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THE STRUCTURE OF RANDOM DISCRETE SPACETIME

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ABSTRACT

The usual picture of spacetime consists of a continuous manifold, together with a metric of Lorentzian signature which imposes a causal structure on the spacetime. We consider a model, first suggested by Bombelli et al., in which spacetime consists of a discrete set of points taken at random from a manifold, with only the causal structure on this set remaining. This structure constitutes a partially ordered set (or poset). Working from the poset alone, we show how to construct a metric on the space which closely approximates the metric on the original spacetime manifold, how to define the effective dimension of the spacetime, and how such quantities may depend on the scale of measurement. We discuss possible desirable features of the model.

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Spacetime is conventionally regarded as a pseudo-Riemannian manifold which provides an arena for the interaction of fundamental particles and fields. Via general relativity, a good low energy theory of gravity, we also have a picture of spacetime as a dynamical object, distorting according to its energy content, and thus interacting with the matter fields it contains. Unfortunately, such a picture has proved problematic to the incorporation of quantum theory. Many problems arise from attempting to probe behaviour at very small scales, scales at which it is generally believed that 'established' physics does not hold. One obvious means of circumventing such difficulties is to assume that there does indeed exist a physical cutoff, by making spacetime discrete. There were some early attempts at discretisation by Das[1], who replaced the continuous manifold by a regular lattice of points in spacetime. However, this approach has the major disadvantage that the resulting models were not Lorentz invariant, and therefore are not suitable for incorporating gravity, which has local Lorentz invariance as a symmetry. A more fruitful area of study has been that of Regge Calculus (see Regge[2]), but the aim of this is to approximate the manifold by a discrete tesselation in order that properties of the manifold can be more readily calculated. In other words, although discrete, the Regge tesselation is still special in that it carries over certain structure (such as dimension, measure etc.) from its parent manifold. We, on the other hand, are interested in calculating the actual properties of the discrete structure, and relating these to the analogous continuous ones.

In this letter, we examine a class of discrete spacetimes recently proposed by Bombelli et al[3]: causal sets. We however adopt a complementary approach; we consider the causal set as the fundamental object, and examine what physical properties one can derive directly from it. As we will illustrate, the lack of continuity means that we need to take great care in choosing quantities that really do measure something of physical interest. Also, not surprisingly, the discreteness introduces phenomena akin to the 'uncertainties' of quantum physics. We begin by reviewing causal sets, before setting up our definitions of structure on the set. We show how to define timelike intervals, geodesics and dimension of the set, as well as discussing measurement of spacelike distance and velocities. We will also report on recent mathematical results paralleling this work. We conclude with some remarks on the future of this line of study.

The fundamental feature of a spacetime manifold is the notion of time, or timelike intervals; time is a preferred direction in the manifold. It is the causality properties of the manifold which determine what sort of physical spacetime it is. The causal structure of a manifold determines the metric structure up to a local conformal factor[4], so that given a causal structure, we have a good idea what manifold we are dealing with. Causality is an example of what is known mathematically as a partial order. A *partial order* on a set X

is a relation $<$ which satisfies *transitivity*, i.e.,

$$x < y \ , \ y < z \quad \Rightarrow \quad x < z$$

and such that $x < x$ is forbidden. The set X, together with the partial order $<$, is known as a *partially ordered set*, or *poset*. Thus a spacetime, with the partial ordering defined by causality, is an example of a poset. Partially ordered sets are studied in their own right by mathematicians, although a general poset has far less structure than a spacetime manifold. The motivation for studying the behaviour of spacetime as a poset is to examine the effects of discreteness in a potentially calculable manner.

We must first discuss what we mean by a discretisation of spacetime. We have already commented on the problem arising from taking a regular lattice as a model for a discrete spacetime: such a lattice loses isotropy. One way of avoiding the phenomenon of a preferred direction is to take as a model a collection of points distributed at random in the manifold: so that in each finite region there are a finite number of points, and the average number of points in a region is proportional to the volume of that region. The causal relation imposes a partial order on this discrete set of points, so we have a random partial order as the basis of a model of spacetime. Such a model has been considered by Bombelli et al.[3], who named it a causal set. In any random model of this type, it is to be expected that small scale phenomena will depend on local (random) effects, while large-scale phenomena will depend only on the "average case" behaviour, which is essentially the behaviour of the manifold we had to start with. Such attributes would be in keeping with a picture of spacetime incorporating quantum behaviour. There are thus two major problems to be considered with this model. One is the task of trying to build a quantum theory on top of this spacetime framework, and the other, more basic, problem is to discover the extent to which we really do recover ordinary physics (i.e., our continuum manifold) on the large scale. In this paper, we take a step towards a resolution of the second of these questions. We will present various physical properties one can derive from a poset, as well as reporting on recent mathematical results concerning their applicability.

