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Parity Breakdown and Induced Fermion Number in the O(3) Nonlinear σ Model

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Abstract

We study the fermion number induced by solitons in the O(3) nonlinear σ model in 2+1 dimensions, in the presence of a parity breaking fermion mass term. The appearance of zero energy modes during the adiabatic evolution of a background of winding number unity is analysed as a function of the relative magnitudes of the explicit, odd parity, fermion mass, m_{odd}, the fermion mass induced by the Yukawa coupling, m_Y, and the inverse soliton width, $1/\rho_s$. We find ρ_c, the maximum value of $\rho = \rho_s m_Y$ for which a fermion zero energy level crossing occurs. For $M_f = m_{odd}/m_Y < 1$ and $\rho > \rho_c(M_f)$ the ground state charge of the soliton is wholly topological. Otherwise, it vanishes.

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Recently, renewed attention has been given to 2+1 dimensional physics and the modelling of systems of statistical mechanics [1]. In particular, in the case of the nonlinear σ model coupled to fermions, this interest has been stimulated by the possibility of providing a field theoretic realization of a system of interacting spin waves and holons, that could describe the behaviour of high T_c superconducting materials. Several aspects of these models remain speculative at present, nevertheless, the nonlinear σ model has interesting physical properties that make it worthy of study in itself. Let us consider the (2+1)-dimensional O(3) non-linear σ model coupled to an isodoublet, ψ, of two component fermions, treating the scalar triplet $\phi_a \rightarrow (\vec{\phi}, \phi_3)$ as a background configuration ($|\phi| = v$),

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a + \bar{\psi} \imath \partial_\mu \gamma^\mu \psi - g_y \bar{\psi} \phi_a \mathcal{T}_a \psi. \tag{1}$$

The above Lagrangian is invariant under discrete parity transformations, under which the fields transform as:

$$\begin{aligned} \psi(t, x_1, x_2) &\rightarrow \mathcal{T}_2 \gamma^1 \psi(t, -x_1, x_2) \\ \phi(t, x_1, x_2) = \phi_a(t, x_1, x_2)\mathcal{T}_a &\rightarrow \mathcal{T}_2 \phi(t, -x_1, x_2)\mathcal{T}_2. \end{aligned} \tag{2}$$

Our conventions for the γ matrices in 2+1 dimensions are: $\gamma^0 = \mathcal{S}_3$ and $\gamma^i = \imath \mathcal{S}^i$, i=1,2, obeying the Clifford algebra $[\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu}$ and $\gamma^\mu \gamma^\nu = g^{\mu\nu} - \imath \epsilon^{\mu\nu\lambda}\gamma_\lambda$. $\mathcal{S}_a$ and $\mathcal{T}_a$, a=i,3, are the Pauli matrices σ_a in the Dirac and weak isospin spaces, respectively. An explicit mass term for the fermions

$$\mathcal{L}_{odd} = -m_{odd} \bar{\psi} \psi \tag{3}$$

is odd under the parity transformation, and its inclusion in the model can lead to interesting consequences.

This nonlinear classical field theory has topologically nontrivial soliton solutions [2] due to the nontriviality of $\pi_2(S_2) = Z$. The expression for the winding number (topological charge) is

$$n(= Q_{top.}) = \frac{1}{8\pi v^3} \epsilon_{ij} \epsilon_{abc} \int d^2 x \phi_a \partial^i \phi_b \partial^j \phi_c. \tag{4}$$

A topological term associated with the nontriviality of the homotopy classes of $\pi_3(S_2) = Z$ can be included in the action, $\Delta S = \theta\ H$. The so-called Hopf term

$$H = \pi \int d^3x \epsilon^{\mu\nu\lambda} j_\mu \partial_\nu \frac{1}{\partial^2} j_\lambda \ , \tag{5}$$

where j_μ is the topological current that leads to eq.(4), breaks P and T invariance. Its appearance may induce nontrivial spin and statistics for the solitons, depending on the parameter θ. In our discussion, we do not introduce a Hopf term in the tree-level action. However, due to the presence of the parity odd mass m_{odd}, a Hopf term can be induced through radiative corrections [3]-[7].

The fermionic charge induced by scalar soliton backgrounds has been extensively studied in even dimensional space times [8] and, more recently, in 2+1 dimensions [3], [7], [9]. The adiabatic method [10] is a powerful technique to evaluate the induced fermion current in powers of derivatives of the background fields[11]. The fermion number induced by a soliton background configuration obtained in the adiabatic limit of the gradient expansion, $Q_{ind.}$, can be used to calculate the fractional part of the ground state charge in all generality. However, to accurately compute the ground state charge of the system, Q_{GS}, the number of zero energy level crossings $n_+(n_-)$ in the positive (negative) direction of the energy axis that occur during the adiabatic evolution must be evaluated [11]-[16],

$$Q_{GS} = Q_{ind.} - (n_+ - n_-) \ . \tag{6}$$

In this work, we study the ground state charge of the system in the background of a topologically nontrivial scalar configuration in the O(3) nonlinear σ model in 2+1 dimensions, in the presence of parity breakdown. A similar analysis has been done recently, for the parity even 2+1 dimensional $O(3)$ nonlinear σ model ($m_{odd} = 0$)[17]. It has been shown that a soliton carries the fermion number of any fermion which is sufficiently heavy compared with the typical mass scale of the topological configuration. The odd parity 2+1 dimensional model has a unique feature, namely an extra dimensionful parameter - the explicit fermion mass responsible for parity breakdown- which makes the analysis more complicated. Our goal in this work will be to obtain constraints on the values of the soliton width, the explicit parity odd fermion mass, and the Yukawa coupling induced fermion mass, for which the ground state charge of the system is wholly determined by the topological charge of the soliton.

