Possible Sources of the Population I Lithium Abundance and Light Element Evolution

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ABSTRACT

We examine one zone numerical models of galactic chemical evolution of the light elements (lithium, beryllium, boron, and deuterium) with a broad sample of possible stellar lithium production sites and star formation histories (including the multiple merger motivated model of Mathews and Schramm (1990)). Following Mathews, Alcock, and Fuller (1990), the question of an initially high ($\sim 10^{-8}$) or low ($\sim 10^{-10}$ as observed in Pop II stars) value of $^7\text{Li} / H$ is examined. Scenarios starting with primordial $^7\text{Li} / H$ values of $\sim 1 \times 10^{-10}$ are well fit by many models. The best fits are achieved for star formation histories with an overall decrease in the star formation rate such as the multiple merger scenario of Mathews and Schramm. Models with high primordial lithium are constrained to have initial Li abundances close to the Pop I abundance of ($\sim 10^{-9}$) by observations involving lithium and potassium in the interstellar medium of the Larger Magellanic Cloud (LMC) (Malaney and Alcock, 1990; Sahu, Sahu, and Ptasch, 1988; Steigman, 1990). $^7\text{Li}$ production in intermediate or high mass stars ($m > 4 M_\odot$) is found to fit observations better than production in low mass ($1 - 5 M_\odot$) stars. Since very elevated levels of lithium have recently been observed consistently in intermediate mass stars in the LMC, it seems likely that this is indeed the major source of the Pop I $^7\text{Li}$ abundance. The current predictions for low-level yields of $^7\text{Li}$ from supernova envelope shocks (Brown, et. al., 1990) have no significant impact.
Introduction

The light elements in general and $^7\text{Li}$ in particular are important indicators of the course of both cosmological and Galactic history. As is well known, lithium is produced in standard models of the Big Bang (e.g. Olive, et. al. (1989)) with an abundance $^7\text{Li} / H \approx 1 \times 10^{-10}$. ($A/B$ as an abundance is used in this paper to indicate the ratio of the number of $A$ nuclei to $B$ nuclei.) This prediction for the primordial starting abundance of $^7\text{Li}$ is based on the assumption that the neutrons and protons formed, as the early Universe cools and expands beyond the temperatures and densities at which quarks exist unbound, are a homogeneous gas. That is, all of the Universe is assumed to be at the same density and temperature with the same relative numbers of neutrons and protons at all points.

The abundances of the light elements produced in the Big Bang ($H, D, ^3\text{He}, ^4\text{He}$, and $^7\text{Li}$) can then be compared to observations of these elements to limit the present density of the universe in baryons to the range $0.02 \leq \Omega_B \leq 0.11$ (Olive, et. al., 1989). In the past few years, however, it has been proposed that density fluctuations could be produced in the Universe if the quark-hadron phase transition were first order. This would divide the Universe at the time of Big Bang Nucleosynthesis (BBN) into neutron-rich and proton-rich regions which would alter primordial light element predictions and total Universal density limits substantially (Applegate, Hogan, and Scherrer, 1987, 1988; Alcock, Fuller, and Mathews, 1987; Fuller, Mathews, and Alcock, 1988; Mslaney and Fowler, 1988). It was originally thought that it might be possible with these models to obtain values of $\Omega_B = 1$. These models would have increased the primordial abundance of $^7\text{Li} / H$ to at least $7 \times 10^{-9}$. Mathews, Alcock, and Fuller (1990, referred to as MAF below) proposed a Galactic evolution model which was capable of fitting this value to observed abundances. We discuss their model in detail below. Kurki-Suonio, et. al. (1989) have since shown that the amount of $^4\text{He}$ predicted by $\Omega_B = 1$ inhomogeneous BBN models is incompatible with observed abundances (see also Terasawa and Sato (1988)). Recent simulations of inhomogeneous nucleosynthesis have focused on $\Omega_B < 1$ and produce less $^7\text{Li}$ as well. Even so, we have included the $^7\text{Li}_{\text{primordial}} = 7 \times 10^{-9}$ model in this work.
because it illustrates the general behavior required of any high initial $^7$Li model and it is important to see just how constrained the primordial lithium abundance is by current observations.

In almost a reflection of the theoretical situation of Big Bang $^7$Li production, there are two main measurements of the abundance of lithium. The metal poor population II stars have an average lithium abundance of $^7$Li /$H \approx 1.6 \times 10^{-10}$ with a dispersion of a factor of $\sim 2$ (Rebolo, Molaro, and Beckmann, 1988), while the main Galactic disk stars (population I) have a maximum abundance $^7$Li /$H = 1.0 \times 10^{-9}$. The naive explanation of these observations would be that the Pop II abundance is the primordial abundance and this is quite possibly the explanation. However, high metallicity stars are observed to deplete lithium at a possibly predictable rate while on the main sequence. This is why the maximum observed Pop I lithium abundance is the important parameter for galactic chemical evolution studies. (See "Observations" section below.) If the Population II stars have undergone main sequence depletion as well, their observed lithium abundances do not represent the primordial value. Lithium is destroyed in stars at temperatures $\sim 2 \times 10^6 \, ^0 K$. The way surface lithium is destroyed in main sequence stars is via deep convective zones which mix surface material to deep within the star where high temperatures are present and then bring the lithium depleted material back to the surface. The depth of convection zones in stars (and hence the temperatures experienced) is a strong function of the metallicity of the star since this determines the opacity of the stellar gas. Michaud (1986) has argued that the small dispersion in the Pop II value, despite a wide variation in metallicities (and hence in convective dynamics) makes it unlikely that these stars have depleted much of their original lithium by stellar surface processes and D'Antona and Mazzitelli (1984) and Deliyannis (1990) have constructed stellar models which explain the Pop II lithium versus stellar surface temperature distribution but which indicate that these low metallicity stars do not destroy lithium while on the main sequence. However, Vauclair (1987) has argued that Pop II stars could have started with much higher lithium abundances if the depletion is caused by rotational mixing. Deliyanis, et.al. (1987, 1990) are engaged in studying both of these possibilities. While it remains uncertain exactly what
the surface depletion history of lithium is in low metallicity stars, Reeves, et. al. (1990), have argued that its origin is almost certainly primordial (i.e. it is not produced by an earlier stellar source, Pop III). All of these questions apply specifically to the evaluation of measurements of the surface abundance of $^7$Li in isolated stars where only the metallicity and not the specific age of the star is known. A selection of these measurements and their sources in the literature are shown in our figures 9(a-f).

In addition to the lithium vs. metallicity ($Fe/H$) relation obtained from observations of individual stars (Rebolo, Molaro, and Beckmann, 1988), one may also examine the abundance of $^7$Li in clusters of stars where the age of the cluster may be derived from its main sequence turn off point and thus the age of the stars can be known independent of their metallicity. With the measurement of $^7$Li /$H$ in stars of different surface temperatures in a cluster, it is possible, in principle, to extrapolate the initial abundance of lithium in the interstellar medium (ISM) at the time of formation of that cluster (see "Observations" for further discussion). In particular, Hobbs and Pilachowski's (1988b) determination of the lithium abundance in the open cluster NGC188 found indications of at least a Pop I initial lithium abundance for that cluster which formed $8.1 \pm 1.4$ Gyr (Hobbs, Thorburn, and Rodriguez-Bell, 1990) prior to today. The presence of this early high Li lithium abundance, this early places a strong constraint on models of Galactic chemical evolution. It is this observation which has been a major motivating factor in several recent discussions of this problem (Audouze and Silk, 1989; Brown, et. al., 1990; Dearborn, et. al. 1989; MAF; Reeves, et. al., 1990).

A final, very important, observation is the non-detection of lithium in the interstellar gas along the line of sight of supernova 1987A by Sahu, Sahu, and Pottasch (1988). From this observation they derived a limit of $Li/H < 1.6 \times 10^{-10}$ in the Larger Magellanic Cloud (LMC). The LMC has an observed metallicity of $.25 < [Fe/H] < .5$ which, in terms of chemical evolution, places it at roughly $1/5$ the age of our Galaxy. (Square brackets around a quantity refer to the ratio of that quantity to its solar system value.) This would rule out a high primordial lithium abundance. However, the amount of depletion of lithium from the ISM
onto dust grains is uncertain and since lithium in dust grains does not appear in optical spectra, Malaney and Alcock (1990) argued that this limit was much more uncertain and proposed a much higher upper limit of $Li/H < 4.4 \times 10^{-9}$. However, this limit is disputed by Steigman (1990) who argues for an upper limit of $Li/H < 5 \times 10^{-10}$. In this work, we have adopted, for discussion purposes the, we feel, conservative constraint of $Li/H < 1 \times 10^{-9}$. See discussion below.

Thus, the questions posed by the lithium measurements are twofold. Is the Pop II abundance primordial, and if so what is the source of the Pop I lithium? It has been known for some time that the elements $^6Li$, $^9Be$, and $^{10}B$ can be produced in the right ratios by the galactic cosmic rays (GCR). The high energy protons of the GCR break up heavier nuclei (spallation) like nitrogen and carbon to produce these light elements. With the assumption of an unobservable low energy component of the GCR, $^{11}B$ can also be produced (Austin, 1981; Reeves, Fowler, and Hoyle, 1970; Walker, Mathews, and Viola, 1985). This assumption then implies a some $^7Li$ production as well, but not enough to explain Pop I abundances (see also Reeves, et. al. (1990)). Thus, some stellar source of lithium must exist if the Pop II $^7Li$ abundance is primordial.

