Fermi National Accelerator Laboratory

December 1990

FERMILAB-PUB-90/267-T

Analyzing the Solutions of Hermitian Matrix Models

Shyamoli Chaudhuri and Joseph D. Lykken

Theory Group
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL 60510

Abstract

We analyze in detail the nonperturbative solutions of both the usual and the bigeneric k-multicritical hermitian matrix models. We show that the methods of Boutroux can be generalized to provide "triply truncated" solutions of the string equations which are unique up to symmetries of the equations. For k even, the solutions to the string equations thus obtained are always complex. For k odd, the usual multicritical one-matrix models always have real solutions. The bigeneric models for k odd have real specific heat, but certain correlators may be complex. We also make explicit the relation between these results and an eigenvalue analysis on the sphere.

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

1. Introduction

Since the initial excitement over the nonperturbative solutions of hermitian matrix models[1]-[4], certain perverse features of these solutions have come to light. The simplest one-matrix model, whose string equation coincides with Painlevé I, was supposed to describe pure 2-D gravity. However David[5]-[7]has shown that the model has no consistent real solutions. It is possible to obtain a consistent real solution for pure gravity by dimensional reduction[8]-[10], but this requires that we abandon the analytic elegance of the KdV hierarchy.

It would thus seem incumbent upon us to grapple further with the k-multicritical string equations, in an attempt to better understand the nature of their solutions. Many authors have already made progress in this direction. Brézin, Marinari, and Parisi(BMP)[11] showed that the k=3 model does have a consistent, possibly unique, real solution, which they found numerically. Futhermore, numerical investigation showed that the renormalization group flow from the k=3 to the k=2 model is singular[12]. Several considerations have since led to the speculation that all of the k even models will share the unfortunate property of k=2, while all of the k odd models should possess analogs of the BMP solution. One consideration is that the simplest matrix potentials giving k even criticality are unbounded below. This leads to instanton-like barrier penetration effects[6][13][14]. For k=2 David has shown[6] that such single instanton contributions seem to match up with the nonperturbative terms in the asymptotic solution of Painlevé I. Since these terms occur with complex coefficients, it is tempting to identify these instanton-like effects as the source of the problem, which should then, by extension, occur for all even k. A related consideration is that the even k models exhibit an instability to variations of the steepest descent eigenvalue configuration on the sphere[6][14]. This can be avoided by complex analytic continuation, but this in turn diminishes the prospects for real solutions. A third consideration is that, while all of the odd k one-matrix models exhibit Borel summability[13], the even k models do not.

In this paper we demonstrate, by explicit analytic methods, that all of the odd k one-matrix models have unique[1] consistent real solutions, and that none of the even k models have consistent real solutions. We give an analytic expression for the BMP solution (though not, of course, in closed form), and show that, while real, it is of the same form as the complex "triply truncated" solution of Painlevé I. Further, we show how to obtain a unique triply truncated solution for any one-matrix model,

[1] We will define "unique" more precisely in the sequel.

and how the presence or absence of certain symmetries in the development determine whether this solution is real or complex.

Our analysis applies to models with asymmetric matrix potentials as well as the simpler even potential models. This allows us to include the bigeneric models introduced by us in [15]. These models, despite their bizarre appearance in perturbation theory, fit in very naturally with the string equation analysis outlined above. This indicates that the bigeneric models may not be mere mathematical accidents, but may rather correspond to unusual continuum theories yet to be identified.

We also provide the detailed correspondence between the string equation analysis and an eigenvalue analysis on the sphere. This extends the eigenvalue analysis of David[6] to arbitrary asymmetric potentials, and extends the results of Dalley, Johnson, and Morris[19]. As is now well known, asymmetric potentials lead to a doubling of the string equations[16]-[18], [15]. As shown in [19], the critical behavior at the two endpoints of the eigenvalue density determines, respectively, the form of the string equations for the two universal scaling functions. We show further, that the range of $arg(x)$ (x is the scaling variable) for which a consistent double-scaling limit exists at each endpoint matches precisely to the pole-free regions of the triply truncated solutions to the two string equations. This is nontrivial in our examples since these regions differ between the two solutions.

2. The Analysis of Boutroux

Many years ago the mathematician Pierre Boutroux developed a number of powerful techniques for analyzing the analytic properties of the solutions to the Painlevé I equation[20]. In this section we review Boutroux's methods and results preparatory to extending them to the string equations of general hermitian matrix models. A brief discussion of Boutroux's work can already be found in the papers of David [5]-[7], following the book by Hille[21].

The Painlevé I equation (with appropriate rescalings) can be written as

$$-\frac{1}{6}\rho'' + \rho^2(x) = x \tag{2.1}$$

The equation is invariant under the transformation

$$x \to \omega x, \quad \rho \to \omega^3 \rho, \quad \omega = exp\frac{2\pi i}{5} \tag{2.2}$$

It is thus convenient to make the change of variables

$$z = \frac{4}{5}x^{5/4}, \qquad w(z) = x^{-1/2}\rho \tag{2.3}$$

The equation then becomes

$$w'' - 6w^2 + 6 = -\frac{w'}{z} + \frac{4}{25}\frac{w}{z^2} \tag{2.4}$$

This is asymptotic to the elliptic equation

$$w_0'' - 6w_0^2 + 6 = 0 \tag{2.5}$$

whose general solution is given by the Weierstrass elliptic function $\mathcal{P}(z - z_0; 12, c)$, where z_0 and c are arbitrary. This function is doubly periodic and, in general, has an infinite double array of second order poles in the complex z-plane. We thus expect that solutions of Painlevé I will, in general, inherit a similar network of double poles, at least asymptotically.

On the other hand, Boutroux shows that (2.4) has an infinite number of so–called *truncated* solutions which have no poles at all over half of the complex z-plane. To see this, introduce an artificial expansion parameter λ

$$w'' - 6w^2 + 6 = \lambda\left(-\frac{w'}{z} + \frac{4}{25}\frac{w}{z^2}\right) \tag{2.6}$$

and develop the solution as a power series in λ:

$$w(z) = 1 + \lambda w_1(z) + \lambda^2 w_2(z) + \cdots \tag{2.7}$$

where we have employed the fact that $w(z)$=1 is the trivial pole–free solution of the elliptic equation (2.5).

The functions $w_j(z)$ all obey second order *linear* equations of the same form:

$$w_j'' - 12w_j = h_j \tag{2.8}$$

where:

$$h_j(z) = -\frac{w_{j-1}'}{z} + \frac{4}{25}\frac{w_{j-1}}{z^2} + 6\sum_{i=1}^{j-1} w_i w_{j-i} \tag{2.9}$$

(note the last term above was inadvertently dropped(!) by Boutroux). In any domain where $h_j(z)$ is analytic, the general solution of (2.8) is given by

$$w_j(z) = \frac{1}{2}\left[e^{\sqrt{12}z}\int_{z_1}^{z} dz' e^{-\sqrt{12}z'} h_j^p(z') + e^{-\sqrt{12}z}\int_{z_2}^{z} dz' e^{\sqrt{12}z'} h_j^p(z')\right] \tag{2.10}$$

where z_1 and z_2 are arbitrary and $h_j^p(z)$ is the primitive of $h_j(z)$.

In particular, for j=1 we have

$$h_1 = \frac{4}{25}\frac{1}{z^2} \tag{2.11}$$

and

$$w_1(z) = -\frac{2}{25}\left[e^{\sqrt{12}z}E_1(\sqrt{12}z) + e^{-\sqrt{12}z}E_1(-\sqrt{12}z)\right] + d_1 e^{-\sqrt{12}z} + d_2 e^{\sqrt{12}z} \tag{2.12}$$

where d_1 and d_2 are arbitrary constants, and $E_1(z)$ is the exponential integral function[22][23]. Now $E_1(z)$ can be rewritten as a logarithm plus an entire function:

$$E_1(z) = Ein(z) - log(z) - \gamma \tag{2.13}$$

where γ is Euler's constant and the entire function $Ein(z)$ is known as the complementary exponential integral function. In addition, $E_1(z)$ has the following asymptotic expansion:

$$E_1(z) \to \frac{e^{-z}}{z}\left[1 - \frac{1}{z} + \cdots\right] \qquad (|argz| < \frac{3}{2}\pi) \tag{2.14}$$

Thus $w_1(z)$ is an analytic function away from the log branch point, and it has smooth asymptotic behavior given by

$$w_1(z) \to \frac{1}{150z^2} + d_1 e^{-\sqrt{12}z} + d_2 e^{\sqrt{12}z} \qquad (|argz| < \frac{3}{2}\pi) \tag{2.15}$$

If we chose d_2=0 (d_1 still arbitrary) then $w_1(z)$ is analytic, bounded and tends to zero like $1/z^2$ in the entire right half plane.