Following [16], we choose an interpolating configuration that builds up the soliton adiabatically from the trivial vaccum

$$
\begin{aligned}
\phi_3 &= v\cos[h(t)(\pi - \arccos f_1(r))] \ , \\
\vec{\phi}(r,t) = (\phi_1, \phi_2) &= v\sin[h(t)\arcsin f_2(r)]\hat{r} \qquad\qquad r \le \rho_s \ , \\
&= -v\sin[h(t)(\pi + \arcsin f_2(r))]\hat{r} \qquad r \ge \rho_s \ ,
\end{aligned}
\tag{7}
$$

where $\hat{r} = \vec{r}/r$, and ρ_s is the soliton width. $f_1(r)$ goes monotonically from -1 at $r = 0$ to 1 at $r \to \infty$, $f_2(r) \le 0$ vanishes at $r \to 0, \infty$, but is otherwise negative, $h(t)$ being a function which varies slowly and monotonically from 0 at $t = -\infty$ to 1 at $t = \infty$ ($h(t) + h(-t) = 1$) and $\arcsin f_2$ and $\arccos f_1$ taking values in the intervals $[-\pi/2, \pi/2]$ and $[0, \pi]$, respectively. The configuration (7) gives a soliton of winding number unity at $h(t) = 1$, $\phi(h(t) = 1) = v \ (\ f_2(r)\hat{r},\ -f_1(r)\ )$ and the boundary condition at spatial infinity is $\phi(h(t) = 1, r \to \infty) = (0, 0, -1)$. The expression for the adiabatic current has been calculated in ref. [3] using the gradient expansion,

$$
< j^\mu(x) >= \frac{g_y}{|g_y|}\theta(|m_Y| - |m_{odd}|)\frac{1}{8\pi|\phi|^3}\epsilon_{\mu\nu\lambda}\epsilon_{abc}\phi_a\partial^\nu\phi_b\partial^\lambda\phi_c \ , \tag{8}
$$

This expression is reminiscent of the analogous results in even dimensions and in the parity invariant 2+1 dimensional case, however, it has a peculiar discontinuity at the critical value of $M_f \equiv m_{odd}/m_Y = \pm 1$, where the θ function changes value from 0 to 1. The adiabatic method breaks down at $M_f = \pm 1$ where, due to the presence of a massless mode, the condition of slowly varying background configurations over the Compton wavelength of the fermion can no longer be satisfied. In the present work we will not discuss this particular limiting case. In the adiabatic limit of the gradient expansion, and for positive g_y, the fermion number induced by a soliton background is found to be identical to the topological charge of the soliton whenever $|M_f| < 1$ and to vanish whenever $|M_f| > 1$. In the following we will assume a positive value of g_y. In order to evaluate the ground state charge of the soliton we must analyse the existence of zero energy level crossings during the adiabatical evolution of the scalar configuration. In the presence of the scalar background, the Dirac equation for the eigenstates of energy E is,

$$E\psi = -\mathcal{S}_i\partial_i\left(\mathcal{S}_3\psi\right) + \varphi_a\mathcal{T}_a\left(\mathcal{S}_3\psi\right) + M_f\left(\mathcal{S}_3\psi\right) \tag{9}$$

Here, we have defined new variables in terms of the fermion mass scale induced through the Yukawa coupling to the scalars, $m_Y = vg_y$, which are $E \to E/m_Y$, $m_{odd} \to m_{odd}/m_Y \equiv M_f$, $\vec{x} = \vec{r}m_Y$ and $\rho = \rho_s m_Y$. We have also rescaled the fields, $\varphi_3 = \phi_3/v$, and $\varphi_i = \phi_i/v$.

Since $\varphi_3 = \varphi_3(r)$ and $\vec{\varphi} = \varphi(r)\hat{r}$, the "grand" momentum operator, defined by $M_3 = j_3 + I_3$, where $j_3 = -\imath\partial_\theta + \frac{\mathcal{S}_3}{2}$ is the ordinary angular momentum and $I_3 = \frac{\mathcal{T}_3}{2}$ is the isospin, commutes with the Hamiltonian, Eq. (9). Therefore, as in ref. [17], we make an ansatz for the fermion that is a simultaneous eigenstate of energy and "grand" momentum,

$$\psi_{(m)} = \exp(\imath\theta m)\begin{pmatrix} \begin{bmatrix} g_1(x)\exp(-\imath\theta) \\ g_2(x) \end{bmatrix} \\ \\ \begin{bmatrix} g_3(x) \\ g_4(x)\exp(\ \imath\theta\ ) \end{bmatrix} \end{pmatrix} \tag{10}$$

where $M_3\psi_{(m)} = m\psi_{(m)}$, $\hat{x} = (cos\theta, sin\theta)$ and $x = |\vec{x}|$. In terms of the radial functions g_1, g_2, g_3 and g_4 eq. (9) reduces to a set of first-order, coupled, differential equations:

$$\begin{aligned}
\partial_x g_2 &= \frac{m}{x}g_2 - \varphi g_3 + \left(E - M_f - \varphi_3\right)g_1 \\
\partial_x g_3 &= -\frac{m}{x}g_3 - \varphi g_2 - \left(E + M_f - \varphi_3\right)g_4 \\
\partial_x g_1 &= -\frac{(1+m)}{x}g_1 - \varphi g_4 - \left(E + M_f + \varphi_3\right)g_2 \\
\partial_x g_4 &= -\frac{(1-m)}{x}g_4 - \varphi g_1 + \left(E - M_f + \varphi_3\right)g_3
\end{aligned} \tag{11}$$

