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Renormalization of Four-Fermion Operators Determining B–$\overline{B}$ Mixing on the Lattice

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Abstract

The B–$\overline{B}$ mixing parameter can be determined from lattice gauge theory by treating the heavy quarks using the static effective field theory. We determine to order α_S the linear combination of discretized four-fermion operators whose matrix element determines this parameter.

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1. Introduction

Experimental measurements of B–$\overline{B}$ mixing, characterized by x_d, the ratio of the mass difference to the (nearly) common lifetime for the non-strange neutral B meson [1], constrain the top quark mass and the magnitude of the td element of the KM matrix in the standard model. The largest uncertainty arises from the occurrence of a factor $B_B f_B^2$ in the usual expression for x_d arising from a box diagram with internal t quark exchange. Here, B_B is the ratio of the true matrix element of the B–$\overline{B}$ mixing operator to its value in the vacuum insertion approximation, and f_B is the B meson decay constant. Neither quantity is known well but it is hoped that lattice calculations will yield accurate values. In this paper we concentrate on the relation of the continuum B–$\overline{B}$ mixing operator to its lattice counterpart in order to determine the corrections to the combination $B_B f_B^2$ measured on the lattice.

Since b quarks are heavy relative to the cutoff scale of lattices currently being used for numerical simulations, an expansion (see reference [2] and references therein) which analytically removes the dependence on the b quark mass, m, must be used [3][4]. The relationship between operators in the full theory and their counterparts in the theory built around the zeroth order term in this expansion [4]–[10], termed the static effective field theory, is perturbatively calculable.

We perform a two stage matching procedure. First we relate the full theory operator to a combination of operators in the static effective theory. This is much like the computation of the full order α_S contributions to the static effective field theory operator determining f_B performed in reference [7]. The ultraviolet divergences in the continuum full and static theories are regulated with dimensional regularization and modified minimal subtraction ($\overline{\mathrm{MS}}$). The second step is much like the matching of the continuum operator determining f_B to its lattice counterpart [11][12]. (A different approach to this computation was carried out in [13], but see references [12] and [14].) The two step matching is useful conceptually and to disentangle technical issues. For example, it is most convenient to regulate the infrared divergences in the matrix elements in the first stage of the matching with dimensional regularization, and to use a gluon mass for the second stage. This two step procedure would also be useful in extending this calculation to next-to-leading logarithmic order. However, the logarithm of ma, where a is the lattice spacing, is not so large as to require this extension.

The paper is organized as follows. We begin by writing down the static effective field theory action for heavy quarks and antiquarks in the continuum, and determining at tree level the operator which fixes $B_B f_B^2$ in the effective theory. We then perform the one loop matching between the operators in the full theory and the continuum static theory. The discretized Euclidean action for heavy quarks is considered next, and then we perform the one loop matching between the operators in the continuum static theory and the lattice regulated theory. In the conclusion, we discuss our results for some typical values of the parameters involved, emphasizing the uncertainty associated with the value of α_S. Discussion of the numerically evaluated constants arising from the one loop integrals on the lattice is relegated to an appendix.

Before proceeding, we note our renormalization prescription. This is necessary because the theory with the $\Delta B = 2$ operator present, which we have been referring to as the "full" theory, is also an effective theory with the W and Z bosons and the t quark eliminated. $\Delta B = 2$ matrix elements in this theory have additional divergences not present in the standard model. For the one loop calculations in the "full" theory we use modified minimal subtraction with the convention that γ_5 is always anticommuting; there are no parity violating fermion loops in this calculation. This is the same scheme used in the next-to-leading-logarithmic computation of the coefficient of the $\Delta B = 2$ operator performed by Buras, Jamin and Weisz [15], thus allowing direct use of their results. In the calculation of matching conditions at order α_S, evanescent operators (whose tree level matrix elements vanish in four dimensions) are produced. The renormalization prescription is to eliminate these evanescent terms in the one loop finite parts. This means that the evanescent terms appear as counterterms for higher order calculations, such as the two loop anomalous dimension of the B–$\overline{B}$ mixing operator, but they need not concern us further here [15]–[17].