We can now obtain Boutroux's truncated solutions by iterating the above analysis for all of the w_j's, adjusting the integration constants to obtain the following behaviors. At each iteration, $h_j(z)$ is a polynomial function of the w_i's ($i<j$), their derivatives, and $1/z$. Each $h_j(z)$ is analytic, bounded, and tends to zero like $1/z^{2j}$ in the right half plane. In fact the only singularities of the h_j's correspond to those of the log and powers of logs. It follows that each $w_j(z)$ is analytic, bounded, and tends to zero like $1/z^{2j}$ in the right half plane; also the only singularities of the w_j's correspond to those of the log and powers of logs. Boutroux shows, in addition, that for sufficiently large $Re(z)$ greater than some $z_c>0$, the bound on $|w_j(z)|$ decreases with each iteration like $1/|z^{2-\epsilon}|$, where ϵ is arbitrarily small[2]. We can thus arrange

[2] Since Boutroux dropped $O(j)$ number of terms on the r.h.s. of (2.9), one might worry that this spoils his bounds on the w_j. However this $O(j)$ enhancement is offset by a $1/(2j-1)$ suppression in the asymptotic series for h_j^p.

that the series (2.7) converges for $\lambda=1$, and we obtain solutions of (2.4) which are pole–free in the half plane $Re(z)>z_c$. There is a one–parameter infinity of such solutions, which Boutroux calls solutions truncated in the direction of the positive real axis. Obviously one can also construct an infinite family of solutions which are truncated in the direction of the negative real axis. Each truncated solution has an infinite array of double poles, but the array terminates for some $Re(z)<z_c$.

There is a unique *triply truncated* solution of (2.4) which is pole–free asymptotically in the entire (cut) z-plane[3]. It is obtained by setting both constants d_1 and d_2 equal to zero in the expression (2.12) for $w_1(z)$, and making analogous tunings for the other w_j's, such that all the w_j's tend to zero like $1/z^{2j}$ in the entire (cut) z-plane. In terms of the original scaling variable x, the triply truncated solution is asymptotically pole–free for all $|arg x|<4\pi/5$. By employing the transformation (2.2), we can in fact obtain ten distinct triply truncated solutions of Painlevé I, five of which have the asymptotic behavior $\rho(x)\rightarrow\sqrt{x}$, and five of which go like $\rho(x)\rightarrow-\sqrt{x}$. Each such solution has poles asymptotically only in a wedge of interior angle $2\pi/5$.

By choosing the wedge $2\pi/5<arg x<4\pi/5$, we have a solution of Painlevé I with (at most) a finite number of poles on the real axis. Unfortunately, this solution is complex for real x [5]. This can be seen at the level of $w_1(z)$ as given by (2.12) with d_1 and d_2 set to zero. From (2.13) it is clear that for real positive x, which implies real positive z, $w_1(z)$ acquires an imaginary part from a term proportional to $log(-\sqrt{12}z)$. The situation is even worse for real negative x, since z itself is complex. David[6] has interpreted this as an instanton–like instability in what would otherwise seem the unique consistent solution of the $k=2$ pure gravity matrix model.

3. The Bigeneric $k=2$ String Equations

As discussed in [15], hermitian matrix models with one–cut eigenvalue distributions but asymmetric potentials can exhibit $k=2$ double-scaling behavior which is quite different from the pure gravity model. For the *bigeneric* $k=2$ model the contributions to the specific heat from even genera vanish. One might imagine that such bizarre solutions are somehow sick, and that this can be seen from a careful analysis of the $k=2$ bigeneric string equations and eigenvalue density. However, this

[3] Note that, throughout this paper, "triply truncated" merely denotes solutions with this property.

is not the case; the bigeneric solution is in fact slightly better behaved than its pure gravity sibling.

Let us examine the k=2 bigeneric string equations. These consist quite simply of a pair of Painlevé I 's:

$$-\frac{1}{6}\rho''_{\pm} + \rho_{\pm} = x \tag{3.1}$$

where the specific heat $\rho(x)$=ρ_++ρ_- and we have suppressed irrelevant rescalings. In addition, the matrix model analysis that produces these equations gives a prescription for the asymptotics of the solutions:

$$\rho_+(x) \to +\sqrt{x}; \quad \rho_-(x) \to -\sqrt{x} \tag{3.2}$$

Thus the ρ_+ equation is identical to the pure gravity equation for ρ. To avoid an infinite number of double poles on the real x axis, we can take the appropriate triply truncated solution for ρ_+, as discussed in the previous section. This solution is of course complex.

The ρ_- equation requires a different kind of solution, due to its differing asymptotics. As discussed above, triply truncated solutions of Painlevé I with $\rho_-(x)$→$-\sqrt{x}$ asymptotics can be obtained by applying the transformation (2.2) to the original triply truncated solution for $w(z)$. One can easily show that (2.2) is equivalent to the following transformation on $w(z)$:

$$z \to iz; \qquad w \to -w \tag{3.3}$$

This is an obvious invariance of (2.4). Thus if we denote by $w_{trip}(z)$ the triply truncated solution, $-w_{trip}(iz)$ is also a triply truncated solution, with the expansion

$$-1 - w_1(iz) - w_2(iz) - \cdots \tag{3.4}$$

This gives a solution for $\rho_-(x)$ which is asymptotically pole–free for real x and satisfies (3.2). Furthermore this solution is real for real z! To see this, observe from (2.8) and (2.9) that all the $w_j(z)$'s have (for the triply truncated case) a z→$-z$ symmetry. Since

$$\left(log(i\sqrt{12}z)\right)^* = log(-i\sqrt{12}z) \tag{3.5}$$

for real z, and since all the $w_j(iz)$'s can be written as sums of powers of these logs times entire functions, this z→$-z$ symmetry guarantees that $-w_{trip}(iz)$ is real for real z. Thus $\rho_-(x)$ is real for real positive x. Of course ρ=ρ_++ ρ_- is still complex.

We see then that, although the k=2 bigeneric model inherits the same instanton–like difficulty as the pure gravity model, the new features of the string equation analysis are quite reasonably behaved.

4. The Solutions of the k=3 String Equation

In this section we will show how to extend the techniques of Boutroux to an analysis of the k=3 string equation. In particular, we will demonstrate that this equation has a unique triply truncated solution which is quite similar to that of pure gravity. In this case, however, we will obtain a solution which is real for real x, and corresponds in fact to the real pole–free solution found numerically by Brezin, Marinari, and Parisi [11].

The k=3 string equation can be written

$$\frac{1}{10}\rho^{(4)} + \frac{1}{2}(\rho')^2 + \rho\rho'' + \rho^3 = x \tag{4.1}$$

The equation is invariant under the transformation

$$x \rightarrow \omega x, \quad \rho \rightarrow \omega^5 \rho, \quad \omega = exp\frac{2\pi i}{7} \tag{4.2}$$

It is thus convenient to make the change of variables

$$z = \frac{6}{7}x^{7/6}, \qquad w(z) = x^{-1/3}\rho \tag{4.3}$$

The equation then becomes

$$\begin{aligned}\frac{1}{10}w^{(4)} + \frac{1}{2}(w')^2 + ww'' + w^3 - 1 \\ = -\frac{1}{5}\frac{w'''}{z} + \frac{41}{490}\frac{w''}{z^2} - \frac{41}{490}\frac{w'}{z^3} \\ + \frac{128}{2401}\frac{w}{z^4} - \frac{ww'}{z} + \frac{6}{49}\frac{w^2}{z^2}\end{aligned} \tag{4.4}$$

This is asymptotic to the equation

$$w^{(4)} + 5(w')^2 + 10ww'' + 10w^3 - 10 = 0 \tag{4.5}$$

which turns out to be equivalent to the elliptic equation

$$(w_0')^2 = -2w_0^3 + cw_0^2 - c^2 w_0 - \frac{1}{2}c^3 - 10 \tag{4.6}$$

where c is an arbitrary constant. The solutions are Weierstrass elliptic functions plus a constant, and depend on c plus one additional arbitrary parameter.Thus, just as for Painlevé I, we expect solutions of (4.1) to have an infinite double array of second order poles in the complex z-plane.