It is possible to argue quite generally that there is a critical value of the soliton width above which zero energy modes cannot exist. The argument is similar in spirit to that given for the O(4) model in four dimensions in ref. [15] and for the $m_{odd} = 0$ case in ref.[17]. A normalizable zero energy mode is a solution to the equation

$$\imath\gamma^i\partial_i\psi = g_y\phi_a\mathcal{T}^a\psi + m_{odd}\psi. \tag{12}$$

In order to show the absence of zero energy modes for sufficiently wide solitons, it is convenient to define the field $U(x)$, which belongs to the fundamental representation of $SU(2)$, and such that $\phi_a\mathcal{T}^a = vU\mathcal{T}^3U^\dagger$. Once the field U is rotated away, eq.(12) reads

$$\imath\gamma^i\left(\partial_i - iA_i\right)\chi - (m_Y\mathcal{T}^3 + m_{odd})\chi = 0 \tag{13}$$

where $A_i = \imath U^\dagger\partial_i U$ and $\chi = U^\dagger\psi$. The characteristic energy scale of the gauge field A_i is given by $1/\rho_s$. Hence, the perturbations of the free Dirac spectrum will be characterized by $1/\rho_s$, and no solution to eq.(13) will exist whenever $|\pm m_Y + m_{odd}| >> 1/\rho_s$.

The exact value of the soliton width is not relevant as long as it is sufficiently smaller than the critical width. Thus, we will analyse, using analytic arguments, the existence of zero energy modes during the evolution of an infinitely narrow soliton of winding number unity. In the limit $\rho_s = 0$ the expression for the scalar field eq. (7) reads $\varphi_3(t) = cos[h(t)\pi]$, $\varphi(t) = -sin[h(t)\pi]$. Therefore, it obeys the relation $\varphi_3(t) = -\varphi_3(-t)$ and $\varphi(t) = \varphi(-t)$. Using this relation, and redefining $E \to -E$ and $m \to -m$ in eqs.(11), we observe that for each solution $\psi^T_m = \exp(\imath\theta m)\left([g_1\exp(-\imath\theta),\ g_2],\ [g_3,\ g_4\exp(\imath\theta)]\right)$ of energy E in the orbital m at time t, there is a solution $\psi^T_{-m} = \exp(-\imath\theta m)\left([g_4\exp(-\imath\theta), g_3], [g_2, g_1\exp(\imath\theta)]\right)$ of energy $-E$ in the orbital $-m$ at time $-t$. Therefore, at the time $t = 0$ and in the scalar background with $\rho_s = 0$, there is a symmetry in the Hamiltonian $H_{t=0}$, which gives a one to one correspondence between states of positive and negative energy. Hence, the spectral asymmetry, defined as the ς-function regularization of the difference between the number of positive and negative eigenvalues of the Dirac Hamiltonian, vanishes, $\eta_{[H_{t=0}]} = 0$.

The $h(t) = 1$ infinitely narrow soliton background is equivalent to the vacuum configuration and, consequently, its ground state fermion number vanishes. The induced fermion number, as given by the adiabatic method, is equal to the soliton winding number, whenever the explicit parity odd mass is smaller than the fermion mass induced through the Yukawa coupling. Therefore, whenever $M_f < 1$, one zero energy level crossing must occur during the adiabatical evolution in order to get consistency between the induced and the ground state fermion numbers. From the symmetry considerations above it is clear that if a zero energy level crossing occurs at any time t in the orbital m, another zero energy level crossing occurs at time $-t$ in the orbital $-m$. Thus, a single zero energy level crossing can only occur if a zero mode is present in the $m=0$ orbital, and at the time $t=0$, for which a symmetry exists in the Hamiltonian. Eq.(7) tells us that $\varphi_3(r,t=0)|_{\rho_s=0}=0$ and $\varphi(r,t=0)|_{\rho_s=0}=-1$.

Consider the equations (11) in the background of the interpolating scalar field from the vacuum to the step soliton final state, at the time $t=0$, and with $\rho_s=0$, $m=0$ and $E=0$. We observe that upon identifying $g_1^{(-)}=g_1=-g_4$, and $g_2^{(-)}=g_2=-g_3$, the set of four coupled equations reduces to the equations

$$\begin{aligned} \partial_x^2 g_2^{(-)} &= -\partial_x g_2^{(-)}\left(2+\frac{1}{x}\right)-g_2^{(-)}\left(\frac{1}{x}-M_f^2+1\right) \\ M_f g_1^{(-)} &= -g_2^{(-)}-\partial_x g_2^{(-)} \end{aligned} \tag{14}$$

Another possibility, is to define $g_1^{(+)}=g_1=g_4$ and $g_2^{(+)}=g_2=g_3$. In this case the equations reduce to

$$\begin{aligned} \partial_x^2 g_2^{(+)} &= \partial_x g_2^{(+)}\left(2-\frac{1}{x}\right)+g_2^{(+)}\left(\frac{1}{x}+M_f^2-1\right)\ , \\ M_f g_1^{(+)} &= -\partial_x g_2^{(+)}+g_2^{(+)} \end{aligned} \tag{15}$$

