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SU(N) Gauge Theories with C-Periodic Boundary Conditions: I. Topological Structure

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Abstract

C-periodic boundary conditions are introduced to SU(N) gauge theory on a torus. C-periodic fields are replaced by their charge conjugates when they are shifted over the boundary. As for periodic boundary conditions the most general C-periodic boundary condition includes twist. The topological structure with C-periodic boundary conditions is quite different from the periodic case. In the periodic case twist leads to the $\mathbb{Z}_N$ 't Hooft flux sectors. In the C-periodic case with even N the symmetry of the flux sectors is reduced to $\mathbb{Z}_2$. For odd N the flux sectors are eliminated completely. Furthermore, the topological charge is an integer when N is odd, whereas it can be a half-integer when N is even.

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1 Introduction

Field theories are often formulated with periodic boundary conditions, i.e. on a torus. The finite volume regulates infrared divergencies without compromising global symmetries or gauge invariance. Because the torus has no curvature it is locally identical to the infinite volume flat space, and hence the infrared regulator does not introduce geometrical artifacts. However, since the torus differs topologically from the infinite volume, various topological peculiarities arise. For example, SU(N) Yang-Mills theories possess $\mathbb{Z}_N$ electric and magnetic flux sectors [1], which decouple in the infinite volume limit.

Another topological peculiarity of periodic boundary conditions has to do with charged particles. A charged state cannot exist in a periodic volume, because its flux cannot go to infinity. The flux must end in anti-charges such that the total periodic system is neutral. More formally, this is a consequence of Gauss' law for spaces with no boundary. In Abelian theories this property can be avoided with anti-periodic boundary conditions for the gauge field, because then the flux lines emanating from a charge can end in its (charge-conjugate) image on the "other" side of the boundary. Anti-periodic boundary conditions have been used to facilitate a recent study of magnetic monopole charges [2] in an Abelian $\mathbb{Z}$ lattice gauge theory.

This paper investigates a new boundary condition for non-Abelian gauge fields, which is a generalization of anti-periodic boundary conditions for Abelian fields. We call these boundary conditions C-periodic, because gauge invariant operators with $C = +1$ are periodic and those with $C = -1$ are anti-periodic.[1] The new boundary condition identifies the field at $\boldsymbol{x} + L\boldsymbol{e}_i$ with its charge conjugate at $\boldsymbol{x}$, where the unit vectors $\boldsymbol{e}_i$ and the length L generate the torus. The various components of the gauge potential A^a acquire different phases at the boundary. To be specific the most general C-periodic boundary condition for the gauge field is

$$\boldsymbol{A}(\boldsymbol{x} + L\boldsymbol{e}_i) = \Omega_i(\boldsymbol{x})[\boldsymbol{\nabla} + \boldsymbol{A}^*(\boldsymbol{x})]\Omega_i^{-1}(\boldsymbol{x}), \tag{1.1}$$

where $*$ denotes complex conjugation, and we write $A = A^a T^a$, where the T^a are anti-Hermitian generators of su(N).[2] The SU(N) functions Ω_i are "twists," as introduced by 't Hooft for periodic boundary conditions. Eq. (1.1) specifies connections in $\mathbb{Z}_2 \cdot \mathrm{SU}(N)/\mathbb{Z}_N$ bundles over the torus with fiber $\mathrm{SU}(N)/\mathbb{Z}_N$. The $\mathbb{Z}_2$ factor in the semi-direct product describes charge conjugation. With C-periodic boundary conditions there is a non-trivial $\mathbb{Z}_2$ transition function at the boundary, whereas with periodic boundary conditions all $\mathbb{Z}_2$ transition functions are trivial. If $N = 2$ the gauge transformation

$$h = \exp\left(\frac{i\sigma_2}{2L}\sum_i x_i\right), \tag{1.2}$$

converts periodic boundary conditions to C-periodic, and *vice versa*, so the distinction is illusory. Indeed we find the C-periodic formulation reproduces the physics of the periodic formulation. However, for $N \geq 3$ C-periodic boundary conditions are different.

Note that C-periodic boundary conditions are necessary for charged matter fields, even in anti-periodic Abelian gauge theory [2], as the picture of flux lines and image charges suggests. Hence, C-periodic boundary conditions are the natural generalization to non-Abelian gauge fields, which are charged themselves.

