m_{\mathcal{P}}^2}{8\pi}\right)^{1/2}\left[\frac{1}{2}{\pi^{\phi_k}}^2 + V_0\gamma\right]^{1/2} \tag{3.15a}$$

$$\frac{\dot{\phi}_k}{N} = \gamma^{-1/2}\pi^{\phi_k} \tag{3.15b}$$

$$\frac{\dot{\gamma}}{N} = -\frac{16\pi}{m_{\mathcal{P}}^2}\gamma^{1/2}\pi^\gamma \tag{3.15c}$$

$$\frac{\dot{\pi}^{\phi_k}}{N} = 0. \tag{3.15d}$$

to find H as some function of time. Note that π^γ is actually negative for an expanding Universe. The SHJE will then be integrated through the method of characteristics. Each point $\vec{\phi}$ in the vicinity of $\bar{\vec{\phi}}$ will have a trajectory passing through it, and one associates with it the value of the Hubble parameter that a trajectory has when it passes through that point, $H \equiv H(\vec{\phi})$. To determine the trajectory, one notes that the solution of eq.(3.15d) is trivial implying that

$$\pi^{\phi_k} = \pi^{\bar{\phi}_k} \tag{3.16}$$

where $\pi^{\bar{\phi}_k}$ are constant parameters. In order to integrate the remaining equations, one must decide on a choice of time parameter, and the natural choice is γ itself because the Hubble parameter is a function of it, eq. (3.15a). In this case, eq.(3.15c) which defines the momentum of γ becomes the definition of the lapse function,

$$N = -\frac{m_{\mathcal{P}}^2}{16\pi}\gamma^{-1/2}/\pi^\gamma. \tag{3.17}$$

Equation (3.15b),

$$d\phi_k = -\left(\frac{m_P^2}{24\pi}\right)^{1/2} \frac{\pi^{\tilde{\phi}_k} d(\gamma^{-1/2})}{\left[\frac{1}{2}\left(\pi^{\tilde{\phi}_k}\right)^2 \gamma^{-1} + V_0\right]^{1/2}}$$

may then be integrated giving ϕ_k as a function of γ, eq.(3.12b). I now wish to determine the Hubble parameter at point ϕ_k given that the trajectory started at $\tilde{\phi}_k$. This amounts to finding an expression for γ along a fixed trajectory. Subtracting $\tilde{\phi}_k$ from both sides of (3.12b), squaring and summing both sides, one rederives (3.14) which when substituted into (3.15a) leads immediately to $H \equiv H(\phi_k)$, eq.(3.12a).

3. Reduction of the Number of Degrees of Freedom

From the solutions of Sec. 1 and 2, one sees that m scalar fields essentially act as a single scalar field. This holds more generally even if there are additional scalar fields, ϕ_k, $k = m+1, m+n$ which interact through some potential, $V \equiv V(\phi_{m+1}, ..., \phi_{m+n})$. One may reduce the m massless degrees of freedom to single one by considering solutions for the Hubble function of the form

$$H \equiv H(u, \phi_{m+1}, ..., \phi_{m+n}), \quad \text{where} \quad u = [\sum_{k=1}^{m} (\phi_k - \tilde{\phi}_k)^2]^{1/2}. \tag{3.18a}$$

The reduced SHJE then depends on one massless field, u, and the n additional fields,

$$H^2 = \frac{m_P^2}{12\pi}\left[\left(\frac{\partial H}{\partial u}\right)^2 + \sum_{k=m+1}^{m+n} \left(\frac{\partial H}{\partial \phi_k}\right)^2\right] + \frac{8\pi}{3m_P^2} V(\phi_{m+1}, ...\phi_{m+n}). \tag{3.18b}$$

One need only find a solution of (3.18b) depending on n independent parameters to define a complete solution of the original SHJE because (3.18a) already depends on m parameters, $\tilde{\phi}_1, ..., \tilde{\phi}_m$, through u. Such a reduction of the number of variables may be effected whenever there is a symmetry in the system. In this case, the massless scalar fields are rotationally symmetric (as well as translationally symmetric, although this would suggest a different class of solutions; see, for example, Sec. E).

Massless degrees of freedom may become important in an epoch after inflation if they develop a potential at some lower energy scale. These fluctuations are called isothermal fluctuations. However, at late times during the inflation epoch, when all decaying modes are no longer important, the evolution is trivial. All the massless fields have an arbitrary spatial dependence which is constant in time. The field that drives inflation has a non-trivial potential. After the decaying modes have died, it evolves independently of the massless degrees of freedom and may be treated as a single scalar field.

C. General Solutions and Green's Function Solutions of the Separated Hamilton-Jacobi Equation

An n-parameter solution of the SHJE for n scalar fields is complete in the sense that it characterizes all possible solutions to the equations of motion. For example, it will be shown how almost all solutions of the nonlinear SHJE may be derived from such a solution. This section is more abstract than the others and should perhaps be skipped over on first reading. It is nonetheless important because it links the analysis of this paper with that of SB1.

Given a solution, eq (3.1) of the SHJE equation which depends on n constant parameters, $\tilde{\phi}_l$, $l = 1, ..., n$, one may obtain another solution by allowing the $\tilde{\phi}_l$ to depend on ϕ_k in a very special way. The $\tilde{\phi}_l$ will be chosen to minimize (or maximize) the Hubble function, $H(\phi_k, \tilde{\phi}_l)$, holding ϕ_k fixed, provided that $\tilde{\phi}_l$ is constrained to lie on some fixed surface, $g(\tilde{\phi}_l) = 0$. Hence for small arbitrary variations $d\tilde{\phi}_l$ of the parameters, one requires that

$$dH = \left(\frac{\partial H}{\partial \tilde{\phi}_l}\right)_\phi d\tilde{\phi}_l = 0. \tag{3.19}$$

The resulting Hubble function $H(\phi_k, \tilde{\phi}_l(\phi_k))$ is also a solution of the SHJE because its partial derivative with respect to ϕ_k,

$$\frac{\partial}{\partial \phi_k} H(\phi, \tilde{\phi}(\phi)) = \left(\frac{\partial H}{\partial \phi_k}\right)_{\tilde{\phi}} + \left(\frac{\partial H}{\partial \tilde{\phi}_l}\right)_\phi \frac{\partial \tilde{\phi}_l}{\partial \phi_k} = \left(\frac{\partial H}{\partial \phi_k}\right)_{\tilde{\phi}}, \tag{3.20}$$

reduces to the standard partial derivative holding $\tilde{\phi}_l$ fixed by virtue of (3.19).

The extrema procedure is a powerful method of generating solutions of the SHJE. It will be illustrated using the complete solution for m massless scalar fields, eq.(3.12a). I wish to find a solution of the SHJE (3.9a) which has the value H_ψ at the point $\vec{\psi}$ and which has trajectories emanating from ψ in all possible directions. In analogy to the terminology of linear differential equations, such a solution will be referred to as the Green's function, and will be denoted by $H(\phi|\psi, H_\psi)$. The parameters $\vec{\tilde{\phi}}$ appearing in the complete solution (3.12a) will therefore be constrained such that the Hubble parameter has the value H_ψ at point ψ. Thus, $\vec{\tilde{\phi}}$ must lie along a sphere centered about $\vec{\psi}$ with radius,

$$r = |\vec{\psi} - \vec{\tilde{\phi}}| = \frac{m_P}{\sqrt{12\pi}} \cosh^{-1}(H_\psi / H_0). \tag{3.21}$$

The Hubble parameter $H(\phi|\psi, H_\psi)$ at point $\vec{\phi}$ is just the minimum (or maximum) value of (3.12a) given that $\vec{\tilde{\phi}}$ is restricted by (3.21). Since in this particular case, $H(\vec{\phi}, \vec{\tilde{\phi}})$ is a function only of the distance between $\vec{\phi}$ and $\vec{\tilde{\phi}}$, it is clear geometrically that $\vec{\tilde{\phi}}$ must be collinear with $\vec{\phi}$ and $\vec{\psi}$, and the point giving the minimum (maximum) value of the Hubble parameter is (see Fig. 2)

$$\vec{\tilde{\phi}} = \vec{\psi} \pm r(\vec{\phi} - \vec{\psi})/|\vec{\phi} - \vec{\psi}|. \tag{3.22}$$

The value of the Hubble parameter is then

$$H(\vec{\phi}|\vec{\psi}, H_\psi) \equiv H(\vec{\phi}, \vec{\tilde{\phi}}(\vec{\phi})) = H_0 \cosh\left[\sqrt{\frac{12\pi}{m_P^2}} |\vec{\phi} - \vec{\psi}| \mp \cosh^{-1}(H_\psi / H_0)\right]. \tag{3.23}$$

which may be shown explicitly to be a solution of the SHJE satisfying $H(\vec{\phi} = \vec{\psi}|\vec{\psi}, H_\psi) = H_\psi$.

The Green's function solution allows one to solve the initial value problem for the corresponding SHJE. Given that the Hubble parameter is constant, $H = H'$, on some $n-1$ dimensional surface, $g(\phi) = 0$, what is the solution of the SHJE equation in the vicinity of this surface? At the point $\vec{\phi}$, the required Hubble parameter is the minimum (or maximum) value of of the Green's function $H(\phi|\psi, H_\psi = H')$ where one considers all variations of the parameters ψ which lie on the surface $g(\psi) = 0$. The resulting solution of the SHJE satisfies the initial data, but it need not be unique, as the extrema problem may have several solutions.

The extrema method of obtaining solutions to the SHJE is not an obvious result, and one may wonder how it was motivated. In fact, the momentum constraint (neglecting gravitational radiation) suggested such a property. Using (3.3), one may write the momentum constraint as $(\frac{\partial H}{\partial \tilde{\phi}_k}) d\tilde{\phi}_k = 0$, which is just (3.19). The interpretation is clear: assuming that ϕ_k is fixed, H is minimized respect to variations of $\tilde{\phi}_k$ which are consistent with some constraint on $\tilde{\phi}_k$, say (3.7a).

In SB1, it was shown that if one neglected gravitational radiation, then the Hubble parameter was only a function of the scalar fields, $H \equiv H(\vec{\phi})$. There was no explicit time or spatial dependence. However, the solution (3.1) of the SHJE depends on parameters $\tilde{\phi}_k \equiv \tilde{\phi}_k(x)$ which are allowed to have spatial dependence through the integration of the momentum constraint (3.7b). How does one reconcile the two approaches? First one must realize that the function of two arguments $H(\phi, \tilde{\phi})$ that appears in (2.14a) and (3.1) actually generates a canonical transformation which solves the long wavelength problem. In Sec. II of SB1, the Hubble parameter actually referred to $H(t,x) = -8\pi/(3m_P^2)\gamma^{-1/2}\pi^\gamma$ expressed as a function time and space. In general, these two functions are different (for example, they require different input arguments), but only *after* one solves for the evolution equations as well as the constraints are the two the same. For example, one may use the canonical transformation (2.14f) to write the integral of the momentum constraint (3.7a,b) as

$$\frac{\partial H}{\partial \tilde{\phi}_k} / \frac{\partial H}{\partial \tilde{\phi}_k} = -\frac{\partial f}{\partial \tilde{\phi}_j}, \quad \text{with} \quad \tilde{\phi}_1 = f(\tilde{\phi}_2, \tilde{\phi}_3, ..., \tilde{\phi}_n). \tag{3.24a}$$

Consequently, one may solve for $\vec{\tilde{\phi}} \equiv \vec{\tilde{\phi}}(\vec{\phi})$ in terms of $\vec{\phi}$, and then substitute the result into (3.1) to find a solution of the Hubble parameter which depends only on the scalar fields,

$$H \equiv H(\vec{\phi}) = H\big(\vec{\phi}, \vec{\tilde{\phi}}(\vec{\phi})\big). \tag{3.24b}$$

In fact the resulting Hubble function is just that given by minimizing (3.1) where $\tilde{\phi}$ is allowed to vary on the surface given by (3.7a), $0 = g(\tilde{\phi}) = -\tilde{\phi}_1 + f(\tilde{\phi}_2, \tilde{\phi}_3, ..., \tilde{\phi}_n)$. To see this, it is useful to introduce a Lagrange multiplier, λ and then minimize

$$H(\vec{\phi}, \vec{\tilde{\phi}}) - \lambda g(\vec{\tilde{\phi}}).$$

Variations in $\tilde{\phi}$ holding $\vec{\phi}$ fixed imply that

$$\frac{\partial H}{\partial \tilde{\phi}_1} = -\lambda, \quad \frac{\partial H}{\partial \tilde{\phi}_k} = \lambda \frac{\partial f}{\partial \tilde{\phi}_k}, \quad k = 2, .., n.$$

Combining these two equations by eliminating the Lagrange multiplier leads directly to eq.(3.24a). Thus after one solves the evolution equations, the Hubble function (3.1) and that used in Sec. II of SB1, eq.(3.24b), are identical.

