Fermi National Accelerator Laboratory

FERMILAB-Pub-90/222-A
October 1990

Statistics of the gravitational microlensing

Piotr Amsterdamski

NASA/Fermilab Astrophysics Center
Fermilab National Accelerator Laboratory, Batavia, Illinois 60510

Nicolaus Copernicus Astronomical Center
00-716 Warszawa, Bartycka 18

ABSTRACT: Using simple statistical method I calculate the average number of images due to microlensing and the probability distribution of their amplification.

I. INTRODUCTION

Almost every large gravitational lens consists of many small clumps of matter that are distributed within the large lens in a more or less random fashion. Each such clump acts as a gravitational lens on its own right. If it happens that such a clump is on the way of the rays forming one of the macroimages, than it should amplify its brightness. Such an amplification may be observed as a sudden change in brightness of this image. On the other hand, one needs to know what is a likely difference of the brightness between macroimages due to the microlensing. Such information is necessary to properly reconstruct the distribution of matter in the large lens.

It is clear that any theory of microlensing must be statistical in nature, since the most one can hope for is to know the statistical distribution of matter in the large lens. One would like to find a direct relation between the statistical properties of the matter distribution and the distribution of images. So far two approaches to this problem have been tried. First, one can do Monte Carlo simulation of the matter distribution and compute the illumination of the observer plane by numerically integrating the ray equation sufficient number of times (Paczyński and Wambsganss 89, Wambsganss, Paczyński and Katz 89). This approach is very straightforward and gives the only chance to check the details of the distribution of images, but it requires a rather formidable numerical work. Also, to use it effectively, one would like to have some statistical measures of the resulting distribution of images. The second approach is based on the idea of treating the microlensing as a multiple scattering problem. (Katz, Balbus and Paczyński 86, Deguchi and Watson 88).

In this paper I would like to present the third method. This method leads to a very direct relation between the statistical measures of the matter distribution and the statistical properties of the images. The two required assumptions are that the matter is distributed in a random way and the lens is far away, so that the paraxial optics applies. The method is hardly new - it was developed some thirty years ago by Longuet - Higgins (Longuet - Higgins 56,58,59,60a,b,c) for the problem of the statistical distribution of the reflexions of the sunlight from the surface of water. The present paper relies very heavily on his work, in fact it would have been unnecessary to write this paper, if people interested in the gravitational lensing had been aware of the Longuet Higgins work. In the paper, I do not give detailed references to his work, but one should be aware that most of these results are his.

II. FERMAT PRINCIPLE AND THE SURFACE OF TIME DELAY

It is a rather easy exercise to derive the lens equation for a thin lens using Fermat principle. (Blandford and Narayan 86, Schneider 84, Blandford and Kochanek 87). The geometrical relations are shown in the Fig. 1. The time required for a light ray to travel from the source to the observer through the point $\vec{x}$ in the lens plane can be written as a sum of two terms. The first is the geometrical length of the path

$$ct_{\text{geo}} = \frac{(1+z_L)D_{OS}}{2D_{OL}D_{LS}}(\vec{x}-\vec{r})^2, \tag{2.1}$$

and the second is due to the gravitational time delay

$$ct_{\text{grav}} = -2(1+z_L)\frac{G}{c^2}\int ds\phi(s), \tag{2.2}$$

where $\phi(s)$ is the newtonian potential along the ray. z_L is the redshift of the lens. Assuming that the lens is thin and projecting matter density on the plane of the lens, one can write ct_{grav} as

$$ct_{\text{grav}} = -4(1+z_L)\frac{G}{c^2}\int d^2x'\Sigma(x')\log(|x-x'|). \tag{2.3}$$

It is convenient to introduce $\tau = \frac{D_{OL}D_{LS}}{(1+z_L)D_{OS}}(ct_{\text{geo}}+ct_{\text{grav}})$ and $\Sigma_{\text{crit}} = \frac{c^2D_{OS}}{4\pi GD_{OL}D_{LS}}$. Σ_{crit} is the uniform surface density required to focus rays on the observer. Now I introduce $\tilde{\Sigma} = \Sigma/\Sigma_{\text{crit}}$, and the surface of the time delay τ is

$$\tau(\vec{x},\vec{r}) = \tfrac{1}{2}(\vec{x}-\vec{r})^2 - \tfrac{1}{\pi}\int d^2x'\tilde{\Sigma}(x')\log(|\vec{x}-\vec{x}'|). \tag{2.4}$$

Now, I define $\psi(\vec{x}) = \frac{1}{\pi}\int d^2x'\tilde{\Sigma}(x')\log(|\vec{x}-\vec{x}'|)$. Since $\triangle\log(x) = 2\pi\delta^{(2)}(x)$, the potential ψ satisfies two dimensional Poisson equation

$$\triangle\psi(\vec{x}) = 2\tilde{\Sigma}(\vec{x}). \tag{2.5}$$

Finally, it is convenient to rescale the coordinates in the lens plane by the characteristic size of the macrolens L, and introduce $\tilde{\psi} = \psi/L^2$. Now the equation of the surface of time delay is

$$\tilde{\tau}(\vec{x},\vec{r}) = \tfrac{1}{2}(\vec{x}-\vec{r})^2 - \tilde{\psi}(\vec{x}), \tag{2.6}$$

where all variables are dimensionless. Since now I drop the tilde to simplify notation. One can make the notation even simpler by introducing $\vec{x}' = \vec{x} - \vec{r}$. I do that, and drop the prime.