Now let us introduce an artificial expansion parameter λ on the right hand side of (4.4), just as we did in (2.6). We want to develop the solutions for $w(z)$ as a power series in λ:

$$w(z) = 1 + \lambda w_1(z) + \lambda^2 w_2(z) + \cdots \tag{4.7}$$

where we have used the fact that $w(z)$=1 is a trivial pole–free solution of the elliptic equation (4.5).

The key observation is that each $w_j(z)$ is given by a *pair* of second order linear equations of similar form:

$$w_j'' = a w_j + g_j(z) \tag{4.8a}$$

$$g_j'' = a^* g_j + 10 h_j(z) \tag{4.8b}$$

where

$$a = -5 + i\sqrt{5} = \sqrt{30} e^{i(\pi-\phi)}; \quad \phi = tan^{-1}\left(1/\sqrt{5}\right) \tag{4.9}$$

and a^* denotes the complex conjugate of a. The functions $h_j(z)$ are polynomial functions of the w_i's ($i<j$), their derivatives, and $1/z$. In particular, for j=1 we have

$$h_1 = \frac{6}{49}\frac{1}{z^2} + \frac{128}{2401}\frac{1}{z^4} \tag{4.10}$$

The general solution for $w_1(z)$ is found to be:

$$\begin{aligned} w_1(z) = &-\gamma\left[e^{\sqrt{a}z}E_1(\sqrt{a}z) + e^{-\sqrt{a}z}E_1(-\sqrt{a}z)\right] \\ &-\gamma^*\left[e^{\sqrt{a^*}z}E_1(\sqrt{a^*}z) + e^{-\sqrt{a^*}z}E_1(-\sqrt{a^*}z)\right] \\ &+ d_1 e^{\sqrt{a}z} + d_2 e^{-\sqrt{a}z} \end{aligned} \tag{4.11}$$

where

$$\gamma = \frac{5}{a^* - a}\left(\frac{6}{49} + \frac{64}{7203}a\right) \tag{4.12}$$

and where d_1 and d_2 are arbitrary constants. We have already fixed two other arbitrary constants in the general solution by selecting the unique solution of (4.8b) such that $g_1(z)$ tends asymptotically to zero like $1/z^2$ in the entire cut z-plane.

The solution (4.11) is very similar to (2.12) obtained for Painlevé I. To obtain the triply truncated solution for $w(z)$, we can employ the same arguments used by Boutroux. The first step is to set d_1 and d_2 equal to zero, so that $w_1(z)$ tends asymptotically to zero like $1/z^2$ in the entire cut z-plane. We observe that $w_1(z)$ can be rewritten as logs and entire functions. Thus $h_2(z)$ can be written as logs, squares of logs, and entire functions; furthermore $h_2(z)$ tends asymptotically to zero like $1/z^4$ in the entire cut z-plane. By Boutroux's arguments, this is sufficient to guarantee that, with suitable choices of the integration constants, $g_2(z)$ and $w_2(z)$ can also be expressed in terms of powers of logs and entire functions, and that they tend asymptotically to zero like $1/z^4$ in the entire cut z-plane. The rest of the derivation proceeds just as for Painlevé I.

The unique triply truncated solution of (4.4) is real for real z. This follows from three facts. The first is that all of the $w_j(z)$'s can be written as finite polynomials of logs times entire functions. The second is that the resolution of (4.4) into the pair of second order equations (4.8)has an $a \leftrightarrow a^*$ symmetry (note also that all the other coefficients in (4.8)are real). The third is that the choices of integration constants required for the triply truncated solution preserves this symmetry. Thus all the $w_j(z)$'s are real for real z. In addition, using the above arguments plus the symmetry of (4.4) under $z \to -z$, it follows that the triply truncated solution of (4.4) is also real when z is pure imaginary.

It is now a straightforward matter to match up the real, triply truncated solution of (4.4) to the results of BMP[11], and David [6] . Although there is considerable branch choice ambiguity in the solution, we can fix this by enforcing the $a \to a*$ and $z \to -z$ symmetries. The branch cuts of the solution correspond to the branch cut of $log(\sqrt{a}z)$ and the complex conjugate branch cut of $log(\sqrt{a^*}z)$. They are located at $arg(z)=\pi/2+\phi/2$ and $-\pi/2-\phi/2$ respectively. Thus the boundaries of the asymptotically pole–free regions in the complex x-plane occur at $arg(x)=\pm 3(\pi+\phi)/7$. Because of the $a \to a^*$ symmetry, this solution is real for real positive z, and thus provides a solution for $\rho(x)$ which is real for real positive x and has asymptotics $\rho(x) \to |x|^{1/3}$ as $x \to +\infty$. This matches precisely to the BMP solution and the eigenvalue analysis of David [6] for real positive x.

To obtain a real solution for real negative x, we must apply the transformation (4.2). In terms of w and z, this transformation becomes:

$$w(z) \to \omega^4 w(\omega^{-1} z), \quad \omega = exp\frac{\pi i}{3} \tag{4.13}$$

Consider the result of applying this transformation twice. The branch cut at *arg* $x=3(\pi+\phi)/7$ is rotated by $4\pi/7$ to $\pi+(3\phi/7)$. By our symmetry assumption, the conjugate branch cut is at $-\pi-(3\phi/7)$. For real negative x, the argument of z is $7\pi/6$. Using this information, the transformed solution can be written

$$\begin{aligned} \rho(x) &= x^{1/3} e^{\frac{8\pi i}{8}} w(e^{-\frac{2\pi i}{8}} z) \\ &= -|x|^{1/3} w\left(i|z|\right) \end{aligned} \tag{4.14}$$

This is real, has the same $x\to-\infty$ asymptotic as the BMP solution, and has branch cuts which match to David's eigenvalue analysis.

5. The Bigeneric $k=3$ String Equations

As shown in [15], the bigeneric $k=3$ string equations are given by

$$\frac{1}{10}\rho_{\pm}^{(4)} + \frac{1}{2}(\rho_{\pm}')^2 + \rho_{\pm}\rho_{\pm}'' + \rho_{\pm}^3 = \pm\frac{6}{c_1}x \tag{5.1}$$

where c_1 is an arbitrary nonzero constant. The eigenvalue analysis of this model, which we present in the next section, shows that a consistent one–cut eigenvalue density requires that c_1 be complex. This is a harmless requirement as long we can still succeed in finding real solutions for the specific heat. For simplicity we take c_1 positive imaginary. After suitable rescalings, and a change of variables $x\to\tilde{x}$, $\rho_{\pm}\to-\rho_{\pm}$, where $x=i\tilde{x}$, the string equations may then be written

$$\frac{1}{10}\rho_{\pm}^{(4)} + \frac{1}{2}(\rho_{\pm}')^2 + \rho_{\pm}\rho_{\pm}'' + \rho_{\pm}^3(\tilde{x}) = \mp\tilde{x} \tag{5.2}$$

Thus for ρ_- we are simply interested in the solutions of the $k=3$ equation (4.1) evaluated along the imaginary axis. For ρ_+, we note the following. If one replaces the term -1 in (4.4) by a term $-\epsilon$, then (4.4) is invariant under the following transformation:

$$z \to iz, \quad w \to -w, \quad \epsilon \to -\epsilon \tag{5.3}$$

Thus $-w(iz)$ corresponds to solutions for ρ_+.