2. Continuum Static Action and Tree Level Operators

The Minkowski space Lagrangian for a field b that annihilates heavy quarks and a field $\tilde{b}$ that annihilates heavy antiquarks is,

$$\mathcal{L} = Zb^\dagger i\mathcal{D}_0 b + Z\tilde{b}i\mathcal{D}_0\tilde{b}^\dagger + Z\delta m b^\dagger b - Z\delta m\tilde{b}\tilde{b}^\dagger. \tag{2.1}$$

The field b is a two-component column vector and the field $\tilde{b}$ is a two-component row vector. Both fields are rotational doublets. Z is the wave function renormalization and δm is the mass counterterm. The gauge-covariant derivative $i\mathcal{D}_0$ is equal to $i\partial_0 + gA_0$ where $A_0 = A_0^a T_a$.

The terms in the Lagrangian for the b and $\tilde{b}$ fields are more similar than they appear. If one re-orders the terms involving $\tilde{b}$, the Pauli matrix structure is less natural, but we see that the action for the two fields is otherwise very similar. The Fermi minus sign in the kinetic term is compensated for by a parts integration. The only difference is then that in the gauge covariant derivative acting on $\tilde{b}$, $-T_a^T$ would appear instead of T_a. This is as it should be since in the static limit there are no kinematical differences between different flavors of quarks [18] or between heavy quarks and heavy antiquarks; the only difference is in the quantum numbers they carry. From the Lagrangian (2.1) we obtain the heavy quark and antiquark propagators and gluon couplings used in our one loop calculations. We now determine the tree level operators in the full and static effective theories which fix $B_B f_B^2$.

To be definite consider a B^0 meson comprising a $\overline{b}$ antiquark and a light quark, q, mixing into $\overline{B}^0$, with quark content $b\overline{q}$ (Particle Data Group convention [19]). The lowest dimension operator in the full theory contributing to this process is

$$\mathcal{O}_L = \tfrac{1}{2}\,\overline{b}\,\gamma^\mu P_L q\,\overline{b}\,\gamma_\mu P_L q. \tag{2.2}$$

This operator can also create two b quarks or annihilate two $\overline{b}$'s so it corresponds to several operators in the effective theory which has independent fields for heavy quarks and heavy antiquarks. We match onto the effective theory operator for the $B^0 \to \overline{B}^0$ mixing described above by matching the matrix element of (2.2) between an incoming light quark and heavy antiquark and an outgoing light antiquark and heavy quark. Let the outgoing heavy quark be described by a spinor u' and have color A, the incoming light quark be described by u and B, the incoming heavy antiquark by v and C, and the outgoing light antiquark by v' and D. The tree level matrix element of $\mathcal{O}_L$ denoted $\langle \mathcal{O}_L \rangle_0$ is then,

$$\langle \mathcal{O}_L \rangle_0 = L\,\delta\delta - \overline{L}\,\overline{\delta\delta}, \tag{2.3}$$

where,

$$L \equiv \overline{u}'\gamma^\mu P_L u\,\overline{v}\gamma_\mu P_L v', \quad \overline{L} \equiv \overline{u}'\gamma^\mu P_L v'\,\overline{v}\gamma_\mu P_L u. \tag{2.4}$$

and,

$$\delta\delta \equiv \delta_{AB}\delta_{CD}, \quad \overline{\delta\delta} \equiv \delta_{AD}\delta_{CB}. \tag{2.5}$$

Of course a Fierz identity is that $L = -\overline{L}$. The operator with the same matrix elements in the effective theory is

$$\mathcal{O}_L^{eff} = b^\dagger(1\ \ 0)\gamma^\mu P_L q\,\tilde{b}(0\ \ 1)\gamma_\mu P_L q. \tag{2.6}$$

Note the absence of a $\frac{1}{2}$ multiplying the operator. The field $b^\dagger$ creates the outgoing heavy quark; it cannot annihilate the incoming heavy antiquark. Likewise, $\tilde{b}$ cannot create the outgoing heavy quark.