The solutions to these equations are given by [18]

$$\begin{aligned} g_1^{(\pm)}(x) &= \exp(\pm x)\left(A_1^{(\pm)}K_1(M_f x)+A_2^{(\pm)}I_1(M_f x)\right) \\ g_2^{(\pm)}(x) &= \exp(\pm x)\left(A_1^{(\pm)}K_0(M_f x)-A_2^{(\pm)}I_0(M_f x)\right) \end{aligned} \tag{16}$$

where $I_j(M_f x)$ and $K_j(M_f x)$ with $j=0,1$ are the modified Bessel functions, and the $A_i^{(\pm)}$ with $i=1,2$ are normalization constants. A normalizable solution must satisfy the condition $|g_i| < 1/x$, in both the limits $x \to 0$, and $x \to \infty$. Consequently, no normalizable solutions exists for $g_1^{(+)}$ and $g_2^{(+)}$. For $g_1^{(-)}$ and $g_2^{(-)}$ a well behaved solution exists only if the condition $M_f < 1$ holds. For $M_f < 1$, the normalizable solution reads

$$\begin{aligned} g_1(x) &= -g_4(x) = A_2^{(-)} \exp(-x) I_1(M_f x) \\ g_2(x) &= -g_3(x) = -A_2^{(-)} \exp(-x) I_0(M_f x). \end{aligned} \tag{17}$$

Therefore a zero energy level crossing occurs during the adiabatical evolution of an infinitely narrow soliton whenever the parity odd fermion mass is smaller than the fermion mass induced through the Yukawa coupling. This result is consistent with our predictions based on the adiabatic result for the induced charge and the known value for the ground state charge for the final soliton of width $\rho_s = 0$.

Let us recall that the spectral asymmetry is related to the ground state charge by the usual expression, $Q_{GS} = -\eta_{[H]}/2$. However, in the presence of zero energy modes of a given Hamiltonian H_t the above expression is modified after taking into account the charge degeneracy and reads[13]

$$Q_{GS} = -\frac{\eta_{[H_t]}}{2} - \frac{1}{2}\left(N^{emp.}_{t_{E=0}} - N^{occ.}_{t_{E=0}}\right) , \tag{18}$$

with $N^{emp.\,(occ.)}_{t_{E=0}}$ being the number of zero energy modes at time t that are empty or occupied, respectively. Therefore, due to the existence of the zero energy mode in the scalar background at time $t = 0$ and for $\rho_s = 0$, the ground state charge of this scalar field is $Q_{GS} = \pm 1/2$. The adiabatic method predicts that at $t = 0$ one has $Q_{ind.} = 1/2$. The value $Q_{GS} = \pm 1/2$ can be obtained by considering the zero energy mode to be occupied (empty), in which case it must (not) be counted as part of the system's ground state and the induced charge is equal to (differs by one unit from) the ground state charge.

We will now evaluate the ground state charge of the soliton configuration of width $\rho_s = \rho/m_Y$ and topological charge unity. We already know that its induced charge is determined by its topology for any $M_f < 1$, and that it vanishes for $M_f > 1$. Thus, to obtain the exact value of its ground state charge we have to compute the number of zero energy level crossings in the adiabatical evolution to the soliton background. We will define the critical width ρ_c as a lower bound of ρ, above which no zero energy level crossings occur. If a zero energy level crossing occurs during the adiabatical evolution, the time t at which it takes place increases monotonically with ρ[17]. Hence, the critical value of the soliton width given in units of $1/m_Y$, $\rho = \rho_c$, is identified with the value of ρ at which a zero energy mode appears in the final soliton background. For $\rho > \rho_c$ the ground state charge of the soliton is completely determined by its topology [17].

In our model, we expect values of ρ_c to appear as a function of the parity odd fermion mass, also given in units of the inverse of the Yukawa type fermionic mass. Based on the results obtained for the infinitely narrow soliton, we will restrict our present analysis to the $m = 0$ orbital. The scalar configuration under consideration should be sufficiently simple to allow the analytic evaluation and sufficiently general so that the conclusions we arrive at are representative of those to be obtained in the background of any scalar soliton of winding number unity. For simplicity, then, we propose step-like approximations to a smooth scalar field configuration of given width as follows (see Figures 1a and 1b):

$$\varphi_3 = \begin{cases} 1 & x < x_1 \\ 0 & x_1 < x < x_2 \\ -1 & x > x_2 \end{cases} \qquad \varphi = \begin{cases} 0 & x < x_1 \\ -1 & x_1 < x < x_2 \\ 0 & x > x_2 \end{cases} , \tag{19}$$

where we define $x_1 = (1-\alpha)\rho$, $x_2 = (1+\alpha)\rho$ and $\rho = (x_1+x_2)/2$. The appropriate value of the parameter α will be determined below. In the following, we first solve the differential equations, eq.(11), for a zero energy eigenvalue separately in the three regions of space defined in eq. (19), and then match the solutions at the boundaries.

For $E = 0$ and $m = 0$, the solutions to eq. (11) in the intermediate region, $x_1 < x < x_2$, ($\varphi_3 = 0$, $\varphi = -1$), are identical to the solutions to eqns. (14) and (15). Since no normalizability restriction has to be considered within this region, the general solutions for the radial functions are a linear combination of those given in eq.(16),

$$
\begin{aligned}
g_1(x) &= e^{(-x)}\left(A_1^{(-)}K_1(M_f x) + A_2^{(-)}I_1(M_f x)\right) + e^{(x)}\left(A_1^{(+)}K_1(M_f x) + A_2^{(+)}I_1(M_f x)\right) \\
g_2(x) &= e^{(-x)}\left(A_1^{(-)}K_0(M_f x) - A_2^{(-)}I_0(M_f x)\right) + e^{(x)}\left(A_1^{(+)}K_0(M_f x) - A_2^{(+)}I_0(M_f x)\right) \\
g_3(x) &= -e^{(-x)}\left(A_1^{(-)}K_0(M_f x) - A_2^{(-)}I_0(M_f x)\right) + e^{(x)}\left(A_1^{(+)}K_0(M_f x) - A_2^{(+)}I_0(M_f x)\right) \\
g_4(x) &= -e^{(-x)}\left(A_1^{(-)}K_1(M_f x) + A_2^{(-)}I_1(M_f x)\right) + e^{(x)}\left(A_1^{(+)}K_1(M_f x) + A_2^{(+)}I_1(M_f x)\right) \quad (20)
\end{aligned}
$$