In addition to the viability of charged states, anti-periodic—or more generally C-periodic—boundary conditions simplify the structure of the magnetic and electric flux sectors. In the Abelian $\mathbb{Z}$ gauge theory with periodic boundary conditions these sectors are described by global $\mathbb{Z}^{d(d-1)/2}$

[1] An example of the latter is $\mathrm{Im}\,\mathrm{Tr}(F_{ij}F_{jk}F_{ki}) = d^{abc}F^a_{ij}F^b_{jk}F^c_{ki}/4$.

[2] As a rule, this paper uses the Lie algebra and group conventions of ref. [3].

and $\mathbb{Z}^d$ symmetries, but with anti-periodic boundary conditions they are drastically reduced to $\mathbb{Z}_2$. The main result of this paper is the analogous statement for SU(N) gauge theories with C-periodic boundary conditions: For odd N there is no 't Hooft magnetic or electric flux, whereas for even N magnetic and electric flux are classified by $\mathbb{Z}_2$, rather than $\mathbb{Z}_N$.

For odd N the absence of 't Hooft electric flux sectors may simplify small-volume "analytic" calculations of the mass spectrum. Because of asymptotic freedom, in small volume Yang-Mills theories one can integrate out the non-zero momentum modes perturbatively, leaving a non-integrable effective Hamiltonian for the constant modes [3]. A (non-perturbative) solution of the constant-mode dynamics yields results for the glueball mass spectrum that are rigorously correct in small volumes [4]. With periodic boundary conditions the electric-flux sectors are degenerate to all orders of perturbation theory. To compute the splitting between the sectors, it is necessary to account for tunneling between gauge field configurations related by gauge transformations with non-trivial winding number. Tunneling also significantly affects the glueball spectrum [5]. For odd N these complications will not arise, because there is only one sector. Further discussion of the small-volume expansion will appear in a forthcoming publication.

This paper details the topological structure of C-periodic boundary conditions. Sect. 2 establishes the gauge invariance, as well as some properties of the transition functions. Sects. 3 and 4 investigate the properties of 't Hooft magnetic and electric flux, respectively. We work in d space dimensions, discussing the physical degrees of freedom in the Hamiltonian formulation. However, the essence of a Lagrangian, space-time formulation can be deduced from the formulae of sect. 3. Sect. 5 examines the "vacuum angle" θ. We switch to the $(d+1)$ dimensional space-time formulation, for $d=3$, and introduce the topological charge in sect. 6. The fractional part is that expected based on experience with periodic boundary conditions, combined with our results for the electric and magnetic flux. Only for even N can the fluxes contribute a half-integer part, and for odd N the topological charge is always an integer.

2 Gauge invariance and transition function properties

Consider the gauge transformation law of the gauge potential:

$$^{g}\!A(x) = g(x)[\nabla + A(x)]g^{-1}(x). \tag{2.1}$$

If one admits arbitrary gauge transformations, it follows immediately that the twist functions Ω_i satisfy the transformation law

$$^{g}\Omega_i(x) = g(x + Le_i)\Omega_i(x)g^{\mathrm{T}}(x). \tag{2.2}$$

The superscript T denotes transpose, and for a unitary matrix $g^{\mathrm{T}} = (g^{-1})^*$.

A precise interpretation of eq. (2.2) is based on expressing Ω_i is terms of transition functions [6]. In d dimensions the torus must be covered with 2^d cells, as depicted for $d=2$ in fig. 1. On the overlaps between cells α and β, the gauge potentials within each cell are related by

$$A^{(\alpha)}(x) = v_{\alpha\beta}(x)[\nabla + A^{(\beta)}(x)]v_{\beta\alpha}(x), \tag{2.3}$$

where $v_{\beta\alpha} = v_{\alpha\beta}^{-1}$. Under a gauge transformation $v_{\alpha\beta} \mapsto {}^{g}v_{\alpha\beta}$ with

$$^{g}v_{\alpha\beta} = g_\alpha v_{\alpha\beta} g_\beta^{-1}. \tag{2.4}$$

The torus is $\mathsf{T}^d = \mathbb{R}^d/\Lambda$, with lattice $\Lambda = \{x \,|\, x = L\sum_i x_i e_i\}$. The $\mathbb{Z}_2$ transition functions can be incorporated implicitly by extending the SU(N) transition functions as