To match the matrix elements we have expanded the amplitude in the full theory to zeroth order in $1/m$, and compared with the static theory matrix element. The amplitudes are in agreement to this order in $1/m$ provided that the spinor u' in the full theory satisfies $\gamma_0 u' = u'$ and the spinor v satisfies $\gamma_0 v = -v$, which is the situation for mixing. With the most common conventions for the normalization of states in the full theory, compensating factors of $\sqrt{2m}$ must be put into the matching condition. We work in the Dirac basis so that the two-by-four matrices $(1\ 0)$ and $(0\ 1)$ appearing in (2.6) have this simple block structure and simply pick out the upper and lower two rows, respectively, of the succeeding four-by-four Dirac matrices.

3. Full Theory-Static Effective Theory Matching at One Loop

It will be useful to define an additional operator which will be generated at order α_S in the continuum owing to the mass of the heavy quark,

$$\mathcal{O}_S = \tfrac{1}{2}\,\overline{b}P_L q\,\overline{b}P_L q. \tag{3.1}$$

For the incoming and outgoing states defined in the previous section, the tree level matrix element of this operator is,

$$\langle \mathcal{O}_S \rangle_0 = S\delta\delta - \overline{S}\,\overline{\delta\delta}, \tag{3.2}$$

where,

$$S \equiv \overline{u}' P_L u \,\overline{v} P_L v', \quad \overline{S} \equiv \overline{u}' P_L v' \,\overline{v} P_L u. \tag{3.3}$$

The effective theory operator $\mathcal{O}_S^{eff}$ is defined analogously to $\mathcal{O}_L^{eff}$ in equation (2.6).

To determine the one loop coefficients of $\mathcal{O}_L^{eff}$ and $\mathcal{O}_S^{eff}$, we use the same matrix element used in the previous section for tree level matching. For calculational convenience, we will set the light quark mass to zero, and take our mass shell point to be the point where the incoming and outgoing momenta of the light quarks are zero and the heavy quark and antiquark are on shell and at rest. Dimensional regularization with modified minimal subtraction will be used to regulate the infrared divergences which appear at this point. Because the result of the matching is infrared finite, dependence on the infrared regulator and scheme will drop out provided the same procedure is followed in both the full theory and the effective theory. Using dimensional regularization for both the ultraviolet and infrared divergences results in the remarkable simplification that all the one-loop diagrams in the continuum static theory which must be calculated vanish. Thus the non-vanishing contributions to the matching come from the full theory matrix element alone.

The full order α_S contribution to the matrix element of $\mathcal{O}_L$ can be written as

$$\frac{\alpha_S}{4\pi}\left(-6\ln\frac{\mu^2}{m^2} + C_L\right)\langle \mathcal{O}_L \rangle_0 + \frac{\alpha_S}{4\pi} C_S \langle \mathcal{O}_S \rangle_0. \tag{3.4}$$

The only subtlety in putting the result into this form, is that to zeroth order in the $1/m$ expansion, when γ_0 is next to $\overline{u}'$ or $\overline{v}$, we can replace it by 1 or -1, respectively, resulting in a reduction in the number of independent amplitudes. For example,

$$\overline{u}'\gamma^\mu\gamma^\nu P_L u \,\overline{v}\gamma_\mu\gamma_\nu P_L v' = -4L. \tag{3.5}$$

With the renormalization scheme discussed in the introduction we find that the constants appearing in the coefficients are,

$$C_L = -14, \quad C_S = -8. \tag{3.6}$$

The operator in the effective theory which corresponds to $\mathcal{O}_L$ is therefore,

$$\left(1+\frac{\alpha_S}{4\pi}\left(-6\ln\frac{\mu^2}{m^2}+C_L\right)\right)\mathcal{O}_L^{\mathit{eff}}+\frac{\alpha_S}{4\pi}C_S\mathcal{O}_S^{\mathit{eff}}. \tag{3.7}$$