Many stellar sources have been proposed for the Pop I $^7Li$ abundance. The measurement of the high lithium value in the very old cluster NGC188, combined with the explosion of Supernova 1987A from a blue progenitor, prompted a reevaluation of Arnould and Norgaard's (1975) proposal of lithium production in supernova shocks since this would be a fast production mechanism. (i.e. it occurs in short lived/ high mass stars and would thus more easily explain the high early lithium abundance of NGC188). It was hoped that the higher density of the compact blue envelope of these types of supernova progenitors would lead to higher temperatures in the gas moving behind the supernova shock which would cause production of significant amounts of lithium from the thermonuclear combination of $^3He$ and $^4He$ in this hot gas. However, it was found that this process is not very efficient (Brown, et. al., 1990). Woosley, et. al. (1990) have also proposed a supernova based neutrino-induced mechanism for $^7Li$ production, but it may overproduce $^{11}B$ and some rare heavy elements. High $^7Li$ abundances have been seen in some red giants for quite some time (see Scala (1976) and references
therein), and in 1972 Cameron and Fowler (Cameron and Fowler, 1971) proposed a mechanism whereby $^7\text{Li}$ is made in red giants by formation of $^7\text{Be}$ at the base of the convective envelope which is then quickly (< 45 days) transported to the surface of the star where it decays to the more fragile $^7\text{Li}$ which is then deposited into the ISM by mass loss. A very important recent development is the observations by Smith and Lambert (1989) which indicate that high $^7\text{Li}$ values are generic to asymptotic giant branch (AGB) stars. These observations are exciting because this elevated lithium abundance is seen in intermediate ($4 - 9M_\odot$) stars as opposed to $1 - 5M_\odot$ stars; i.e.a “fast” source, but perhaps more importantly because this production seems to be present in all stars of the AGB class which have been observed. Finally, $^7\text{Li}$ can be made in novae (Starrfield, et. al., 1988) but this is a very “slow” source and significant production of $^7\text{Li}$ would overproduce $^{13}\text{C}$ and $^{15}\text{N}$. Reeves, et. al. (1990) argue that it is unlikely that $^7\text{Li}$ is produced in high mass stars. Steigman, however, reports (1990) that analytical modeling points toward $^7\text{Li}$ production in oxygen producing stars. With the observations of Lambert and the difficulties of the supernova shock mechanism, red giants are seen as a very attractive production site and one that is observationally known to exist.

Many models of the chemical evolution of the galaxy have been proposed and utilized in different ways. Analytical models based on the instantaneous recycling approximation (IRA) (all stars of mass greater than ~ $1M_\odot$ die as soon as they are formed and all stars of mass less than ~ $1M_\odot$ live forever) are seen to describe the evolution of primary elements extremely well (see e.g.Audouze and Tinsley (1976) and references therein). These models have recently been extended to include a much larger class of models (Clayton, 1985, 1988) making them very useful in the study of nuclear chronology (which we discuss below). Numerical models (e.g.Truran and Cameron (1971), Audouze and Tinsley (1976), and references therein) are often needed in light element studies because low mass stars are involved and the IRA breaks down. Recent studies of this nature include Mathews, Alcock, and Fuller (1990), Audouze and Silk (1989), Reeves, et. al. (1990), and Audouze, et. al. (1983) Since we want to compare production of lithium in different mass ranges of stars we must of necessity use numerical
models, but we note that many of our results for other elements could have been obtained more straightforwardly with the use of analytic models. For simplicity and transparency, and in order to easily cover a wide range of models, we have essentially adopted the simple numerical framework used by Mathews, Alcock, and Fuller (1990). We have also assumed, as in their work, that for the high primordial $^7\text{Li}$ models, a mechanism can be constructed for main sequence stellar lithium destruction which explains the uniform lithium abundance in all stars of metallicity $\lesssim 0.04$.

Our purpose in this paper is threefold. We wish to estimate how high an initial $^7\text{Li}$ abundance can be tolerated by current observations. We wish to determine if there is a preferred stellar source for $^7\text{Li}$ if the Pop II lithium abundance is assumed to be approximately primordial. Finally, we wish to determine if, in the context of our simplified generic models, there is a preferred class of star formation histories. In this paper, we investigate the history of the light element abundances in the galaxy in numerical one zone models. We first review the observed abundances and choose our adopted constraints from these observations. We next describe the models we used and the different stellar birthrates incorporated in them. In particular, we discuss the birthrate function of Mathews and Schramm (1990, referred to as MS below) which is based on the multiple coalescence models of galaxy formation. We discuss the proposed light element production sites which include a broad range of possible stellar $^7\text{Li}$ sources, followed by a description of our treatment of metals production. We then describe analytical checks of our numerical models. Finally, we discuss our results which imply that the observations are best fit by models which have higher past star formation than present and that high lithium scenarios are tightly constrained by current observations.
Light Element Observations

Observations of the light element abundances with which we are concerned in this paper ($D$, $^6Li$, $^7Li$, $^9Be$, $^{11}B$, and $^{10}B$) fall into several different categories and the different methods and sites of observation lead to different types of constraints on chemical evolution models. The most accurate (in a sense) determinations are those made within the solar system and for $^6Li$, $^{11}B$, and $^{10}B$ these are also the only observations available. The two sites for measurements of the solar system abundances of these three elements (which, as we will discuss below, are probably produced solely by Galactic cosmic ray (GCR) spallation), are the meteorites and the photosphere of the sun. The abundance determinations of these two sites, however, differ by factors of $\sim 2 - 3$. Since chemical fractionation can distort meteoritic abundances, and since thermonuclear destruction processes can do the same to the photospheric abundances, it is unclear which provides a more accurate picture of the general interstellar abundance of these elements at the time of formation of the solar system. We have here adopted the abundances used by Walker, Mathews, and Viola which rely primarily on the solar photospheric abundances with that caveat that they are at least uncertain by a factor of 2 (Walker, Mathews, and Viola, 1985). The lower portion of table 2 shows these abundances. For GCR production, we will express these values as ratios by mass with respect to $^9Be$. Table 3 quotes the solar system observed values of the mass ratios of the light elements with respect to $^9Be$. While these ratios are uncertain to a factor of 2, the ratio of $^{11}B$ to $^{10}B$ is much less uncertain since it is basically the same in both solar system sites and has an accepted value $M(^{11}B /^{10}B) = 4.45 \pm 0.10$. The GCR model of Walker, Mathews, and Viola (1985) which we use is tied directly to this ratio (see below).

Since deuterium is so easily destroyed in stars, it can only be directly observed in non-stellar environments. There are only two types of measurements of interstellar $D$ available. We have a measurement of deuterium in the atmospheres of the giant planets (assumed to coincide with the ISM value of 4.6 Gyears ago), and a measurement of the current ISM value. This is obtained by analyzing the shape of Lyman absorption lines in the spectra of bright stars shining through
interstellar gas. These values are given in table 2. (See Boesgaard and Steigman (1985) for discussion and further references).

$^9\text{Be}$ and $^7\text{Li}$ are the only light elements of interest which have been observed in other stars. The observations of $^9\text{Be}$ shown in figure 5 and on which our adopted "recent" value in table 2 are based, are those summarized in Reeves and Meyer (1978). While no time uncertainties are shown, the ages of the stars plotted could be uncertain by up to 50%. These observations are similar in character to the cluster abundances of $^7\text{Li}$ discussed below in that they have roughly solar metallicity, but their ages are independently known. Since it is believed that possible errors in age are not systematic, the overall conclusion can be made that the $^9\text{Be}$ abundance in the ISM has remained relatively unchanged over the last 10 – 12 Gyears and this provides an important constraint on evolutionary theories. The measurements of $^9\text{Be}$ in stars can also be plotted as a function of the metallicity of the stars (see figure 6). These observations provide data for much older times in the case of very low metallicity stars, but the ages in this case are, of course, not independently known since these are isolated stars. Our adopted average values for $^9\text{Be}/H$ for low and high metallicity stars are given in table 2.

There are two types of observations of the lithium abundance. The first and perhaps most widely employed (and less controversial in their actual derivation from the observed spectra if not in their subsequent interpretation) are determinations of the lithium abundance on the surfaces of stars. These are usually plotted versus $Fe/H$ and they show a rise from $^7\text{Li} /H = 1 \times 10^{-10}$ in very low metallicity stars to $^7\text{Li} /H = 1 \times 10^{-9}$ in the highest lithium abundances in main sequence stars at a given metallicity. If the low metallicity stars have not substantially depleted their abundances (for which there is some theoretical explanation (D'Antona and Mazzitelli, 1984; Deliyannis, 1990; Michaud, 1986) from the value they were "born" with, then the Population II lithium abundance is primordial. If the low metallicity stars have uniformly depleted their $^7\text{Li}$ abundances despite their large range in metallicity, this conclusion is weakened. Mathews, Alcock, and Fuller (1990) give a description of this alternate scenario for explaining the lithium versus metallicity relation. See Reeves, et. al. (1990) and the introduc-
tion to this paper for more discussion of this controversy. Figures 9(a-f) plot a large number of these types of observations. Table 2 gives our accepted abundance, but the assumption that this is equivalent to the ISM abundance at times corresponding to metallicity $\lesssim 0.04$ is still not entirely certain. We will see that most of our models fit these observations fairly well, so these numbers are less restrictive than direct observations of sites for which either independent ages or direct observations of the ISM are available.