$$v_{\alpha\beta}(x + Le_i) = v^*_{\alpha\beta}(x), \tag{2.5}$$

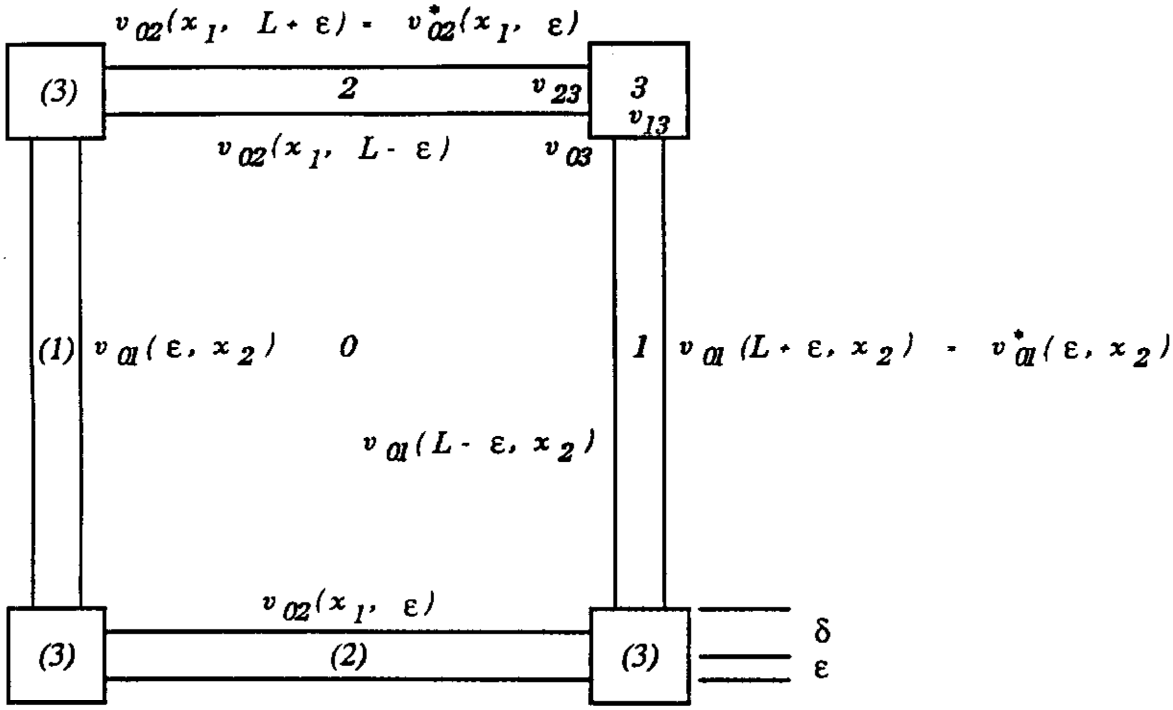

Figure 1: Covering of the torus by 2^d cells, illustrated for $d = 2$. The $v_{\alpha\beta}$ denote the transition functions, where α, $\beta \in \{0, 1, 2, 3\}$. (Cell labels in parentheses indicate periodic copies.)

and the SU(N) gauge transformations similarly as

$$g_\alpha(x + Le_i) = g^*_\alpha(x). \tag{2.6}$$

These properties are illustrated in fig. 1.

The twist (or "multiple transition" [6]) functions are obtained from the transition functions in the limit $\delta \to \varepsilon \to 0$. For $d = 2$

$$\begin{aligned} \Omega_1(x_2) &= \lim_{\epsilon\to 0} v_{01}(L-\epsilon, x_2) v^*_{10}(\epsilon, x_2), \\ \Omega_2(x_1) &= \lim_{\epsilon\to 0} v_{02}(x_1, L-\epsilon) v^*_{20}(x_1, \epsilon), \end{aligned} \tag{2.7}$$

and the generalization to higher d is obvious. The gauge transformation law, eq. (2.2), is then a trivial consequence of eq. (2.4). It is now clear, however, that one admits arbitrary gauge transformations in eqs. (2.1) and (2.2), because $g_0|_{x_i=L-\epsilon}$ need not equal $g_0|_{x_i=\epsilon}$.