Note, that unlike the case of the current determining f_B, the operator in the full theory, O_L, is itself renormalization point dependent. The combination (3.7), which corresponds to O_L, is thus also dependent on renormalization point. The coefficient of the logarithm reflects the difference of the running in the full theory and the static effective theory. The complete leading logarithmic contribution to the running of the operator in the static effective theory was calculated by Voloshin and Shifman and Politzer and Wise [20][6]. The running of O_L in the full theory has been calculated to next-to-leading-logarithmic order in reference [15].

4. Euclidean Discretized Operators and Heavy Quark Action

In Euclidean space, the static effective field theory Lagrangian is

$$\mathcal{L}_E = Zb^\dagger i\mathcal{D}_0 b + Z\tilde{b}i\mathcal{D}_0\tilde{b}^\dagger. \tag{4.1}$$

We have dropped the mass counterterm, which only multiplies the propagator by an exponential that is independent of the gauge field. Since b, $b^\dagger$, $\tilde{b}$ and $\tilde{b}^\dagger$ are independent fields in Euclidean space, the overall phase of the action is arbitrary. However, independent phase redefinitions are compensated for by changes in the phase in the relation between Euclidean Green's functions measured on the lattice and the Minkowskian matrix elements which define quantities like f_B and B_B. With the phase convention used here, the free propagator for heavy quarks in Euclidean space is $1/(p_0+i\epsilon)$.

Many discretizations of the action with the same naive continuum limit are possible. The following choice,

$$S_E = ia^3\sum_n\left[b^\dagger(n)\left(b(n)-U_0(n-\hat{0})^\dagger b(n-\hat{0})\right)+\tilde{b}(n)\left(U_0(n)\tilde{b}^\dagger(n+\hat{0})-\tilde{b}^\dagger(n)\right)\right], \tag{4.2}$$

reproduces the propagator being proposed for numerical simulations [3]. Manipulations like those discussed following equation (2.1) show that the heavy quark and heavy antiquark Lagrangians are more similar than they appear. The Wilson action for the light quark fields is given in equation (A3) of reference [21] which will be frequently compared to in the appendix.

Various choices of four-fermion operators yielding the same naive continuum limit can be made. The lattice operators we will use are the zero-distance operators,

$$\begin{aligned} \mathcal{O}_L^{latt} &= b^\dagger(n)(1\ 0)\gamma_\mu P_L q(n)\, \tilde{b}(n)(0\ 1)\gamma_\mu P_L q(n), \\ \mathcal{O}_S^{latt} &= b^\dagger(n)(1\ 0)P_L q(n)\, \tilde{b}(n)(0\ 1)P_L q(n). \end{aligned} \tag{4.3}$$

corresponding to $\mathcal{O}_L^{eff}$ and $\mathcal{O}_S^{eff}$ which they closely resemble. Our Euclidean Dirac matrices are Hermitian. With this convention, Fierz identities are unaffected.

5. Static Effective Theory-Lattice Matching at One Loop

We first define two additional operators which will be generated at order α_S owing to the chiral symmetry breaking Wilson mass term,

$$\mathcal{O}_N = \tfrac{1}{2}\,(2\bar{b}P_L q\,\bar{b}P_R q + 2\bar{b}P_R q\,\bar{b}P_L q + \bar{b}\gamma^\mu P_L q\,\bar{b}\gamma_\mu P_R q + \bar{b}\gamma^\mu P_R q\,\bar{b}\gamma_\mu P_L q), \tag{5.1}$$

and an operator identical to $\mathcal{O}_L$ except that P_R appears instead of P_L,

$$\mathcal{O}_R = \tfrac{1}{2}\,\bar{b}\gamma^\mu P_R q\,\bar{b}\gamma_\mu P_R q. \tag{5.2}$$