It is not too difficult to obtain bigeneric solutions which are asymptotically pole–free on the real x-axis. Consider first $\rho_-(x)$ for real positive x. This corresponds to $\tilde{x}$ on the negative imaginary axis, and thus $arg(z)=-7\pi/12$. To obtain something close to a real solution for ρ_-, we use the triply truncated solution of (4.4) and apply the *inverse* of transformation (4.13) one time:

$$\begin{aligned} \rho_-(x) &= (-ix)^{1/3} e^{\frac{-4\pi i}{8}} w(e^{\frac{\pi i}{8}} z) \\ &= i|x|^{1/3} w(e^{-\frac{\pi i}{4}} |z|) \end{aligned} \tag{5.4}$$

The rotated "branch cuts" of this solution (which are actually the boundaries of the pole-free region) are located at $arg\ \tilde{x}=\pm(\pi+3\phi)/7$. We will show in the next section that this matches up nicely with our eigenvalue analysis. On the other hand, this solution is clearly not real.

To obtain $\rho_+(x)$, we simply replace $w(z)$ by $-w(iz)$ in the above solution. Thus

$$\rho_+(x) = -i|x|^{1/3} w(e^{\frac{\pi i}{4}}|z|) \tag{5.5}$$

The boundaries of the pole-free region for ρ_+ will be rotated by $3\pi/7$ in $arg(x)$ relative to those of ρ_-.

Now note that, by the arguments of the previous section, $w(z)+w(z^*)$ is real for the triply truncated solution. It follows that, for the bigeneric solution just described, the specific heat $\rho(x)=\rho_+ + \rho_-$ is real! This is the unique real solution in the same sense that BMP is the unique real solution of (4.1). For negative real x, we simply interchange the solutions for ρ_+ and ρ_-, and again obtain a real (in fact, the same) solution for $\rho(x)$.

The sentient reader will also have noted that, for our bigeneric solution, the *other* universal function $\sigma(x)=\rho_+ - \rho_-$ is pure imaginary. This sounds rather unphysical, since it may cause certain correlators to be complex, but we should point out that the physical interpretation of $\sigma(x)$ in matrix models has yet to be elucidated.

6. Eigenvalue Analysis

In this section we sketch the eigenvalue analysis of the bigeneric $k=3$ model and of bigeneric models in general, showing consistency with the results of the previous sections. Since our techniques are lifted wholesale from David [6] (see also [24]), we suppress those details which are not germane to our discussion.

In the naive large N limit (i.e. the spherical limit) a hermitian matrix model with polynomial potential $V(\Phi)$ is dominated by the steepest descent configuration for the eigenvalues $\lambda(x)=\lambda(i/N)=\sqrt{N}\lambda_i$, which is described (in general) by a normalized density measure $d\rho(\lambda)$ which has support on some contour C in the complex λ-plane. Following [24] and [6], the spherical solution is conveniently analyzed in terms of two functions:

$$F(\lambda) = \int d\rho(\mu)\frac{1}{\lambda-\mu} \tag{6.1}$$

and the primitive

$$G(\lambda) = \int^{\lambda} d\mu\,(V'(\mu) - 2F(\mu)) \tag{6.2}$$

In this paper we consider only one-arc solutions, for which the support C consists of a single connected component (multi-arc solutions are discussed extensively in [25][26]). Thus C has two endpoints, at $a=\lambda(0)$ and $b=\lambda(1)$. The function $G(\lambda)$ has branch points at a and b, and is pure imaginary along C :

$$G[\lambda(x)] = \pm 2i\pi x \tag{6.3}$$

Thus, along C , $G'(\lambda)$ is proportional to the eigenvalue density $u(\lambda)=dx/d\lambda$:

$$G'(\lambda) = \pm 2i\pi u(\lambda) \tag{6.4}$$

Furthermore $G(\lambda)$ can be interpreted as the action of a single eigenvalue. Thus the steepest descent solution is stable only if the real part of $G(\lambda)$ is positive along the *entire* integration contour for λ in the original path integral [6].

Let us now consider an eigenvalue analysis of the k bigeneric models introduced in [15]. In [15] the k bigeneric solution on the sphere was expressed as $2k+1$ partial derivative constraints on the functional $\Omega(R,S)$, where $R(x)$, $S(x)$ are the recursion coefficient functions:

$$\Omega_{0,1} = 1, \quad \Omega_{0,2} = \cdots \Omega_{0,k} = 0, \quad \Omega_{1,0} = \cdots \Omega_{1,k-1} = 0, \quad \Omega_{1,k+1} = 0 \tag{6.5}$$

These constraints split nicely into two sets which involve only the coupling constants which multiply even (odd) powers of λ in $V(\lambda)$. More precisely, the constraints on the sphere split into $k+1$ even (odd) and k odd (even) constraints according as k is odd (even). Each set *separately* satisfies the constraints for $k+1$ and k criticality, respectively. We should, of course, see the same constraint structure from the eigenvalue analysis of these models.

Before writing the explicit form of the eigenvalue density, we note the following formal relation between $\Omega(R,S)$ on the sphere and the eigenvalue density:

$$u(\lambda) = \frac{1}{\pi}\int_{R_1}^{R_c} \frac{dR}{\sqrt{4R-(\lambda-S)^2}}\frac{d\Omega_{1,0}}{dR} \tag{6.6}$$

where R_1 is determined implicitly by $4R_1=(\lambda - S(R_1))^2$. In [15] we scaled R_c to one, and shifted S_c to zero. This has the effect that the endpoints a, b of the support C are located at $\lambda=\pm 2$. Note, however, that while the endpoints are symmetrical, the eigenvalue distribution is *not* symmetrical for asymmetric potentials.

Given the discussion above the reader will not be surprised to learn that the eigenvalue density for a one-cut solution of the k bigeneric models has the following form:

$$u(\lambda) = [P_{even}(\lambda) + gP_{odd}(\lambda)] \sqrt{(2-\lambda)^{2k-1}(2+\lambda)^{2k-1}} \tag{6.7}$$

where g is an arbitrary but nonzero coupling. P_{even} and P_{odd} are even (odd) polynomials in λ with the property that $P_{even}(P_{odd})$ has an *extra* zero at both endpoints $\lambda = \pm 2$ when k is odd (even).

Two important facts follow from the form of (6.7). The first is that the branch point behavior of (6.7) implies that the double-scaling continuum limit will be determined by a pair of order $2k-2$ KdV type string equations. This of course agrees with the explicit results of [15].

The second fact appears when we check the sign of the leading order corrections to $u(\lambda)$, as given by (6.7), when we move away from either endpoint along the support $\mathcal{C}$. Suppose k is odd. Then P_{even} has an extra zero at both endpoints. Letting $\lambda = \pm(2-\epsilon)$ in (6.7), we find that the leading order in ϵ behavior has the form

$$u(\pm(2-\epsilon)) \sim \pm g\epsilon^{(k-1/2)} + O\left(\epsilon^{(k+1/2)}\right) \tag{6.8}$$

Thus, if we assume that $\mathcal{C}$ lies along the real axis, then the eigenvalue density given by (6.7) has opposite sign near the two endpoints. This problem does not occur for k even, since in this case the contribution from P_{even} dominates the behavior near the endpoints. We conclude that a consistent one-cut solution for the k odd bigeneric models requires *complex* support $\mathcal{C}$.

This is not too alarming in light of the fact that bigeneric matrix potentials are, typically, not bounded below, and thus the path-integral is ill-defined anyway unless we continue to complex λ (or impose some other regularization [9][10][27][28]). However the above result also implies that for k odd we must approach the double-scaling limit along complex trajectories in complex coupling constant space (the critical couplings are still real). It thus becomes doubtful as to whether the string equations have real solutions.

The situation for odd k bigeneric models is quite similar to that of the standard even k multicritical models. For these models, the potential is also unbounded below. Furthermore, if the support $\mathcal{C}$ is assumed real, one finds that $Re(G)$ turns negative as soon as one moves off one of the endpoints of $\mathcal{C}$ [5][6][14]. We must emphasize that, as in our case, this *does not* imply that there are no stable one-arc solutions, but it does raise the spectre that no such solutions are *real*[4]. Since

[4] The lack of a real solution may itself indicate an instanton-like instability, but this connection is not entirely clear.

David has shown this to be the case explicitly for pure gravity, many people have conjectured that this holds for all even k. We will prove this conjecture in the next section.