Furthermore, using the well-known cocycle condition

$$v_{\alpha\gamma}(x) = v_{\alpha\beta}(x) v_{\beta\gamma}(x), \tag{2.8}$$

one can derive the corresponding identity for the Ω_i:

$$\Omega_i(x + Le_j)\Omega^*_j(x) = \Omega_j(x + Le_i)\Omega^*_i(x). \tag{2.9}$$

The previous discussion has not specified whether there are matter fields in a faithful representation of SU(N). If all fields faithfully represent only SU(N)/$\mathbb{Z}_N$, the equations should be interpreted in the adjoint representation of SU(N). Maps from the circle into SU(N)/$\mathbb{Z}_N$ can be homotopically non-trivial, because it is not simply connected: $\Pi_1(\mathrm{SU}(N)/\mathbb{Z}_N) = \mathbb{Z}_N$. This homotopy is relevant to the torus, because $\mathrm{T}^d = (S^1)^d$. When non-trivial maps are pulled back to the fundamental representation of SU(N), which can only be done locally, $\mathbb{Z}_N$ discontinuities appear. In sects. 3 and 4 we shall use the SU(N) notation with $\mathbb{Z}_N$ jumps, because it is easier to envision SU(N) than the quotient space SU(N)/$\mathbb{Z}_N$.

3 Magnetic Flux

For pure gauge theories the twist function $\Omega_i(x)$ is equivalent to $\zeta_i\Omega_i(x)$, where ζ_i is an N-th root of unity, and $\Omega_i(x)$ is considered to be in the fundamental representation of SU(N). Similarly, the cocycle condition for the twist functions reads

$$\Omega_i(x + Le_j)\Omega_j^*(x) = z_{ij}\Omega_j(x + Le_i)\Omega_i^*(x). \tag{3.1}$$

The "twist tensor" z consists of N-th roots of unity, and $z_{ji} = z_{ij}^*$. 'T Hooft's magnetic flux is given by $n_{ij} \in \mathbb{Z}_N$, where $z_{ij} = \exp(i2\pi n_{ij}/N)$. The twist tensor is gauge invariant, and furthermore, if $\{\Omega_i\}$ and $\{\Omega_i'\}$ produce the same twist tensor, then one can construct a gauge transformation relating them.

Introducing the factors ζ_i and writing $\zeta_i^*\zeta_j = w_{ij}$ shows that the twist tensors z_{ij} and $z_{ij}w_{ij}^2$ are physically equivalent. This is a significant difference with periodic boundary conditions, where identical ζ_i factors would appear on both sides of eq. (3.1).

If $d = 2$ the only non-trivial component of the twist tensor is z_{12}. If N is odd, every N-th root of unity has a square root in the set of N-th roots of unity. Hence, all choices of z_{12} are equivalent to $z_{12} = 1$, and consequently there are no magnetic flux sectors. If N is even, either z_{12} or $z_{12}\exp(-i2\pi/N)$ has a square root in the set of N-th roots of unity. Then, all choices of z_{12} are equivalent to $z_{12} = 1$ or $z_{12} = \exp(i2\pi/N)$, and consequently magnetic flux sectors are classified by $n_{12} \in \mathbb{Z}_2$.

For $d > 2$ the trivial identity $w_{ij}w_{jk}w_{ki} = 1$ implies that not all components of w are independent. However, on the one hand

$$\begin{aligned}
&\Omega_i(x + Le_j + Le_k)\Omega_j^*(x + Le_k)\Omega_k(x) = \\
&\Omega_j(x + Le_i + Le_k)\Omega_i^*(x + Le_k)\Omega_k(x)z_{ij} = \\
&\Omega_j(x + Le_i + Le_k)\Omega_k^*(x + Le_i)\Omega_i(x)z_{ij}z_{ki} = \\
&\Omega_k(x + Le_i + Le_j)\Omega_j^*(x + Le_i)\Omega_i(x)z_{ij}z_{ki}z_{jk},
\end{aligned} \tag{3.2}$$

while on the other hand

$$\begin{aligned}
&\Omega_i(x + Le_j + Le_k)\Omega_j^*(x + Le_k)\Omega_k(x) = \\
&\Omega_i(x + Le_j + Le_k)\Omega_k^*(x + Le_j)\Omega_j(x)z_{kj} = \\
&\Omega_k(x + Le_j + Le_i)\Omega_i^*(x + Le_j)\Omega_j(x)z_{kj}z_{ik} = \\
&\Omega_k(x + Le_j + Le_i)\Omega_j^*(x + Le_i)\Omega_i(x)z_{kj}z_{ik}z_{ji}.
\end{aligned} \tag{3.3}$$