These other observations are of three types and are plotted on figures 8(a-f). The most precise in terms of time is the solar system average value which approximately gives the ISM value of $^{7}Li / H$ 4.6 Gyears ago and was discussed above.

The next method is to measure the lithium abundance on the surfaces of stars in an open stellar cluster. The cluster age can be determined from its main sequence turnoff point. That is, the position of the point in the Hertzsprung-Russell diagram where the cluster stars leave the main sequence can be compared to stellar evolutionary models, and an age for the cluster can be determined. This procedure can be difficult. Hobbs, Thorburn, and Rodriguez-Bell (1990) contains a very recent example of this method and discussion of the difficulties involved for the important and notorious cluster NGC188. The initial lithium abundance of the cluster can be roughly determined by extrapolating the curve of $^{7}Li / H$ versus effective surface temperature to the high effective stellar surface temperature "plateau" region where no destruction of lithium occurs. This plateau occurs because low mass (low surface temperature) stars have deeper convective envelopes and thus deplete more lithium during their lifetime because the deeper convection zones carry the lithium down to higher temperatures where it is destroyed. Thus, the high mass, high surface temperature stars have the highest lithium abundances in a cluster. By examining the shape of the lithium versus surface temperature curve for the stars of a cluster and also by assuming that all stars of the same mass and metallicity deplete their lithium at the same rate while on the main sequence, one can make a rough estimate of the initial lithium abundance associated with a cluster of known age. This is a difficult business for the older clusters since only the low temperature (and thus low mass) stars are
still on the main sequence. In the case of NGC188, since only stars at essentially one temperature can be observed (no \( Li \) vs. surface temperature curve) and since it is a different metallicity from the other more modern clusters used (no well known age-lithium relation) it can only be used as a lower limit. Hobbs and Pilachowski (1988b) give a full discussion of these points and Boesgaard (1990) gives a current discussion of the Pop I lithium question (as well as \(^9 Be \)). We have used for our plotted points the extrapolations of Mathews, Alcock, and Fuller (1990), and our adopted Pop I abundance is given in table 2.

The final, and most uncertain (but most potentially constraining) method of lithium determination is to measure the lithium absorption line in the spectra of a bright object shining through interstellar gas. This is not as straightforward as it appears, since it is very uncertain to what extent interstellar lithium is depleted from the gas onto interstellar dust grains. The chemistry and surface physics involved in the process are not at all well understood. Using supernova 1987A as a light source, Sahu, Sahu, and Pottasch (1988) did not detect lithium in the ISM of the Larger Magellanic Cloud (LMC). Since the LMC has a metallicity of .25(2) (numbers in parentheses represent uncertainty factors) it corresponds in evolutionary terms to about 1/5 the age of our galaxy.) They reported this non-detection as an upper limit to the lithium abundance of \( Li/H < 1.6 \times 10^{-10} \). However, this number has been challenged as not properly accounting for the large uncertainties in lithium depletion factors. Assuming depletion factors for lithium and potassium to be the same, Malaney and Alcock argue that the limit should be much higher at \( Li/H < 4.4 \times 10^{-9} \). Steigman, however, has compared lithium and potassium measurements along several lines of sight in the Galaxy. He finds the ratio to be constant implying that the \( Li/K \) ratio is a constant in the ISM. Thus, the relative depletions of \( Li \) and \( K \) in the LMC should be similar. Since potassium is definitely not produced in the big bang, if lithium starts out at a high inhomogeneous big bang value (i.e. \( \gtrsim 10^{-9} \)) and is depleted over galactic time, the observed \( Li/K \) ratio in the LMC should be higher than that in this galaxy since \( K \) can only increase with galactic evolution and lithium decreases with time in these models. The \( Li/K \) ratio is not higher in the LMC. Steigman argues for an upper limit of \( Li/H < 5.0 \times 10^{-10} \) for the LMC. For our
adopted value we have chosen what seems to us to be a conservative value of $Li/H < 1 \times 10^{-9}$ at some time in the range $10 - 15 \, G\text{years}$ ago.

The Numerical Model Framework

Since we are, in this investigation, interested in examining differences in stellar production sites among different mass stars and since we are also interested in a variety of somewhat mathematically arbitrary star formation rates, we have used strictly numerical models of Galactic chemical evolution. We take as our main sources for the basic equations Mathews, Alcock, and Fuller (1990) and Mathews and Schramm (1990). We begin by making the standard (though unverified) assumption that the starbirth function $B(m, t)$, which is a function of both the age of the galaxy and the mass of the stars being formed can be written as the product of $\psi(t)$, the overall star formation rate (SFR) in $G\text{year}^{-1}$, and $\phi(m)$, the initial mass function (IMF) in $M_\odot^{-1} \, pc^{-2}$ (Audouze and Tinsley, 1976). The actual observed quantity for the IMF is the present day distribution of stellar masses of main sequence stars in the solar neighborhood, the present day mass function (PDMF), $\xi(m)$. However, most larger mass stars which were formed have already evolved off the main sequence, so to derive the IMF from a given PDMF we use the formula:

$$\phi(m) = \xi(m) \int_{T_g - \tau(m)}^{T_g} \psi(t) dt,$$

where $\tau(m)$ is the main sequence lifetime of a star of mass $m$ taken from Scalo (1986), and where we have normalized our SFR to 1 when integrated over the galactic lifetime so that the PDMF will match the IMF for stars with lifetimes $> T_g$ (MS). In our main numerical models, we have used for a PDMF the tabulated function of Scalo (1986). We also evolved test cases with other PDMFs and stellar age functions ($\tau(m)$) but the main features of the results were essentially the same. Figure 1 shows a selection of different initial mass functions all derived using a constant star formation rate for comparison.
For the evolution of the surface gas density of the galactic disk we assume:

\[
\frac{dm_g}{dt} = -b(t) + p(t) + f_i(t)
\]  

(2)

where \(f_i\) is any net infall of gas from outside the disk, \(b\) is the rate at which gas is lost from the ISM to star formation:

\[
b(t) = \int_{m_{\text{low}}}^{m_{\text{high}}} m\phi(m)\psi(t)dm,
\]  

(3)

where we have taken the limits of integration from \(m_{\text{low}} = 0.1 M_\odot\) to \(m_{\text{high}} = 62 M_\odot\) (MS), and \(p\) is the rate at which gas is returned to the ISM by dying stars:

\[
p(t) = \int_{m_{\text{low}}}^{m_{\text{high}}} (m - m_r)\phi(m)\psi(t - \tau(m))dm.
\]  

(4)

Note the appearance of the stellar death rate \(\psi(t - \tau(m)) \neq \psi(t)\) which accounts for the finite lifetimes of stars. The remnant mass left behind in a star of mass \(m\),

\[
m_r(m) = \begin{cases} 
1.4 & m > 6.8 \\
0.15m + 0.38 & 0.7 < m < 6.8 \\
m & m \leq 0.7
\end{cases}
\]  

(5)

is from Iben and Renzini (1983). The present surface density of gas in the Galactic disk has recently been measured as \(13 \pm 3 M_\odot \text{ pc}^{-2}\) (Kulkarni and Heiles, 1987). We have accordingly taken this as the required final value of \(m_g\). The resultant initial values of \(m_g\) are given in table 4. The total galactic disk surface density \((m_g(0) + f_iT_g)\) is determined to be \(M_{\text{tot}} = 46 \pm 9 M_\odot \text{ pc}^{-2}\) (Gilmore, Wyse, and Kuijken, 1989). All of our models are reasonably close to this value.

Our equation for the evolution of the mass per square parsec of the Galactic disk for an element \(i\) is similar:

\[
\frac{dm_i}{dt} = P_i^{\text{cr}} + P_i^\phi + E_i + X_i(0)f_i - bm_i/m_g(t).
\]  

(6)