For the spatial intervals $x < x_1$ ($\varphi = 0, \varphi_3 = 1$) and $x > x_2$ ($\varphi = 0, \varphi_3 = -1$) the equations reduce to

$$
\begin{aligned}
\partial_x^2 g_2 &= -\frac{1}{x}\partial_x g_2 + (\varphi_3 + M_f)^2 g_2 \,, \quad g_1 = -\frac{1}{(\varphi_3 + M_f)}\partial_x g_2 \\
\partial_x^2 g_3 &= -\frac{1}{x}\partial_x g_3 + (\varphi_3 - M_f)^2 g_3 \,, \quad g_4 = \frac{1}{(\varphi_3 - M_f)}\partial_x g_3 \qquad (21)
\end{aligned}
$$

The only normalizable solution to the above set of equations in the region $x < x_1$ ($\varphi_3 = 1$) reads,

$$
\begin{aligned}
g_2 &= C_1 I_0(|1 + M_f|x) \,, \quad g_1 = -C_1 I_1(|1 + M_f|x) \\
g_3 &= C_2 I_0(|1 - M_f|x) \,, \quad g_4 = \pm C_2 I_1(|1 - M_f|x) \;, \qquad (22)
\end{aligned}
$$

where the plus sign in g_4 corresponds to the case $M_f < 1$, while the minus sign stands for the case $M_f > 1$. Analogously, in the interval $x > x_2$ ($\varphi_3 = -1$) the only normalizable solution is given by

$$
\begin{aligned}
g_2 &= D_1 K_0(|1 - M_f|x) \,, \quad g_1 = \mp D_1 K_1(|1 - M_f|x) \\
g_3 &= D_2 K_0(|1 + M_f|x) \,, \quad g_4 = D_2 K_1(|1 + M_f|x) \;, \qquad (23)
\end{aligned}
$$

where the minus sign corresponds to the case $M_f < 1$, while the plus sign holds for the opposite relation. The constants $A_i^{(\pm)}$, C_i, and D_i, i=1,2, are to be determined by the normalization and continuity conditions. The continuity conditions at the boundaries, $x = x_1$, and $x = x_2$ yield eight linear homogenous equations for the above constants. A non-trivial solution requires the vanishing of the determinant of the coefficients of these equations. Using *Mathematica*^{TM}, we have evaluated this determinant numerically. For any given value of $M_f < 1$, we obtain a single critical value of $\rho = \rho_c$ for which the determinant vanishes. For $M_f > 1$, instead, no solution compatible with the vanishing of the determinant has been found.

In order to determine the appropriate step like approximation to a smooth scalar soliton, we analyse the dependence of ρ_c on the value of α, for different values of M_f. We find that ρ_c is almost independent of α for a wide range around the minimum value $\rho_c(\alpha_{min.})$. In Figure 2 we show the above dependence for $M_f = 0$ and $M_f = 0.9$. A reasonable value of α is one close to $\alpha_{min.}$ and we choose $\alpha = 0.5$ to be a suitable value of α for any value of $M_f < 1$.

Our results appear in Figure 3, where ρ_c is plotted as a function of M_f, for $\alpha = 0.5$. Notice that the value of ρ_c is almost stationary for small values of M_f, while it grows very rapidly when $M_f \to 1$. Moreover, the value of ρ_c tends to infinity as $M_f \to 1$. This gives evidence of the presence of a massless mode in the $M_f = 1$ free fermion spectrum.

In conclusion, the ground state charge of the soliton is wholly determined by its topological charge whenever the parity odd fermion mass m_{odd} is smaller than the fermion mass induced by the Yukawa coupling m_Y and the soliton width $\rho_s > \rho_c(M_f)/m_Y$. Otherwise, it vanishes.

References

[1] *See for example*: Y.-H. Chen. F. Halperin, F. Wilczek and E. Witten, Int. J. Mod. Phys. B3 1001 (1989) and Refs. therein. S. Randjbar-Daemi, Abdus Salam and J. Strathdee, Trieste preprint IC/89/283. E. Fradkin, University of Illinois preprint P/89/12/173. J. Lykken, J. Sonnenschein and N. Weiss, Fermilab-Pub-89/231-T (1989).

[2] A. A. Belavin and A. M. Polyakov, Pis'ma Zh. Eksp. Teor. 22 (1975) 503 [JETP Lett. 22 (1975) 245].

[3] Y.-H. Chen and F. Wilczek, Int. Jour. Mod. Phys. B3, 117 (1989).

[4] F. Wilczek and A. Zee, Phys. Rev. Lett. 51 (1983) 2250.

[5] I. Dzyaloshinski, A. Polyakov and P.B. Wiegmann, Phys. Lett. 127A, 112 (1988); P. B. Wiegmann, Phys. Rev. Lett. 60, 821 (1988).

[6] E. Fradkin and M. Stone, Phys. Rev. B38 7215 (1988).

[7] Z. Hlousek, D. Senechal and S.-H. H. Tye, Phys. Rev. D41 (1990) 3773.

[8] R. Jackiw and C. Rebbi, Phys. Rev. D13, (1976) 3398. R. Jackiw, Rev. Mod. Phys. 49 (1977) 681. R. Jackiw and J. R. Schrieffer, Nucl. Phys. B190 (1980) 253. W. P. Su, J. R. Schrieffer and A. J. Heeger, Phys. Rev. Lett. 42 (1979) 1698.