The Fermat principle says that the images are formed at the critical points of the surface $\tau(\vec{x},\vec{r})$

$$\nabla_x \tau(\vec{x},\vec{r}) = 0, \quad \text{or} \quad \nabla\psi(x) = \vec{x}. \tag{2.7}$$

The intensity of the images depends on the external curvature of the surface $\tau(\vec{x},\vec{r})$ at the critical point. The intensity is highest when a caustic is formed, that is when the hessian at the critical point vanishes. In other words, the determinant of the tensor of external curvature vanishes. Consequently, one would like to know the statistical distribution of the critical points of $\tau(\vec{x},\vec{r})$ and the distribution of the curvature Ω at these points.

III. MODELS OF THE SURFACE DENSITY

The only essential requirement that $\Sigma(x)$ must satisfy is that it is random. It also must vanish for $x >> 1$ quickly enough so that the integral of Σ over the entire lens plane is finite. The simplest possibility is to assume that Σ is a superposition of plane waves with random phases

$$\Sigma(x) = \frac{1}{2\pi} \int d^2k a(|k|) \sin(\vec{k} \cdot \vec{x} + \epsilon), \tag{3.1}$$

where ϵ is selected from the interval $(0, 2\pi)$ with a uniform probability. I assume that $a(k)$ depends on the length of the wave vector only. This assumption is not necessary, but it simplifies the algebra very much.

The more realistic assumption is to take $\Sigma(r) = \sum_n f(r - r_n)$, where f is a function that specifies the density profile of the micro lense and $\{r_n\}$ is a set of random positions. Instead of trying to write all formulae in the most general fashion, I take f to be a gaussian function, so that

$$\Sigma(r) = \sum_{n=1}^{M} \frac{\sigma_n}{2\delta^2} \exp - \frac{(r - r_n)^2}{2\delta^2}. \tag{3.2}$$

Now I can take both σ_n and r_n to be random variables, but to make things even simpler I assume that all σ_n are the same. I also assume that $\vec{r}_n$ have random directions, selected in an isotropic way, while the magnitude of r_n is chosen according to a given probability distribution $h(r)$. The function $h(r)$ should be chosen in such a way, that the density profile of the macrolens is reproduced, and is normalized so that $2\pi \int_0^1 dr r h(r) = 1$.

Having decided what the surface density is one can immediately write the corresponding expression for the potential ψ, since it is related to Σ by the Poisson equation. So I have

$$\psi(r) = \frac{-1}{\pi} \int d^2k \frac{a(k)}{k^2} \sin(\vec{k} \cdot \vec{r} + \epsilon), \tag{3.3}$$

for the superposition of plane waves, and

$$\psi(r) = \frac{-\sigma}{2\pi} \sum_{n=1}^{M} \int d^2k \frac{e^{-\frac{1}{2}k^2\delta^2}}{k^2} \cos(\vec{k} \cdot (\vec{r} - \vec{r}_n)). \tag{3.4}$$

for the superposition of gaussian fluctuations. If the distribution of gaussian peaks is uniform in the plane of the macrolens, then $h = 1/\pi$. It is more realistic to assume that the gaussian peaks are distributed uniformly within three dimensional, spherical matter distribution, so that for the projected density of gaussian peaks I have $h(r) = \frac{3}{2\pi}\sqrt{(1 - r^2)}$.

In many situations one may be interested in the lensing due to a surface density Σ that can be written as a sum of a regular, smooth component Σ_s and a random Σ_r. Since the Poisson equation is linear, one can write $\psi = \psi_s + \psi_r$ and redefine the vector $\vec{r} \rightarrow \vec{r} + \nabla\psi_r$, so that only stochastic part ψ_r plays any role in the analysis of the statistical properties of the images.

IV. STATISTICS OF THE CRITICAL POINTS

In this section I want to derive the formula for the number density of the critical points and the their total number. The crucial ingredient in this calculation is the fact, that $\psi(x)$ is a sum of many stochastic terms. According to the central limit theorem, the distribution of all quantities formed from $\psi(x)$ by linear operations is normal. I denote $\xi_1 = x^{-1}\partial_x\psi$, $\xi_2 = y^{-1}\partial_y\psi$, $\xi_3 = \partial^2_{xx}\psi$, $\xi_4 = \partial^2_{yy}\psi$, $\xi_5 = \partial^2_{xy}\psi$. The distribution of ξ_i is

$$p(\xi_1, ..\xi_5) = \frac{1}{(2\pi)^{5/2}\sqrt{\Delta}} \exp -\tfrac{1}{2} M_{ij}\xi_i\xi_j, \tag{4.1}$$

where $\Delta = \det\Theta$, $\Theta_{ij} = \langle\xi_i\xi_j\rangle$, and $M\Theta = 1$. So I have to calculate the required correlations $\langle\xi_i\xi_j\rangle$. The calculation is very simple, in particular for the first model of the surface density. One gets

$$\langle\xi_1\xi_1\rangle = \frac{1}{x^2}\frac{1}{2\pi}\int_0^\infty dk\frac{a^2(k)}{k} = \frac{m_{-1}}{x^2}, \tag{4.2}$$

$\langle\xi_2\xi_2\rangle = \frac{m_{-1}}{y^2}$, and $\langle\xi_1\xi_2\rangle = 0$. The correlations between the first and second derivatives of ψ all vanish. Next

$$\langle\xi_3\xi_3\rangle = \frac{3}{8\pi}\int_0^\infty dk k a^2(k) = m_1, \tag{4.3}$$

and $\langle\xi_4\xi_4\rangle = m_1$, $\langle\xi_5\xi_5\rangle = \langle\xi_3\xi_4\rangle = \frac{1}{3}m_1$. For $a(k) = \sigma(k\delta)^{m/2}e^{-\frac{1}{2}k\delta}$ the moment $m_1 = 3(m+1)!\sigma^2/(8\pi\delta^2)$.