Let us give two examples of eigenvalue densities of the form (6.7). For the k=2 bigeneric model, we obtain

$$u(\lambda)=\frac{1}{60\pi}\left[20+3g\lambda(\lambda^2-4)\right](2-\lambda)^{3/2}(2+\lambda)^{3/2} \tag{6.9}$$

where g is the arbitrary coupling appearing in the potential[15]:

$$V(\lambda)=g\lambda+\lambda^2-\frac{1}{2}g\lambda^3-\frac{1}{12}\lambda^4+\frac{1}{10}g\lambda^5-\frac{1}{140}g\lambda^7 \tag{6.10}$$

(note we have fixed the redundant couplings c_1,c_2 of [15]: c_1=c_2=2).

For the k=3 bigeneric model we obtain

$$u(\lambda)=\frac{1}{3360\pi}\left[48(\lambda^2-4)+14c_1\lambda\right](2-\lambda)^{5/2}(2+\lambda)^{5/2} \tag{6.11}$$

where c_1 is the arbitrary nonzero coupling appearing in the potential[15]:

$$\begin{aligned}V(\lambda)=&\frac{1}{12}c_1\lambda+2\lambda^2-\frac{1}{12}c_1\lambda^3-\frac{1}{2}\lambda^4\\&+\frac{1}{60}c_1\lambda^5+\frac{1}{15}\lambda^6-\frac{1}{840}c_1\lambda^7-\frac{1}{280}\lambda^8\end{aligned} \tag{6.12}$$

(note we have fixed the redundant coupling c_2 of [15]: c_2=$4c_1$). Observe that if we assume a real support then the eigenvalue density (6.11) changes sign somewhere along C , which would be inconsistent.

For the remainder of this section we will focus on the k=3 bigeneric model (6.12), showing how a consistent, stable solution can be defined through complex analytic continuation. With the critical couplings fixed, the only freedom we have to modify the eigenvalue support C is by adjusting the redundant couplings c_1 and c_2. It is easy to see that the unfortunate property of (6.11) persists for any real nonzero value of c_1. Thus to obtain a consistent one-cut solution at criticality we must take c_1 complex. To simplify the algebra, we will take c_1 pure imaginary: c_1=$24i/7$. The function (6.1) then becomes

$$F(\lambda)=\frac{1}{2\gamma}V'(\lambda)+\frac{1}{\gamma}\left[(\lambda^2-4)+i\lambda\right](\lambda^2-4)^{5/2} \tag{6.13}$$

where γ is the overall coupling introduced in [15](essentially, the cosmological constant) and equals one at criticality.

Straightforward integration allows us to compute the primitive $G(\lambda)$ defined by (6.2). We fix the integration constant by requiring $G(-2)=0$ (this is equivalent to taking $\lambda=-2$ as the lower endpoint of the integral). The result is

$$G(\lambda) = -\frac{i\gamma}{245}t^7 - \frac{1}{840}\lambda(3\lambda^2 - 26)t^5 - \frac{1}{12}\lambda t^3 + \frac{1}{2}\lambda t \\ - 2log\left(\frac{\lambda + t}{2}\right) + 2\pi i \tag{6.14}$$

where

$$t = \sqrt{\lambda^2 - 4} \tag{6.15}$$

While it is not at all obvious from (6.14), the singularities of $G(\lambda)$ are two 7/2 branch points at $\lambda=\pm 2$, and a log branch point at infinity. In fig. 1, we have plotted the contours of $Re(G)=0$ in the complex λ-plane. Note that the contour C which connects the two branch points is now a curve below the real axis. $Im(G)$ does not change sign along C , in fact it increases monotonically from 0 to $2\pi i$, as implied by (6.3).

To examine the question of stability of the one-cut solution, we compute $G(\lambda)$ in the double scaling limit. Because both endpoints of C contribute, we must compute $G(\lambda)$ separately for each endpoint. The scaling at the upper endpoint is given by:

$$\lambda = 2 - \delta^{2/3}z; \quad \gamma = 1 - \delta^2 x \tag{6.16}$$

where δ is the "lattice spacing" (note we employ the same notation z, x for the scaling variables as [6], but with opposite sign). After some algebraic computing, we obtain:

$$G(z) = -\frac{16}{245}\delta^{7/3}\left[8z^2 - 12yz + 15y^2\right](z+y)^{3/2} \tag{6.17}$$

where

$$y = \left(\frac{6x}{c_1}\right)^{1/3} = \left(\frac{7x}{4i}\right)^{1/3} \tag{6.18}$$

The expression for $G(z)$ scaled to the lower endpoint is obtained from (6.17) by letting $y \to -y$ and changing the overall sign.

Since we have now scaled away from precise criticality, the 7/2 branch point of G at $\lambda=2$ now splits up into a 3/2 branch point and two first-order zeros (the other branch point is shifted off to $z=+\infty$). This explains the generic appearance of the $Re[G(z)]=0$ contours plotted in fig 2. For much of the range of $arg(x)$, the support C does not terminate at the 3/2 branch point, or is "pinched" by intersecting a zero of

$G'(z)$. In these cases the one-cut solution is simply inconsistent. For values of $arg(x)$ that pass this test, we then impose the stability criterion that it must be possible to draw a smooth curve which includes C and goes off to infinity without traversing any region with $Re(G)<0$ [6] . By infinity we mean precisely that $|\lambda|\to\infty$ with $arg(\lambda)$ equal to an odd multiple of $\pi/8$, so that the original path integral converges. As is the case for the usual $k=3$ model[6], there are three distinct sectors of $Re(G)>0$ as $\lambda\to\infty$. This ambiguity provides three types of stable one-cut solutions. All three can be seen from fig 2. One solution corresponds to a contour which includes C and goes to infinity through the sector in the lower right half plane. It satisfies the stability and consistency criteria for $|arg(y)|<(5\pi+\phi)/7$, which includes figs 2(c-h). This corresponds to $|arg(\tilde{x})|<(\pi+3\phi)/7$.

This matches up exactly with the solution (5.4) for $\rho_-(x)$ of the string equations presented in the previous section. A similar analysis for G scaled at the lower endpoint should then match up to our solution for $\rho_+(x)$. It is gratifying that this analysis makes even more explicit the connection between the two endpoints of C and the doubled string equations, as discussed in [19].

There is also a solution which corresponds to a contour which goes to infinity through the sector in the lower left half plane. This solution, however, has complex specific heat; it results from applying the transformation (4.13) in an infelicitous manner. Lastly, there is a solution corresponding to a contour which goes to infinity through the sector in the upper half plane. It satisfies the stability and consistency criteria for $-(3\pi+\phi)/7 > arg(y) > -\pi+\phi/7$, which includes figs 2(a-e). This corresponds to $-\pi+3\phi/7 < arg(\tilde{x}) < (5\pi-3\phi)/7$. Now recall that the triply truncated solution for $w(z)$ has a $z\to -z$ symmetry. Thus we expect to find a second solution identical to (5.4) but with its pole-free region rotated by π in $arg(z)$, which is $6\pi/7$ in $arg(\tilde{x})$. This produces limits on $arg(\tilde{x})$ identical to those which we have just given. We conclude that our eigenvalue analysis is in perfect agreement with our analysis of the triply truncated solutions to the string equations.