[9] T. Jaroszewicz, Phys. Lett. 146B (1984) 337.

[10] J. Goldstone and F. Wilczek, Phys. Rev. Lett. 47 (1981) 986.

[11] R. MacKenzie and F. Wilczek, Phys. Rev. D30 (1984) 2260. Phys. Rev. D30 (1984) 2194.

[12] A. J. Niemi and G. W. Semenoff, Nucl. Phys. B269 (1986) 131. A. J. Niemi and G. W. Semenoff, Phys. Rep. 135 (1986) 99. A. J. Niemi, Nucl. Phys. B235 (1985) 14. S. Forte, Nucl. Phys. B 301, 69 (1988).

[13] M. Carena, DESY 89-084, July 1989.

[14] S. Kahana and G. Ripka, Nucl. Phys. A429 (1984) 462.

[15] E. D'Hoker and E. Farhi, Nucl. Phys. B241 (1984) 109.

[16] M. Carena, Phys. Lett. B217 (1989) 135; Phys. Lett. B211 (1988) 117.

[17] M. Carena, S. Chaudhuri and C.E.M. Wagner, Fermilab-Pub-90/90-T (May 1990), to appear in Phys. Rev. D.

[18] M. Abramowitz and I. Stegun, eds., *Handbook of Mathematical Functions*, Dover Publications, New York, 1965.

FIGURE CAPTIONS

Figure 1: Scalar configuration as a function of x (solid line), and its step like approximation as given in Eq.(19) (dashed line), for a) $\varphi_3(x)$, b) $\varphi(x)$.

Figure 2: Critical value of the soliton width, ρ_c, normalized to its minimum value $\rho_c(\alpha_{min.})$, as a function of the parameter α, for $M_f = 0$ (solid line) and $M_f = 0.9$ (dashed line).

Figure 3: Critical value of the soliton width, ρ_c, as a function of M_f, for $\alpha = 0.5$.