Comparing the accumulated factors of z reveals a constraint on z,

$$z_{ij}^2 z_{jk}^2 z_{ki}^2 = 1, \tag{3.4}$$

which compensates for the restriction on w. The square root argument again leads to the conclusion that there is no 't Hooft magnetic flux if N is odd, and that 't Hooft magnetic flux is described by $\mathbb{Z}_2^{d(d-1)/2}$ if N is even.

4 Electric Flux

To study the electric flux sectors we consider the homotopy properties of gauge transformations. Those homotopic to the identity must be represented by unity on the Hilbert space of states, but in general they are represented by a unitary operator. 'T Hooft electric flux parametrizes the eigenvalues that arise from gauge transformations' first homotopy. A C-periodic $\mathrm{SU}(N)/\mathbb{Z}_N$ gauge transformation, written in the fundamental representation of $\mathrm{SU}(N)$, can have an explicit $\mathbb{Z}_N$ jump:

$$g_{\boldsymbol{z}}(\boldsymbol{x} + L\boldsymbol{e}_i) = z_i g_{\boldsymbol{z}}^*(\boldsymbol{x}), \tag{4.1}$$

where the $z_i = \exp(i2\pi n_i/N)$ are N-th roots of unity, and n_i is the winding number of $g_{\boldsymbol{z}}$ viewed as a map into $\mathrm{SU}(N)/\mathbb{Z}_N$. C-periodic boundary conditions require the complex conjugation. Compare

$$g_{\boldsymbol{z}}(\boldsymbol{x} + L\boldsymbol{e}_i + L\boldsymbol{e}_j) = z_i z_j^* g_{\boldsymbol{z}}(\boldsymbol{x}) \tag{4.2}$$

with

$$g_{\boldsymbol{z}}(\boldsymbol{x} + L\boldsymbol{e}_j + L\boldsymbol{e}_i) = z_j z_i^* g_{\boldsymbol{z}}(\boldsymbol{x}); \tag{4.3}$$

the $\mathbb{Z}_N$ jumps are consistent if and only if

$$z_i^2 = z_j^2. \tag{4.4}$$

Hence, all z_i must be equal when N is odd, and the various z_i are equal up to a sign when N is even.

Let $T_{\boldsymbol{z}}$, sometimes called a central conjugation, be the unitary operator in Hilbert space implementing the gauge transformation $g_{\boldsymbol{z}}$. Because the Hamiltonian and all central conjugations commute, these operators can be simultaneously diagonalized. Thus,

$$T_{\boldsymbol{z}}(\boldsymbol{x})|\Psi_{\mathbf{e}}\rangle = \exp\left(i\frac{2\pi}{N}\omega_{\mathbf{e}}(\boldsymbol{n})\right)|\Psi_{\mathbf{e}}\rangle, \tag{4.5}$$

where $|\Psi_{\mathbf{e}}\rangle$ is an eigenstate of the Hamiltonian labeled by quantum numbers $\mathbf{e}$, which will turn out to be 't Hooft electric flux. The reader should not confuse $\mathbf{e}$ with the spatial unit vectors $\boldsymbol{e}_i$; both notations are conventional. The quantity $\omega_{\mathbf{e}}(\boldsymbol{n})$ must be linear in $\boldsymbol{n}$, and, since $T_{\boldsymbol{z}}^N = 1$, $\omega_{\mathbf{e}}(\boldsymbol{n}) \in \mathbb{Z}_N$.

With periodic boundary conditions $\omega_{\mathbf{e}}(\boldsymbol{n}) = \mathbf{e}\cdot\boldsymbol{n}$, and all states with $\mathbf{e} \in \mathbb{Z}_N^d$ appear in the Hilbert space, but not so with C-periodic boundary conditions. First, $\boldsymbol{z}$ is contrained by eq. (4.4). Second, note that the constant gauge transformation w, where w is an N-th root of unity, obeys eq. (4.1) with $z_i = w^2$, $\forall i$. Consequently, the phases induced by $T_{\boldsymbol{z}}$ and $T_{w^2\boldsymbol{z}}$ must be the same, because any constant map is homotopic to the identity.