For convenient comparison with equations (2.2) and (3.1), we have introduced these operators in their full theory form in Minkowski space. The corresponding tree level amplitudes are,

$$\langle \mathcal{O}_N\rangle_0 = N\delta\delta - \overline{N}\,\overline{\delta\delta}, \quad \langle \mathcal{O}_R\rangle_0 = R\delta\delta - \overline{R}\,\overline{\delta\delta}, \tag{5.3}$$

where,

$$N \equiv 2\,\overline{u}'P_L u\,\overline{v}P_R v' + 2\,\overline{u}'P_R u\,\overline{v}P_L v' + \overline{u}'\gamma^\mu P_L u\,\overline{v}\gamma_\mu P_R v' + \overline{u}'\gamma^\mu P_R u\,\overline{v}\gamma_\mu P_L v', \tag{5.4}$$

and $\overline{N}$, R and $\overline{R}$ are defined in by now familiar analogous fashions, as are $\mathcal{O}_N^{eff}$, $\mathcal{O}_R^{eff}$, $\mathcal{O}_N^{latt}$ and $\mathcal{O}_R^{latt}$. Fierz identities show that $\overline{N} = N$ and $\overline{R} = -R$.

The full order α_S difference between the continuum effective theory and lattice amplitudes we write as

$$\frac{\alpha_S}{4\pi}(4\ln\mu^2 a^2 + D_L)\langle \mathcal{O}_L\rangle_0 + \frac{\alpha_S}{4\pi}D_N\langle \mathcal{O}_N\rangle_0 + \frac{\alpha_S}{4\pi}D_R\langle \mathcal{O}_R\rangle_0. \tag{5.5}$$

In equation (5.5), $\langle\mathcal{O}_L\rangle_0$, $\langle\mathcal{O}_N\rangle_0$ and $\langle\mathcal{O}_R\rangle_0$ are the Euclidean counterparts of the matrix elements in equations (2.3) and (5.3). D_L, D_N and D_R are numerically evaluated constants given in Table 1. Errors are at most $\mathcal{O}(1)$ in the third significant figure. Analytical expressions for and numerical values of the various contributions to these constants from each type of Feynman diagram, are given in the appendix.

r	D_L	D_N	D_R
1.00	−65.5	−14.4	−1.61
0.75	−63.8	−14.5	−1.77
0.50	−62.0	−14.0	−1.92
0.25	−59.8	−11.4	−1.59
0.00	−58.1	0.0	0.00

Table 1. Operator Coefficients as a Function of Wilson Parameter.

The discretized operator which corresponds to $\mathcal{O}_L^{eff}$,

$$1+\frac{\alpha_S}{4\pi}(4\ln\mu^2a^2+D_L)\mathcal{O}_L^{latt}+\frac{\alpha_S}{4\pi}D_N\mathcal{O}_N^{latt}+\frac{\alpha_S}{4\pi}D_R\mathcal{O}_R^{latt}, \tag{5.6}$$

can thus be obtained to one loop order once the renormalization point, lattice spacing, and value of α_S are given. To obtain the discretized operator corresponding to $\mathcal{O}_L$, the additional one loop contributions coming from equation (3.7),

$$\frac{\alpha_S}{4\pi}\left(-6\ln\frac{\mu^2}{m^2}+C_L\right)\mathcal{O}_L^{latt}+\frac{\alpha_S}{4\pi}C_S\mathcal{O}_S^{latt}, \tag{5.7}$$

must be included.