\(P_i^{\text{cr}}\) is the galactic cosmic ray production term which we will describe below. \(P_i^\phi\)
is the stellar production term:

$$P_i^* = \int_{m_{\text{low}}}^{m_{\text{high}}} (m - m_r)\phi(m)\psi(t - \tau(m))X_i(m)dm$$  \hspace{1cm} (7)

where $X_i(m)$ is the overall mass fraction of element $i$ in the ejected material of a star of mass $m$. $E_i$ is the mass of element $i$ present in a star at its birth which is returned to the ISM by a star at its death, given by the numerically obnoxious formula:

$$E_i = \int_{m_{\text{low}}}^{m_{\text{high}}} (m - m_r)\phi(m)\psi(t - \tau(m))\frac{m_i(t - \tau(m))}{m_g(t - \tau(m))}A_i(m)dm,$$  \hspace{1cm} (8)

where $A_i(m)$ is the factor by which element $i$ is depleted in a star of mass $m$. Since Dearborn (1990), has shown that all of the light elements ($D, ^{6}Li, ^{7}Li, ^{9}Be, ^{10}B, ^{11}B$) have very small $A_i$'s of $\lesssim 1/100$ we have approximated $A_i$ by zero for the light elements and by 1 for iron. The factor $X_i(0)$ in equation (6) is the primordial mass fraction of element $i$. See table 1 for the values of $X_i(0)$ used for the two BBN scenarios. The presence of the infall term $f_i$ is not necessitated by observations, but is traditionally used to adjust for “overastration” of the light elements (particularly deuterium) by providing a continuing source of primordial, unastrated material. In any case, Tinsley (1977) has argued that observations of the Oort clouds place an upper limit on the present infall rate of $f_i < 2M_\odot pc^{-2} Gyr^{-1}$. We have accordingly adopted a constant infall rate of whatever magnitude is necessary to force the current deuterium abundance in our models to be $D/H \geq 0.8 \times 10^{-5}$, the lower limit to the current observed deuterium abundance (Boesgaard and Steigman, 1985). Table 4 shows the actual values used and they are small or zero in all cases. Since infall is not an important feature of any of our models, we have not considered the likely possibility that any real infall is probably not strictly constant over the whole galactic lifetime. In fact, there is some indication (e.g. Clayton, 1988) that time varying rates of infall could be very important in galactic chemical evolution, particularly in the area of radioactive isotope dating schemes.
Before discussing our adopted star formation rates, we should note that other numerical frameworks are possible. In particular, it has been proposed that the IMF is not constant in time. Basically, these models assume that stars form differently in a low metallicity environment such that high mass stars are formed in a greater percentage than at present. This results in a solution to the "G-dwarf problem" since it predicts fewer low mass/low metallicity stars than are predicted by constant IMF models. These models also would provide a faster rise in the cosmic ray produced light elements (since the Galactic cosmic ray flux is assumed to be proportional to the supernova rate which would be higher at early times with greater numbers of high mass stars). These models would possess many of the same desirable features of the decreasing star formation rates and the multiple coalescence model discussed below. Scala (1986) gives an exhaustive analysis of the present state of knowledge on the constancy of the IMF in time, space, and metallicity and finds no evidence for any variation in the IMF. Models have also been proposed which treat the Galactic halo and disk separately; for a recent example see Pagel (Pagel and Simonson, 1989).

**STAR FORMATION RATES**

The choice of the star formation rate (SFR), the function $\psi(t)$, is perhaps the most decisive basic input parameter in our galactic chemical evolution models. We have consequently tried to cover the major types of SFR in our survey.

There are two major constraints on the SFR. First, there is no direct evidence that the SFR has ever been different than it is today and there is an upper limit to how much greater the SFR can have been in the past. From the observed age metallicity relation, Twarog (1980) obtains the limit $\langle \psi(t) \rangle / \psi(T_g) < 2.5$, which reduces to $\psi(T_g) > 1/(2.5 T_g)$. Another constraint is less certain and is based on the physical assumption that stars of mass $\gtrsim 1 M_\odot$ are not formed by mechanisms of a significantly different type than stars of mass $\lesssim 1 M_\odot$. This is translated into the requirement that there not be too great a discontinuity in the derived IMF at the point where the IMF deviates from the PDMF. In other words, there is assumed to be nothing "special" about stars with main sequence
lifetimes of exactly the current galactic age. This continuity constraint is given
by \( 0.18 \leq \gamma T_g(T_g) \leq 2.5 \) (Scalo, 1986). All of our SFR's obey these constraints.

One further constraint on the star formation rate is that of nuclear chronology. This method relies on comparisons of the current meteoritic abundances of pairs of heavy radioactive isotopes. By combining a long lived isotope (i.e. one with a half-life close to the age of the Galaxy) with a short lived one (i.e. a half-life of 1 to a few Gyears) one can constrain the allowed variations in star formation rate over time since each isotopic abundance averages the stellar production over its respective lifetime. Any variation in the star formation rate (or more accurately the effective nucleosynthesis rate) averaged over the time shortly before the solar system formed and the SFR averaged over the whole lifetime of the galaxy is observable if we know the starting (i.e. production) ratio of the isotope pair. If we then include several pairs involving short, medium, and long lifetime isotopes (e.g. various isotopes of \( U, Th, \) and \( P \)) we can obtain a limit on the parameter \( t_\nu/T_e \) (defined below). (For full discussions of this method see Meyers and Schramm (1986), Clayton (1988), and Reeves (1990).) Using the instantaneous recycling approximation, since most of these elements are made in reasonably high mass stars, we can write (see Meyer and Schramm (1986)):

\[
T_\nu = \frac{\int_0^T \! \! \! \! \int t \psi(t) e^{-\int_0^t \! \! \! \! \int \omega(t') dt'} dt}{\int_0^T \! \! \! \! \int \psi(t) e^{-\int_0^t \! \! \! \! \int \omega(t') dt'} dt},
\]

where \( \omega \), the rate of movement of metals out of the ISM is given by:

\[
\omega(t) = \frac{\int_{m_{\text{high}}}^{m_{\text{low}}} m \phi(m) \psi(t) dm}{m_f(t)}, \tag{9}
\]

and \( T_e \) is the age of the galaxy at the formation of the solar system. Meyer and Schramm find that \( t_\nu \) is confined to the range \( 0.43 \leq t_\nu/T_e \leq 0.59 \). The values of this parameter for the various star formation rates described below are shown in table 4 and the only one which seriously violates this constraint is the rate DSFR which has several other difficulties as well but does represent the behavior of an extreme case. It can also be seen that the MSSFR cases are at the very low end of
the range. As might be expected, the parameter \((t_Y)\) is very sensitive to the exact height of the "spike" in this SFR. We found that small changes in the adopted IMF or the adopted present deuterium abundance (see discussion below) could change \(t_Y\) by \(\pm \sim 0.04\) with no significant change in the chemical evolutionary predictions of these models so \(t_Y\) should not be seen as a tight constraint on these models.

First, we used the three most straightforward possible choices of SFR (see Miller and Scalo (1979)): a constant SFR (CSFR):

\[
\psi(t) = \frac{1}{T_g}, \quad (10)
\]

a maximally (within the continuity constraint) exponentially increasing SFR (ISFR):

\[
\psi(t) = \frac{0.41e^{(1.6t/T_g)}}{T_g}, \quad (11)
\]

and a maximally exponentially decreasing SFR (DSFR):

\[
\psi(t) = \frac{2.6e^{-2.4t/T_g}}{T_g}. \quad (12)
\]

This decreasing star formation rate is too extreme in several senses. It violates the nuclear chronology constraints as mentioned above, and we shall see that it violates other constraints below. We will see that a high early star formation rate is attractive for light element evolution however, so we also use the more modestly decreasing SFR (DSFR2):

\[
\psi(t) = \frac{1.7e^{-1.2t/T_g}}{T_g}. \quad (13)
\]

This particular SFR is used because it is the most steeply decreasing SFR which fits the deuterium constraints discussed below. In addition to these generic functions, other functions have been proposed for one reason or another. It has been suggested that there may have been a hiatus in star formation between the initial formation of the stars in the galactic halo and the formation of the galactic disk.
We have tried to schematically represent this by a constant star formation rate with a gap of zero star formation from 16 to 12 Gyears ago (Wheeler, Sneden, and Truran, 1989).

It has been realized for the last few years that a potentially much more accurate way of modeling chemical evolution is to use dynamical simulations of galaxy formation to trace elemental evolution. As a preliminary step, and as a means of performing simplified tests on one of these models, Mathews and Schramm (1990) have proposed an analytical SFR (MSSFR) based on their dynamical arguments of the following form:

\[
\psi(t) = \begin{cases} 
A[(\mu_t/\mu_i)e^{-t/to} - 1]e^{5t/3t_0} & t/to < ln(\mu_t/\mu_i) \\
B & t/to \geq ln(\mu_t/\mu_i),
\end{cases}
\]

(14)

where

\[
A = \frac{1 - B(T_g - t_0ln(\mu_t/\mu_i))}{3t_0(0.3(\mu_t/\mu_i)^{5/3} - 0.5\mu_t/\mu_i + 0.2)}
\]

(15)

and B is the present star formation rate, \(\psi(T_g)\). Within the bounds on \(\psi(T_g)\) mentioned above, we are free to choose B. This model is based on the assumption of no infall, so we have chosen B so that the current deuterium abundance matches the lower limit of the observed D abundance in the ISM. The other two parameters are associated with the dynamics of the model which is based on the growth of primordial gas clouds through collisions. The ratio of initial cloud mass to total galactic mass is believed to be \(\mu_t/\mu_i \approx 1.0 \times 10^5\) and the mean growth time of a cloud is \(t_0 \approx 0.4\) Gyears. We found that reasonable changes in these parameters did not affect the overall course of Galactic chemical evolution to a noticeable extent. We have used these values in our SFR. This SFR is similar to previously proposed initial enrichment SFR's in that is has a very sharp "spike" in the first few Gyears of Galactic evolution during which most of the initial star formation takes place, followed by a long period of relatively low constant star formation until the present time. As we will show, this model provides a natural solution to many of the problems associated with the light elements since the early star formation spike can cause "fast" production (or astringation) of these sensitive elements.
COSMIC RAY PRODUCTION

It has been known since the early 1970's that the observed ratios of the light elements ($^7\text{Li}$, $^6\text{Li}$, $^{11}\text{B}$, $^{10}\text{B}$, and $^9\text{Be}$) can be produced by spallation reactions in the Galactic cosmic rays (GCR) (Reeves, Fowler, and Hoyle, 1970). $^6\text{Li}$, $^{10}\text{B}$ and $^9\text{Be}$ are naturally produced by the observed GCR flux in the same ratios as their observed solar abundances to within observed error which suggests that the GCRs are the sole source of these elements (Walker, Mathews, and Viola, 1985). The origin of $^{11}\text{B}$ is less clear. The observed GCR flux does not produce $^{11}\text{B}$ and $^{10}\text{B}$ in the right ratio. It is possible, by postulating a low energy component of the GCR (one which would be unobservable from a terrestrial vantage), to produce $^{11}\text{B}$ in the correct, well known ratio to $^{10}\text{B}$, while retaining acceptable, less tightly constrained ratios of $^6\text{Li}$, $^{10}\text{B}$, and $^9\text{Be}$. On the other hand, there is a possibility that $^{11}\text{B}$ is made via neutrino spallation and thermonuclear shock processing in supernovae (Woosley, et. al., 1990). However, the details of this mechanism are still uncertain. For this work, we have used the smallest low energy component GCR model ($\gamma = 7$) of Walker, Mathews and Viola (1985); i.e. we have assumed that present $^{11}\text{B}$ is produced entirely by cosmic rays. This implies twice as much $^7\text{Li}$ production by the GCRs but it is still much less than the production necessary to explain Pop I abundances if Pop II abundances are primordial.