This section has shown how to derive solutions of the SHJE. From a complete solution, one constructs the Green's function, $H(\vec{\phi}|\vec{\psi}, H_\psi)$ describing the solution with trajectories of all possible directions emanating from the point $\vec{\psi}$ with $H(\vec{\phi} = \vec{\psi}|\vec{\psi}, H_\psi) = H_\psi$. The Green's function then allows one to solve the initial value problem, where the Hubble parameter is constant on some initial $n-1$ dimensional surface. In this way, one can obtain all solutions of the SHJE from an n-parameter solution. In addition, one may use the extrema method to show that the technique of solution applied in SB1 is actually consistent with this paper.

D. Scalar Field Interacting via Exponential Potentials

The evolution of a single scalar field will be considered when the Universe undergoes a phase transition from an inflation epoch to a typical Friedman-Robertson-Walker era where the scale factor varies as $a(t) \equiv \gamma^{1/6} \propto t^p$, $p < 1$. The transition will be modelled by patching together two exponential potentials.

In Sec. D.1, the evolution of a scalar field under an exponential potential will be quickly reviewed. In Sec. D.2, one will apply this result to model a phase transition.

1. Review of a Scalar Field with an Exponential Potential

The SHJE of a single scalar field interacting with an exponential potential,

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{16\pi}{p}}\frac{\phi}{m_{\mathcal{P}}}\right), \tag{3.25}$$

can be solved exactly.[8] Here the constant p describes the flatness of the potential. In the limit that $p \to \infty$, the slow-roll approximation, $H_{SR}(\phi) = (8\pi V(\phi)/(3m_{\mathcal{P}}^2))^{1/2}$ is an exact solution. A one parameter solution which is valid for all positive p was given in SB1:

$$H(\phi, \tilde{\phi}, p) = \left(\frac{8\pi V_0}{3m_{\mathcal{P}}^2}\right)^{1/2} \exp\left(-\sqrt{\frac{4\pi}{p}}\frac{\phi}{m_{\mathcal{P}}}\right)\cosh(u). \tag{3.26a}$$

where u is a function of ϕ, $\tilde{\phi}$ and p defined through,

$$\phi = \tilde{\phi} - \frac{m_{\mathcal{P}}}{\sqrt{12\pi}}\frac{1}{1-1/(3p)}\left[u + \frac{1}{\sqrt{3p}}\ln|\cosh(u) - \sqrt{3p}\sinh(u)|\right]. \tag{3.26b}$$

Once again, the canonical transformation is given by differentiation with respect to ϕ and $\tilde{\phi}$,

$$\pi^\phi = \sqrt{2V_0}\gamma^{1/2}\exp\left(-\sqrt{\frac{4\pi}{p}}\frac{\phi}{m_{\mathcal{P}}}\right)\sinh(u), \qquad \pi^{\tilde{\phi}} = \pi^\phi - \frac{m_{\mathcal{P}}}{\sqrt{4\pi p}}\gamma^{1/2}H.$$

As u approaches $\tanh^{-1}(1/\sqrt{3p})$, the Hubble function approaches the attractor solution,

$$H_{att}(\phi, p) = \left(\frac{8\pi V_0}{3m_{\mathcal{P}}^2(1-1/(3p))}\right)^{1/2}\exp\left(-\sqrt{\frac{4\pi}{p}}\frac{\phi}{m_{\mathcal{P}}}\right).$$

This solution which is valid for $p > 1/3$ describes the power-law evolution of the Universe, $a(t) \propto t^p$, where the time parameter corresponds to a choice of unit lapse function in the metric (2.2a) (synchronous gauge).

The parameter u has physical significance as it measures the deviation of π^ϕ from the attractor solution momentum density:

$$u = \sinh^{-1}\left[(2V_0)^{-1/2}\gamma^{-1/2}\exp\left(\sqrt{\frac{4\pi}{p}}\frac{\phi}{m_\mathcal{P}}\right)\pi^\phi\right]. \tag{3.27a}$$

For $u < \tanh^{-1}(1/\sqrt{3p})$, the momentum of the field is below that of the attractor, while the opposite is true for $u > \tanh^{-1}(1/\sqrt{3p})$. Using (3.26a), one obtains the following expression for the new canonical variables $(\tilde{\phi}, \pi^{\tilde{\phi}})$,

$$\tilde{\phi} = \phi + \frac{m_\mathcal{P}}{\sqrt{12\pi}}\frac{1}{1-1/(3p)}\left[u + \frac{1}{\sqrt{3p}}\ln|\cosh(u) - \sqrt{3p}\sinh(u)|\right] \tag{3.27b}$$

$$\pi^{\tilde{\phi}} = \pi^\phi\left[1 - \frac{1}{\sqrt{3p}\tanh(u)}\right]. \tag{3.27c}$$

The solution (3.26) of the SHJE is the prototype for numerous others.

2. Phase Transition in the Universe

A phase transition in the Universe may be modelled by considering a single scalar field interacting through the potential,

$$\begin{aligned} V(\phi) &= V_0\exp\left(-\sqrt{\frac{16\pi}{p_-}}\frac{\phi}{m_\mathcal{P}}\right), \quad \text{for } \phi \le 0, \\ &= V_0\exp\left(-\sqrt{\frac{16\pi}{p_+}}\frac{\phi}{m_\mathcal{P}}\right), \quad \text{for } \phi > 0, \end{aligned} \tag{3.28}$$

composed by joining two exponentials with $p_- > 1$ for $\phi < 0$ and $p_+ < 1$ for $\phi < 0$. For negative values of ϕ, the Universe inflates, $a(t) \propto t^{p_-}$, whereas for positive values it evolves with a different power-law index, $a(t) \propto t^{p_+}$ which imitates a matter dominated ($p_+ = 2/3$) or a radiation dominated ($p_+ = 1/2$) Universe. A solution of the SHJE which depends on single parameter $\tilde{\phi}_-$ is just

$$\begin{aligned} H_{pt}(\phi, \tilde{\phi}_-) &= H(\phi, \tilde{\phi}_-, p_-), \quad \text{for } \phi \le 0, \\ &= H(\phi, \tilde{\phi}_+(\tilde{\phi}_-), p_+), \quad \text{for } \phi > 0, \end{aligned} \tag{3.29}$$

where the function H was given in eq.(3.26). $\tilde{\phi}_+(\tilde{\phi}_-)$ is a function of $\tilde{\phi}_-$ which is parametrically given in terms of u_0,

$$0 = \tilde{\phi}_- - \frac{m_\mathcal{P}}{\sqrt{12\pi}}\frac{1}{1-1/(3p_-)}\left[u_0 + \frac{1}{\sqrt{3p_-}}\ln|\cosh(u_0) - \sqrt{3p_-}\sinh(u_0)|\right]; \tag{3.30a}$$

$$0 = \tilde{\phi}_+ - \frac{m_\mathcal{P}}{\sqrt{12\pi}}\frac{1}{1-1/(3p_+)}\left[u_0 + \frac{1}{\sqrt{3p_+}}\ln|\cosh(u_0) - \sqrt{3p_+}\sinh(u_0)|\right]. \tag{3.30b}$$

Thus given $\tilde{\phi}_-$, one solves for $u_0(\tilde{\phi}_-)$ and then substitutes in (3.30b) to find $\tilde{\phi}_+ \equiv \tilde{\phi}_+(\tilde{\phi}_-)$.

This solution is found by joining continuously across $\phi = 0$ the solutions, $H(\phi, \tilde{\phi}_-, p_-)$ and $H(\phi, \tilde{\phi}_+, p_+)$, for two exponential potentials. Since these solutions are parametrized by u in (3.26), it proves convenient to assume that the same parameter describes both solutions, and that it is continuous at the junction $\phi = 0$ where $u = u_0$. Continuity of the Hubble parameter at $\phi = 0$ then leads to relation between $\tilde{\phi}_-$ and $\tilde{\phi}_+$, eq.(3.30).

In Fig. 3, the Hubble parameter $H_{pt}(\phi, \tilde{\phi}_-)$ is plotted for the phase transition between $p = 2$ and $p = 1/2$. For $\phi < 0$, the scalar field evolution was given by the attractor solution, $\tilde{\phi}_- = -\infty$. After $\phi = 0$, the Hubble parameter quickly decays to the new attractor solution. The derivative of the Hubble parameter, $(\partial H/\partial\phi)$, is continuous at $\phi = 0$.

It should be emphasized that $\pi^{\tilde{\phi}_-}(x) = m_P^2/(4\pi)(\partial H_{pt}/\partial\tilde{\phi}_-)$ is constant in time; there are no jumps at $\phi = 0$. This is a general feature for Hamilton-Jacobi theory. In particular, for a single scalar field, one should note that if the fluctuations are initially Gaussian, then they remain Gaussian in evolution independent of the choice of potential $V(\phi)$. The treatment of a phase transition presented here has proven to be relatively simple. Analogous analyses using linear perturbation theory[22,23] are actually more complicated because one must make a gauge choice.

E. Multiple Scalar Fields with Linear $\ln V(\phi_k)$

If the logarithm of the potential for multiple scalar fields is linear, $\ln V(\phi_k) = \sum_k a_k\phi_k$, where the a_k are constants, one can derive a complete solution of the SHJE. For n scalar fields, the canonical transformation (2.14e,f) yields $2n$ constants of integration. By considering the asymptotic behaviour of the fields after all decaying modes have died away, one obtains a nonlinear generalization for ζ for multiple scalar fields that characterizes the adiabatic initial conditions for structure formation.

For simplicity, I will consider only two scalar fields interacting through the potential,

$$V(\phi_1, \phi_2) = V_0 \exp\left(-\sqrt{\frac{16\pi}{p}}\frac{\phi_2}{m_P}\right). \tag{3.31}$$

Of course, by rotating the fields (ϕ_1, ϕ_2), one can obtain the general form for linear $\ln V(\phi_1, \phi_2)$ but the expression (3.31) clearly identifies the inflaton, ϕ_2, and the massless scalar field, ϕ_1. The resulting SHJE is,

$$H^2 = \frac{m_P^2}{12\pi}\left[\left(\frac{\partial H}{\partial\phi_1}\right)^2 + \left(\frac{\partial H}{\partial\phi_2}\right)^2\right] + \frac{8\pi V_0}{3m_P^2}\exp\left(-\sqrt{\frac{16\pi}{p}}\frac{\phi_2}{m_P}\right). \tag{3.32}$$

One of the reasons for considering a massless scalar field is that one may reduce the classical gravitational degrees of freedom to a single massless scalar field (Sec. IV). The analytic solution presented here may also be used to construct models with non-Gaussian fluctuations for structure formation.[24]

1. Complete Solution of the SHJE

A complete solution the SHJE (3.32) which depends on two constant parameters, b and m is:

$$H(\phi_1,\phi_2;b,m) = \left(\frac{8\pi V_0}{3m_P^2}\right)^{1/2}\sqrt{\frac{(3p)(m^2+1)}{m^2(3p-1)+3p}}\exp\left(-\sqrt{\frac{4\pi}{p}}\frac{\phi_2}{m_P}\right)\cosh(u) \qquad (3.33a)$$

where u is a function of b, m, ϕ_1 and ϕ_2 which is defined implicitly through,

$$\sqrt{12\pi}(\phi_2 - m\phi_1 - b)/m_P = -\frac{\sqrt{3p}}{3p-1}\times$$
$$\left[u\sqrt{m^2(3p-1)+3p} + \ln\left|\cosh(u) - \sinh(u)\sqrt{m^2(3p-1)+3p}\right|\right]. \quad (3.33b)$$

For $m = 0$, this solution reduces to the single scalar field result, eq. (3.26). The Hamilton-Jacobi map for $p = 3$ is shown in Fig. 4, where the solid lines are surfaces of constant Hubble parameter and the broken lines are the trajectories. The trajectories start at the lower right hand corner and move to the top of the figure. Initially, the kinetic energy of the fields is much larger than the potential energy, and the fields evolve like two massless fields. The slope of the trajectory for large ϕ_1 is $-1/m$. In Fig. 4, $m = 1$ is plotted. Because the Universe is expanding, the velocities are damped, and the massless scalar field ϕ_1 reaches a constant value while ϕ_2 moves to larger values as the potential drags it downward. The surfaces of constant Hubble parameter are orthogonal to the trajectories.