6. Conclusions

Our perturbative results are contained in the combination of four lattice operators corresponding to $\mathcal{O}_L$ given by equations (5.6), (5.7) and (3.6) and Table 1. For B–$\overline{B}$ mixing, the matrix element of $\mathcal{O}_R^{latt}$ is the same as that of $\mathcal{O}_L^{latt}$. From the numerical results in Table 1, which will be multiplied by $\alpha_S/4\pi$, we see that the correction factor for the coefficient of $\mathcal{O}_L^{latt}$ in $\mathcal{O}_L^{eff}$ will be large. Of the operators which appeared at one loop, the only one whose matrix element has been measured on the lattice is $\mathcal{O}_L^{latt}$ [22]. We illustrate the use of our perturbative results by computing the coefficient of this operator. We will see that there is considerable uncertainty in the application of these results.

The coefficient of $\mathcal{O}_L^{eff}$ in $\mathcal{O}_L$, where both operators are renormalized at scale $\mu = 2$ GeV (see the discussion below equation (3.7)) is $1+(C_L - 6\ln(\mu^2/m^2))\alpha_S/4\pi$ which is 0.94. Here we have taken the b quark mass to be 5 GeV and the QCD scale for four active quarks, $\Lambda^{(4)}_{\overline{\rm MS}}$, to be 200 MeV (the central value from reference [19]) resulting in a two loop value of α_S of 0.25.

For the second step of the matching, which gives the coefficient of $\mathcal{O}_L^{latt}$ in $\mathcal{O}_L^{eff}$, we take the $r = 1$ value for D_L from Table 1 which is -65.5. A typical inverse lattice spacing in current calculations is $a^{-1} = 2$ GeV. With μ also already chosen to be 2 GeV, the logarithm in the coefficient of $\mathcal{O}_L^{latt}$ vanishes. This is the point where a large source of uncertainty appears. The choice of lattice or continuum definitions of α_S [23] makes a significant numerical difference, despite being technically a higher order effect in α_S. One choice is to use the lattice value of α_S in this step of the matching (see reference [24] for a review of lattice perturbation theory). Using $\beta = 6.0$ which corresponds to a 2 GeV inverse lattice spacing, the relation $4\pi\alpha_S = 6/\beta$ yields the value 0.080 for α_S. We find that $1 + \alpha_S D_L/4\pi = 0.59$, a rather large reduction. Thus, with the lattice coupling, the product of 0.94 and 0.59 gives 0.55 for the coefficient of $\mathcal{O}_L^{latt}$ in $\mathcal{O}_L$.

Another choice is to use a renormalized value of α_S, which is roughly the continuum value at the scale π/a [25] and at an inverse lattice spacing of 2 GeV is 1.8 times the lattice value [25]. This results in a correction to the coefficient of $\mathcal{O}_L^{latt}$ in $\mathcal{O}_L$ so large that perturbation theory clearly cannot be trusted. Of course, this uncertainty in α_S is not peculiar to this computation although recent discussion [25] and the large numerical coefficient D_L highlight the difficulty.

The magnitude of the perturbative corrections of course depends a great deal on the quantity being calculated. In the case of K–$\overline{K}$ mixing, factoring f_K^2 out of the matrix element cancels a large part of the perturbative correction. Similarly, it can be seen that the lattice part of the one loop corrections to the coefficient of $\mathcal{O}_L^{latt}$ in B_B is not as large as the correction to $B_B f_B^2$. The coefficients of the other three operators are unaffected at this order. After subtracting the perturbative corrections to f_B, the factor multiplying $\langle \mathcal{O}_L \rangle_0$ in equation (3.4) is changed from $-6\ln(\mu^2/m^2)-14$ to $-2\ln(\mu^2/m^2)-26/3$. Instead of 0.94, the factor in the continuum part of the matching becomes 0.90. The constant D_L appearing in the second stage of the matching is changed to

$$\frac{2}{3}(1-d_1)-\frac{8}{3}d_2-\frac{1}{3}c+\frac{1}{3}(-5-v). \tag{6.1}$$

Taking the numerical values of these quantities for $r=1$ from the appendix, we find that this combination is 15.4, much smaller in magnitude than the value of D_L which was -65.5 at $r=1$. For the first choice of α_S, the combination (6.1) yields a factor 1.10 for this part of the correction to B_B, compared to 0.59 for $B_B f_B^2$. The product of the coefficients from both stages of the matching for B_B is 0.99. With the second choice for the value of α_S, the product for B_B would be 1.06.