For the term, $P^\sigma_i$, we have assumed, as in Mathews, Alcock, and Fuller (1990) that the GCR flux is proportional to the core collapse supernova rate so that

$$P^\sigma_i = a_i \int_{m_{\text{low}}}^{m_{\text{high}}} \phi(m)\psi(t)dm.$$  \hspace{1cm} (16)

The limits of integration are the lowest and highest core collapse supernova masses which we have taken to be $9M_\odot$ and $62M_\odot$ respectively. The factor $a_i$ is determined for all GCR produced elements except $^9\text{Be}$ by the ratios derived in Walker, Mathews, and Viola (1985) adjusted to give ratios by mass rather than abundance (see table 3). For an overall normalization, we have assumed that the $^9\text{Be}$ abundance at the formation of the Earth (assumed to be 4.6 Gyears ago) is
equal to the solar system value of \( \frac{^9Be}{H} = 1.4 \times 10^{-11} \). This assumption could be unreasonable for some of our models. To maintain a check on this possible difficulty, we report the required enhancement in GCR production efficiency for a given SFR with respect to a constant star formation rate (see table 4). Specifically, we give the ratio of the present required rate of GCR element production for a given SFR to that of the CSFR model.

LITHIUM PRODUCTION SITES

Since, as we have discussed above, it is exceedingly unlikely that enough \( ^7Li \) is produced in cosmic rays to account for an increase from Pop II to Pop I abundances; if the Pop II abundance is primordial, then there must be a stellar source of \( ^7Li \) which does not produce significant \( ^6Li \). As mentioned in the introduction, there are reasons to suspect that \( ^7Li \) might be produced in supernovae. However, Brown, et. al. (1990) have shown that current state of the art models of envelope shocks do not produce mass fractions in their ejecta of greater than \( \sim 10^{-10} \). This amount of production is completely negligible. Since the \( \nu \)-process (Woosley, et. al., 1990) or some other unknown core collapse supernova process could conceivably produce significant \( ^7Li \), we have taken as one of our production sites stars of mass \( 9 - 62 M_\odot \).

However, red giants have become the generally favored site. Since they have actually been observed to be enriched in lithium and thus require minimal theoretical machinations. Our red giant production takes two distinct forms. Following Mathews, Alcock, and Fuller (1990) we use 1-5\( M_\odot \) stars as a red giant source. However, Smith and Lambert’s (1989) observations indicate that it is likely that a majority of asymptotic giant branch stars of intermediate mass \( 4 - 9 M_\odot \) may be extremely surface enriched with lithium \( (Li/H \sim 10^{-7}) \). With this higher mass range, the motivation for using supernovae to obtain the rapid rise in lithium abundance required by the observations of NGC188 is no longer present since, as we will show, intermediate mass stars produce lithium on timescales essentially identical to those associated with core collapse supernovae progenitors. It is hypothesized that this red giant \( ^7Li \) is the product of thermonuclear burning of \( ^4He \) and \( ^3He \) to form \( ^7Be \) at the hot base of the deep convective envelopes of
these stars. This $^7\text{Be}$ is then brought to the surface of the star in a short amount of time, $\lesssim 49$ days. The $^7\text{Be}$ then decays via electron capture to $^7\text{Li}$ and it is presumed that much of this now very enriched (100-1000 times Pop I abundance) material is injected into the interstellar medium via mass loss which is common in AGB stars.

For each mass range (1 - 5, 4 - 9, and 9 - 62$M_\odot$) we have assumed, for simplicity that all stars have a single average $^7\text{Li}$ mass fraction in their ejecta. We also included as a possible site 1 - 5$M_\odot$ and 9 - 62$M_\odot$ stars with the supernova ejected mass fractions set to the maximum values of Brown, et. al. (1990),

$$X_i = \begin{cases} 
5 \times 10^{-10}/m & 9 \leq m \leq 20M_\odot \\
2 \times 10^{-9}/m & 20 \leq m \leq 62M_\odot.
\end{cases} \tag{17}$$

We then solved for the average red giant enrichment needed to produce the current value of $^7\text{Li}/H$. The required ejecta enrichment for each production site is given in table 4. This number represents the total mass of new $^7\text{Li}$ released by the star into the ISM throughout its lifetime divided by the total gas mass released into the ISM during its lifetime ($m - m_r$). In our models, this production is assumed to occur instantaneously at the end of the star's lifetime.

**Metallicity Evolution**

Since essentially all lithium abundance measurements are tied to $Fe/H$ as their metallicity parameter, we need to perform evolution of the $Fe$ abundance as well. We note, however, that $O/H$ may be a more valid indicator of the metallicity evolution than $Fe/H$. Since $Fe$ can be made in various places and in very uncertain amounts (see below), while $O$ is made only in high mass stars in reasonably well predicted quantities, it is almost certainly a more straightforward tracker of Galactic time, particularly in models with star formation rates that have abrupt changes over short time periods such as our MSSFR and HIASFR models. Wheeler, Sueden, and Truran and references therein (1989) give an overall view of the problems of using $Fe/H$ as the “clock” in chemical evolution theories and observations. These concerns are born out by observations.
of non-stellar ratios of $O/Fe$ in low metallicity stars (see references in Wheeler, Sneden, and Truran). Since we will see that observations of light elements as a function of $Fe/H$ are not, in general, the strongest constraints on our models (they are reasonably fit by most of our scenarios) this is not a fatal problem for our purposes, but caution should be used.

$Fe$ is almost certainly produced in supernovae, but how much $Fe$ is produced in which types of supernovae is still unclear. Fortunately, much can be said about $Fe$ production without a complete knowledge of all of the details (e.g. acc Arnett, Schramm, and Truran, 1989). This is, however, one of the good reasons for preferring $O$ as a metallicity parameter. The actual mass fraction of $Fe$ in the ejecta of core collapse supernovae is a very difficult quantity to model precisely, because it depends strongly on how much stellar core mass is left behind. Since in pre-supernova models, the iron core of the star extends to a mass roughly the same as the predicted remnant (neutron star or black hole) mass, the relatively small amount of iron produced in these events can vary greatly in terms of its mass fraction in the ejecta. There are indications that the remnant core mass (i.e. the “mass cut”) in core collapse may even be grossly non-monotonic (Barkat, 1990). For detonation/deflagration supernovae, the amount of $Fe$ produced is probably within a factor of $\sim 2$ of $1.4M_\odot$, but it is still unclear exactly what the progenitor star or stars are for this “Type Ia” event. In this case, a thermonuclear detonation or deflagration (the exact mechanism is still under debate) wave burns essentially all of the material of the progenitor (which probably has a mass of around $1.4M_\odot$ since it is the collapse of a white dwarf type object which is presumed to trigger the detonation/deflagration) into $Fe$. In this case, however, the fraction of stars of the right mass which undergo this process is entirely unknown empirically since no consensus exists on the mechanism for triggering these events. For our $Fe$ production, we follow the treatment of Mathews and Schramm (1990). We divide the different types of supernovae into three classes: Type Ia (assumed to be detonation or deflagration occurring in some fraction of low mass stars) with progenitor masses $2.5 - 7.5M_\odot$, Type II with progenitors $9 - 30M_\odot$, and Type Ib (assumed to be large progenitor star core collapses occurring in stars which have lost their hydrogen envelope) with progenitors
30 – 62 $M_\odot$. The various supernova Types are observational classifications; see Van den Bergh, McClure, and Evans (1987) for explanations. We assume that Ia's produce $0.6M_\odot$ of $Fe$ (Nomoto, Thielemann, and Wheeler, 1984), II's produce $0.013(m - 12)$ for progenitor mass $> 12M_\odot$ and no $Fe$ otherwise (Woosley and Weaver, 1986), and Ib's produce $0.2M_\odot$ (Schaeffer, Casse, and Cahen, 1986). Since we really have no clear idea of how many 2.5 – 7.5$M_\odot$ stars are Type Ia supernova progenitors (a number of different mechanisms have been suggested), we now scale the Type II and Type Ib production relative to Ia using the observed ratios of Van den Bergh, McClure, and Evans (1987) of $R_{II}(T_g)/R_{Ia}(T_g) = 4/3$ and $R_{Ib}(T_g)/R_{Ia}(T_g) = 3.7$, where

$$R(T_g) \propto \int_{m_{low}}^{m_{up}} \phi(m) \psi(t - \tau(m)) dm,$$  \hspace{1cm} (18)

where $m_{low}$ and $m_{up}$ are the progenitor mass limits for the given type of SN. We should note that these ratios are uncertain because they do not account for the possibility of a much higher Type II rate if there are many dim Type II supernovae like SN1987A which are unobservable in other galaxies. Fortunately, we found our results to be very insensitive to variations in these relative rates. We then normalize the resultant Fe production so that $Fe/H$ equals the solar value at the formation of the solar system (taken to be 4.6 $G$years ago).