7. Solutions of the String Equations in General

In this section we extend our analysis to the solutions to the arbitrary, k-th, differential equation in the KdV heirarchy of Painlevé I,

$$R_k[\rho(x)] = x \tag{7.1}$$

where the KdV potentials[30], R_k, are derived from the recurrence relation

$$R_{l+1}^{(1)} = \frac{1}{4}R_l^{(3)} - \rho R_l^{(1)} - \frac{1}{2}\rho^{(1)}R_l, \quad R_0 = \frac{1}{2} \tag{7.2}$$

The equations are invariant under the symmetry transformation

$$x \to \omega x, \quad \rho \to \omega^{2k-1}\rho, \quad \omega = exp\{2\pi i/(2k+1)\} \tag{7.3}$$

which suggests the change of variables

$$z = (\frac{2k}{2k+1})x^{(2k+1)/2k}, \quad w(z) = x^{-1/k}\rho \tag{7.4}$$

The equation becomes

$$R_k[w(z)] = 1 + (\text{terms of } O[1/z]) \tag{7.5}$$

which is asymptotic to the equation[5] $R_k[w_0] = 1$. As before we choose the trivial (pole-free) solution w_0=1, and introducing an artificial expansion parameter, λ, develop solutions for $w(z)$ as a power series in λ

$$w(z) = 1 + \lambda w_1(z) + \lambda^2 w_2(z) + \cdots \tag{7.6}$$

where each of the $w_j(z)$'s satisfies a *linear* inhomogenous $(2k-2)$-th order differential equation of the form

$$R_k^{lin}[w_j] \equiv \sum_{r=0}^{k-1} A_{k-r} w_j^{(2r)}(z) = h_j(z) \tag{7.7}$$

Notice that the linearized KdV operator, R_k^{lin}, contains only *even* powers of derivatives and that the coefficients, A_r, are normalized such that

$$A_k = \frac{(-1)^{k+1}2^{k-1}(2k-1)!!}{(k-1)!} A_1 = k \tag{7.8}$$

These $(2k-2)$-th order differential equations (we suppress the index j in what follows) can be solved recursively by reduction to a system of $k-1$ second order differential equations

$$\begin{aligned} w''(z) &= a_0 w(z) + f_1(z) \\ f_m''(z) &= a_m f_m(z) + f_{m+1}(z), \quad m = 1, \cdots, k-2 \end{aligned} \tag{7.9}$$

[5] The general solution to $R_k[w_0]$=1 is presumably an elliptic function, but we have not encountered a proof of this.

where $f_{k-1}(z) = h(z)$. To see this, note that the ansatz

$$w''(z) = a_0 w(z) + f_1(z) \tag{7.10}$$

implies that

$$w^{(2r)}(z) = a_0^{r-1} w''(z) + \sum_{s=1}^{r-1} a_0^{s-1} f_1^{(2r-2s)}(z) \tag{7.11}$$

Substituting back in the differential equation yields a $(k-1)$-th order polynomial constraint on a_0, and a $(2k-4)$-th order differential equation for $f_1(z)$, namely:

$$Q_{k-1}[a_0] \equiv \sum_{r=0}^{k-1} A_{k-r} a_0^r = 0 \tag{7.12a}$$

$$\sum_{r=1}^{k-1} \sum_{m=0}^{r-1} A_{k-r} a_0^{r-m-1} f_1^{(2m)}(z) = h(z) \tag{7.12b}$$

Iterating the procedure above, we make the ansatz

$$f_1''(z) = a_1 f_1(z) + f_2(z) \tag{7.13}$$

Consistency requires the $(k-2)$-th order polynomial constraint

$$Q_{k-2}[a_1; a_0] \equiv \sum_{r=1}^{k-1} \sum_{m=0}^{r-1} A_{k-r} a_0^{r-m-1} a_1^m = 0 \tag{7.14}$$

and the $(2k-6)$-th order differential equation

$$\sum_{r=1}^{k-1} \sum_{m=1}^{r-1} \sum_{l=0}^{m-1} A_{k-r} a_0^{r-m-1} a_1^{m-l-1} f_2^{(2l)}(z) = h(z) \tag{7.15}$$

Succeeding stages yield the polynomial constraints

$$Q_{k-3}[a_2; a_0, a_1] \equiv \sum_{r=1}^{k-1} \sum_{m=1}^{r-1} \sum_{l=0}^{m-1} A_{k-r} a_0^{r-m-1} a_1^{m-l-1} a_2^l = 0 \tag{7.16}$$

$$Q_{k-4}[a_3; a_0, a_1, a_2] \equiv \sum_{r=1}^{k-1} \sum_{m=1}^{r-1} \sum_{l=1}^{m-1} \sum_{n=0}^{l-1} A_{k-r} a_0^{r-m-1} a_1^{m-l-1} a_2^{l-n-1} a_3^n = 0 \quad , \quad \cdots$$

The final step of the iteration gives an expression for a_{k-2} in terms of all the preceeding coefficients, and the second order differential equation

$$f_{k-2}''(z) = a_{k-2} f_{k-2}(z) + h(z) \tag{7.17}$$

The second order differential equations can all be solved by the methods described previously. By a suitable choice of integration constants a unique triply truncated solution can be developed. Each contribution to this solution can be expressed in terms of entire functions and powers of logarithms. The analysis of Boutroux can be applied just as in our example of section 4.

It remains to solve the set of $(k-1)$ coupled polynomial equations for the coefficients $a_0, a_1, \cdots, a_{k-2}$. We note, however, that the polynomial functions, Q_n, are related as follows

$$Q_{k-1}[a_0] = (a_0 - a_0^{(1)})Q_{k-2}[a_0; a_0^{(1)}] = (a_0 - a_0^{(1)})(a_0 - a_0^{(2)})Q_{k-3}[a_0; a_0^{(1)}, a_0^{(2)}] \cdots \tag{7.18}$$

where $a_0^{(1)}, \cdots, a_0^{(k-1)}$ are the roots of the polynomial $Q_{k-1}[a_0]$. Therefore the only consistent solution for the coefficients, $a_0, \cdots, a_{k-2}$, are the $k-1$ roots of Q_{k-1}! Furthermore, (7.18) is clearly symmetric under permutations of the roots among themselves, which ensures that the triply truncated solution also has this symmetry.

It is important to note that when k is even the polynomial Q_{k-1} has *at least* one real root, (positive when A_1 and A_k are taken to have opposite signs). This *guarantees* that the triply truncated solution is complex. To see why, let a_{real} denote the real root. Then in the development we will have $log(-\sqrt{a_{real}}z)$ terms, which for real positive z have imaginary part $\pm\pi$; these imaginary parts occur order by order in the development and clearly do not cancel out. By contrast, roots which are complex conjugate pairs only give contributions which are themselves complex conjugate pairs, for real z; this follows from the symmetry of the solution under interchange of the roots. Thus, leaving aside the slightly more complicated case of bigeneric models, we have arrived at the following simple criterion: *a real triply truncated solution is obtained if and only if the polynomial $Q_{k-1}[a_0]$ has no real roots*[6]. Since for odd k all coefficients in the linearized KdV operator enter with the same sign[13], it follows that $Q_{k-1}[a_0]$ has no real roots. Thus all of the odd k models have real triply truncated solutions.

As an example, we apply this analysis to the $k = 4$ equation in the KdV heirarchy

$$\frac{1}{35}\rho^{(6)} + \frac{2}{5}\rho\rho^{(4)} + \frac{4}{5}\rho'\rho^{(3)} + \frac{3}{5}(\rho'')^2 + 2\rho^2\rho'' + 2\rho\rho'^2 + \rho^4 = x \tag{7.19}$$

Letting

$$\rho = x^{1/4}w(z) \quad , \quad x = (9z/8)^{8/9} \tag{7.20}$$

[6] Many statements in this section are reminiscent of the asymptotic analysis of Ginsparg and Zinn-Justin[13].

and developing the solution $w(z) = 1 + \lambda w_1 + \lambda^2 w_2 + \cdots$, gives the system of linearized sixth order equations

$$\frac{1}{35} w_j^{(6)} + \frac{2}{5} w_j^{(4)} + 2 w_j^{(2)} + 4 w_j = h_j(z) \tag{7.21}$$

We make the ansatz (suppressing the index j)

$$\begin{aligned} w''(z) &= a_0 w(z) + f_1(z) \\ f_1''(z) &= a_1 f_1(z) + f_2(z) \\ f_2''(z) &= a_2 f_2(z) + h(z) \end{aligned} \tag{7.22}$$

Consistency requires that the coefficients satisfy the conditions

$$\begin{aligned} a_0^3 + 14 a_0^2 + 70 a_0 + 140 &= 0 \\ a_1^2 + (a_0 + 14) a_1 + (a_0^2 + 14 a_0 + 70) &= 0 \\ a_2 + a_1 + a_0 + 14 &= 0 \end{aligned} \tag{7.23}$$

There are two (complex conjugate) roots, and one real root. The presence of the real root guarantees that the triply truncated solution is complex.

8. Conclusion

It would naturally be of interest to extend this analysis to the string equations obtained from multiple arc phases of one-matrix models, unitary matrices, etc. Of particular interest are the hermitian matrix chain models, certain of which are known to describe the *unitary* minimal models dressed by gravity [4]. An asymptotic analysis of these equations indicates that *all* of the unitary solutions are non-Borel-summable [13]. This would lead us to suspect that the specific heat for these models is always *complex*; it would be nice to demonstrate this explicitly.