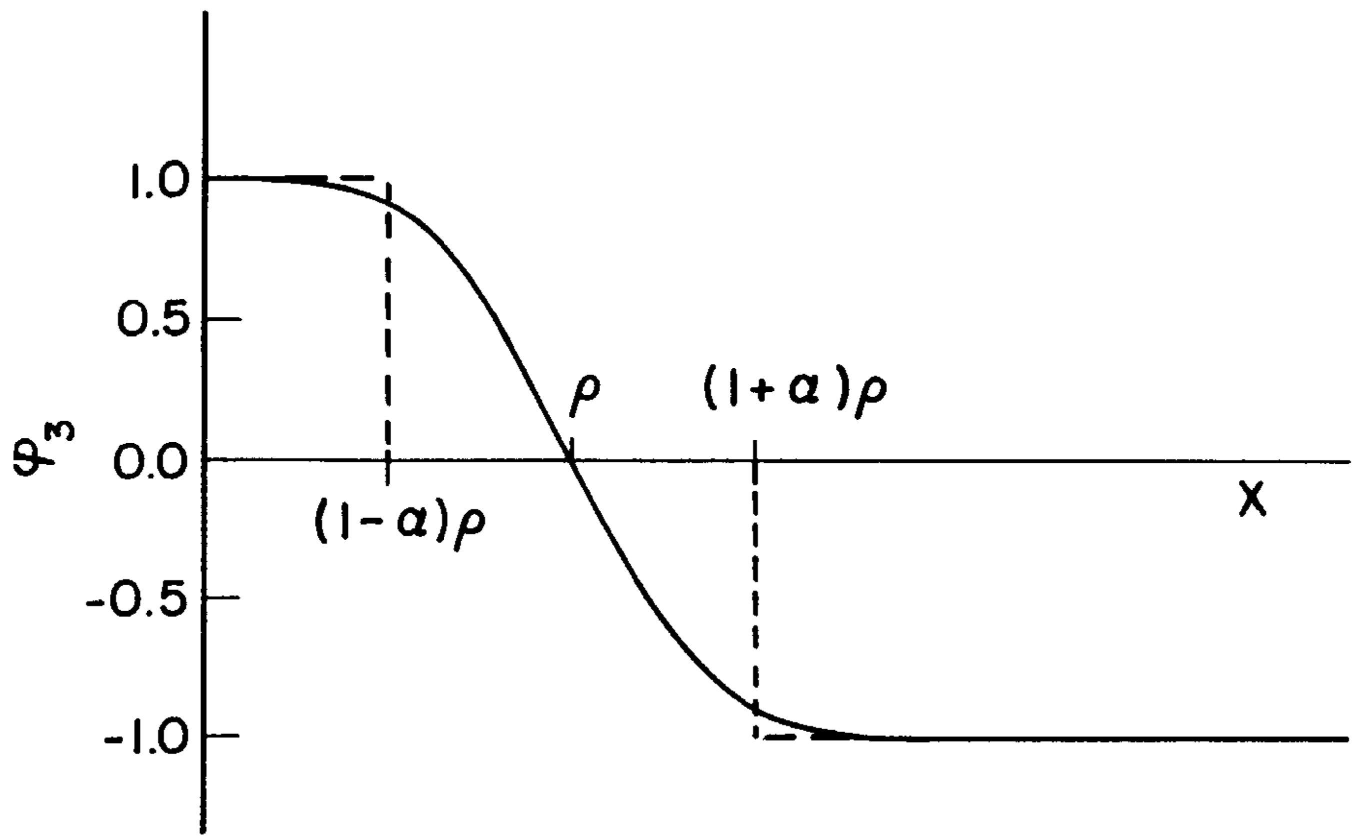

Fig.1a

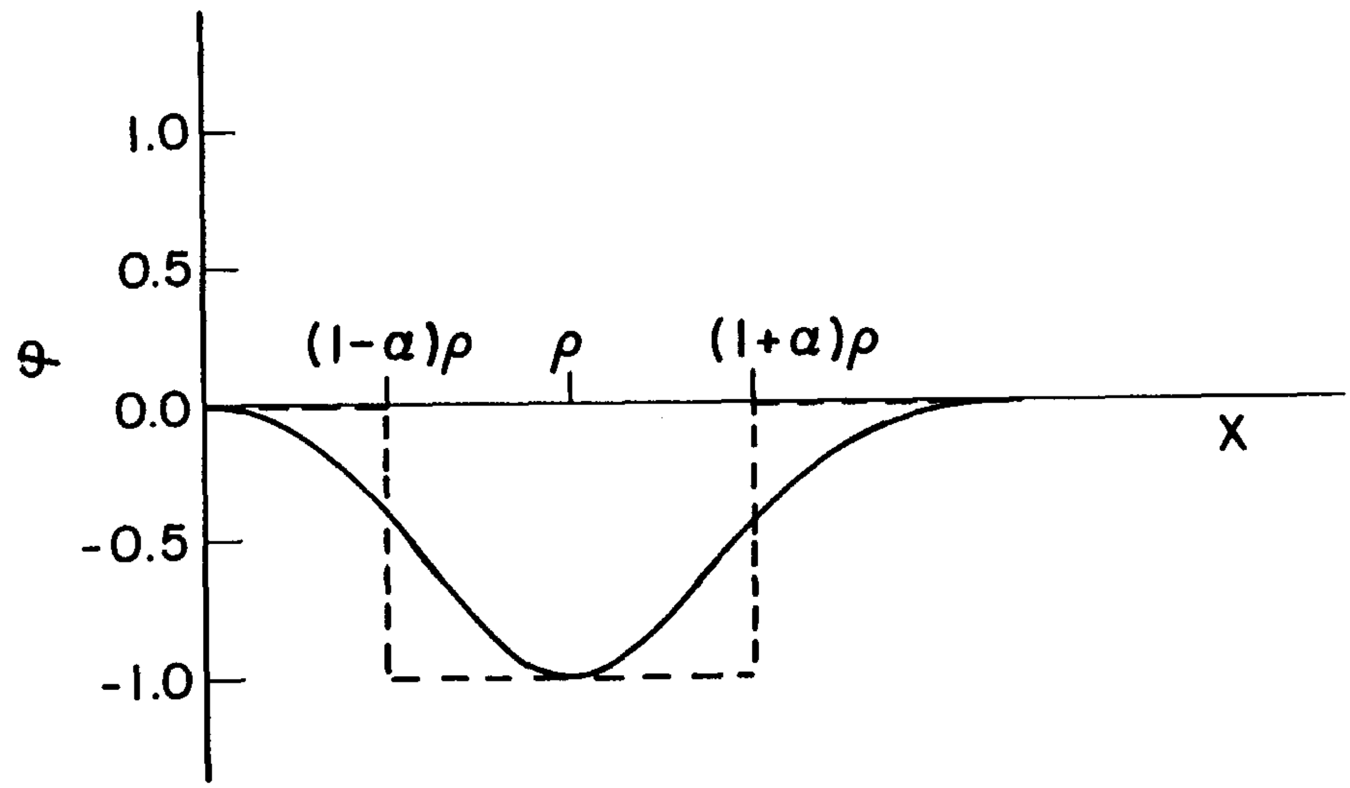

Fig.1b