The structure of the correlations matrix is the identical for the second model of the surface density, but the expressions for the moments m_{-1} and m_1 are different

$$m_{-1} = \frac{\sigma^2 M^2}{4\pi}\int_0^\infty dk\frac{e^{-k^2\delta^2}}{k}I(k), \tag{4.4}$$

$$m_1 = \frac{3\sigma^2 M^2}{16\pi}\int_0^\infty dk k e^{-k^2\delta^2} I(k), \tag{4.5}$$

where

$$I(k) = \left[2\pi\int_0^1 r dr h(r) J_0(kr)\right]^2. \tag{4.6}$$

($J_0(x)$ is the Bessel function.)

The matrix of correlations has a block diagonal form, so the probability distribution factors

$$p(\xi_1 ... \xi_5) = p(\xi_1, \xi_2) p(\xi_3, \xi_4, \xi_5), \tag{4.7}$$

$$p(\xi_1, \xi_2) = \frac{|xy|}{2\pi m_{-1}} \exp -\frac{(x^2\xi_1^2 + y^2\xi_2^2)}{2m_{-1}}, \tag{4.8}$$

and

$$p(\xi_3, \xi_4, \xi_5) = \frac{1}{(2\pi)^{3/2}\sqrt{\Delta}} \exp -\tfrac{1}{2} M_{ij}\xi_i\xi_j. \tag{4.9}$$

where $i, j = 3, 4, 5$, $\Delta = \frac{8}{27} m_1^3$, and

$$M = \frac{3}{8m_1} \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Now I am interested in the number of points with a specified values of ξ_1 and ξ_2 in the cell $dxdy$. It is given by the expression

$$\begin{aligned} n =& p(\xi_1, \xi_2) \int d\xi_3 d\xi_4 d\xi_5 p(\xi_3, \xi_4, \xi_5) \left| \frac{\partial(\xi_1, \xi_2)}{\partial(x, y)} \right| \\ =& \frac{p(\xi_1, \xi_2)}{|xy|} \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{\Delta}} \int d\xi_3 d\xi_4 d\xi_5 \left| (\xi_3 - 1)(\xi_4 - 1) - \xi_5^2 \right| \exp -\tfrac{1}{2} M_{ij}\xi_i\xi_j \end{aligned} \tag{4.10}$$

The next step is to evaluate this integral. There are two quadratic forms present in this integral. It is possible to make a linear change of variables $\xi_i = a_{ij}\eta_j$, such that both forms are diagonalized

$$\begin{aligned} M_{ij}\xi_i\xi_j =& \eta_1^2 + \eta_2^2 + \eta_3^2 \\ \Omega_{ij}\xi_i\xi_j = \xi_3\xi_4 - \xi_5^2 =& l_1\eta_1^2 + l_2\eta_2^2 + l_3\eta_3^2, \end{aligned} \tag{4.11}$$

where the numbers l_i are the roots of the equation

$$\det(\Omega - lM) = 0. \tag{4.12}$$

They are $l_i = m_1(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$. The roots satisfy the following relations

$$\begin{aligned} & l_1 l_2 l_3 = \tfrac{1}{4}\Delta, \\ & l_1 + l_2 + l_3 = 0, \\ & l_1 l_2 + l_2 l_3 + l_3 l_1 = -\tfrac{1}{3} m_1^2 = -\tfrac{3}{4} H, \end{aligned} \tag{4.13}$$

and $|\partial(\xi_i)/\partial(\eta_i)| = \Delta^{1/2}$. One can also prove that

$$(\xi_3 - 1)(\xi_4 - 1) - \xi_5^2 = \sum_j l_j(\eta_j + y_j), \tag{4.14}$$

where in general

$$y_j = -\frac{a_{3j} + a_{4j}}{2l_j},$$

but in the isotropic case $y_1 = \frac{-1}{\sqrt{l_1}}$ and $y_2 = y_3 = 0$. Therefore

$$n = \frac{1}{|xy|} p(\xi_1, \xi_2) \frac{1}{(2\pi)^{3/2}} \int\int\int_{-\infty}^{\infty} d\eta_1 d\eta_2 d\eta_3 \left| \sum_j l_j(\eta_j + y_j)^2 \right| e^{-\frac{1}{2}\eta^2}. \tag{4.15}$$

One can check that without the modulus sign, the integral over η_j gives 1; adding such an integral to both sides of the equation above and integrating over $dxdy$ I get

$$N + 1 = \frac{2}{(2\pi)^{3/2}} \int\int\int d\eta_1 d\eta_2 d\eta_3 \sum_j l_j(\eta_j + y_j)^2 e^{-\frac{1}{2}\eta^2}, \tag{4.16}$$

where the integral now is over the region in the η space, such that the expression under the sum is positive. One can easily see that this region can be parametrized by

$$\sqrt{l_1}(\eta_1 + y_1) = r, \quad -\infty < r < +\infty,$$

$$\sqrt{|l_2|}\eta_2 = r \sin\theta \cos\phi, \quad 0 < \theta < \tfrac{1}{2}\pi,$$

$$\sqrt{|l_3|}\eta_3 = r \sin\theta \sin\phi, \quad 0 < \phi < 2\pi$$

The integration is now straightforward and the result is

$$N = 1 + \frac{4m_1}{3\sqrt{3}} \exp - \frac{1}{2m_1}. \tag{4.17}$$

Inserting the moment m_1 of the superposition of plane waves, one gets

$$N = 1 + \frac{(m+1)!\sigma^2}{2\sqrt{3}\pi\delta^2} \exp - \frac{4\pi\delta^2}{3(m+1)!\sigma^2}. \tag{4.18}$$

In the Fig.2 and Fig. 3 I plot N as a function of σM for several values of δ, both for the uniform distribution of gaussian fluctuation in the plane of the macrolense and in the sphere of the macrolensing object. The number of images decreases when δ increases. This seems correct: when δ is small the potential ψ changes on smaller scales and the chance of finding a critical point increases.