The solution (3.33) may be verified by straightforward differentiation, but the derivation proves instructive. The first step is to remove the explicit ϕ_2 dependence in the SHJE by defining a new dependent parameter, $h(\phi_1,\phi_2)$:

$$H(\phi_1,\phi_2) = \left(\frac{8\pi V_0}{3m_P^2}\right)^{1/2}\exp\left(-\sqrt{\frac{4\pi}{p}}\frac{\phi_2}{m_P}\right) h(\phi_1,\phi_2). \qquad (3.34a)$$

which leads to the following equation:

$$h^2 = \frac{m_P^2}{12\pi}\left[\left(\frac{\partial h}{\partial\phi_1}\right)^2 + \left(\frac{\partial h}{\partial\phi_2} - \sqrt{\frac{4\pi}{p}}\frac{h}{m_P}\right)^2\right] + 1. \qquad (3.34b)$$

This equation is translationally invariant which suggests a complete solution of the form,

$$h \equiv h(v), \quad \text{with,} \quad v = \sqrt{12\pi}(\phi_2 - m\phi_1 - b)/m_P, \qquad (3.35a)$$

where b and m are constant parameters, leading to an equation in a single variable, v:

$$h^2 = m^2\left(\frac{\partial h}{\partial v}\right)^2 + \left(\frac{\partial h}{\partial v} - \frac{h}{\sqrt{3p}}\right)^2 + 1. \qquad (3.35b)$$

Again, I have reduced the effective number of degrees of freedom by taking advantage of a symmetry. After solving for $\partial h/\partial v$,

$$\frac{\partial h}{\partial v} = \frac{h}{\sqrt{3p}(m^2+1)} \pm (m^2+1)^{-1/2}\left[h^2\frac{m^2(3p-1)+3p}{3p(m^2+1)} - 1\right]^{1/2},$$

one can make a change of variables from h to u,

$$h = \sqrt{\frac{(3p)(m^2+1)}{(m^2(3p-1)+3p)}}\cosh(u), \tag{3.36}$$

leading to the integral

$$\begin{aligned} v(u) = &-\frac{\sqrt{3p}}{3p-1}\times \\ &\left[u\sqrt{m^2(3p-1)+3p} + \ln\left|\cosh(u) - \sinh(u)\sqrt{m^2(3p-1)+3p}\right|\right]. \end{aligned} \tag{3.37}$$

Combining (3.34a), (3.35a), (3.36) and (3.37), one obtains the final expression for the Hubble parameter, eq.(3.33).

A surface of constant Hubble parameter, H may be plotted, if, given, ϕ_2, one solves u through

$$u = \cosh^{-1}\left[\frac{H}{\left(\frac{8\pi V_0}{3m_{\mathcal{P}}^2}\right)^{1/2}}\exp\left(\sqrt{\frac{4\pi}{p}}\phi_2/m_{\mathcal{P}}\right)\sqrt{\frac{m^2(3p-1)+3p}{3p(m^2+1)}}\right], \tag{3.38a}$$

and then ϕ_1 is calculated through

$$\phi_1 = \left(\phi_2 - b - \frac{m_{\mathcal{P}}}{\sqrt{12\pi}}v(u)\right)/m \tag{3.38b}$$

where $v \equiv v(u)$ was given in (3.37).

2. Constants of Integration for Cosmology with Linear $\ln V(\phi_k)$

The canonical transformation is obtained found by straightforward although tedious differentiation using (2.14e,f) and the results are given in Appendix A, eq.(A1). For future applications, it will be useful to invert the transformation to obtain expressions for the new canonical variables b, m, π^b, π^m in terms of the old variables ϕ_1, ϕ_2, π^{ϕ_1}, π^{ϕ_2}:

$$\pi^b = \pi^{\phi_2} - \frac{1}{\sqrt{3p}}\left[{\pi^{\phi_1}}^2 + {\pi^{\phi_2}}^2 + 2\gamma V_0 \exp\left(-\sqrt{\frac{16\pi}{p}}\frac{\phi_2}{m_{\mathcal{P}}}\right)\right]^{1/2}, \tag{3.39a}$$

$$m = -\frac{\pi^{\phi_1}}{\pi^b}; \tag{3.39b}$$

It is convenient to introduce auxiliary fields, u, v,

$$u = \sinh^{-1}\frac{1}{\sqrt{(3p-1)(m^2+1)}} - \sinh^{-1}v,$$

$$v = [2(3p-1)V_0]^{-1/2}\sqrt{m^2(3p-1)+3p}\,m^{-1}\gamma^{-1/2}\exp\left(\sqrt{\frac{4\pi}{p}}\frac{\phi_2}{m_{\mathcal{P}}}\right)\pi^{\phi_1},$$

in which case the remaining new canonical variables are,

$$b = \phi_2 - m\phi_1 + \sqrt{\frac{pm_P^2}{4\pi}}\frac{1}{(3p-1)}$$
$$\left[u\sqrt{m^2(3p-1)+3p} + \ln\left|\cosh(u) - \sinh(u)\sqrt{m^2(3p-1)+3p}\right|\right]. \quad (3.39c)$$

$$\pi^m = \pi^b\{\phi_1 - \sqrt{\frac{pm_P^2}{4\pi}}(\frac{m}{m^2(3p-1)+3p} + \frac{mu}{\sqrt{m^2(3p-1)+3p}})\}. \quad (3.39d)$$

Eqs.(3.39a-d) are the 4 integration constants that completely characterize the evolution of the two scalar field system with potential (3.31).

The equations for the trajectories in scalar field space, $\phi_2 \equiv \phi_2(\phi_1)$, follow by eliminating the auxiliary variable u from eqs.(3.39c,d). Given ϕ_1, one first defines u (which now is simply interpreted as some intermediate parameter) and then one determines ϕ_2:

$$u = u_0 + \sqrt{\frac{4\pi}{pm_P^2}}\sqrt{m^2(3p-1)+3p}\,m^{-1}(\phi_1 - \phi_{1\,min}),$$
$$\phi_2 = m\phi_1 + b + \frac{m_P}{\sqrt{12\pi}}v(u), \quad (3.40a)$$

where once again, $v(u)$ was defined in (3.37) and $u_0 = \tanh^{-1}[m^2(3p-1)+3p]^{-1/2}$, and

$$\phi_{1\,min} = \frac{\pi^m}{\pi^b} + \sqrt{\frac{pm_P^2}{4\pi}}[\frac{m}{m^2(3p-1)+3p} + \frac{m}{\sqrt{m^2(3p-1)+3p}}u_0]. \quad (3.40b)$$

The late time evolution of the fields determines microwave background fluctuations as well as the initial conditions for structure formation. As $\gamma \to \infty$, the decaying modes are no longer dynamically important. ϕ_1 approaches $\phi_{1\,min}$, eq.(3.40b), and ϕ_2 evolves according to the attractor solution,[8]

$$\ln(\sqrt{\gamma}) = 3\sqrt{4\pi p}\frac{\phi_2}{m_P} + f(b, m, \pi^b, \pi^m), \quad (3.41)$$

where f is a constant along the trajectory and hence is a function of the new canonical variables; its form is written explicitly in Appendix A, eq.(A3). At late times, the metric fluctuation on a uniform ϕ_2 slice is then given by, ζ (see SB1),

$$\zeta(x) \equiv \Delta_{\phi_2}\ln(\sqrt{\gamma}) = \Delta f(b(x), m(x), \pi^b(x), \pi^m(x))$$
$$= \Delta\Big[\ln|\pi^b| + \frac{1}{2}\ln[(m^2+1)(m^2(3p-1)+3p)]$$
$$- \frac{\sqrt{36\pi p}}{m_P}(1 - \frac{1}{3p})(m\phi_{1\,min} + b) + \sqrt{m^2(3p-1)+3p}\,u_0\Big] \quad (3.42a)$$

where

$$\phi_{1min} = \frac{\pi^m}{\pi^b} + \sqrt{\frac{pm_P^2}{4\pi}}\left[\frac{m}{m^2(3p-1)+3p} + \frac{m}{\sqrt{m^2(3p-1)+3p}}\, u_0\right]. \qquad (3.42b)$$

and

$$u_0 = \tanh^{-1}\frac{1}{\sqrt{m^2(3p-1)+3p}}. \qquad (3.42c)$$

Here it is understood that b, m, π^b and π^m are spatially dependent constants, and that the difference Δ_{ϕ_2} is taken between x and some arbitrary but fixed fiducial point x_0, *e.g.*,

$$\Delta_{\phi_2} \ln|\pi^b| = \Delta \ln|\pi^b| = \ln|\pi^b(x)| - \ln|\pi^b(x_0)|. \qquad (3.43)$$

Eq.(3.42) is the nonlinear generalization of ζ to multiple fields interacting via an exponential potential. It is the quantity of primary interest for structure formation. For example, in the Cold Dark Matter Model, microwave background anisotropies at angular scales greater than $\sim 1^0$ are proportional to ζ,[13]

$$\Delta T_{cmb}/T_{cmb} = \zeta/15. \qquad (3.44)$$

Eq. (3.42) will be play a important role in developing non-Gaussian models for galaxy formation from nonlinear long wavelength evolution.[24]

This example illustrates the power of the Hamilton-Jacobi formalism. Using the evolution equations (2.7), it would have been very difficult indeed to obtain this exact general solution. The biggest stumbling block is the choice of time which is readily resolved using the SHJE. In addition, the SHJE can be solved because one can take advantage of symmetries (see eq.(3.34b)) that are not apparent in the equations of motion.

IV. LONG WAVELENGTH SCALAR FIELD SOLUTIONS INCLUDING GRAVITATIONAL RADIATION

It is shown how the complicated interaction of gravitational degrees of freedom appearing in the separated Hamilton-Jacobi eq.(2.14b) may be reduced to that of single massless scalar field. This result is motivated by linear perturbation theory where one can reduce the gravitational radiation equations to those describing massless fields,[25] independent of the wavelength of the fluctuations (see, for example, Sahni[26] who has given elegant exact solutions). However, the results given here are proven in a nonlinear context for long wavelength fields. The canonical transformation linking the old and new gravitational variables is also derived. The momentum constraint may be simply expressed in terms of the new variables. Explicit complete solutions are given for the case of a pure cosmological constant as well as for the case of a single scalar field interacting through an exponential potential.

A. Canonical Transformation for Gravitational Degrees of Freedom

In Sec. III.B, it was shown that if the Hamilton-Jacobi equation describes m massless scalar fields, then these degrees of freedom may essentially be reduced to a single massless scalar field, even if there are other interacting fields present. Similarly, it is now shown that the gravitational radiation degrees of freedom may be reduced to a single massless scalar field. The momentum constraint may be conveniently written in terms of the new canonical variables and some solutions are discussed.

1. Solving the SHJE for Gravitational Radiation

One attempts the following solution to the separated Hamilton-Jacobi eq.(2.14b),

$$H(\phi_k, h_{ij}; \tilde{\phi}_k, \tilde{h}_{ij}) \equiv H(\phi_k, \tilde{\phi}_k, z), \quad \text{where} \quad z^2 \equiv \frac{m_P^2}{32\pi} \mathrm{Tr}\{ \ln\left([h][\tilde{h}]^{-1}\right) \ln\left([h][\tilde{h}]^{-1}\right) \}. \tag{4.1a}$$

Here, $[h]$ and $[\tilde{h}]^{-1}$ symbolize matrices with components, h_{ij} and $\tilde{h}^{ij}$, respectively, while Tr refers to a trace. The expression z may be loosely thought of as the distance in field space between the old conformal 3-metric $[h]$ and the new one $[\tilde{h}]$, each having unit determinant. No information is lost in this step. In the ansatz (4.1a), one has introduced 6 constants of integration through $\tilde{h}_{ij}$ which are sufficient to describe the dynamics of the gravitational field. The ansatz (4.1a) is analogous to the complete solution for m massless scalar fields, eq. (3.11a), where one introduced m constants of integration by utilizing the rotational symmetry of the SHJE. The separated Hamilton-Jacobi equation reduces to that of a single massless scalar field, z, as well as n interacting scalar fields,

$$H^2 = \frac{m_P^2}{12\pi}\left[\left(\frac{\partial H}{\partial z}\right)^2 + \left(\frac{\partial H}{\partial \phi_k}\right)^2\right] + \frac{8\pi}{3m_P^2}V(\phi_k). \tag{4.1b}$$

Here, one has applied the following expression for the derivative of the Hubble parameter with respect to h_{ij},

$$\frac{\partial H}{\partial h_{ij}} = \frac{\partial H}{\partial z}\frac{\partial z}{\partial h_{ij}} = \frac{m_P^2}{32\pi}\frac{\partial H}{\partial z}\, z^{-1}\left([h]^{-1}\ln\left([h][\tilde{h}]^{-1}\right)\right)^{ij} \tag{4.2a}$$

which is derived in detail in Appendix B. *After differentiation*, one sets $det(h) = 1$, and one concludes that $h_{ij}(\partial H/\partial h_{ij}) = 0$, which simplifies the analysis enormously. Similarly, one may show that the derivative with respect to $\tilde{h}_{ij}$ is given by,

$$\frac{\partial H}{\partial \tilde{h}_{ij}} = \frac{\partial H}{\partial z}\frac{\partial z}{\partial \tilde{h}_{ij}} = -\frac{m_P^2}{32\pi}\frac{\partial H}{\partial z}\, z^{-1}\left([h]^{-1}\ln\left([h][\tilde{h}]^{-1}\right)[h][\tilde{h}]^{-1}\right)^{ij}, \tag{4.2b}$$

and that $\tilde{h}_{ij}(\partial H/\partial \tilde{h}_{ij}) = 0$. Since eq.(4.1b) always admits solutions, the ansatz (4.1a) is justified.