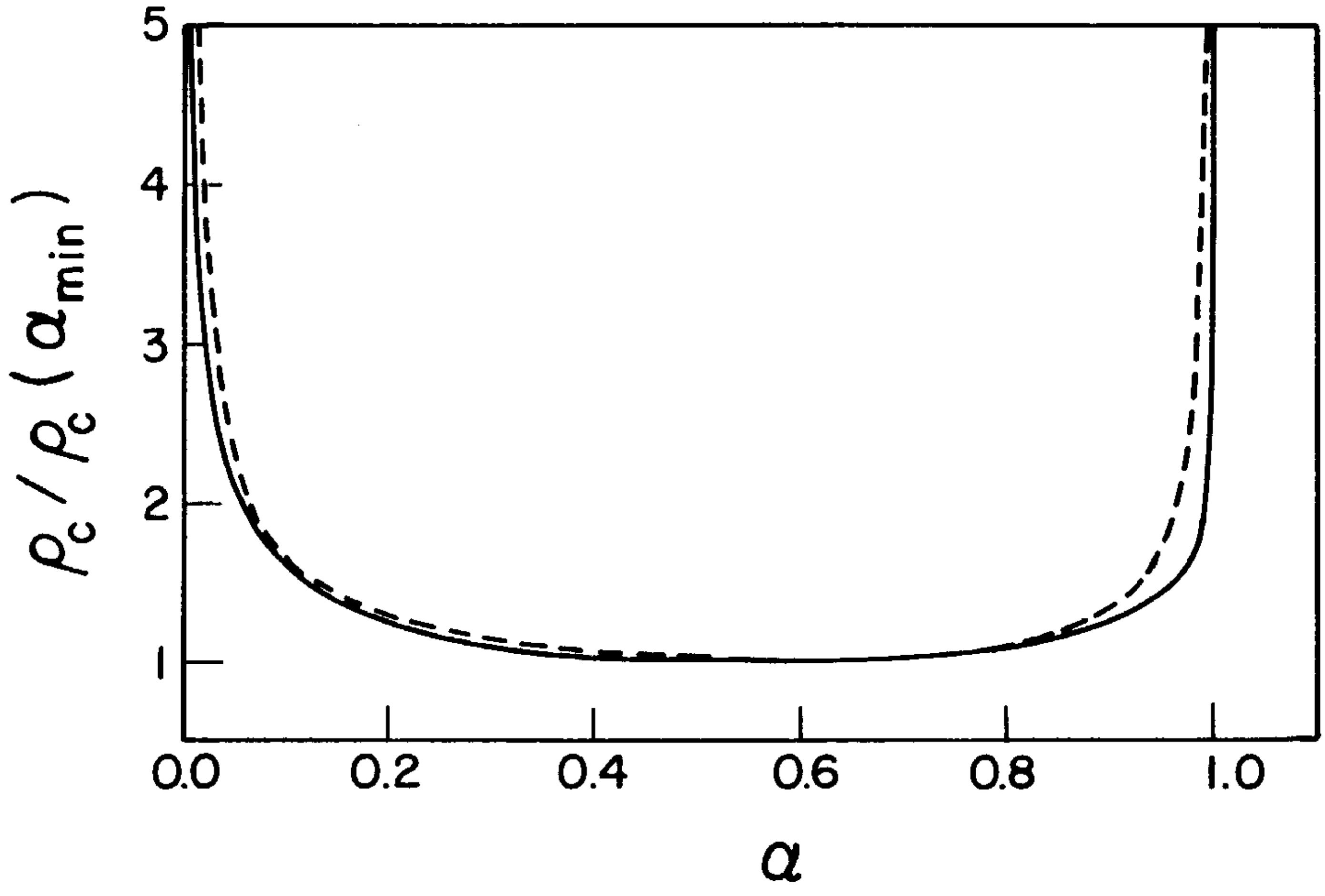

Fig.2

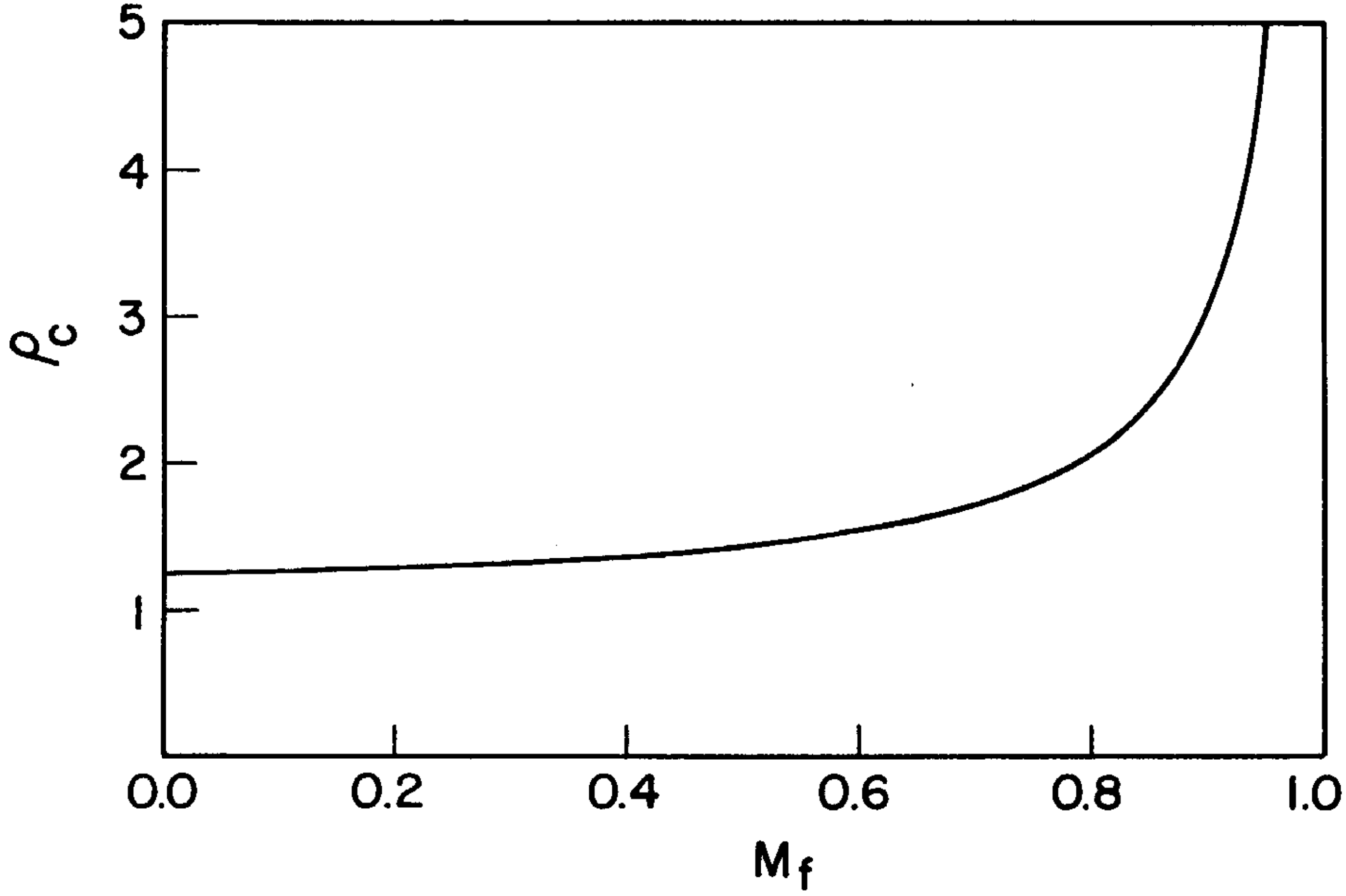

Fig.3