V. DISTRIBUTION OF THE CURVATURE

There are two measures of the curvature of the surface τ

$$\begin{aligned} J(\tau) &= \nabla^2\tau = \kappa_1 + \kappa_2, \\ \Omega(\tau) &= \mathrm{Hess}(\tau) = \kappa_1\kappa_2, \end{aligned} \tag{5.1}$$

where κ_i are the two principal radii of the curvature of the surface τ. One can easily check, that $\Omega(\tau) = 1 - J(\psi) + \Omega(\psi)$. If $J(\psi)$ and $\Omega(\psi)$ are not correlated, I can compute their distribution separately.

$J(\psi)$ is a sum of two random variables, so its distribution is normal.

$$\langle J^2 \rangle = \tfrac{8}{3} m_1$$

so

$$p(J) = \frac{1}{4}\sqrt{\frac{3}{\pi m_1}} \exp -\frac{3J^2}{16m_1}. \tag{5.2}$$

Since the gradient of ψ are not correlated with the second derivatives, $p(J)$ gives the correct distribution of J at the critical points.

The distribution of $\Omega(\psi)$ is more difficult, since there are no reasons to expect that this distribution is normal. The trick is to use the characteristic function (Fourier transformation)

$$\phi(t) = \int_{-\infty}^{\infty} p(\Omega) e^{i\Omega t} d\Omega. \tag{5.3}$$

Now, I can use the variables η_i defined in the previous section

$$p(\Omega(\psi)) = p(l_1\eta_1^2 + l_2\eta_2^2 + l_3\eta_3^2). \tag{5.4}$$

The η's are independent stochastic variables, so the probability of their sum is a product of probability for each η_i

$$p(l_1\eta_1^2 + l_2\eta_2^2 + l_3\eta_3^2) = p_1(\eta_1)p_2(\eta_2)p_3(\eta_3), \tag{5.5}$$

where

$$p_i(\eta_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\eta_i^2}$$

The corresponding distribution for the variable $\chi_1 = l_1\eta_1^2$ is

$$p(\chi_1) = \begin{cases} 0, & \text{if } \chi < 0 \\ \frac{1}{\sqrt{2\pi l_1 \chi_1}} \exp -\frac{\chi_1}{2l_1}, & \text{if } \chi > 0 \end{cases} \tag{5.6}$$

and the expressions for the probability distribution of χ_2 and χ_3 are similar. Now I can easily compute the characteristic function of $p(\chi_1)$

$$\begin{aligned} \phi_1(t) &= \int_{-\infty}^{\infty} p(\chi_1) e^{i\chi_1 t} d\chi_1 \\ &= \frac{1}{\sqrt{1 - 2il_1 t}}, \end{aligned} \tag{5.7}$$

and the characteristic function of the $p(\Omega)$ is the product of ϕ_i

$$\phi(t) = \frac{1}{\sqrt{1 + 3Ht^2 + 2i\Delta t^3}}. \tag{5.8}$$

Defining $\lambda = 2\Delta/(3H)^{3/2}$ and $\omega = \Omega/\sqrt{3H}$ I can write find the probability distribution of curvature as

$$\begin{aligned} p(\Omega(\psi)) &= \frac{1}{\sqrt{3H}} \int_{-i\infty}^{+i\infty} \frac{e^{-\alpha\omega}}{\sqrt{\lambda\alpha^3 + \alpha^2 - 1}} \\ &= \frac{1}{\sqrt{3H}} f(\omega, \lambda). \end{aligned} \tag{5.9}$$

The function $f(\omega, \lambda)$ cannot, in a general, be calculated analytically. However, in the present case the two negative roots of the the expression under the square root coincide and the calculation is very easy

$$f(\omega) = e^{\sqrt{3}\omega} \begin{cases} 1, & \text{if } \omega < 0 \\ \operatorname{erfc}\left(\sqrt{\frac{3\sqrt{3}\omega}{2}}\right), & \text{if } \omega > 0 \end{cases} \tag{5.10}$$

$f(\omega)$ has a very sharp peak at $\omega = 0$ and is not symmetric around the maximum.

If $J(\psi)$ and $\Omega(\psi)$ are correlated it is necessary to compute $\Omega(\tau)$ directly. Besides, such a calculation should be useful in any case and is not much more difficult than the computation of the distribution of $\Omega(\psi)$. Using the η_i variables and restricting myself to the case of an isotropic matter distribution, I can write

$$\Omega(\tau) = l_1(\eta_1 - l_1^{-1/2})^2 + l_2\eta_2^2 + l_3\eta_3^2. \tag{5.11}$$

So now the variable $\chi_1 = l_1(\eta_1 - l_1^{-1/2})^2$ and the characteristic function of its distribution is

$$\phi_1(t) = \frac{\exp\left(\frac{it}{1-2il_1t}\right)}{\sqrt{1-2il_1t}}. \tag{5.12}$$

Consequently, the probability distribution of $\Omega(\tau)$ is

$$p(\Omega(\tau)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp\left(\frac{it}{1-2il_1t} - i\Omega t\right)}{\sqrt{1-2il_1t}} \frac{dt}{1-2il_2t}. \tag{5.13}$$

This integral can be evaluated analytically for $\omega \le 0$. I write, as before, $p(\Omega(\tau)) = \frac{1}{\sqrt{3H}} f_H(\omega, H)$, where the function $f_H(\omega, H)$, for $\omega \le 0$, is

$$f_H(\omega, H) = \exp\left(\sqrt{3}\omega - \frac{1}{3\sqrt{H}}\right).$$

In the Fig. 4 I plotted $f_H(\omega, H)$ for several values of H. When $H \to \infty$, $f_H(\omega, H) \to f(\omega)$, as defined before.