Bombelli et al.[3] proposed the idea of first recovering the manifold (approximately), with its associated volume measure, from the partial order and then deriving the Lorentzian metric and other properties from the manifold. The problem of recovering the manifold led them to consider the question of which (small) partial orders could be embedded in a given spacetime manifold. We wish to suggest that it is more natural to concentrate on the poset itself as the more fundamental object, and to derive basic physical properties of spacetime from the poset. Implicit in this is the assumption that the poset does indeed correspond with some physical manifold, which is guaranteed if we consider the discretisation already mentioned. We first derive a timelike distance directly from the partial order. Given this distance, we can then, roughly, determine the manifold. Of course, since our model is inherently random, we cannot expect to recover the metric exactly. What we can do is to specify a function of the causal structure which is a good approximation of the timelike distance.

We first give a few definitions in order to set up our nomenclature and conventions. We will then define analogues of geodesics for discrete spacetimes, and thence the metric; we prove that this has the required properties for a Lorentzian metric. We then show how to derive an effective dimension for the set and discuss how this varies according to scale. We conclude with a few remarks about the relevance of the work.

As a first step, let us be more precise about the random nature of the model we are considering. We begin with a spacetime manifold M, with an associated causal structure ($x < y$ for events x and y if y is in the future light-cone of x), and a metric and volume measure on the manifold. We will also take as fixed a parameter ρ, the *density*. We now take a Poisson distribution with density ρ of points in M: that is, we take a set $X(\equiv X(M))$ of points at random in M, so that the number of points of X in each subset of M which has volume A, say, is a Poisson random variable with mean ρA. This defines the discretisation of M. The causal structure on M then induces a partial order $<$ on X, whereby $x_1 < x_2$ if, considered as points in M, x_1 is to the past of x_2. This defines our causal set or poset.

For N a subset of M, we will write the random set $N \cap X$ as $X(N)$. We shall be particularly interested in the *Alexandrov sets*, which form a basis for the topology on our manifold[4]. These are the sets of the form $[x, y] \equiv \{z : x < z < y\}$, i.e., all events lying between x and y, (for x and y events in X). Note that each Alexandrov set has finite volume, so, with probability 1, there are only finitely many points in each $X([x, y])$. Note also that the set of points in M which are null with respect to an event x has measure zero, so almost surely there is no pair (x, y) of points chosen for X such that y lies on the null cone of x.

Having explained the discretisation process, we now show how to construct timelike geodesics and distance. We start by defining a *chain*, C, in a partial order as a set of points in X such that each pair of points from C is related by $<$. Translated into the language of relativity, a chain in the causal structure of a spacetime manifold is a set of events such that every pair of events are causally connected, in other words, for each x and y , x is either to the past or to the future of y. If a chain C has a minimal element x (i.e., an element x such that every element of C is above x) and a maximal element y, we say that C is a *chain from x to y*.

Now, if $X(M)$ is our random poset derived from a manifold M, and C is a chain from x to y in X, then there are almost surely only a finite number of elements in the chain, since otherwise there would be infinitely many points in the discrete Alexandrov set $X([x,y])$. Thus C is a sequence $x = x_1 < x_2 < \dots x_{s-1} < x_s = y$ of points in X. Now, if there is another point z in one of the Alexandrov sets $[x_i, x_{i+1}]$, then we can always form a 'longer' chain by adding z to C. If there is no such point in any of the sets, then we say that C is a ***maximal chain*** or ***path*** from x to y. Given such a path, $C(= (x_1, x_2, \dots, x_s))$, we then define its *length* to be $s - 1$.

Another way of thinking about this is in terms of nearest neighbours. If x and y are points of X with $x < y$ but no other point of X in $[x,y]$, then we say that x and y are ***nearest neighbours***. A path can then be thought of as a sequence of steps from one point to a nearest neighbour to its nearest neighbour and so on, with the length being the number of such steps in the path. The maximal chain or path corresponds approximately to a curve in M. However, whilst the length of a curve in M is given by the integral of proper time elapsed along it, here in $X(M)$ we merely count the points in C.