Once again, $\tilde{\gamma}_{ij} \equiv \tilde{\gamma}^{1/3}\tilde{h}_{ij}$ may be considered as a new canonical variable. Given a solution, $H(\phi_k, \tilde{\phi}_k, z)$ of eq.(4.1b) which depends on n parameters, the new variables are related to the old by differentiation through (2.14c,d)

$$[\pi^\gamma]^{ij} = -\frac{m_P^2}{4\pi}\sqrt{\gamma}\left[\frac{1}{2}H\gamma^{ij} + \gamma^{-1/3}\frac{\partial H}{\partial h_{ij}}\right], \tag{4.3a}$$

$$[\pi^{\tilde{\gamma}}]^{ij} = \frac{m_P^2}{4\pi}\sqrt{\tilde{\gamma}}\tilde{\gamma}^{-1/3}\frac{\partial H}{\partial \tilde{h}_{ij}}. \tag{4.3b}$$

$[\pi^{\tilde{\gamma}}]^{ij}$ is traceless. Since the Hubble parameter depends symmetrically upon γ_{ij} and $\tilde{\gamma}_{ij}$, one may show that their respective partial derivatives are related through a reciprocity relation,

$$[\pi^{\gamma}(x)]^{ij}\gamma_{jl} = [\pi^{\tilde{\gamma}}(x)]^{ij}\tilde{\gamma}_{jl} + \frac{1}{3}\pi^{\gamma}(x)\delta^i_l. \tag{4.4}$$

2. Momentum Constraint Expressed Using New Variables

For the special case considered in Sec. III.A, it was found that the momentum constraint admitted a simple expression in terms of the new canonical variables, eq.(3.3). One can generalize this result to include gravitational radiation. The gravitational momenta can be decomposed into a trace part and a traceless part (denoted by a bar):

$$[\pi^{\gamma}]^{ij} = \frac{\pi^{\gamma}}{3}\gamma^{ij} + [\bar{\pi}^{\gamma}]^{ij} \tag{4.5}$$

in which case the momentum constraint (2.5b) becomes,

$$-\frac{2}{3}\pi^{\gamma}_{,i} - 2\left([\bar{\pi}^{\gamma}]^{jl}\gamma_{li}\right)_{,j} + [\pi^{\gamma}]^{lm}\gamma_{lm,i} + \pi^{\phi_k}\phi_{k,i} = 0. \tag{4.6}$$

The generating functional (2.14a) may be rewritten in terms of the trace of the gravitational momenta,

$$S = \frac{2}{3}\int d^3x\, \pi^{\gamma}\big(\phi_k(x), h_{ij}(x); \tilde{\phi}_k(x), \tilde{h}_{ij}(x)\big).$$

Hence one finds that the new and old canonical variables may be expressed as *partial* derivatives of π^{γ},

$$[\pi^{\gamma}]^{ij} = \frac{2}{3}\frac{\partial \pi^{\gamma}}{\partial \gamma_{ij}}, \quad [\pi^{\tilde{\gamma}}]^{ij} = -\frac{2}{3}\frac{\partial \pi^{\gamma}}{\partial \tilde{\gamma}_{ij}}, \quad \pi^{\phi_k} = \frac{2}{3}\frac{\partial \pi^{\gamma}}{\partial \phi_k}, \quad \pi^{\tilde{\phi}_k} = -\frac{2}{3}\frac{\partial \pi^{\gamma}}{\partial \tilde{\phi}_k},$$

and the spatial derivative of π^{γ} may be written as,

$$\begin{aligned}
\pi^{\gamma}{}_{,i} &= \frac{\partial \pi^{\gamma}}{\partial \gamma_{lm}}\gamma_{lm,i} + \frac{\partial \pi^{\gamma}}{\partial \tilde{\gamma}_{lm}}\tilde{\gamma}_{lm,i} + \frac{\partial \pi^{\gamma}}{\partial \phi_k}\phi_{k,i} + \frac{\partial \pi^{\gamma}}{\partial \tilde{\phi}_k}\tilde{\phi}_{k,i} \\
&= \frac{3}{2}[\pi^{\gamma}]^{lm}\gamma_{lm,i} - \frac{3}{2}[\pi^{\tilde{\gamma}}]^{lm}\tilde{\gamma}_{lm,i} + \frac{3}{2}\pi^{\phi_k}\phi_{k,i} - \frac{3}{2}\pi^{\tilde{\phi}_k}\tilde{\phi}_{k,i}.
\end{aligned} \tag{4.7}$$

When (4.7) is substituted into the momentum constraint (4.6), one effectively performs a Legendre transformation between the new and old variables,

$$0 = \tilde{\mathcal{H}}_i = -2\left([\pi^{\tilde{\gamma}}]^{jl}\tilde{\gamma}_{li}\right)_{,j} + [\pi^{\tilde{\gamma}}]^{lm}\tilde{\gamma}_{lm,i} + \pi^{\tilde{\phi}_k}\tilde{\phi}_{k,i}. \tag{4.8}$$

In deriving this result, one has also applied the reciprocity relation (4.4), $[\bar{\pi}^{\gamma}]^{jl}\gamma_{li} = [\pi^{\tilde{\gamma}}]^{jl}\tilde{\gamma}_{li}$.

It is fortuitous that the momentum constraint (4.8) admits such a simple expression in terms of the new canonical variables. This would not in general occur if one chose a solution of the SHJE other than (4.1a) The evolution equations for the new metric variables are given by variation of the new variables in the new action written in (2.10),

$$\tilde{I} = \int d^4x \left(\pi^{\tilde{\phi}_k}\dot{\tilde{\phi}}_k + [\pi^{\tilde{\gamma}}]^{ij}\dot{\tilde{\gamma}}_{ij} - N^i\tilde{H}_i\right) : \tag{4.9}$$

$$\frac{\partial}{\partial t}\tilde{\gamma}_{ij} - \tilde{\gamma}_{il}N^l_{,j} - \tilde{\gamma}_{jl}N^l_{,i} - N^l\tilde{\gamma}_{ij,l} = 0, \tag{4.10a}$$

$$\frac{\partial}{\partial t}[\pi^{\tilde{\gamma}}]^{ij} + [\pi^{\tilde{\gamma}}]^{il}N^j_{,l} + [\pi^{\tilde{\gamma}}]^{jl}N^i_{,l} - \left(N^l[\pi^{\tilde{\gamma}}]^{ij}\right)_{,l} = 0. \tag{4.10b}$$

The corresponding equations for the new scalar field variables were already given in eq.(3.5). Since the trace of $[\pi^{\tilde{\gamma}}]^{ij}$ vanishes, the evolution of the determinant $\tilde{\gamma}$ is unrestricted: in (4.10a), one should actually subtract out the trace. However, one is then free to choose $\tilde{\gamma}$ arbitrarily and one can just assume for simplicity that it evolution is determined by the trace of (4.10a).

If the shift vanishes, then the new canonical variables are independent of time but they are spatially dependent,

$$\tilde{\gamma}_{ij} \equiv \tilde{\gamma}_{ij}(x), \quad [\pi^{\tilde{\gamma}}]^{ij} \equiv [\pi^{\tilde{\gamma}}]^{ij}(x); \tag{4.11}$$

they are restricted only by the momentum constraint (4.8). The evolution of the fields h_{ij} and $[\pi^{\gamma}]^{ij}$ may be found by inverting (4.3) in terms of γ and the new variables. Hence, if one finds the complete solution of the separated Hamilton-Jacobi equation, and then independently obtains a solution of the momentum constraint, then the long wavelength problem has been completely solved.

Eq.(4.8) may be simplified in two important ways. Since the canonical transformation (4.1a) depends on the new metric only through $\bar{h}_{ij}$, one can explicitly write the momentum constraint in terms of $\bar{h}_{ij}$:

$$0 = \tilde{\mathcal{H}}_i = -2\left(\tilde{\gamma}^{1/3}[\pi^{\tilde{\gamma}}]^{jl}\bar{h}_{li}\right)_{,j} + \tilde{\gamma}^{1/3}[\pi^{\tilde{\gamma}}]^{lm}\bar{h}_{lm,i} + \pi^{\tilde{\phi}_k}\tilde{\phi}_{k,i}. \tag{4.12}$$

Here, it was important to note that $[\pi^{\tilde{\gamma}}]^{ij}$ was traceless. Thus the momentum constraint only restricts the quantity $\tilde{\gamma}^{1/3}[\pi^{\tilde{\gamma}}]^{ij}$ and not the full momentum degrees of freedom. Secondly, because the theory does not depend on the parametrization of the spatial coordinates, one may write the momentum constraint in terms of a covariant derivative with respect to $\bar{h}_{ij}$,

$$0 = \tilde{\mathcal{H}}_i = -2\left(\tilde{\gamma}^{1/3}[\pi^{\tilde{\gamma}}]^j_i\right)_{|j} + \pi^{\tilde{\phi}_k}\tilde{\phi}_{k,i}. \tag{4.13}$$

This form is perhaps the most useful for general discussions. Unfortunately, the momentum constraint does not admit an explicit general solution which was the case when gravitational radiation was neglected, eq.(3.7).

3. Solutions of the New Momentum Constraint

In order to illustrate the meaning of the new momentum constraint, I will now consider a special class of solutions.

One set of solutions arise if the two terms appearing in (4.13) vanish separately:

$$0 = \left(\bar{\gamma}^{1/3}[\pi^{\bar{\gamma}}]^j_i\right)_{|j}, \quad 0 = \pi^{\bar{\phi}_k}\bar{\phi}_{k,i}. \tag{4.14}$$

The first equation requires that the new momentum tensor which is traceless be divergenceless as well. The solution of the second equation was given in (3.7) and is well understood.

More generally, the momentum constraint (4.13) may be solved using the York prescription.[27] One formally decomposes the new gravitational momentum tensor into a traceless transverse part, $[P^{TT}]^{ij}$, and a longitudinal part, $[P^L]^{ij}$

$$\bar{\gamma}^{1/3}[\pi^{\bar{\gamma}}]^{ij} = [P^{TT}]^{ij} + [P^L]^{ij}. \tag{4.15}$$

Here $[P^{TT}]^{ij}$ is divergenceless (with respect to $\bar{h}_{ij}$) whereas $[P^L]^{ij}$ is derived from a vector potential, W^i:

$$[P^{TT}]^{ij}_{|j} = 0, \tag{4.16a}$$

$$[P^L]^{ij} = W^{i|j} + W^{j|i} - \frac{2}{3}W^l_{|l}\bar{h}^{ij}. \tag{4.16b}$$

The momentum constraint restricts the longitudinal degrees of freedom,

$$-2[P^L]^j_{i|j} + \pi^{\bar{\phi}_k}\bar{\phi}_{k,i} = 0, \tag{4.17}$$

whereas the traceless transverse part is arbitrary.

B. Gravitational Radiation Evolving under a Cosmological Constant

The canonical transformation generated by (4.1) and (4.3) was written with little physical motivation. I will now give a derivation for the case of a cosmological constant, $V(\phi_k) = V_0$, neglecting scalar fields. In this case, the evolution of the metric may be found by direct integration of the equations of motion. The Hubble parameter, now written as a function of the gravitational fields, is the solution of the 6-dimensional separated Hamilton-Jacobi equation.