### Analytical Checks

Equations (2) and (6) must, in general, be solved numerically, but in the case of a constant star formation rate and no stellar production they can be approximately solved analytically to a very high degree of accuracy if we use the analytical form of the IMF from Miller and Scalo (1979) for CSFR:

$$\phi(m) = \begin{cases} 
18.0m^{-1.4} & 0.1 \leq m \leq 1 \\
18.0m^{-2.5} & 1 \leq m \leq 10 \\
104.0m^{-3.3} & 10 \leq m.
\end{cases} \hspace{1cm} (19)$$

This IMF assumes that $T_g = 15.0$ $G$years. This solution is possible because $\psi(t - \tau(m))$ is a constant over the galactic lifetime in this case (CSFR) despite
the complex form of $\tau(m)$. The other approximation which must be made enters because $\psi(t - \tau(m)) = 0$ for $\tau(m) > t$. To treat this, we make the assumption that:

$$\psi(t - \tau(m)) = \begin{cases} 1/T_g & \text{if } \tau(m) < T_g \\ 0 & \text{if } \tau(m) > T_g. \end{cases} \quad (20)$$

Thus for $T_g = 15$, using Scalo’s (1986) stellar age function, $\psi(t - \tau(m)) = 0$ if $m < 0.85M_\odot$.

We will here treat the case of deuterium for both high and low initial values, and will assume an integrated infall mass of $10 M_\odot$ (in practice, it turns out that no infall is needed to produce a satisfactory final $D/H$ abundance for CSFR see (Table 4). For equation (2) we need to evaluate $b(t)$ and $p(t)$ which are, in this case constants. We find:

$$b = 3.381 \, M_\odot \, Gyr^{-1}, \quad p = 1.412 \, M_\odot \, Gyr^{-1}, \quad f_i = \frac{10 \, M_\odot}{15 \, Gyr},$$

which gives us the final value problem:

$$\frac{dm_g}{dt} = -b + p + f_i, \quad m_g(T_g) = 13.0$$

$$\Rightarrow m_g(t) = -1.3023t + 32.535. \quad (21)$$

For deuterium, most of the terms in equation (6) are zero. Since $D$ is not produced except in the Big Bang (Epstein, Lattimer, and Schramm, 1976; Reeves, et. al., 1973), and since it is completely destroyed in stars, we are left with

$$\frac{dm_d}{dt} = X_d(0)f_i - bm_d/m_g(t).$$

If we now let:

$$a = X_d(0)f_i, \quad m_g(t) = c + et,$$

then we have

$$\frac{dm_d}{dt} = a + bm_d/(et + c). \quad (22)$$
Solving this gives

\[ m_d(t) = \frac{u}{(e+b)} m_g(t) + m_g(t)^{-b/e} c^{b/e} (m_d(0) - \frac{ac}{(e+b)}). \]  

(23)

Since

\[ m_d(0) = X_d(0)m_g(0) = (5.4 \times 10^{-5} \text{ or } 7.5 \times 10^{-5}) \times 32.535 \]

\[ = 0.00176 \text{ or } 0.00244, \]

(24)

and likewise

\[ a = 3.6 \times 10^{-5} \text{ or } 5.0 \times 10^{-5}, \quad c = 32.535, \quad e = -1.3023, \]

(25)

we can substitute in these actual values to get

\[ m_d(t) = \begin{cases} 
1.417 \times 10^{-7} m_g(t)^{2.596} + 1.731 \times 10^{-5} m_g(t) & \text{standard BBN} \\
1.964 \times 10^{-7} m_g(t)^{2.596} + 2.4053 \times 10^{-5} m_g(t) & \text{inhomogeneous BBN} 
\end{cases} \]

(26)

We can transform these into abundances by dividing by \( m_g(t) \) to obtain a mass fraction and then dividing by \( 2 \times 0.75 \) to get

\[ D/H(t) = \begin{cases} 
9.451 \times 10^{-8} m_g(t)^{1.596} + 1.154 \times 10^{-5} & \text{standard BBN} \\
1.310 \times 10^{-7} m_g(t)^{1.596} + 1.604 \times 10^{-5} & \text{inhomogeneous BBN} 
\end{cases} \]

(27)

Figure 3 compares this result with our numerical integration of the equations, and there is little significant difference between them. The analytic results are slightly lower, because our approximation has allowed for some stars to be formed before the Galaxy exists (i.e. \( \psi(t - \tau(m)) \) is assumed non-zero for some negative values of \( t - \tau(m) \)). These "pseudo-stars" then explode early in the galactic lifetime and dilute the ISM with deuterium depleted gas. With this same approach, we have also tested \(^7\text{Li}\) evolution. In addition, using the instantaneous recycling approximation, we have verified our derived age metallicity relations.
Results

The results of our integrations of equation (6) are shown in figures 4–9. In all cases we used a galactic age of 17 Gyears following Mathews and Schramm (1990). This value is chosen because it is the age for the galaxy currently implied by observations of the globular clusters. We have chosen this value because it represents a reasonable upper limit to the age of the galaxy. Variations in the galactic age in the range 12 – 17 Gyears produce only small changes in these results. Shorter ages have the effect of strengthening our major conclusions described below. In all but the MSSFR case, we added infall only if necessary to bring the predicted present day deuterium abundance up to $D/H = 0.8 \times 10^{-5}$ (Boesgaard and Steigman, 1985). In the MSSFR case, as described above, the current star formation rate was adjusted to match the deuterium to the present measured abundance. Our IMF was derived from the observed PDMF of Scalo (1986). The adopted observational numbers we wish to fit are given in table 2 and are shown (along with a selection of representative observational data points) as shaded areas on the results graphs.

**Deuterium**

The deuterium evolution is pictured in figures 4(a-b) for the various star formation rates. Deuterium is only destroyed in stars (Epstein, Lattimer, and Schramm, 1976; Reeves, et. al., 1973) and thus tracks the general astation properties of the models. The two starting values in the two graphs are for the two BBN scenarios. All of the SFR’s which do not decrease with time (CSFR, ISFR, and hiatus models) have some difficulty astating enough $D$ to account for the present $D$ abundance, however, particularly in our standard BBN scenario, they are not more than 2σ above the accepted value, and a glance at figure 3 (which was derived with a different IMF) will show that if we modified our IMF within contested limits we could ameliorate this problem. The high primordial $D$ scenario, as expected, has more difficulty with this limit than the standard one. The DSFR rate does have some problem matching the solar system limit on $D/H(T_e)$. The best fit is obtained with the DSFR2 and MSSFR rates.
BERYLLIUM-9 AND THE OTHER GCR ELEMENTS

$^9\text{Be}$ is, in general, the least easily destroyed of the light elements that have been observed in stars (although we are making the approximation that all of the light elements are completely destroyed). As discussed above, $^9\text{Be}$ is seen to be essentially constant over the last $\sim 12$ G-years. The CSFR model is just within acceptable limits. Both of the decreasing and the MSSFR rates perform very well on this test. The hiatus scenario and the ISFR do poorly here. If our assumption that the GCR flux is proportional to the supernova rate is wrong these conclusions are, of course, suspect.

Because of our method of treating GCR production (see above), the other cosmic ray produced elements ($^7\text{Li}$, $^6\text{Li}$, $^{11}\text{B}$, $^{10}\text{B}$, and $^9\text{Be}$) would have evolutionary curves of identical shape with mass ratios to $^9\text{Be}$ as given in table 3. Since all of the other evolutionary behavior of these elements (i.e. astation processes) is the same, these elements will fit the observed solar system abundances to the degree that the observed and theoretical ratios in table 3 agree. We also note that the required GCR production efficiency is roughly half as much for the overall decreasing models (DSFR, DSFR2, MSSFR) as it is for constant star formation as shown by the GCR efficiencies reported in table 4.

Figure 6 shows our results for $^9\text{Be}$ vs metallicity compared to observed values where all rates perform about equally well. This is an important result inasmuch as it supports the general assumption of our models that GCR production of the light elements should be proportional to early metals production. This, at the very least, does not contradict the idea that the GCR flux is tied to the core collapse supernova rate.

METALLICITY

While our main concern in this paper is with the light elements, we must generate an appropriate age-metallicity relation for a comparison of $^7\text{Li}$ vs $\text{Fe/H}$ for this observed data. Figure 7 shows our evolutionary curves for $\text{Fe/H}$ plotted with two determinations of the local age-metallicity relation. The first is that of Twarog (1980) and the other is a reevaluation of Twarog's data using different
stellar models by Carlberg, et. al. (1985) (a current discussion of these two and other determinations of this important function appears in Wheeler, Sneden, and Truran (1989)). It is interesting that while the ISFR, CSFR, and hiatus rates match the Twarog function reasonably well, the DSFR and MSSFR rates match the Carlberg function almost exactly and the DSFR2 rate falls in between. This could represent a useful test of either star formation rates or the age metallicity relation.