Clearly, however, for any given x and y, there can be many different paths between x and y with varying lengths. For instance, we could choose a point z, almost on the future light cone of x and the past light cone of y; such a point could be a nearest neighbour of both x and y, leading to a path of length 2. On the other hand, we could take what intuitively would correspond to the 'straight line' path between x and y which would have considerably more points. This is exactly analogous to the paths between x and y in the continuum case. There, we define a geodesic to be the path of maximal length between x and y, and the distance to be that length. Here we do exactly the same: if x and y are events in X with $x < y$, we define the ***distance*** $d(x,y)$ from x to y to be the maximum length of a path from x to y†. We then automatically have the triangle inequality, for suppose we have three points x, y, z in X, with $x < y < z$. Then the longest path from x to z is certainly no shorter than the longest path from x to z via y. Thus we have the appropriate form of the triangle inequality: $d(x,z) \geq d(x,y) + d(y,z)$.

Perhaps we should stress that our definition of distance as the height of a suitable poset does not rely on the fact that our poset is derived from a manifold. However, if the poset does arise in this way, we might hope that this 'distance' is somehow related to the usual distance in the underlying manifold. (Of course, we cannot hope to read any similar

† Note that, unlike the continuum case, this maximal length path most likely will not be unique; the distance however, is well defined. This is the first example of ambiguity in the discrete case.

meaning into the distance function for an arbitrary poset.) What we shall now show is that the distance function $d(x,y)$ is a close approximation to the continuum distance (times a fixed scale factor).

For convenience, we shall assume for the moment that our manifold M is n-dimensional Minkowski spacetime M_n. Notice that, provided the scale on which spacetime is curving is much greater than the typical M-distance between neighbouring points of $X(M)$, this should not affect the arguments, since we shall only be looking at pieces of M which are almost isomorphic to M_n anyway. Also for convenience, we may as well restrict ourselves to a fixed Alexandrov set $[x,y]$ of (finite) volume V in M_n. Recently Bollobás and Brightwell[5] considered properties of random posets in the partially ordered measure space $([x,y],<)$. We highlight a special case of one of the main results (Theorem 12) which not only shows the relationship between poset height and manifold distance, but also gives us an initial handle on defining the dimension of our poset.

Let $[x,y]$ be an Alexandrov set of volume V in M_n. The length L of a longest chain in $X([x,y])$ satisfies $L(\rho V)^{-1/n} \to m_n$ in probability as $\rho V \to \infty$, for some constant m_n.

Observe that ρV is the mean number of points in $[x,y]$, and that $V^{1/n}$ is proportional to the manifold distance from x to y. Therefore, this result says that the distance between x and y becomes proportional to the continuum distance in the limit that $d(x,y) \to \infty$. This is rather encouraging, since one property we would require of our discretisation is that the 'continuum limit' ($\rho \to \infty$) is indeed recovered. Unfortunately the methods of ref. 5 do not tell us anything about the rate of convergence of $L(\rho V)^{1/n}$ to m_n. Moreover, we do not know the numerical values of m_n. However we do know[5] that $m_2 = 2$ and that

$$1.77 \leq \frac{2^{1-1/n}}{\Gamma(1+1/n)} \leq m_n \leq \frac{2^{1-1/n} \mathrm{e} \Gamma(n+1)^{1/n}}{n} \leq 2.62$$

for n an integer at least 3, which implies that $m_n \to 2$ as $n \to \infty$.

The fact that we do not know m_n precisely is not crucial, the main point is that, for large distances, the parameter L of $(X,<)$ is a good approximation to the manifold-distance, up to some fixed factor κ. Calculating it from $(X,<)$ requires no knowledge of the manifold from which we derived X, not even the dimension n. Thus this result proves that the distance function we have defined is not only internally consistent, but actually does correspond to the manifold distance in the continuum limit.

Once we have the distance function for timelike (x,y), and we accept that this is a good approximation to the manifold-distance, we can recover most of the crude structure of the manifold. Of course, from a continuum point of view, once we have the causal structure we can recover the full metric up to a conformal factor, but since we are remaining with the discrete structure precisely to fix that conformal factor, we must show that we can recover this structure. In particular, we should certainly be able to determine the dimension of the original manifold. One particularly straightforward way of going about this would be to count the number N of points of X in an Alexandrov set $[x, y]$, where $L = d(x, y)$ is moderately large. If M is approximately isomorphic to M_n, then we should have $N \simeq (L/m_n)^n$, and, since m_n is known to be about 2, we should in practice have no difficulty in distinguishing M_n from M_{n+1}.