The next step is to find the probability distribution of curvature at the critical point of τ. This turns out to be rather straightforward. The probability of finding a critical point in the cell $dxdy$ is

$$n dx dy = p(\xi_1, \xi_2) dx dy \int d\xi_3 d\xi_4 d\xi_5 p(\xi_3, \xi_4, \xi_5) \frac{1}{|xy|} |\Omega(\tau)|, \tag{5.14}$$

where the triple integral is exactly the same as one that I evaluated in the section IV. Therefore, the distribution of ξ_i, $i = 3, 4, 5$ at the critical point is

$$p^{\star}(\xi_3, \xi_4, \xi_5) = \frac{p(\xi_3, \xi_4, \xi_5)|\Omega|}{N}. \tag{5.15}$$

Now, integrating over all ξ_i, $i = 3,4,5$ between Ω and $\Omega + d\Omega$ I can get the probability distribution of Ω at the critical point

$$\begin{aligned} p^\star(\Omega(\tau))d\Omega =& \int_\Omega^{\Omega+d\Omega} p^\star(\xi_3,\xi_4,\xi_5)d\xi_3 d\xi_4 d\xi_5 \\ =& \frac{1}{N}\int_\Omega^{\Omega+d\Omega} p(\xi_3,\xi_4,\xi_5)|\Omega| d\xi_3 d\xi_4 d\xi_5, \end{aligned} \tag{5.16}$$

but between the limits of integration Ω is approximately constant, so

$$\begin{aligned} p^\star(\Omega(\tau))d\Omega =& \frac{|\Omega(\tau)|}{N} p(\Omega(\tau))d\Omega \\ p^\star(\Omega(\tau)) =& \frac{|\omega|}{N} f_H(\omega, H) \end{aligned} \quad . \tag{5.17}$$

Finally, since the amplification is proportional to the inverse of $|\Omega(\tau)|$ it is interesting to find the distribution of $|\Psi| = 1/|\Omega|$ ($\psi = 1/\omega$). An elementary derivation gives

$$p^\star(|\Psi|)d\Psi = \frac{\sqrt{3H}}{N|\psi|^3}[f_H(\omega, H) + f_H(-\omega, H)] = g(\psi, H)d\psi. \tag{5.18}$$

The function $g(\psi, H)$ is plotted in the Fig.5 for several values of H.

VI. CONCLUSIONS

The formalism presented here allows one to derive explicit formulas for the average number of images and the distribution of curvature at the critical points. It provides a direct relation between the statistical distribution of matter and that of images. It is interesting that both the number of the images and the distribution of curvature are governed by the same statistical measure of of matter distribution, namely by the m_1 moment. It would be interesting to find the observable features of the images that depend on the other moments.

The results obtained here can be - and will be extended to the more complicated situation. First of all, one can consider time dependence of the microlensig by allowing for the time dependence of the matter distribution. The only new difficulty that should be expected in such case is due to the fact that the random field becomes three dimensional and the corresponding integrals become more difficult.

ACKNOWLEDGEMENTS

I am very obliged to Rocky Kolb and Josh Frieman for their hospitality while I was at Fermilab. My work there was supported by NASA grant number NAGW - 1340, and in part by DOE grant to Fermilab. I would like to thank Bohdan Paczyński for several helpful conversations during my work on this project.

REFERENCES

R. Blandford, R. Narayan, *Ap. J.* **310**, 568, (1986)

R. Blandford, S. Kochanek, *Gravitational Lenses*, CALTECH preprint GRP 138, 1987.

S. Deguchi, W. D. Watson, *Ap. J.* **335**, 67, (1988)

N. Katz, S. Balbus, B. Paczyński, *Ap. J.* **306**, 2, (1986)

M. S. Longuet-Higgins, *Proc. Cam. Phil. Soc.* **52**, 234, (1956).

M. S. Longuet-Higgins, *Proc. Cam. Phil. Soc.* **54**, 439, (1958).

M. S. Longuet-Higgins, *Proc. Cam. Phil. Soc.* **55**, 91, (1959).

M. S. Longuet-Higgins, Jour. *Optical Soc. of* America, **50**, 838, (1960)

M. S. Longuet-Higgins, *Jour. Optical Soc. of America*, **50**, 845, (1960)

M. S. Longuet-Higgins, *Jour. Optical Soc. of America*, **50**, 851, (1960)

B. Paczyńskim J. Wambsganss *Ap. J.* **337**, 581, (1989)

J. Wambsganss, B. Paczyński, N. Katz, preprint POP 315, (1989).

FIGURE CAPTIONS

Fig. 1 The geometry of the lens.

Fig. 2 The number of images as a function of the surface density σ, for $\delta = 0.05$, 0.10, 0.15, 0.20, 0.25, 0.30. The smallest value of δ corresponds to the largest number of images. Uniform distribution of gaussian fluctuations on the lens plane is assumed.

Fig. 3 The number of images as a function of the surface density σ, for $\delta = 0.05$, 0.10, 0.15, 0.20, 0.25, 0.30. The smallest value of δ corresponds to the largest number of images. Uniform distribution of gaussian fluctuations in a sphere containing the lens is assumed.

Fig.4 Function $f_H(\omega, H)$ for $H = 0.01, 1., 10., 100$. The hightest value of H corresponds to the curve with the highest peak at $\omega = 0$.

Fig.5 Function $g(\omega, H)$ for $H = 0.1, 1., 10., 100$. The curve with the highest peak corresponds to the smallest value of H.