The cancellation of part of the contribution to D_L will remain true at any order in perturbation theory, because the contribution due to heavy and light quark wave function renormalization automatically drops out of the corrections to B_B. Even if f_B was independently known, it is not clear that the magnitude of the higher order corrections to B_B would be reduced so significantly. Furthermore, the combination $B_B f_B^2$ is the important one phenomenologically, and unlike in the case of neutral K mesons, there is no independent method for determining the decay constant. Since the matrix element measured on the lattice directly determines the phenomenologically interesting combination, for the neutral B mesons there is little reason for separating the product, and thus little reason, except that the one loop corrections are much smaller, for reporting the perturbative corrections to f_B and B_B separately.

Because the perturbative corrections to the phenomenologically interesting combination are large, we can be confident only in the qualitative statement that these corrections significantly reduce the contribution of $\mathcal{O}_L^{latt}$ to $B_B f_B^2$. Lattice spacings fine enough to reduce α_S significantly are not realistically attainable. Thus, progress towards a more precise statement based on our results for the combination of lattice operators corresponding to $\mathcal{O}_L$ must come from further study of the reliability of lattice perturbation theory [25], the disentanglement of these from order a corrections [26], and additional higher order calculations which could help to set the scale in α_S.

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Appendix. Numerically Evaluated Constants

In this appendix, we express the numerically evaluated factors in the operator coefficients in terms of contributions coming from the various types of diagrams. We will give analytical and numerical expressions for the various contributions. Since the logarithms in the one loop calculation are easily reproducible, we make a simplification in reporting the differences by setting $\mu a = 1$.

Most of the numerically evaluated constants have arisen in previous computations, so we first rewrite D_L, D_N and D_R in terms of these quantities,

$$\begin{aligned} D_L &= \frac{10}{3}(1-d_1) - \frac{1}{3}c - \frac{4}{3}e + \frac{4}{3}(\frac{1}{2} - f) + \frac{1}{3}(-5-v), \\ D_N &= 2d_2, \\ D_R &= \frac{4}{3}w. \end{aligned} \tag{A.1}$$

The constants D_N and D_R are nonzero because the Wilson mass term for the light fermions breaks chiral symmetry. More specifically, lattice diagrams with the gluon coupling one of the heavy quark lines to one of the light quark lines (to be referred to as heavy-light radiative corrections) produced D_N and the lattice light-light radiative corrections produced D_R. The constant d_2 was tabulated in reference [12]. The constant $w = (\Delta_{\gamma_\mu} - \Delta_{\gamma_5\gamma_\mu})/4$ was tabulated in [27] and is the same as the constant K_1 defined in reference [21].

The contribution $1 - d_1$ to D_L arose from heavy-light radiative corrections. The first term in this contribution came from the continuum graph with a gluon mass as an infrared regulator and the d_1 term from the lattice graph. Of course the only quantities independent of the infrared regulator are differences of continuum and lattice graphs. The constant d_1 was tabulated in reference [12]. The constant c arose from the lattice heavy-heavy radiative corrections, and thus is independent of r, as is the constant e which arises from heavy quark wave function renormalization on the lattice and has previously been computed [11][12]. The contribution $\frac{1}{2} - f$ comes from the difference of light quark renormalization in the continuum and on the lattice and was tabulated as Δ_{Σ_1} in reference [27]. Finally, the constant $v = -8\ln\pi - K_0 - 16K_2'$ in the notation of reference [21] where contact with previous work [27][28] is made in their equation (4.29). However with their

$\overline{\mathrm{EZ}}$ renormalization prescription (described above equation (2.44) in [21]), instead of the combination $(-5-v)$ they would have $(-6-v)$ in our equation (A.1).