When N is odd, eq. (4.4) implies that

$$\omega_{\mathbf{e}}(\boldsymbol{n}) = En, \tag{4.6}$$

where n is the value of the components, all equal, of $\boldsymbol{n}$, and $E \in \mathbb{Z}_N$. Substituting $\boldsymbol{z}$ and $w^2\boldsymbol{z}$ into eqs. (4.5) and (4.6) and equating the phases yields

$$2E = 0 \bmod N. \tag{4.7}$$

But since $N/2$ is not an integer, the only possibility is that $E = 0$. In other words, there are no electric flux sectors for N odd.

When N is even, eq. (4.4) implies that

$$n_i = n + \nu_i \frac{N}{2}, \tag{4.8}$$

where $n \in \mathbb{Z}_N$ and $\nu_i \in \mathbb{Z}_2$, and one can choose, say, the I component $\nu_I = 0$. By linearity in $\boldsymbol{n}$ one has

$$\omega_{\mathbf{e}}(\boldsymbol{n}) = En + \mathbf{e} \cdot \boldsymbol{\nu} \frac{N}{2}, \tag{4.9}$$

where $E \in \mathbb{Z}_N$ and $e_i \in \mathbb{Z}_2$, except e_I is not (yet) defined. Equating the phases induced by $T_{\boldsymbol{x}}$ and $T_{\omega^3 \boldsymbol{x}}$ again yields eq. (4.7), but now E is either 0 or $N/2$. Defining $e_I = 2E/N$ produces a 't Hooft electric flux vector $\mathbf{e} \in \mathbb{Z}_2^d$. The $\mathbb{Z}_2^d$ electric flux sectors share most of the properties of the $\mathbb{Z}_N^d$ sectors familiar from periodic boundary conditions. In particular, they are superselection sectors: no local operator can connect two distinct sectors, although operators that wrap around the torus can. In the periodic case the Polyakov loops winding around the torus connect the various flux sectors. With C-periodic boundary conditions only the real part of the Polyakov loop is gauge invariant, and, for even N, the real part of the Polyakov loop changes sign under central conjugation. Thus, it has non-zero matrix elements between states whose $\mathbb{Z}_2$ electric-flux vectors differ.

5 θ-Vacuum

The third homotopy group of SU(N) is also non-trivial, so for $d = 3$ states are classified by yet another topological quantum number. On the sphere or with periodic boundary conditions, the quantum number is the "vacuum angle" θ. Using the map h defined in eq. (1.2), one finds that the topology is the familiar one also in the C-periodic case.

Take $-L/2 < x_i \leq L/2$ and consider the periodic maps

$$k_1(\boldsymbol{x}) = \exp\left(i \frac{\boldsymbol{\sigma} \cdot \boldsymbol{x}}{L} f(z)\right), \; k_n = k_1^n, \; n \in \mathbb{Z}, \tag{5.1}$$

into SU(2), where $z = |\boldsymbol{x}|/L$ and the function $f(z)$ is twice differentiable on $[0, \sqrt{3}/2]$, and $\phi(z) = z f(z)$ satisfies $\phi(0) = 0$ and $\phi(z) = \pi$, for $z \geq 1/2$. The map k_n has a winding number

$$\frac{1}{24\pi^2} \int_{\mathrm{T}^3} d^3x \, \epsilon_{ijk} \mathrm{Tr}\{k_n^{-1}(\boldsymbol{x}) \nabla_i k_n(\boldsymbol{x}) k_n^{-1}(\boldsymbol{x}) \nabla_j k_n(\boldsymbol{x}) k_n^{-1}(\boldsymbol{x}) \nabla_k k_n(\boldsymbol{x})\} = n \tag{5.2}$$

over the torus, and all maps with the same winding number can be smoothly deformed into k_n. However, for C-periodic boundary conditions we would like a C-periodic function with winding number n. Such a function is $\tilde{k}_n \equiv h k_n h^{-1}$. Further, let g_n ($\tilde{g}_n$) be k_n ($\tilde{k}_n$) embedded into SU(N). By a theorem of Bott all periodic (C-periodic) SU(N) maps with winding number n can be deformed into g_n ($\tilde{g}_n$).