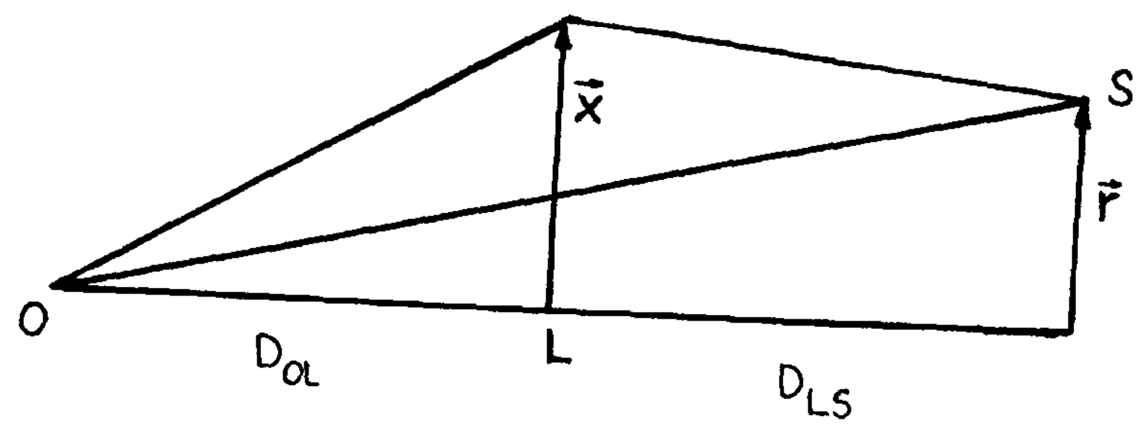

Fig1

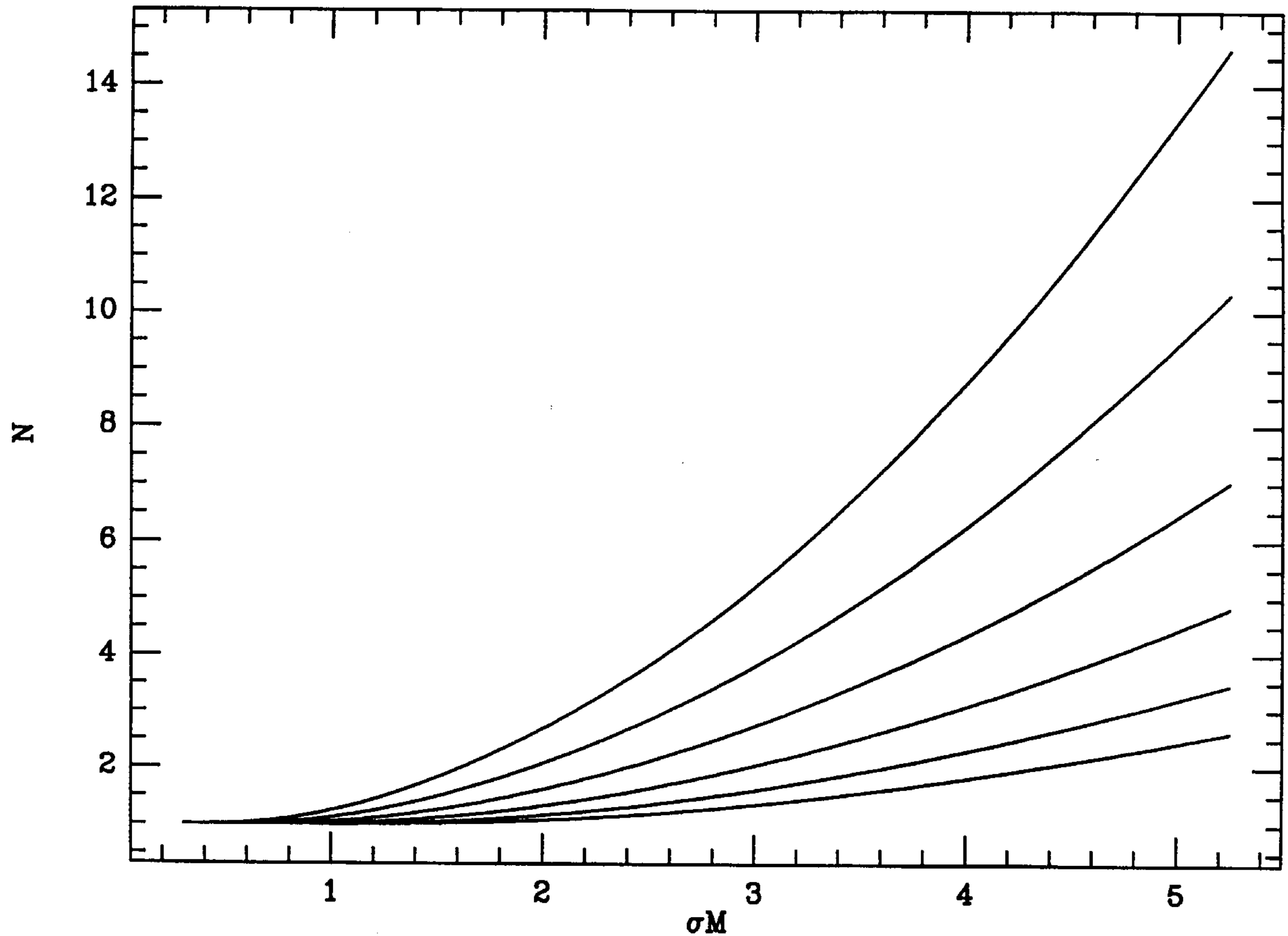