Following reference [27] we introduce some notation to give analytical expressions for the seven numerical constants,

$$\begin{aligned} \Delta_1 &= \sum\nolimits_\mu \sin^2 \frac{q_\mu}{2}, \\ \Delta_2 &= \sum\nolimits_\mu \sin^2 q_\mu + 4r^2 \Delta_1^2, \\ \Delta_4 &= \sum\nolimits_\mu \sin^2 q_\mu, \\ \Delta_5 &= \sum\nolimits_\mu \sin^2 q_\mu \sin^2 \frac{q_\mu}{2}. \end{aligned} \tag{A.2}$$

The sums on μ run from 1 to 4. Let $\Delta_1^{(3)}$ and $\Delta_2^{(3)}$ be identical to Δ_1 and Δ_2 respectively except with q_4 set to zero.

The r independent quantities c and e are given by

$$\begin{aligned} c &= -2 + \frac{1}{\pi^2} \int d^4 l \left[2 \left(\frac{1}{16\Delta_1^2} - \theta(1-l^2)\frac{1}{l^4} \right) - \frac{1}{8}\frac{1}{4\Delta_1} \right], \\ e &= c + \frac{1}{\pi} \int d^3 l \frac{1}{4\Delta_1^{(3)}}. \end{aligned} \tag{A.3}$$

Each integration variable is in the range $[-\pi, \pi]$. The quantities d_1 and d_2 [12] involving both light and heavy quarks are,

$$\begin{aligned} d_1 &= \frac{1}{\pi^2} \int d^4 l \left[\frac{1}{4\Delta_1\Delta_2} - \theta(1-l^2)\frac{1}{l^4} - \frac{1-4r^2}{16\Delta_2} \right], \\ d_2 &= -\frac{1}{\pi} \int d^3 l \frac{r}{2} \frac{1}{\Delta_2^{(3)}}. \end{aligned} \tag{A.4}$$

The constant d_2 is the only quantity odd in the Wilson mass parameter. The constants involving only light quarks are,

$$\begin{aligned} f &= 1 + \frac{1}{\pi^2} \int d^4 l \left[\frac{1}{8\Delta_1} + \theta(1-l^2)\frac{1}{l^4} - \frac{\Delta_4+\Delta_5}{16\Delta_1^2\Delta_2} + \frac{r^2}{4}\frac{2-\Delta_1}{\Delta_2} \right], \\ v &= \frac{1}{\pi^2} \int d^4 l \left[4\theta(1-l^2)\frac{1}{l^4} - \frac{\Delta_4}{\Delta_1\Delta_2^2} + \frac{1-2r^2}{4}\frac{\Delta_4}{\Delta_2^2} - r^4 \frac{\Delta_1^2}{\Delta_2^2} \right], \\ w &= \frac{1}{\pi^2} \int d^4 l \left[\frac{r^2}{4}\frac{\Delta_1^2 - 4\Delta_1}{\Delta_2^2} + \frac{r^2}{16}\frac{\Delta_4}{\Delta_2^2} \right]. \end{aligned} \tag{A.5}$$

r	d_1	d_2	f	v	w
1.00	5.46	−7.22	13.35	−6.92	−1.20
0.75	5.76	−7.23	11.96	−9.34	−1.33
0.50	6.30	−7.00	10.22	−13.43	−1.44
0.25	7.37	−5.72	8.07	−21.99	−1.19
0.00	8.79	0.00	6.54	−35.16	0.00

Table 2. Numerically Evaluated Constants versus Wilson Mass Parameter.

All r-dependent quantities are tabulated in Table 2. Numerically, the r independent quantities c and e are $e = 24.48$ [11][12] and $c = 4.53$. These and all other numerically calculated quantities in this appendix were evaluated to two decimal place accuracy using the Monte Carlo integration routine VEGAS [29]. Errors are at most $\mathcal{O}(1)$ in the last decimal place. Plugging the various constants into equation (A.1), we obtain the results for D_L, D_N and D_R given in Table 1.

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