LITHIUM-7

In the graphs of this section, while we made models in which the maximum production rates from supernova envelope shocks were included with small red giant models, their curves did not differ from those of the small red giant production only models and from table 4 it can be seen that they did not noticeably reduce the required amount of red giant $^7\text{Li}$ production. We chose to add small rather than intermediate mass red giants to the shock production since they represent a "slower" production source and should be most likely to show any deviation due the shock production.

When we compare the models to the observations of $^7\text{Li}$ as a function of time, we see that ISFR, CSFR, and hiatus star formation scenarios have a good deal of difficulty producing enough $^7\text{Li}$ quickly enough, although the CSFR model is just barely within the observational constraints set by NGC188 for intermediate to high mass stellar production sources (AGB stars and supernovae). The MSSFR, DSFR and DSFR2 functions seem tailor made for stellar lithium production. These SFR's all have the very desirable characteristic that they produce a quick rise in $^7\text{Li}$ followed by a relatively constant abundance over the last $\sim 8$ Gyears in accordance with observations. With the MSSFR rate, the abundance remains low at first due to the lack of pre-burst star formation and then rises quickly with the burst to provide a nearly Pop I abundance for the remainder of galactic history, thus it could even explain the very restrictive LMC lithium limit quoted by Sahu, Sahu, and Pottasch (1988).

The other set of observational constraints on $^7\text{Li}$ are addressed by plotting
$^7\text{Li}$ abundance versus $\text{Fe/H}$ . In this case we can only compare our stellar source models to the observations since high primordial lithium scenarios rely on the presence of main sequence stellar destruction of lithium to explain the Pop II lithium observations and thus our derived values for the ISM $^7\text{Li}$ abundance are irrelevant for these low metallicity stars. Very old stars (with $\text{Fe/H} \lesssim 0.04$) in those models will have destroyed their $^7\text{Li}$ by a universally uniform function of stellar age, and thus their measured surface $\text{Li}$ abundances will not reflect the ISM $^7\text{Li}$ abundance at their birth (see Mathews, Alcock, and Fuller (1990) for a full description of a generic model of this process). For our stellar production models, we assume that the observed surface abundance in Pop II stars does reflect the ISM value at their birth. Thus we assume for these models that the arguments of D'Antona and Mazzitelli (1984) against main sequence lithium depletion in low metallicity stars and the recent results of Deliyannis, et. al. (1987, 1990) are a correct picture. Figures 9(a-f) show our predicted ISM lithium values plotted versus $[\text{Fe/H}]$. We expect that a correct model will follow the upper edge of the observed lithium abundances since main sequence destruction most definitely does occur in high metallicity stars (owing to the deeper surface convection envelopes caused by the higher opacity of the gas in these stars). All of our models do about equally well in fitting these tests.

For the high $^7\text{Li}$ scenarios, which are admittedly somewhat Draconian in terms of current research as described in the introduction, none of our models is capable of matching our adopted observational limits for $^7\text{Li}$ versus time. However, for our adopted value of the LMC lithium limit, we can not rule out roughly Pop I primordial values of $^7\text{Li}$ if the small dispersion in Pop II lithium abundances can be explained with main sequence stellar destruction models.
Conclusions

We have evolved models in constant IMF, one-zone galactic chemical evolution models and found that direct and indirect observations of the interstellar medium abundances of $^7$Li, $^D$, and $^9$Be as functions of time and metallicity can be well fit by some of these models. The basic conclusion from these observations, that $^9$Be and $^7$Li have had roughly constant abundances over the past $\sim 10$ Gyears is the most stringent constraint, as essentially all of our models can fit the $^7$Li vs Fe/H observations equally well. The fact that our $^9$Be vs Fe/H predictions all fit the observations as well, validates our assumption that galactic cosmic ray production is proportional to the core collapse supernova rate.

We conclude, primarily from our adopted upper limit to the Larger Magellanic Cloud lithium abundance, that models with high primordial lithium (e.g. inhomogeneous BBN models) are very tightly constrained. In confirmation of the results obtained by Kurki-Suonio, et. al. (1989) using $^4$He we conclude that inhomogeneous models of BBN using $\Omega_B = 1$ are ruled out. We find these decreasing lithium models to be constrained to have primordial $^7$Li/H $\lesssim 10^{-9}$ provided they include a mechanism for uniform main sequence depletion of lithium over time for low mass main sequence stars of low metallicity and no significant stellar production of $^7$Li.

We find, particularly in the case of a strictly constant star formation rate over galactic history, that intermediate mass (4$-9M_\odot$) red giant stars are an efficient source of the Pop I lithium abundance. Since these sources have actually been observed consistently by Smith and Lambert (1989) we see this source as the most likely explanation of this problem. In decreasing star formation models, any stellar source which produces enough $^7$Li to explain the observed Pop I abundance can fit the observational time constraints.

Our primary conclusion, described in the results section above, is that our predictions for the evolution of $D$, $^7$Li, and, to a lesser degree, $^9$Be, all point strongly towards galactic star formation rate histories which decrease on average with time. These rates provide the observed quick $\sim 10$ Gyear rise in $^7$Li and $^9$Be abundances followed by essentially constant abundances of these elements.
They also allow for the required amount of astration of deuterium. The averaged required decrease in star formation with time is not so great as to violate the constraints imposed by nuclear chronology. In particular, the multiple coalescence model star formation rate of Mathews and Schramm (1990) fits all of these requirements extremely well. The very low early star formation rate allows for a sizable star formation burst about 10 – 12 Gyears ago to coincide with the birth of the Galactic disk while obeying nuclear chronological constraints. This same burst creates the bulk of the present light element abundances and the low, constant subsequent star formation associated with the spiral density wave in the Galactic disk keeps these abundances constant over most of the life of the disk.

As mentioned above, shorter galactic lifetimes produce similar results. In a qualitative sense, one can essentially just "relabel" the time axes of the graphs. The only major difference is that stellar abundance measurements for which the age (as opposed to only the metallicity) is known (see table 2) cover a larger overall fraction of the total galactic lifetime. In particular, for $^7\text{Li}$ and $^9\text{Be}$, the essentially constant abundance of these elements over the last 8 and 12 Gyears respectively, is a constraint on the last $\sim 50$ or 70 % of the galactic lifetime. In a 12 Gyear old galaxy, these constraints cover 66% and essentially all of the galactic lifetime. This would considerably strengthen our conclusions about the need for an overall decreasing star formation rate, and would discriminate against small red giant stellar production of $^7\text{Li}$ as well.

With the excellent success of the Mathews and Schramm rate based on dynamical arguments in these, admittedly limited, models, we encourage and await the efforts now being made to develop full dynamical galactic chemical evolution simulations.

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for conversations and advice. The author also thanks Pedro Colin for double checking much algebra in the course of his own work. This work was supported by the DOE and NASA (NAGW-1321 and 1340) at the University of Chicago and Fermilab.
REFERENCES


Dearborn, D.S. 1990 personal communication.


Reeves, H. 1990, preprint.


Steigman, G. 1990, personal communication.


TABLE CAPTIONS

1. This table shows the initial primordial mass fractions of $D$ and $^7\text{Li}$ used in the standard and inhomogeneous big bang scenarios (MS). All other elements are assumed to have initial mass fraction zero except hydrogen and helium which are approximated with constant mass fractions of .75 and .25 respectively.

2. This table shows our adopted constraints for the observed abundances of the light elements. The second column gives the age of the site and the third gives the metallicity of the site. If a number is enclosed in parentheses it means that the number in the other of these two columns is the only one directly associated with the observation and that the number in parentheses is merely derived using the age metallicity relation of our constant star formation rate model and should thus not be taken too seriously. The numbers in the bottom half of the table are those adopted by Walker, Mathews, and Viola (1985) for the solar system observed values. They are all uncertain by a factor of $\sim 2$ since there is a large discrepancy between the light element values for the meteorites and for the solar photosphere and it is unclear which more closely approximates the solar system initial value. The fourth column gives the major source (or review) of the observations used in determining these constraints. RMB-Rebolo, Molaro, and Beckman (1988), ST-Steigman (1990), SSP-Sahu, Sahu, and Pottasch (1988), B-Boesgaard (1990), BS-Boesgaard and Steigman (1985), RETAL-Rebolo, et. al. (1988), RM-Reeves and Meyer (1978), WMV-Walker, Mathews, and Viola (1985).

3. These are the production factors for light element production in the GCR taken from Walker, Mathews, and Viola (1985). We are using their $\gamma=7$ model. The $\alpha_i$ are the production ratios by mass with respect to $^9\text{Be}$. We also give for comparison their adopted solar system observed mass ratios for these elements. It should be remembered that these observations are uncertain to factors of $\sim 2$. The mass ratio of $^{11}\text{B}$ to $^{10}\text{B}$ is defined to be 4.45 for these models of GCR production.