The Hamiltonian commutes with the unitary operator T_n implementing the gauge transformation $\tilde{g}_n$, so

$$T_n(\boldsymbol{x}) |\Psi_\theta\rangle = e^{in\theta} |\Psi_\theta\rangle, \tag{5.3}$$

where $|\Psi_\theta\rangle$ is an eigenstate of the Hamiltonian, and, like the 't Hooft electric flux $\mathbf{e}$, the angle θ labels superselection sectors.

For $d = 5, 7, 9, \ldots$ there is more topology to discuss, because $\Pi_{2m+1}(\mathrm{SU(N)}) = \mathbb{Z}$ for $N > m$. For example, it is easy to see that $\Pi_5(\mathrm{SU}(3)) = \mathbb{Z}$, because SU(3) is a non-trivial bundle over S^5 with fiber S^3. We shall not discuss these issues here, because they follow straightforwardly, and because we are most interested in $d = 3$.

6 Topological charge

We end the topological discussion of C-periodic gauge fields by considering the Pontryagin index in four-dimensional space-time. We consider a Euclidean time interval $0 < t < T$ with periodic boundary conditions. In the spatial directions the time-like component A_0 satisfies a boundary condition analogous to eq. (1.1).

The Pontryagin index is intimately related to the homotopy classes of the Ω_μ:

$$P = -\frac{1}{32\pi^2}\int_{\mathrm{T}^4} d^4x\,\epsilon_{\lambda\mu\rho\sigma}\mathrm{Tr}F_{\lambda\mu}F_{\rho\sigma} = \nu + \frac{N-1}{2N}\epsilon_{ijk}n_i n_{jk}, \tag{6.1}$$

where Greek indices run from 1 to 4 and Latin indices from 1 to 3. As in sect. 3, Ω_j and Ω_k produce the magnetic-flux winding numbers n_{jk}. Analogously to sect. 3, Ω_i and Ω_0 produce the electric-flux winding numbers n_i. This definition of electric flux, in a $(d+1)$-dimensional Lagrangian formulation, is the same as the definition in sect. 4. Finally, the meaning of the integer ν is the same as with periodic boundary conditions.

Eq. (6.1) was derived in ref. [6] for twisted periodic boundary conditions. The proof follows almost unchanged for twisted C-periodic boundary conditions because the Pontryagin density has $C = +1$ and is therefore periodic. The difference with C-periodic boundary conditions is that the values of n_i and n_{jk} are restricted by the constraints derived in sects. 3 and 4.

For odd N eqs. (4.4) and eq. (3.4) imply $n_1 = n_2 = n_3$ and $n_{23} + n_{31} + n_{12} = 0 \bmod N$, respectively. Thus

$$\frac{N-1}{2N}\epsilon_{ijk}n_i n_{jk} = \frac{N-1}{N}n_1(n_{23}+n_{31}+n_{12}) \in \mathbb{Z}; \tag{6.2}$$

the 't Hooft electric and magnetic fluxes contribute an integer, but not a fraction, to the Pontryagin index. Alternatively, one could argue that there can be no fractional part to P, because all choices of n_i and n_{jk} are gauge equivalent to $n_i = 0$ and $n_{jk} = 0$. Note that this structure is the same as when the infinite volume is compactified (following a dynamical assumption on the fields at infinity) to S^4.

For even N eqs. (4.4) and eq. (3.4) imply

$$\begin{aligned} \frac{1}{N}(n_2 - n_1)n_{31} &= 0 \bmod \frac{1}{2}, \\ \frac{1}{N}(n_3 - n_1)n_{12} &= 0 \bmod \frac{1}{2}, \\ \frac{1}{N}n_1(n_{23}+n_{31}+n_{12}) &= 0 \bmod \frac{1}{2}. \end{aligned} \tag{6.3}$$

Hence

$$\frac{N-1}{2N}\epsilon_{ijk}n_i n_{jk} = \frac{N-1}{N}\left[n_1(n_{23}+n_{31}+n_{12}) + (n_2-n_1)n_{31} + (n_3-n_1)n_{12}\right] = 0 \bmod \frac{1}{2}. \tag{6.4}$$

the 't Hooft electric and magnetic fluxes contribute a half-integer to the Pontryagin index. This structure is like that of SU(2) gauge theory with periodic boundary conditions.