To simplify the derivation, I will assume a vanishing shift function, $N^i = 0$. Eq.(2.7d) implies that the traceless part of the gravitational momentum is constant in time. One can thus write,

$$[\pi^\gamma(t,x)]^i{}_j = q^i{}_j(x) + \frac{1}{3}\pi^\gamma(t,x)\delta^i_j, \tag{4.18}$$

where $q^i{}_j(x)$ which depends only on spatial coordinates is traceless and symmetric (the last point will be discusses in more detail later). Substituting into the energy constraint (2.5a), one finds π^γ as a function of γ:

$$\pi^\gamma = -\sqrt{6}\left[|q|^2 + \frac{m_P^2}{16\pi}\gamma V_0\right]^{1/2}, \tag{4.19}$$

where $|q|^2 = q^i{}_j q^j{}_i$. The evolution of the 3-metric is given by (2.7b),

$$\frac{\dot{\gamma}_{ij}}{N} = \frac{16\pi}{m_{\mathcal{P}}^2}\gamma^{-1/2}\big[-\frac{1}{3}\pi^\gamma\gamma_{ij} + 2\gamma_{il}\, q^l{}_j\big]. \tag{4.20}$$

One can choose any parameter as time, and since (4.19) is expressed in terms of γ, it is the natural choice. The lapse function may be expressed in terms of π^γ,

$$\frac{1}{N} = \frac{\dot{\gamma}}{N} = \gamma\gamma^{ij}\frac{\dot{\gamma}_{ij}}{N} = -\frac{16\pi}{m_{\mathcal{P}}^2}\gamma^{1/2}\pi^\gamma, \tag{4.21}$$

which implies that

$$\frac{d[h]}{d\gamma} = \frac{32\pi}{m_{\mathcal{P}}^2 V_0}\frac{\gamma^{-3/2}d\gamma\ [h][q]}{\big[1 + \frac{16\pi}{m_{\mathcal{P}}^2}V_0^{-1}\gamma^{-1}|q|^2\big]^{1/2}}, \tag{4.22}$$

where $[h]$ and $[q]$ denote matrices with components $h_{ij} = \gamma^{-1/3}\gamma_{ij}$ and $q^i{}_j$, respectively. This equation may be solved by making a change of variables from γ to z:

$$\sinh\big(\frac{\sqrt{12\pi}}{m_{\mathcal{P}}}z\big) = \big[\frac{16\pi}{m_{\mathcal{P}}^2 V_0}\big]^{1/2}\gamma^{-1/2}|q| \tag{4.23}$$

which leads to

$$d[h] = -\frac{\sqrt{32\pi}}{m_{\mathcal{P}}}|q|^{-1}[h][q]dz, \tag{4.24}$$

whose solution

$$[h] = [\tilde{h}]\exp\big\{-\frac{\sqrt{32\pi}}{m_{\mathcal{P}}}\frac{[q]}{|q|}z\big\}, \tag{4.25}$$

may be verified by direct differentiation. This solution is analogous to (3.11b); it describes the evolution of the conformal unimodular 3-metric as a function of γ which has been interpreted as time.

One can find a solution of the corresponding separated Hamilton-Jacobi equation by expressing the Hubble parameter as a function of the metric variables. Assuming that $\tilde{h}_{ij}$ is fixed, one requires the trajectory that passes through h_{ij}. More simply, one wishes to eliminate $z \equiv z(\gamma)$ from the expression

$$H = \big(\frac{8\pi V_0}{3m_{\mathcal{P}}^2}\big)^{1/2}\cosh\big(\frac{\sqrt{12\pi}}{m_{\mathcal{P}}}z\big), \tag{4.26}$$

which follows from (2.15), (4.19) and (4.23). To this aim, note that (4.25) may be solved for $[q]$,

$$\frac{[q]}{|q|} = -\frac{m_{\mathcal{P}}}{\sqrt{32\pi}}z^{-1}[\tilde{h}]^{-1}\ln\big([h][\tilde{h}]^{-1}\big)\,[\tilde{h}]. \tag{4.27}$$

Squaring both sides, and taking the trace, one finds

$$z^2 = \frac{m_{\mathcal{P}}^2}{32\pi}\mathrm{Tr}\{\ \ln\big([h][\tilde{h}]^{-1}\big)\ \ln\big([h][\tilde{h}]^{-1}\big)\ \} \tag{4.28}$$

Thus the solution of the SHJE (2.14b) is just

$$H(h_{ij},\tilde{h}_{ij}) = \left(\frac{8\pi V_0}{3m_{\mathcal{P}}^2}\right)^{1/2} \cosh\left(\frac{\sqrt{12\pi}}{m_{\mathcal{P}}} z\right), \quad \text{with} \quad z^2 = \frac{m_{\mathcal{P}}^2}{32\pi} \mathrm{Tr}\{ \ln([h][\tilde{h}]^{-1}) \ln([h][\tilde{h}]^{-1}) \}, \tag{4.29}$$

The expression (4.29) for z motivates the general ansatz of eq.(4.1a); in fact, this solution for H is what was expected from the reduced SHJE (4.1b) with $V \equiv V_0$.

The canonical transformation (4.3) may be inverted to give the evolution of the metric degrees of freedom as functions of time γ:

$$[h] = [\tilde{h}] \exp\left(-\frac{\sqrt{32\pi}}{m_{\mathcal{P}}} z \frac{[\pi^{\tilde{\gamma}}][\tilde{h}]}{|[\pi^{\tilde{\gamma}}]|} \right), \tag{4.31a}$$

$$[\pi^{\gamma}] = -\left(\frac{m_{\mathcal{P}}^2}{24\pi}\right)^{1/2} \gamma^{1/6} \cosh\left(\frac{\sqrt{12\pi}}{m_{\mathcal{P}}} z\right) [h]^{-1} + \gamma^{-1/3} \tilde{\gamma}^{1/3} [\pi^{\tilde{\gamma}}][\tilde{h}][h]^{-1} \tag{4.31b}$$

where

$$\frac{\sqrt{12\pi}}{m_{\mathcal{P}}} z = \sinh^{-1}\left(\frac{\sqrt{16\pi}}{m_{\mathcal{P}} V_0^{1/2}} \gamma^{-1/2} \tilde{\gamma}^{1/3} |[\pi^{\tilde{\gamma}}]| \right). \tag{4.31c}$$

Here $[\pi^{\gamma}]$ and $[\pi^{\tilde{\gamma}}]$ denote matrices whose components are $[\pi^{\gamma}]^{ij}$ and $[\pi^{\tilde{\gamma}}]^{ij}$, respectively, and $|[\pi^{\gamma}]| = (\tilde{h}_{ij}\tilde{h}_{lm}[\pi^{\tilde{\gamma}}]^{jl}[\pi^{\tilde{\gamma}}]^{mi})^{1/2}$.

In classical Hamilton-Jacobi theory,[19] one may invert the canonical transformation to describe the evolution of all degrees of freedom. In the long wavelength gravitational system, however the trace $\tilde{h}_{ij}[\pi^{\tilde{\gamma}}]^{ij}$ vanishes, and one has consequently lost one degree of freedom. For this reason, the determinant of the 3-metric γ has been adopted as time, and one then solves for the conformal 3-metric h_{ij}, eq.(4.31). (To relate the transformation (4.31) with the integration of the equations of motion, one can identify the field $[q]$ introduced in (4.5) to the new gravitational momenta through,

$$q^i{}_l = \tilde{\gamma}^{1/3} [\pi^{\tilde{\gamma}}]^{ij} \tilde{h}_{jl};$$

$[q]^i{}_l$ is clearly a symmetric matrix, when its indices are raised and lowered by $\tilde{h}_{ij}$.)

The evolution of the metric is shown in Fig. 5. Every 3-metric may be represented graphically by an ellipsoid with variables, y^i, $i = 1,2,3$, satisfying,

$$1 = h_{ij} y^i y^j. \tag{4.32}$$

Since $det(h) = 1$, the volume of the ellipsoid is just $\pi = 3.14$.. Eq. (4.31a) may written in a form that is more amenable for calculation,

$$[h] = [\tilde{h}]^{1/2} \exp\left(-\frac{\sqrt{32\pi}}{m_{\mathcal{P}}} z \frac{[\tilde{h}]^{1/2}[\pi^{\tilde{\gamma}}][\tilde{h}]^{1/2}}{|[\pi^{\tilde{\gamma}}]|} \right) [\tilde{h}]^{1/2}, \tag{4.33}$$

which may be justified using a Taylor series expansion. Since the argument of the exponential is a symmetric, it may be diagonalized,

$$\frac{[\tilde{h}]^{1/2}[\pi^{\gamma}][\tilde{h}]^{1/2}}{|[\pi^{\tilde{\gamma}}]|} = [R]^T [D][R] \tag{4.34}$$

where $[R]$ is an orthogonal matrix and $[D]$ is diagonal. One can thus eliminate $[\pi^{\bar{\gamma}}]$ from (4.33) to obtain an expression

$$[h] = [\bar{h}]^{1/2}[R]^T \exp\left(-\frac{\sqrt{32\pi}}{m_{\mathcal{P}}} z[D]\right)[R][\bar{h}]^{1/2} \tag{4.35}$$

which is useful for numerical methods, because the argument of the exponential is a diagonal matrix. For example, the solution of the ellipsoid equation (4.32) which depends on the parameters θ and ψ is,

$$\vec{y} = [\bar{h}]^{-1/2}\,[R]^T \exp\left(-\frac{\sqrt{8\pi}}{m_{\mathcal{P}}} zD\right)[\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta]^T,$$
$$\text{where} \quad 0 \le \theta \le \pi \quad \text{and} \quad 0 \le \psi \le 2\pi. \tag{4.36}$$

If $[\bar{h}]$ is diagonal, then (4.36) stetches or contracts the i^{th} component of a unit vector by $\exp\left(-\frac{\sqrt{8\pi}}{m_{\mathcal{P}}} zD_{ii}\right)$, rotates the resulting vector and then finally stretches or contracts the final components.

In Fig. 5, a two dimensional example is given where an ellipse which was initially at an angle 45^0 to the y^1 axis evolves to its final position given by the broken curve. Because the evolution of h_{ij} is basically that of a decaying mode, it is quite simple. The ellipsoid stetches and rotates for less than 90^0 to its final form. The curves shown are equally spaced in z with the final position corresponding to $z = 0$.

C. Gravitational Radiation Interacting with a Scalar Field with an Exponential Potential

The variable z defined in eq.(4.1a) behaves as a massless scalar in the SHJE (4.1b) even when the potential $V(\phi_k)$ is nontrivial. This analogy made may be made even stronger by noting that z evolves in time according,

$$(\dot{z} - N^i z_{,i})/N = -\frac{m_{\mathcal{P}}^2}{4\pi}\frac{\partial H}{\partial z}. \tag{4.37}$$

Here, I have applied eqs. (4.10a), (4.3a), (4.2a) and (2.7b). Eq.(4.37) is just the evolution equation for a massless scalar field interacting with n additional scalar fields without any gravitational radiation (see eq.(2.9a)).

Hence, in general, it there are n scalar fields interacting through a potential $V(\phi_k)$, one can describe the evolution of the complete gravitational system by solving the SHJE for $n+1$ scalar fields, (4.1b). If one determines that the scalar fields evolve in time, α, according to,

$$z \equiv z(\alpha), \quad \phi_k \equiv \phi_k(\alpha),$$

then the evolution of the gravitational degrees of freedom is given by a formula analogous to (4.31),

$$[h] = [\bar{h}] \exp\left(-\frac{\sqrt{32\pi}}{m_{\mathcal{P}}} z \frac{[\pi^{\bar{\gamma}}][\bar{h}]}{\|[\pi^{\bar{\gamma}}]\|}\right), \tag{4.38a}$$

$$[\pi^{\gamma}] = -\left(\frac{m_{\mathcal{P}}^2}{24\pi}\right)^{1/2} \gamma^{1/6} \cosh\left(\frac{\sqrt{12\pi}}{m_{\mathcal{P}}} z\right) [h]^{-1} + \gamma^{-1/3}\tilde{\gamma}^{1/3}[\pi^{\tilde{\gamma}}][\tilde{h}][h]^{-1}, \tag{4.38b}$$

which is valid even the potential is not a constant, provided one interprets z as a massless field that interacts with the n scalar fields. (For simplicity, $N^i = 0$.) Of course, the new canonical coordinates $[\tilde{h}]$ and $[\pi^{\tilde{\gamma}}]$ must satisfy the momentum constraint (4.13). For example, one may apply the solution of Sec. III.E for 2 scalar fields, one massless, ϕ_1, and the other, ϕ_2, interacting through an exponential potential if one identifies ϕ_1 with z. Eqs.(3.38a-d) may be then inverted to give the general evolution of the 2 scalar field system. I will omit the details.