4. The model parameters. GCR production efficiency required (constant star
formation \( \equiv 1 \), total time integrated infall mass, initial galactic gas surface density, required lithium production, average stellar nucleosynthesis time over age of the galaxy at solar system formation (for nucleocosmochronology limits), and present star formation rate for the MSSFR models. The required \( ^7\)Li production \( (X_i) \) is the average mass fraction of lithium produced in the ejecta of stars of the given mass range. The ranges of stellar sources are: red giants- \( 1 - 5M_\odot \), shocking- (see text), supernovae- \( 9 - 62M_\odot \), and AGB stars- \( 4 - 9M_\odot \).

FIGURE CAPTIONS

1. Shown are four different IMFs derived from the PDMFs of Scala (1986), Rana (1988), and Miller and Scala (1979) using a constant star formation rate (CSFR). The curve marked analytic is a three straight line (actually power law) fit to the Miller and Scala curve. We have used the Scala IMF except for the analytical tests of the models for which we used the analytic fit of Miller and Scala.

2. These are the six different star formation rates used. The constant, decreasing, and increasing functions (csfr, dsfr, and isfr) are taken from Miller and Scala (1979). The hiatus function (hiasfr) is in agreement with the limits quoted in Wheeler, Sneden, and Truran (1989). The multiple coalescence model, starburst type functions (mssfr) are from Mathews and Schramm (1990). The (mssfr) curve with the taller peak corresponds to the inhomogeneous big bang (i.e. high primordial lithium) models.

3. This compares the analytical formulae of equation (27) with the output of our numerical simulations. The galactic age is assumed to be 15 Gyears, and the analytic IMF used in both is taken from Miller and Scala (1979).

4. (a-b) This shows the deuterium evolution for the two possible starting values (homogeneous and inhomogeneous big bang) and the six SFR's. The data points represent the meteoritic and current interstellar values for \( D/H \) taken from Boesgaard and Steigman (1985).
5. This shows the $^6\text{Be}$ evolution for the six SFR's. The data points represent the stellar observations summarized in Reeves and Meyer (1978) which imply that the $^6\text{Be}$ abundance has changed very little in the past 10 $G$years. $^6\text{Be}$ production has been normalized to match the solar system value at 4.6 $G$years ago. The shaded areas are our adopted general constraints.

6. This shows the ISM abundance of $^6\text{Be}$ plotted versus metallicity for our models. The data points are from the summary of Rebolo et. al. (1988) The shaded areas are our adopted general constraints.

7. This figure plots the iron abundance as a function of time. The data points are taken from the age-metallicity relations of Twarog (1980) and from Carlberg, et. al.'s (1985) revised evaluation of Twarog's data. Our values for $[\text{Fe}/H] = \frac{\text{Fe}}{\text{Fe}_0}$ have been normalized so that $[\text{Fe}/H] = 1.0$ at 4.6 $G$years ago as described in the text.

8. (a-f) These show the evolution of the interstellar $^7\text{Li}$ abundance with time. The four source sites plotted are 'rg'- small red giants ($1 - 5 M_\odot$), 'sn'- supernovae ($9 - 62 M_\odot$), 'rgbig'-intermediate mass AGB stars ($4 - 9 M_\odot$), and 'ibbn' indicates a high primordial $^7\text{Li}$ scenario with no stellar production of $^7\text{Li}$. The five data points with x-error bars represent extrapolated initial lithium abundances for the open clusters discussed in Hobbs and Pilachowski (1988b). The age for NGC188 (the oldest cluster) is from Hobbs, Thorburn, and Rodriguez-Bell (1990). The point at 4.6 $G$years ago is the meteoritic value. The three lines at the left side represent three determinations of the upper limit to $\text{Li}/H$ in the LMC determined along the line of sight to SN1987a; see text for discussion. The shaded areas are our adopted general constraints.

9. (a-f) These figures show, for each of the SFR's, the evolution of the lithium abundance with respect to the metallicity. The three stellar source sites plotted are the same as in figure 8 for increasing $^7\text{Li}$. The log scale of the x-axis "telescopes" the course of the very early Galactic $^7\text{Li}$ evolution. The data points are stellar values of $^7\text{Li}/H$ taken from the references in Rebolo, et. al. (1988) and from Hobbs and Pilachowski (1988a) and Spite, et. al.
The greater scatter with increasing $Fe/H$ is certainly due, in part, to the very different ages of stars with the same metallicity (there is more time per inch as we move to the right along the x-axis of the graph) leading to more or less main sequence destruction. It is also possibly due to a lack of any main sequence destruction of $^7Li$ at lower metallicities (D'Antona and Mazzitelli, 1984; Deliyannis, 1990) (this would imply Pop II $^7Li / H = $ primordial).
**TABLE 1: Primordial Mass Fractions**

<table>
<thead>
<tr>
<th>Isotope</th>
<th>Standard BBN</th>
<th>Inhomogeneous BBN</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>$5.4 \times 10^{-5}$</td>
<td>$7.5 \times 10^{-5}$</td>
</tr>
<tr>
<td>$^7\text{Li}$</td>
<td>$5.25 \times 10^{-10}$</td>
<td>$3.68 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

**TABLE 2: Light Element Abundances**

<table>
<thead>
<tr>
<th>Isotope</th>
<th>Age</th>
<th>$[\text{Fe}/\text{H}]$</th>
<th>$X_i/\text{H}$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^7\text{Li}$</td>
<td>$(\sim T_p)$</td>
<td>$&lt; 0.01$</td>
<td>$1.6(2) \times 10^{-10}$</td>
<td>RMB</td>
</tr>
<tr>
<td></td>
<td>$(12 - 15)$</td>
<td>$0.25 \pm 0.2$</td>
<td>$&lt; 1 \times 10^{-9}$</td>
<td>SSP, ST</td>
</tr>
<tr>
<td></td>
<td>$&lt; 8.1$</td>
<td>$&gt; 0.6$</td>
<td>$1.02 \pm 0.34 \times 10^{-9}$</td>
<td>B, RMB</td>
</tr>
<tr>
<td>$D$</td>
<td>4.6</td>
<td>$\equiv 1$</td>
<td>$1 - 4 \times 10^{-5}$</td>
<td>BS</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>$(1.6)$</td>
<td>$0.8 - 2 \times 10^{-5}$</td>
<td>BS</td>
</tr>
<tr>
<td>$^9\text{Be}$</td>
<td>$(15 - 17)$</td>
<td>$0.05 - 0.1$</td>
<td>$0.4 - 4 \times 10^{-12}$</td>
<td>RETAL</td>
</tr>
<tr>
<td></td>
<td>$0 - 12$</td>
<td>$0.3 - 1.6$</td>
<td>$0.7 - 3 \times 10^{-11}$</td>
<td>RM</td>
</tr>
<tr>
<td>$^{11}\text{B}$</td>
<td>4.6</td>
<td>$\equiv 1$</td>
<td>$1.2(2) \times 10^{-10}$</td>
<td>WMV</td>
</tr>
<tr>
<td>$^{10}\text{B}$</td>
<td>4.6</td>
<td>$\equiv 1$</td>
<td>$3.0(2) \times 10^{-11}$</td>
<td></td>
</tr>
<tr>
<td>$^6\text{Li}$</td>
<td>4.6</td>
<td>$\equiv 1$</td>
<td>$7.0(2) \times 10^{-10}$</td>
<td></td>
</tr>
<tr>
<td>$^7\text{Li}$</td>
<td>4.6</td>
<td>$\equiv 1$</td>
<td>$9.0(2) \times 10^{-10}$</td>
<td></td>
</tr>
<tr>
<td>$^9\text{Be}$</td>
<td>4.6</td>
<td>$\equiv 1$</td>
<td>$1.4(1.6) \times 10^{-10}$</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 3: GCR Production Ratios by Mass**

<table>
<thead>
<tr>
<th>Isotope</th>
<th>$\alpha_i$</th>
<th>Observed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^7\text{Li} / ^9\text{Be}$</td>
<td>9.3</td>
<td>50</td>
</tr>
<tr>
<td>$^6\text{Li} / ^9\text{Be}$</td>
<td>3.7</td>
<td>3.3</td>
</tr>
<tr>
<td>$^{10}\text{B} / ^9\text{Be}$</td>
<td>6.1</td>
<td>2.3</td>
</tr>
<tr>
<td>$^{11}\text{B} / ^9\text{Be}$</td>
<td>27</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>CSFR</td>
<td>DSFR</td>
</tr>
<tr>
<td>---------------</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td></td>
<td>IBBN</td>
<td>BBN</td>
</tr>
<tr>
<td>GCR efficiency</td>
<td>≡ 1</td>
<td>0.37</td>
</tr>
<tr>
<td>Total Infall</td>
<td>0</td>
<td>5.53</td>
</tr>
<tr>
<td>$m_0 (0)$</td>
<td>40.4</td>
<td>50.0</td>
</tr>
<tr>
<td>$X_{i,Li} (\times 10^{-8})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>shock</td>
<td>1.35</td>
</tr>
<tr>
<td></td>
<td>red giant</td>
<td>1.36</td>
</tr>
<tr>
<td></td>
<td>AGB stars</td>
<td>6.15</td>
</tr>
<tr>
<td></td>
<td>supernovae</td>
<td>5.30</td>
</tr>
<tr>
<td>$t_c/T_*$</td>
<td>.500</td>
<td>.363</td>
</tr>
<tr>
<td>$\psi(T_2)$</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
Comparison of Analytic and Numeric Solutions

FIGURE 3

Deuterium Evolution Low

FIGURE 4a
FIGURE 8e

FIGURE 8f