It would also be interesting to identify continuum models in the same universality class as the scaling limits of our bigeneric models. In addition we note that this paper has ignored the more general (p, q) bigeneric models of [15], which exhibit p/q-th order criticality for the specific heat. The double–scaling limit in these cases involves both a fine tuning of certain couplings relative to each other, in addition to the overall gross tuning of the potential due to the renormalization of the cosmological constant.

The most important issue is whether it is possible to obtain consistent real solutions of the pure gravity model (and other models) by analytic methods. It is not hard to imagine that some clever prescriptions applied to our previous discussion might be sufficient to do this. One would hope, then, to make contact with the results obtained by dimensional reduction. Further, these "technical difficulties" may provide significant insight into the nonperturbative physics of string theories.

Acknowledgments: We are grateful to S. Dalley, G. Mandal, T. R. Morris, and P. Ginsparg for helpful information and discussions. S.C. would also like to acknowledge the hospitality of the Aspen Center for Physics, where this work was begun.

References

[1] E. Brézin and V. Kazakov, *Phys. Lett.* **236B** (1990) 144.

[2] M. Douglas and S. Shenker, *Nucl. Phys.* **B335** (1990) 635.

[3] D. Gross and A. A. Migdal, *Phys. Rev. Lett.* **64** (1990) 127.

[4] M. Douglas, *Phys. Lett.* **238B** (1990) 176.

[5] F. David, *Mod. Phys. Lett.* **A5** (1990) 1019.

[6] F. David, "Phases of the Large N Matrix Model and Non-perturbative Effects in 2d Gravity", preprint SPhT/90-090, July 1990.

[7] F. David, "Matrix Models and 2-D Gravity", Saclay preprint SPhT/90-127, 1990.

[8] E. Marinari and G. Parisi, *Phys. Lett.* **240B** (1990) 375.

[9] J. Ambjørn, J. Greensite and S. Varsted, "A Non-Perturbative Definition of 2D Quantum Gravity by the Fifth Time Action", Niels Bohr preprint NBI-HE-90-39, July 1990;
J. Ambjørn and J. Greensite, "Non-Perturbative Calculation of Correlators in 2D Quantum Gravity", preprint NBI-HE-90-54, Sep. 1990.

[10] M. Karliner and A. B. Migdal, "Nonperturbative 2D Quantum Gravity via Supersymmetric String", Princeton preprint PUPT-1191, July 1990.

[11] E. Brézin, E. Marinari, and G. Parisi, *Phys. Lett.* **242B** (1990) 35.

[12] M. Douglas, N. Seiberg and S. Shenker, *Phys. Lett.* **244B** (1990) 381.

[13] P.Ginsparg and J. Zinn-Justin, "Large Order Behaviour of Non-Perturbative Gravity", Los Alamos preprint LA-UR-90-3830, October 1990.

[14] S. Dalley, "Instability of Even m Multicritical Matrix Models of 2D Gravity", Univ. of Southampton preprint SHEP-89/90-16, Sept. 1990.

[15] S. Chaudhuri, J. Lykken, and T. Morris, *Phys. Lett.* **251B** (1990) 393.

[16] C. Bachas and P. Petropoulos, *Phys. Lett.* **247B** (1990) 363.

[17] P. Petropoulos, "New Critical Behaviors for One-Hermitian-Matrix Models", preprint CPTH-A993.0890, August 1990; "Doubling Versus Non-doubling of Equations and Phase Space Structure in One-Hermitian-Matrix Models", preprint CERN-TH-5951/90, November 1990.

[18] C. Marzban and R. Vishwanathan, "Matrix Models with Non-even Potentials", Trieste preprint ICTP 90-0493, July 1990.

[19] S. Dalley, C. Johnson, and T.R. Morris, "Classification of Critical Hermitian Matrix Models", Univ. of Southampton preprint SHEP-90/91-5, Nov. 1990.

[20] P. Boutroux, *Ann. École Norm. Supér.* **30** (1913) 255.

[21] E. Hille, "Ordinary Differential Equations in the Complex Domain", J. Wiley & Sons, 1976.

[22] M. Abramowitz and I. Stegun, "Handbook of Mathematical Functions", Dover, 1965.

[23] F. Olver, "Asymptotics and Special Functions", Academic Press, 1974.

[24] E. Brézin, C. Itzykson, G. Parisi, and J.B. Zuber, *Comm. Math. Phys.* **59** (1978) 35.

[25] G. Cicuta, L. Molinari and E. Montaldi, J. Phys. **A23** (1990) L421.

[26] K. Demeterfi, N. Deo, S. Jain, and C.-I. Tan, "Multi-Band Structure and Critical Behavior of Matrix Models", Brown Univ. preprint Brown-HET-764, July 1990.

[27] J. Jurkiewicz, *Phys. Lett.* **245B** (1990) 178.

[28] G. Bhanot, G. Mandal, and O. Narayan, *Phys. Lett.* **252B** (1990) 388.

[29] S. Dalley, "Critical Conditions for Matrix Models of String Theory", Univ. of Southampton preprint SHEP-90/91-6, Nov. 1990.

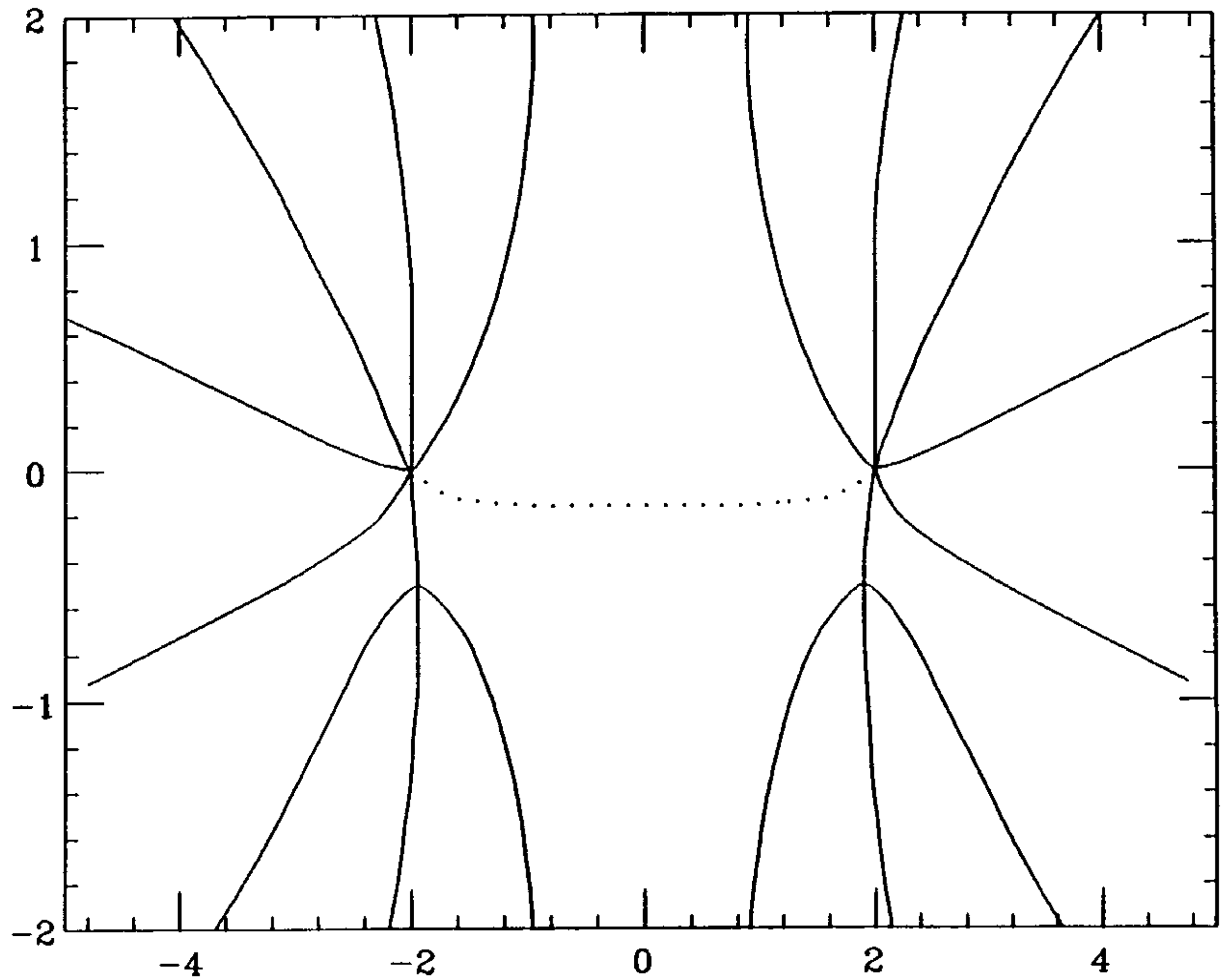
[30] I. M. Gelfand and L. A. Dikii, Russian Math. Surveys, 30:5 (1975) 77.