Fig 2

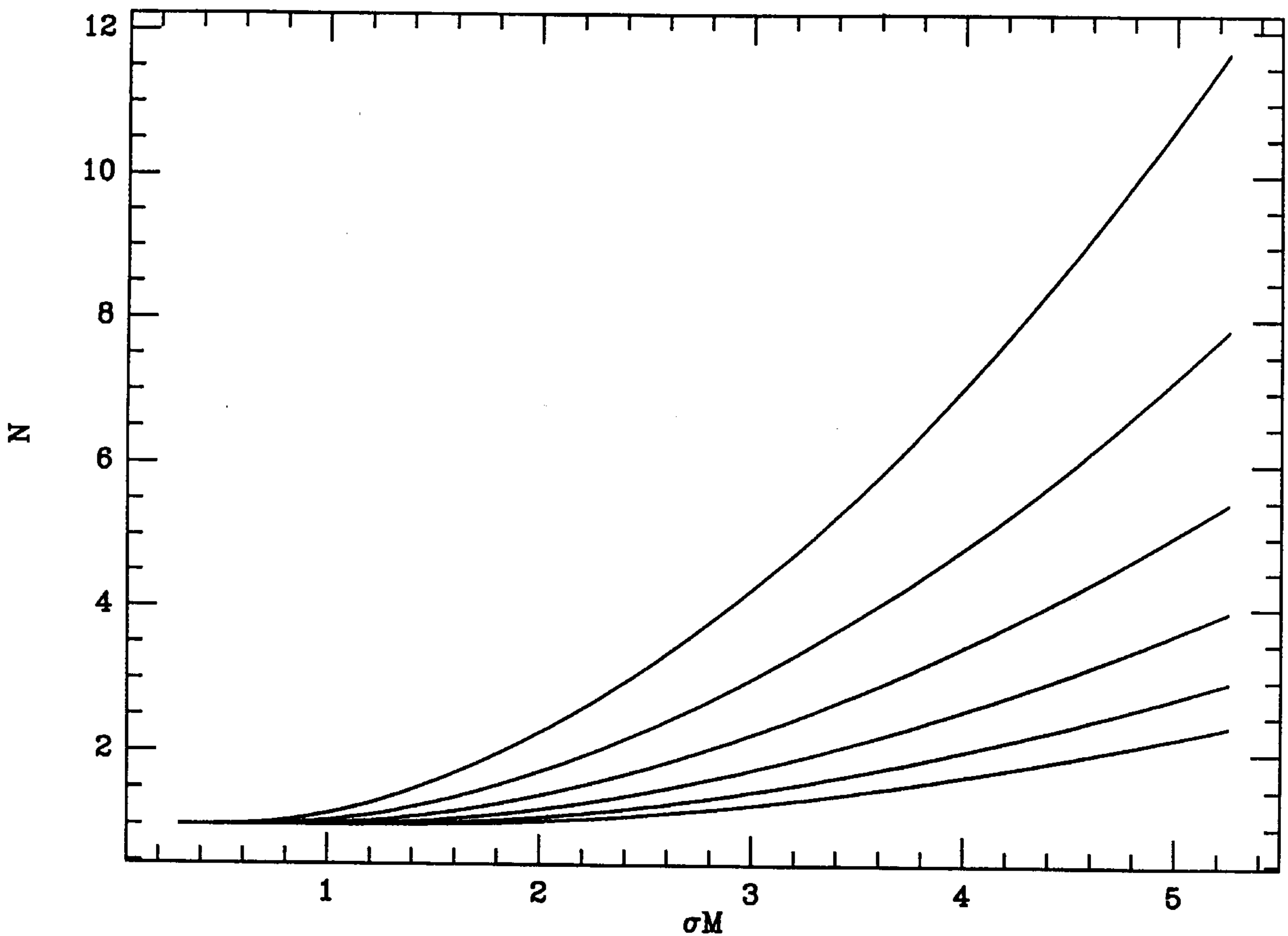

Fig3