However, the analogy cannot be taken too far because one cannot rewrite the momentum constraint simply in terms of the massless degree of freedom z. In this sense, the gravitational radiation degrees of freedom differ from massless fields in that their polarization affects the direction of momentum transport.

V. SUMMARY AND CONCLUSIONS

In an important generalization of homogeneous mini-superspace models, it has been shown that the evolution of the long wavelength metric and scalar fields is tractable, including the evolution of gravitational radiation. One invokes a transformation to new canonical variables where the Hamiltonian density vanishes strongly. Since the evolution of fields is generated through Poisson brackets, the new variables are constants in time if the shift function vanishes although they may be spatially dependent.

The separated Hamilton-Jacobi equation (2.14b), the canonical transformation (2.14a-f) and the new momentum constraint (4.8) are the most important equations in this paper. The SHJE does not depend on the time parameter nor on the spatial coordinates: it yields a covariant formulation of the long wavelength problem. In the SHJE, the gravitational degrees of freedom may be reduced to that of a single massless scalar field. As a result, one can obtain complete solutions for gravitational radiation with n scalar fields for two important cases: (1) when a cosmological constant is present, and (2) when the scalar fields interact through an exponential potential. However, the gravitational field is fundamentally different from massless scalar fields in that it carries spin angular momentum. For example, the momentum constraint restricts the longitudinal modes of the gravitational momentum tensor. Fortunately, the momentum constraint admits a simple expression in terms of the canonical variables.

For many applications, the gravitational radiation modes are not dynamically important, and one may neglect them as in Sec. III. In this case, one can obtain a general solution of the momentum constraint, eq.(3.7). For m massless scalar fields, it is easy to produce a complete solution of the SHJE which depends on m arbitrary parameters because the field space is rotationally symmetric. More generally, from a complete solution one may obtain all solutions of the SHJE by the extrema method of Sec. III.C where the arbitrary integration parameters $\tilde{\phi}_k$ are chosen to be functions of the scalar fields. Exact solutions of the SHJE for single scalar field are given when the potential is formed by joining two exponential functions together (Sec. III.D). In this way, one may model the transition from the inflation epoch to a radiation-dominated era. The complete solution of two scalar fields interacting through and exponential potential given in Sec. III.E constitutes a major advance for practical calculations of long wavelength universes. The new canonical variables which are in fact constants of integration were explicitly given. From the late time evolution, one may then determine the nonlinear generalization of ζ which is a measure of adiabatic primordial fluctuations. These results may be applied to models that produce non-Gaussian fluctuations for structure formation.[24]

There are several extensions of this paper which could prove interesting. Since Hamilton-Jacobi theory has proved fruitful in solving the long wavelength problem, one wonders whether it can also be profitably applied to other gravitational systems where short wavelengths are not neglected. For example, the greatest uncertainty in inflation models lies in the treatment of short wavelength fluctuations. Any improvements here would necessarily have an important impact on the primordial fluctuations to form galaxies. Another possibility is that the Hamilton-Jacobi formalism may provide clues to the quantum theory of the gravitational field, although currently the long wavelength problem does not admit a totally satisfactory quantum formulation.[8] Nonetheless, models of long wavelength universes are a significant improvement over those of homogeneous mini-superspace, particularly when one incorporates the stochastic generation of initial conditions.[13,28]

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APPENDIX A. CANONICAL TRANSFORMATION FOR TWO SCALAR FIELDS WITH $V(\phi_1,\phi_2) = V_0\exp(-\sqrt{\frac{16\pi}{p}}\frac{\phi_2}{m_P})$

A complete solution (3.33) of the SHJE for two scalar fields interacting through an exponential potential was given in Sec. III.E. In this appendix, I will derive the expression, eq.(3.39a-d), for the new canonical variables b, m, π^b, π^m as a function of the old variables, ϕ_1, ϕ_2, π^{ϕ_1} and π^{ϕ_2}. In addition, the expression (3.42) for ζ will be justified.

The canonical transformation is given by differentiation of the Hubble parameter (3.33) through eq.(2.14e,f),

$$\pi^b = -\sqrt{2V_0}\,\frac{\cosh u - \sinh u\sqrt{m^2(3p-1)+3p}}{\left[m^2(3p-1)+3p\right]^{1/2}(m^2+1)^{1/2}}\,\gamma^{1/2}\exp\left(-\sqrt{\frac{4\pi}{p}}\frac{\phi_2}{m_P}\right), \qquad (A1a)$$

$$\pi^{\phi_1} = -m\pi^b, \qquad (A1b)$$

$$\pi^{\phi_2} = \frac{m_P}{\sqrt{4\pi p}}\gamma^{1/2}H + \pi^b, \qquad (A1c)$$

$$\pi^m = \pi^b\left\{\phi_1 - \sqrt{\frac{pm_P^2}{4\pi}}\left(\frac{m}{m^2(3p-1)+3p} + \frac{mu}{\sqrt{m^2(3p-1)+3p}}\right)\right\}, \qquad (A1d)$$

where u was defined implicitly in (3.33b),

$$\frac{\sqrt{12\pi}}{m_P}(\phi_2 - m\phi_1 - b) = -\frac{\sqrt{3p}}{(3p-1)}\times \left[u\sqrt{m^2(3p-1)+3p} + \ln\left|\cosh(u) - \sinh(u)\sqrt{m^2(3p-1)+3p}\right|\right]. \qquad (A2)$$

After some algebra, one can invert eqs.(A1a-d) and express the new variables in terms of the old ones, eqs.(3.39a-d). (3.39a) follows from (A1c) and the definition of the Hubble parameter whereas (3.39b) arises directly from (A1b). The expression for u, v are derived by solving (A1b,a) for u.

It is desirable to determine the late time evolution of the fields in order to determine microwave background fluctuations as well as the initial conditions for structure formation. Eq.(A2) may be rewritten as

$$\begin{aligned}&|\cosh u - \sinh u\sqrt{m^2(3p-1)+3p}| = \\ &\qquad \exp\Big[-(1-\frac{1}{3p})\frac{\sqrt{36\pi p}}{m_P}(\phi_2 - m\phi_1 - b) - u\sqrt{m^2(3p-1)+3p}\Big]\end{aligned}$$

and then substituted into (A1a). In the limit that $\gamma \to \infty$, the first scalar field ϕ_1 approaches a constant given by (3.40b) and one finds that γ evolves in ϕ_2 according to

$$\ln(\sqrt{\gamma}) = 3\sqrt{4\pi p}\frac{\phi_2}{m_P} + f(b, m, \pi^b, \pi^m), \tag{A3a}$$

where

$$\begin{aligned}f(b, m, \pi^b, \pi^m) =& \ln|\pi^b| + \frac{1}{2}\ln[(m^2+1)(m^2(3p-1)+3p)] \\ &- \frac{\sqrt{36\pi p}}{m_P}(1-\frac{1}{3p})(m\phi_{1min} + b) + \sqrt{m^2(3p-1)+3p}\,u_0 - \ln\sqrt{2V_0}.\end{aligned} \tag{A3b}$$

At late times, the metric fluctuation on a uniform ϕ_2 slice is then given by ζ,

$$\begin{aligned}\zeta(x) \equiv \Delta_{\phi_2}\ln(\sqrt{\gamma}) = \Delta\Bigg[&\ln|\pi^b| + \frac{1}{2}\ln[(m^2+1)(m^2(3p-1)+3p)] \\ &- \frac{\sqrt{36\pi p}}{m_P}(1-\frac{1}{3p})(m\phi_{1min} + b) + \sqrt{m^2(3p-1)+3p}\,u_0\Bigg]\end{aligned} \tag{A4a}$$

where

$$\phi_{1min} = \frac{\pi^m}{\pi^b} + \sqrt{\frac{pm_P^2}{4\pi}}\Big[\frac{m}{m^2(3p-1)+3p} + \frac{m}{\sqrt{m^2(3p-1)+3p}}u_0\Big]. \tag{A4b}$$

and

$$u_0 = \tanh^{-1}\frac{1}{\sqrt{m^2(3p-1)+3p}}. \tag{A4c}$$

It is understood that b, m, π^b and π^m are spatially dependent constants, and that the difference Δ_{ϕ_2} is taken between two spatial points x and some fiducial point x_0, *e.g.*,

$$\Delta_{\phi_2}\ln|\pi^b| = \Delta\ln|\pi^b| = \ln|\pi^b(x)| - \ln|\pi^b(x_0)|.$$

Eq.(A4) is the nonlinear generalization of ζ to multiple fields interacting via an exponential potential.

APPENDIX B. MATHEMATICAL NOTES

In this appendix, I will derive the algebraic results (4.2a,b) which were necessary for the analysis of gravitational radiation in Sec. IV.

In eq.(4.1a), the quantity

$$z^2 = \frac{m_P^2}{32\pi}\mathrm{Tr}\{\ \ln[A]\ \ln[A]\ \}, \quad \text{where} \quad [A] = [h][\bar{h}]^{-1}. \tag{B1}$$

was defined. (From now throughout this appendix, braces will be deleted; *e.g.* A will denote a matrix.) One requires the derivative of z with respect to h_{ij} and $\bar{h}_{ij}$. By considering differentials of both sides, one finds,

$$dz = \frac{m_P^2}{32\pi}z^{-1}\mathrm{Tr}\{\ \ln A\, d\ln A\ \}. \tag{B2}$$

The primary complication here is that A and dA need not commute, and one cannot write immediately the desired result,

$$dz = \frac{m_P^2}{32\pi}z^{-1}\mathrm{Tr}\{\ \ln A\ A^{-1}dA\ \}. \tag{B3}$$

Instead, one should expand the $\ln A$ in a power series,

$$\begin{aligned} d\ln A =& d\ln\Big(I + (A-I)\Big) = d\Big((A-I) - \frac{1}{2}(A-I)^2 + \frac{1}{3}(A-I)^3 + ...\Big) \\ =& dA - \frac{1}{2}\big(dA(A-I) + (A-I)dA\big) \\ &+ \frac{1}{3}\big(dA(A-I)^2 + (A-I)dA(A-I) + (A-I)^2dA\big) + ..., \end{aligned} \tag{B4}$$

where I is the identity matrix. Since Tr $CD=$ Tr DC and since I and A commute, one may rewrite (B2) as

$$dz = \frac{m_P^2}{32\pi}z^{-1}\mathrm{Tr}\{\ \ln A\Big(I - (A-I) + (A-I)^2 + ...\Big)dA\ \}$$

which leads to the desired result (B3). Finally, letting $A = h\bar{h}^{-1}$, one finds

$$dz = \frac{m_P^2}{32\pi}z^{-1}\mathrm{Tr}\{\ h^{-1}(h\bar{h}^{-1})\ln(h\bar{h}^{-1})(\bar{h}h^{-1})dh\} = \mathrm{Tr}\{h^{-1}\ln(h\bar{h}^{-1})dh\},$$

which yields the required relation (4.2a); in the last step, one applied the identity $C^{-1}\ln(D)C = \ln(C^{-1}DC)$. Eq. (4.2b) is proved similarly.

Using eq.(4.2a), one can readily show that $h_{ij}(\partial H/\partial h_{ij}) = 0$:

$$\begin{aligned} h_{ij}\frac{\partial H}{\partial h_{ij}} &= \frac{m_P^2}{32\pi}\frac{\partial H}{\partial z}z^{-1}\ \mathrm{Tr}\{\ln A\} \\ &\propto \ln\big[\exp(\mathrm{Tr}\{\ln A\})\big] = \ln\big[det\exp(\ln A)\big] = 0, \end{aligned}$$

since $det\, A = det\, h = det\, \tilde{h} = 1$ after differentiation.

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FIGURE CAPTIONS

Fig. 1. A complete solution, $H \equiv H(\phi_1, \phi_2; \bar{\phi}_1, \bar{\phi}_2)$ (eq.(3.12a)), of the separated Hamilton-Jacobi equation (SHJE) is shown for two scalar fields evolving under the influence of a cosmological constant. The surfaces of constant Hubble parameter (solid curves) are circles concentric with the origin. The broken lines, which are orthogonal to the uniform Hubble surfaces are the trajectories of the scalar fields which all end at the origin. Because the SHJE is invariant under translations in field space, $(\phi_1, \phi_2) \rightarrow (\phi_1 + \bar{\phi}_1, \phi_2 + \bar{\phi}_2)$, the origin, $(\bar{\phi}_1, \bar{\phi}_2)$, in this figure is arbitrary, and hence the solution depends on two free parameters which are then interpreted as new canonical variables. Differentiation of the Hubble function, $H(\phi_1, \phi_2; \bar{\phi}_1, \bar{\phi}_2)$, with respect to the new canonical variables yields the new conjugate momenta, eq.(3.13). The momentum constraint may be simply expressed in terms of these new variables. This trivial example illustrates the basic principles behind more general solutions of the SHJE which include the effects of gravitational radiation.