Figure Captions

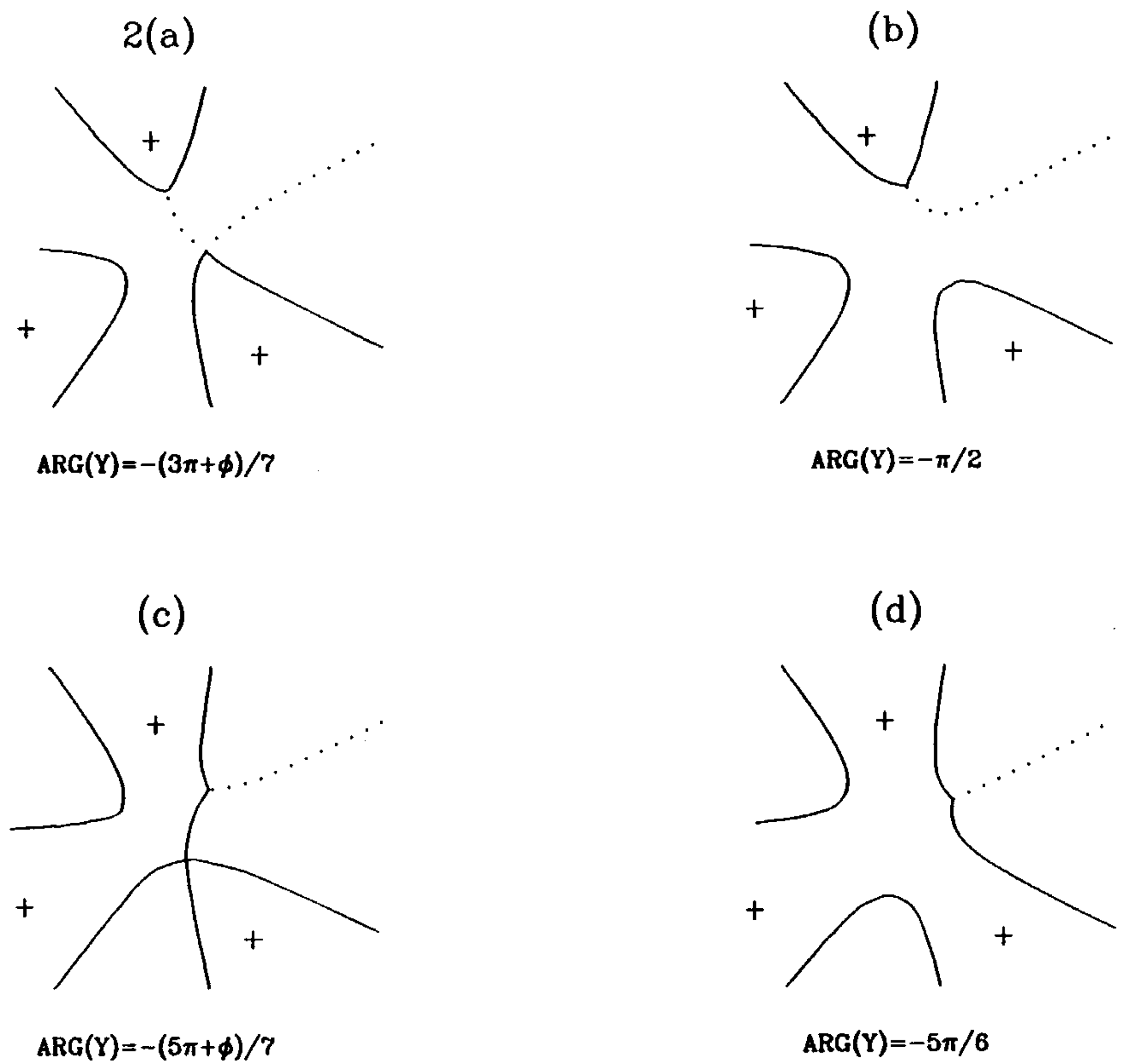
Fig. 1: Contours of $Re[G(\lambda)]$, plotted in the complex λ-plane, for the $k=3$ bi-generic model with $c_1=24i/7$. The dotted contour is the eigenvalue support $\mathcal{C}$.

Fig. 2: Contours of $Re[G(z)]$, plotted in the complex z-plane, for the same model scaled to the upper endpoint. The dotted contour in each figure is the eigenvalue support $\mathcal{C}$. The figures correspond to various (decreasing) values of $arg(y)$, which is related to $arg(x)$ via (6.18).

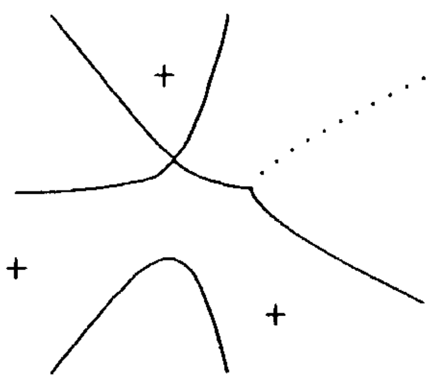
FIG. 1



CONTOURS OF RE[G(λ)]=0

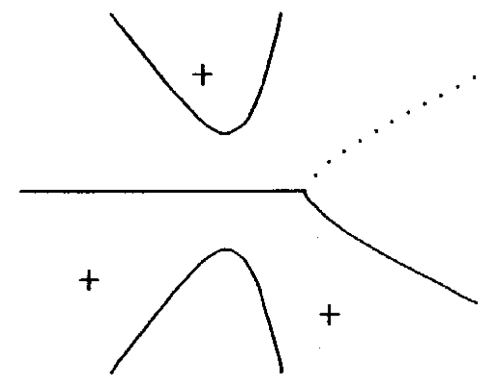

2(a)
+
+
+
ARG(Y)=−(3π+ϕ)/7
(b)
+
+
+
ARG(Y)=−π/2
(c)
+
+
+
ARG(Y)=−(5π+ϕ)/7
(d)
+
+
+
ARG(Y)=−5π/6

(e)



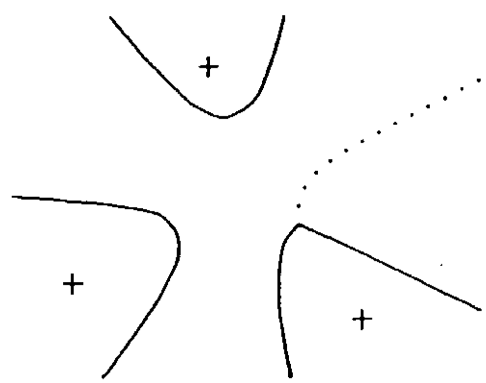
ARG(Y)=$-\pi+\phi/7$

(f)



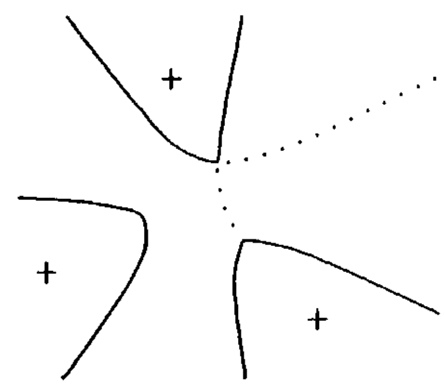
ARG(Y)=$-\pi$

(g)



ARG(Y)=$-7\pi/6$

(h)



ARG(Y)=$-(9\pi-\phi)/7$