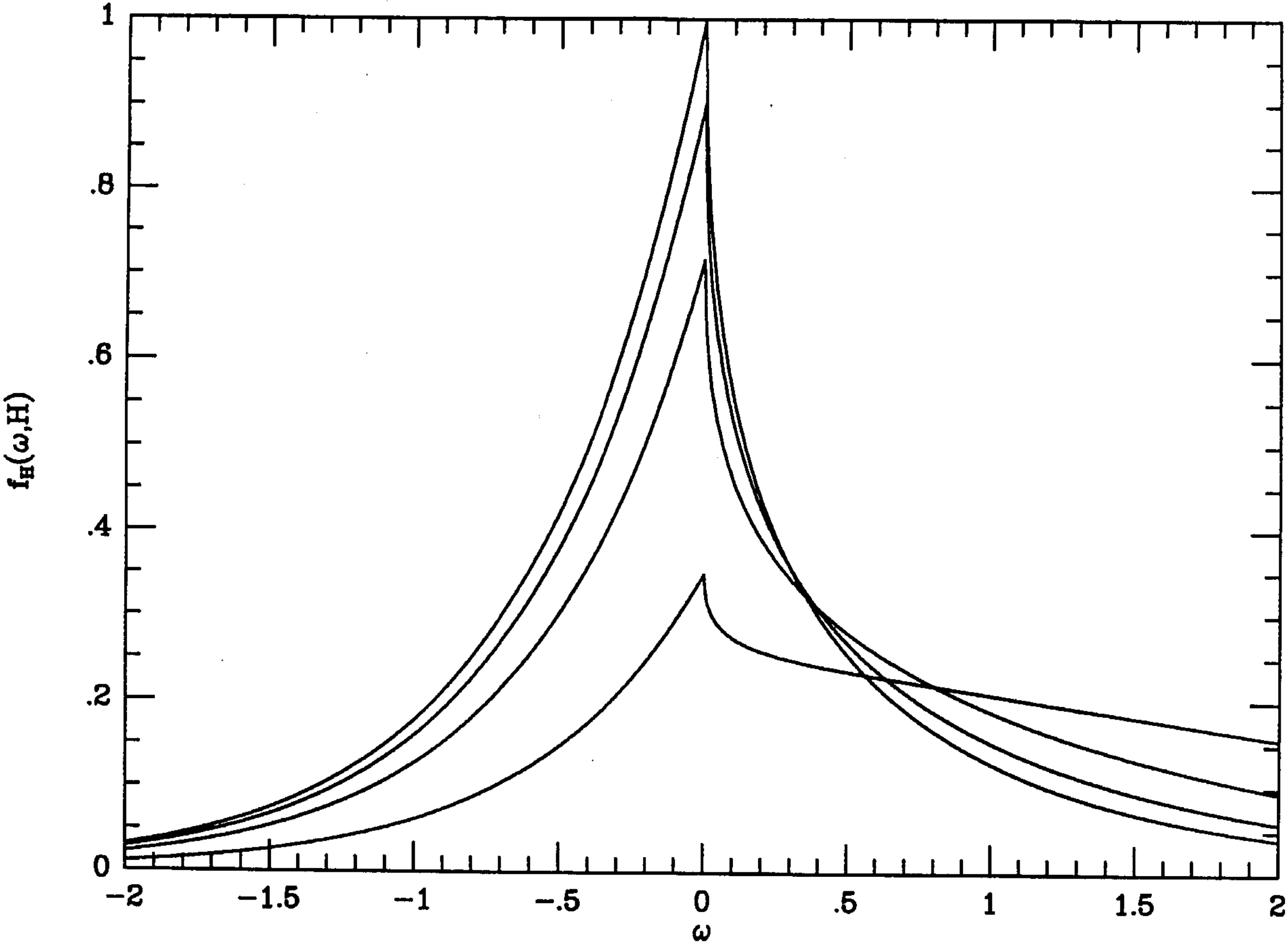

Fig 4

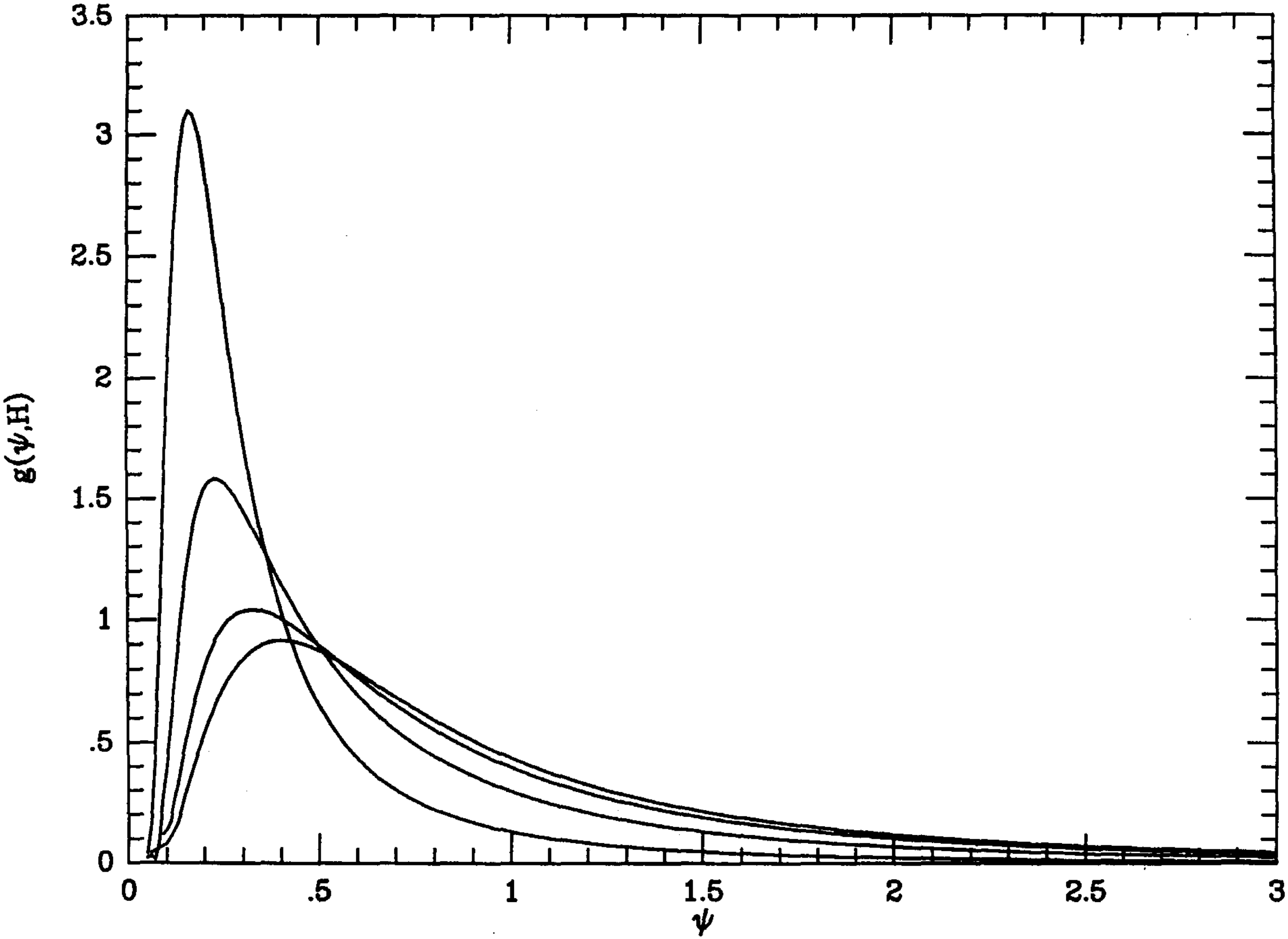

fig 5