Fig. 2. Given the complete solution of the SHJE shown in Fig. 1, one may generate all other solutions using the extrema method of Sec. III.C where the parameters $\vec{\bar{\phi}}$ are chosen to be a function of $\vec{\phi}$. Here, it is shown graphically how to produce the Green's function solution where all trajectories emanate from a single point ψ where the Hubble parameter has value H_ψ. The parameters $\vec{\bar{\phi}}$ are restricted to lie on a circle of radius $r = |\vec{\bar{\phi}} - \vec{\psi}| = \frac{m_{\mathcal{P}}}{\sqrt{12\pi}} \cosh^{-1}(H_\psi / H_0)$ (eq.(3.22)). Holding the observation point $\vec{\phi}$ fixed, one determines the constrained parameters $\vec{\bar{\phi}}$ which extremizes $H(\vec{\phi}, \vec{\bar{\phi}}) \equiv H(|\vec{\phi} - \vec{\bar{\phi}}|)$ (eq. (3.12a)) which is function only of the distance between ϕ and $\vec{\bar{\phi}}$. Hence, $\vec{\bar{\phi}}$ must be collinear with $\vec{\psi}$ and $\vec{\phi}$, and the points which give the minimum and the maximum values are shown. The parameters are thus functions of the scalar fields, $\vec{\bar{\phi}} \equiv \vec{\bar{\phi}}(\vec{\phi})$; substitution into (3.11a) leads to the Green's function solution, eq.(3.23).

Fig. 3. The exact Hubble function (solid curve) is shown for single scalar field with a potential that is defined by continuously joining two exponential functions at $\phi = 0$, eq.(3.28). The broken curve is the slow rollover approximation, $H_{SR} = [8\pi V(\phi)/(3m_{\mathcal{P}}^2)]^{1/2}$, which is effectively a plot of the potential. This system imitates the transition from an inflation epoch to a radiation-dominated era. For $\phi < 0$, the Universe inflates with the scale factor evolving as $a(t) \propto t^2$, whereas for $\phi >> 0$, $a(t) \propto t^{1/2}$. The exact Hubble function and its derivative $\partial H / \partial \phi$ are continuous at $\phi = 0$. The variable ζ, eq.(3.8), which is a measure of the metric fluctuations, is a strict constant for all times that the wavelength of the fluctuation exceeds the Hubble radius.

Fig. 4. A complete solution, $H \equiv H(\phi_1, \phi_2; b, m)$ (eq.(3.33)), of the SHJE is shown for two scalar fields; ϕ_1 is massless whereas ϕ_2 interacts through an exponential potential, $V(\phi) = V_0 \exp(-\sqrt{16\pi/p}\,\phi_2/m_{\mathcal{P}})$, $p = 3$. Once again, the solid curves are surfaces of uniform Hubble parameter; the set of orthogonal broken curves are the trajectories which move up the page. This complete solution depends on two arbitrary parameter, b and m. b reflects the translational invariance of the SHJE, eq.(3.32), in the ϕ_1 direction, $\phi_1 \rightarrow \phi_1 + \bar{\phi}_1$. A typical trajectory begins at large ϕ_1 with slope given by $-1/m$ ($m = 1$ is shown), where the fields behave effectively as two massless scalars because their kinetic energies dominate over the potential. As the Universe expands, the decaying modes no longer become important; ϕ_1 approaches a constant and ϕ_2 evolves according to the attractor solution for a single scalar field, $\phi_2 = \ln(\gamma)\, m_{\mathcal{P}}/(6\sqrt{4\pi p})$ + const. One can then write down an explicit expression for ζ, eq.(3.42), which characterizes the adiabatic fluctuations for structure formation. In addition, this example may be used

to solve completely the evolution of gravitational radiation with a scalar field ϕ_2 that interacts with an exponential potential (see Sec. IV.C).

Fig. 5. The evolution of the long wavelength 3-metric at a fixed spatial point is shown for a system with a cosmological constant (see eq.(4.36)). The conformal metric of unit determinant, $h_{ij} \equiv \gamma^{-1/3}\gamma_{ij}$, may be represented graphically as an ellipsoid with coordinates (y^1, y^2, y^3) satisfying, $1 = h_{ij}y^iy^j$. For plotting purposes, the third coordinate will be suppressed. From its initial starting position, the principle axes of the ellipsoid rotates by less than 90^0 to its final position. At the same time, the eigenvalues of the ellipsoid which measure the lengths of the principle axes are stretched. Even when there are scalar fields that interact with a potential, the evolution is qualitatively the same because gravitational radiation degrees of freedom may be reduced to that of a single scalar field (see eq.(4.1)).

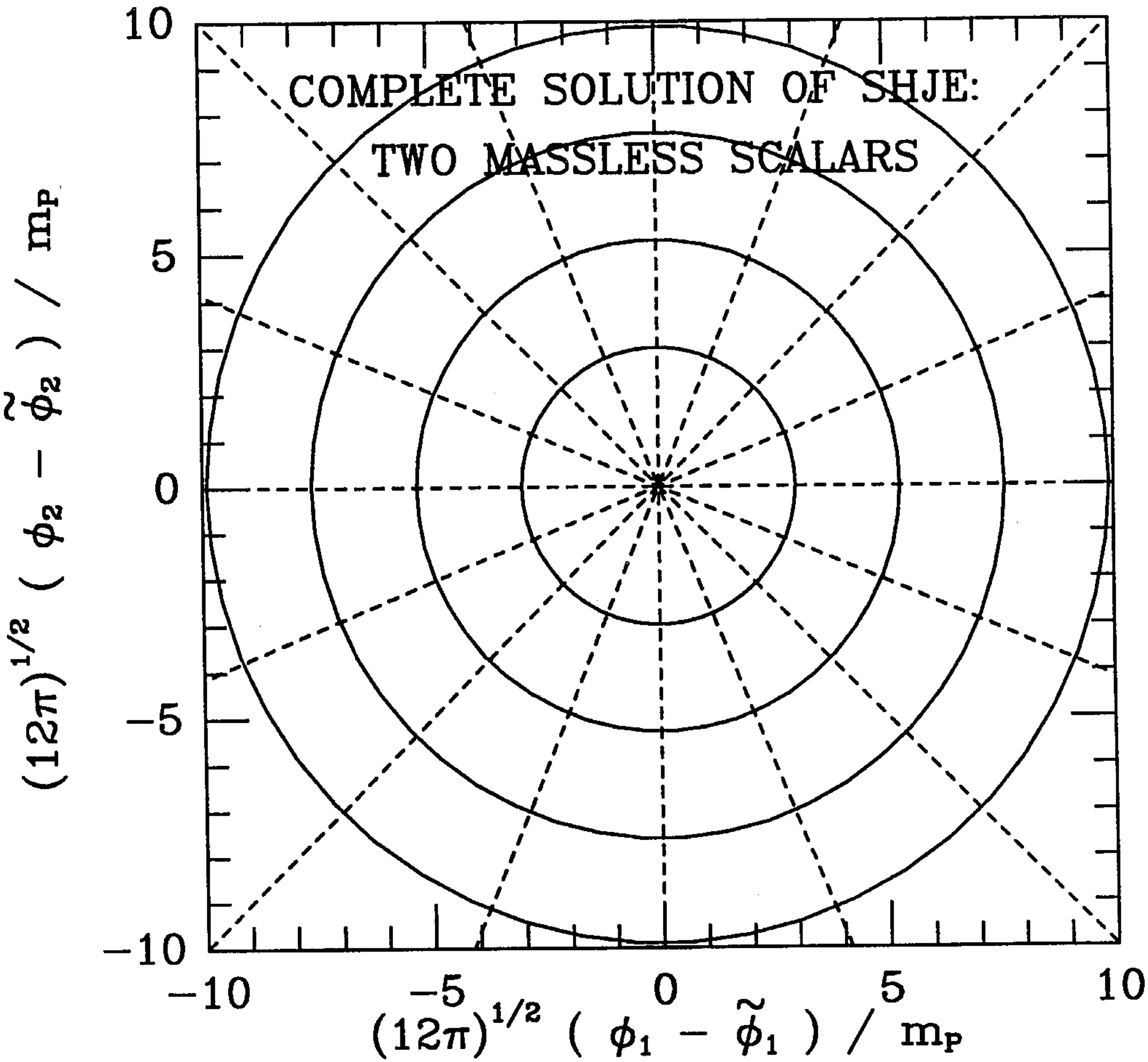
COMPLETE SOLUTION OF SHJE:
TWO MASSLESS SCALARS
10
5
0
−5
−10
−10
−5
0
5
10
$(12\pi)^{1/2}\ (\ \phi_2 - \tilde{\phi}_2\ )\ /\ m_P$
$(12\pi)^{1/2}\ (\ \phi_1 - \tilde{\phi}_1\ )\ /\ m_P$

Fig.1

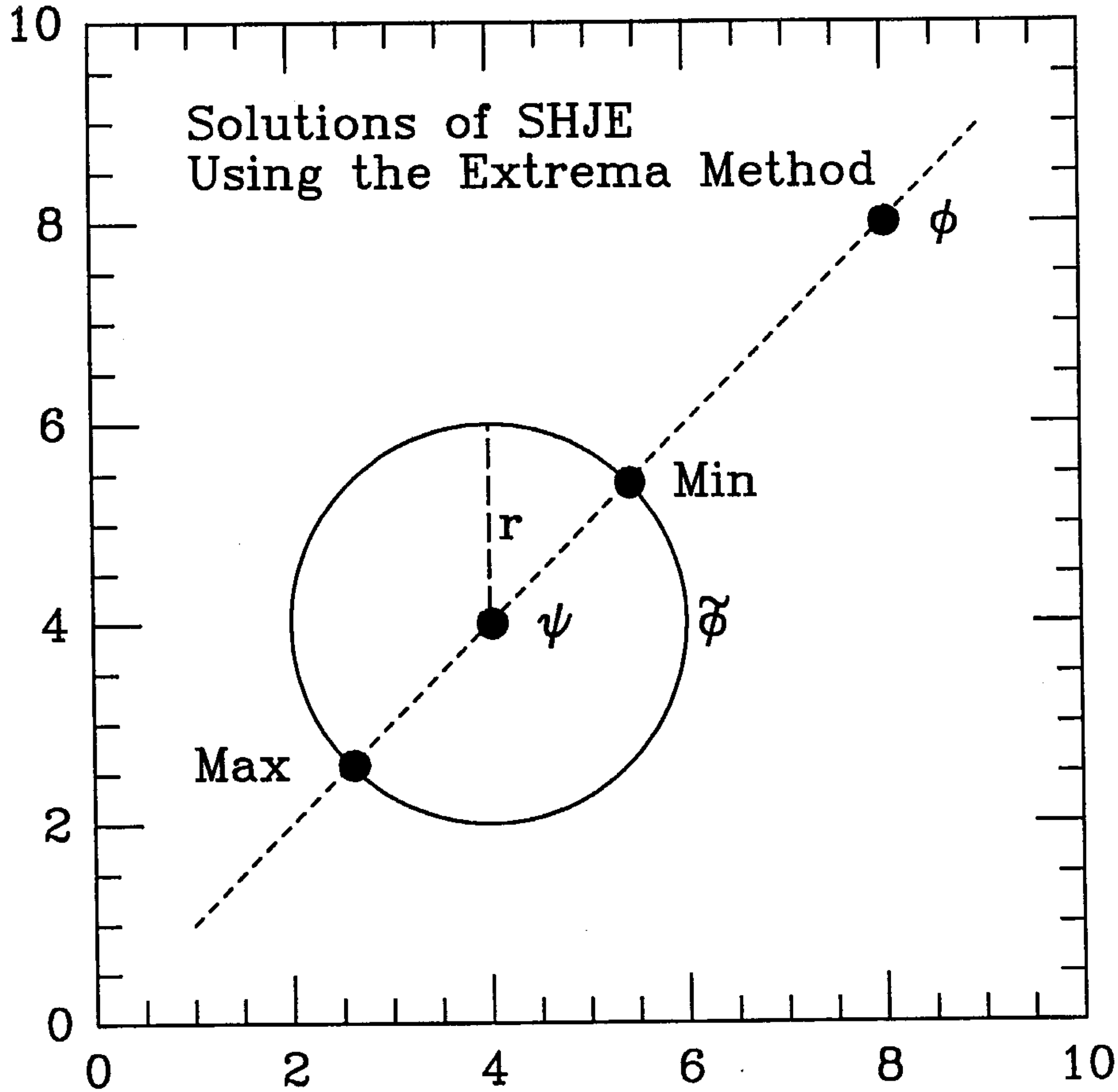
Solutions of SHJE
Using the Extrema Method
φ
Min
r
ψ
$\tilde{\phi}$
Max
0
2
4
6
8
10