7 Conclusions

C-periodic boundary conditions are an alternative to periodic ones for SU(N) gauge fields with $N > 2$. As a first step in studying these boundary conditions, this paper has shown how topology affects the classification of physical states in a C-periodic box. Our major results are as follows:

For all N, states are characterized by the quantum number θ. For SU(3) and for all other odd N the topology of C-periodic fields is the same as in the infinite volume. In particular, there are no electric or magnetic flux sectors and the topological charge is always an integer. For even N the symmetry of the flux sectors of C-periodic fields is reduced to $\mathbb{Z}_2$ compared to the $\mathbb{Z}_N$ symmetry of the periodic case. Consequently the topological charge may assume half-integer values.

There are several applications of C-periodic boundary conditions that come to mind. First, the generalization to matter fields is straightforward: the fields—be they scalars or spinors, bosons or fermions—are replaced by their charge conjugates when they are shifted over the boundary. With fermions an additional minus sign may also be contemplated.

Second, it is possible to develop the small volume expansion as in refs. [3, 4]. Owing to the absence of flux sectors the SU(3) glueball spectrum in a small C-periodic box is not affected by tunneling transitions as in the periodic case. A thorough treatment of the tunneling problem required a series of impressive papers [5, 8], yielding the important result that the tunneling between distinct central conjugates of the perturbative vacuum alters the glueball spectrum in dramatic ways. In a forthcoming publication [7], we will show that there is only one stable perturbative vacuum with C-periodic boundary conditions, substantiating the claim that there is no tunneling to complicate the computation of the spectrum. There are other complications: For example, a breakdown of full rotational invariance at lower order in $g_0^{2/3}$, but it remains to be seen how much these effects change the glueball spectrum.

Third, it is interesting to apply C-periodic boundary conditions to Yang-Mills theories at non-zero physical temperatures. In the periodic case the $\mathbb{Z}_N$ center symmetry breaks spontaneously in the high-temperature gluon-plasma phase. C-periodic boundary conditions break the center symmetry explicitly, for $N > 2$. The explicit symmetry breaking acts as a source coupled to the order parameter, but its strength vanishes as the volume is taken to infinity. In the confined phase, the vacuum expectation value of the order parameter will be non-zero, but its value will decrease with increasing volume. In the plasma phase, the order parameter will maintain a non-zero value, for all volumes. In finite volumes the source-like nature of the explicit breaking should inhibit the tunneling that occurs with periodic boundary conditions. Consequently, the order parameter will not average to zero.

Fourth, C-periodic boundary conditions can also be used in lattice gauge theories. Numerical simulations of field theories are necessarily limited to finite volumes, and it is essential that the boundary conditions do not introduce artifacts that only disappear slowly in the infinite volume limit. Periodic boundary conditions do indeed introduce such artifacts via the $\mathbb{Z}_N$ flux sectors. But C-periodic boundary conditions eliminate the flux sectors from SU(N) lattice gauge theory when N is odd, just as in the continuum, and they reduce them to $\mathbb{Z}_2$ when N is even. Hence, it is not unreasonable to speculate that the spectrum of a C-periodic box may better approximate the infinite-volume spectrum than a periodic box of equal size. Of course, only detailed simulations with both boundary conditions can validate or discredit this speculation.

Finally, a Coulomb-gauge charged state,[3] such as a quark or a gluon, can be defined, because the flux can escape beyond C-periodic boundaries. Of course, the Coulomb gauge suffers from a Gribov ambiguity [9]. A deeper understanding of the Gribov problem's effect on charge states will plausibly require non-perturbative insight. However, non-perturbative calculations in gauge theories, either using a lattice or the methods of refs. [4, 5], require a finite volume to regulate infrared effects. To make progress on this front one needs a finite volume that can support (something like[3]) a charged

[3]Strictly speaking, a state with definite C but indefinite charge can be defined. In the infinite-volume limit, a pair of states, one with $C = +1$ and the other with $C = -1$, becomes degenerate, and particle and anti-particle linear combinations can be formed. See ref. [2] for details.

state: one possibility is the torus, but with C-periodic boundary conditions.

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