Fig. 2

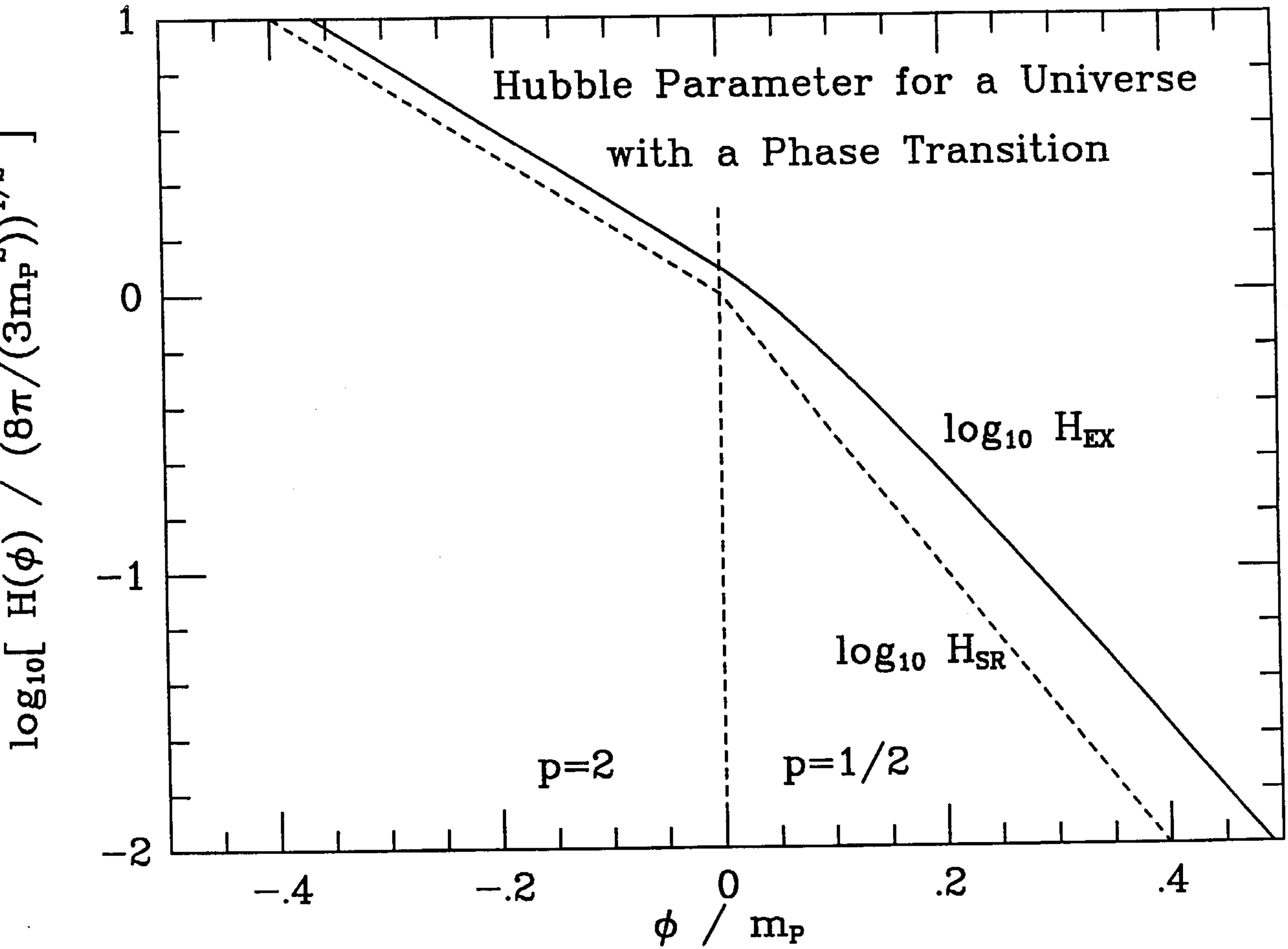
Hubble Parameter for a Universe
with a Phase Transition
log10 HEX
log10 HSR
p=2
p=1/2
log10[H(φ) / (8π/(3mP²))^1/2]
φ / mP
1
0
−1
−2
−.4
−.2
0
.2
.4

Fig.3

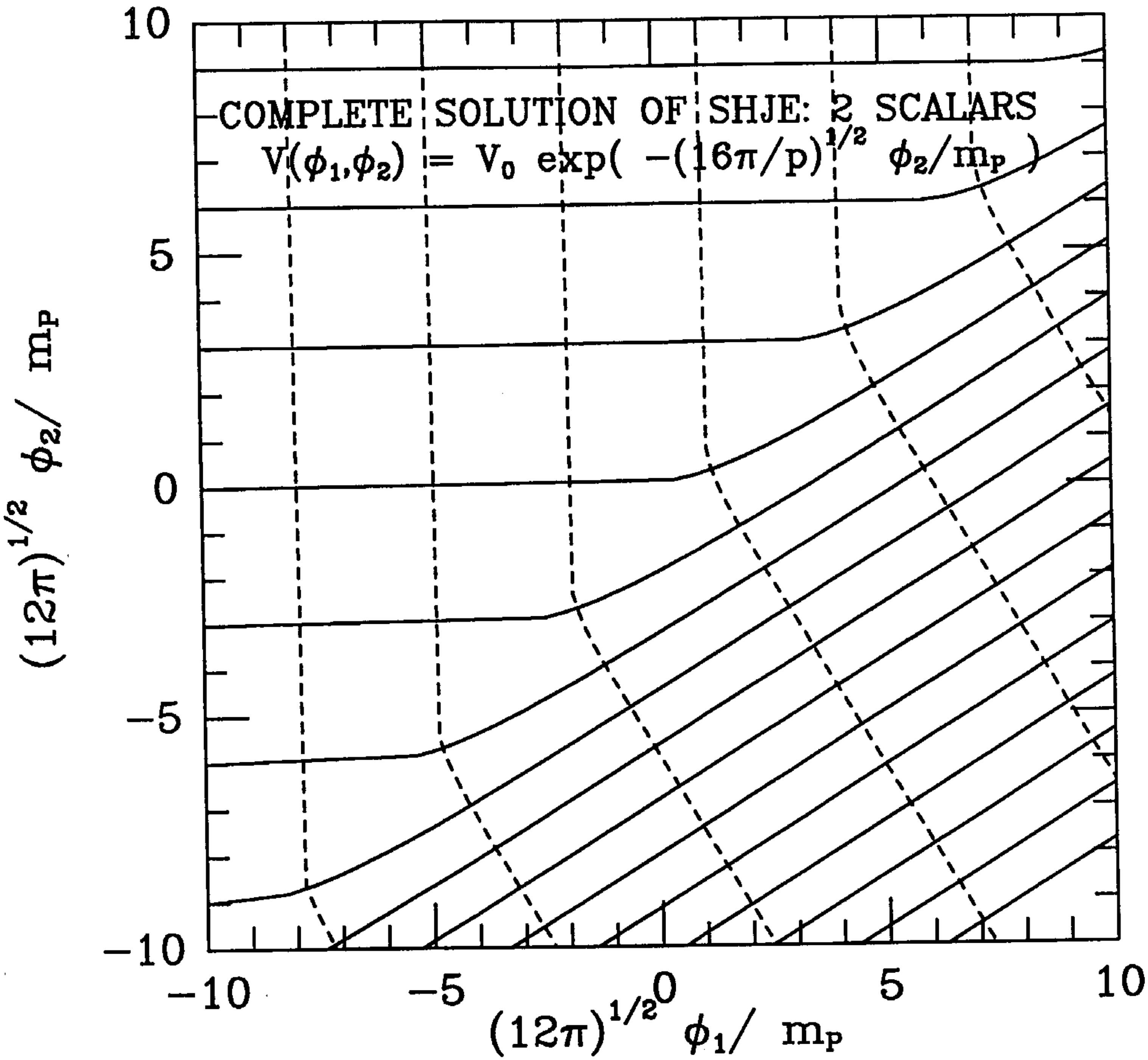
COMPLETE SOLUTION OF SHJE: 2 SCALARS
$V(\phi_1,\phi_2) = V_0 \exp(-(16\pi/p)^{1/2} \phi_2/m_P)$
$(12\pi)^{1/2} \phi_2/ m_P$
$(12\pi)^{1/2} \phi_1/ m_P$
10
5
0
−5
−10

Fig. 4

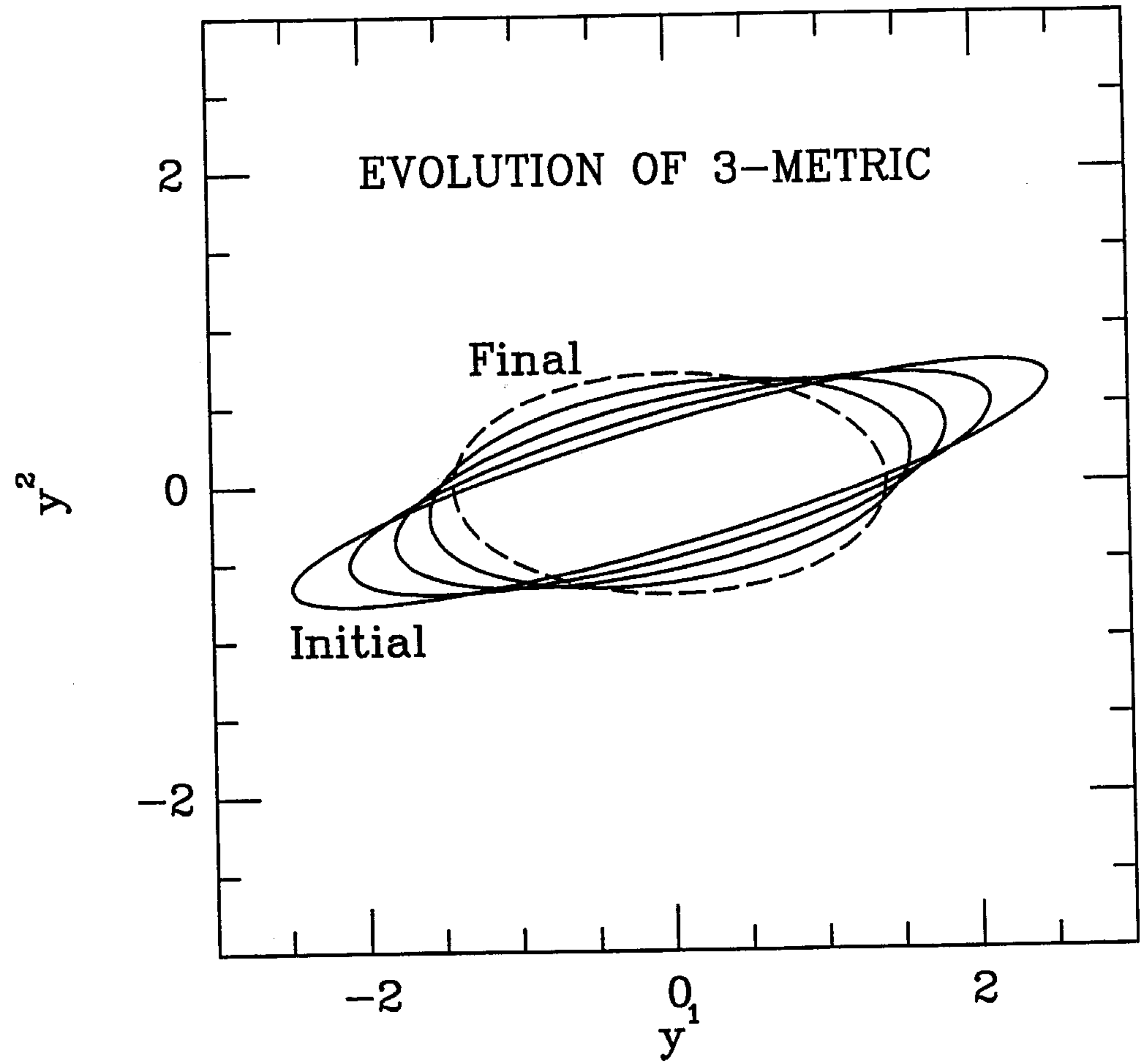

Fig. 5