Let us now consider a slightly more subtle approach, which eliminates the potentially awkward dependence on m_n. Given a (large) Alexandrov set $[x, y]$, with say N points of X in it, find a point z in $[x, y]$ such that the minimum of the number of points of X in $[x, z]$ and the number of points in $[z, y]$ is as large as possible. Denote this number by N_1. If the original manifold was M_n, then the best choice for z will usually be near the point of the manifold half-way between x and y. Therefore we can expect that $N_1 \simeq 2^{-n}N$, for large N. An approximation to n is thus given by $\log_2(N/N_1)$. Unfortunately this will not normally give an integer value even if our manifold is just Minkowski space, so this is best interpreted as a *measurement* of the dimension rather than as a definition. (See the definition of the dimension of a box-space in reference 5, which is a continuous version of the same idea.)

One advantage of the above method is that it does give sensible answers in the case when the dimension is somehow dependent on the "scale", i.e., the size of the original Alexandrov set $[x, y]$. For instance, if the spacetime manifold consists of n_1 "global" dimensions and a further $n_2 - n_1$ "compact" dimensions, then measuring the dimension using a large Alexandrov set will almost always give an answer close to n_1, whereas if the Alexandrov set $[x, y]$ is small compared with the scale of the compact dimensions, then a measurement of dimension using $[x, y]$ would give an answer of approximately n_2, at least provided that $[x, y]$ still contains many points from X. Measurements using Alexandrov sets of various intermediate sizes should, of course, indicate dimensions between n_1 and n_2.

Meyer[6] has succeeded in capturing the dimension in a slightly different way, by comparing the number of points in an Alexandrov set to the number of *covering pairs* in that set. (A covering pair in a poset is a pair (x, y) of elements with $x < y$ but no z between the two.) This seems to us to be rather less satisfactory, since the number of covering pairs has no obvious interpretation in terms of the original manifold.

The approach to dimension suggested by Bombelli et al.[3], making use of the finite subposets of X, is as follows. For each n, one takes a finite poset Y_n which can be embedded in M_n (and therefore in higher dimensional manifolds), but not in M_{n-1} (and therefore not in lower dimensional manifolds). Then the dimension of $(X, <)$ is defined to be the largest n such that Y_n occurs as an induced subposet. Suitable posets were discovered by Brightwell and Winkler[7]. The principal advantage of this approach is that it gives an integer value for the dimension. One possible drawback is that, although Y_n cannot occur in M_{n-1}, it might occur in another $(n-1)$-dimensional manifold with high curvature. Also, if our space does have compact dimensions, it may actually not be appropriate to force the dimension to an integer value. Whatever approach we use, what we are doing is taking a fixed (not too large) Alexandrov set $[x, y]$, and using the structure of $X([x, y])$ to give us a *measurement* of the dimension. If the "real" dimension depends on the size of $[x, y]$, we may well prefer the measured dimension to vary "smoothly" as we change the size of our sample Alexandrov sets.

Another aspect of the manifold structure that we might at first expect to be able to recover is the spacelike distance function. However, it seems that there is no convenient way of abstracting a definition of the distance between two spacelike points x and y so as to approximate the manifold distance between x and y. Let us give some idea of why this is so, before going on to see what we can do instead.

Let x and y be two spacelike points in $X(M_n)$, where $n \geq 3$, and let l denote the manifold-distance between x and y. Perhaps the most obvious way of defining the distance between x and y in $X(M_n)$ is to take the minimum, over all pairs (w, z) with $w \leq x, y \leq z$, of $d(w, z)$. We shall show that this definition spectacularly fails to approximate l. Let P_- (P_+) be the intersection of the past (future) light-cones of x and y: an $(n-2)$-dimensional manifold. For every point p on P_-, there is a point p' on P_+ such that $d(p, p')$ is equal to l (divided by the speed of light). (In fact p' is at the intersection of the plane defined by x, y, p with P_+.) Furthermore, if p and q are points of P which are far apart, then the volume of $[p, p'] \cap [q, q']$ is very small. Now, if w is a point of X just below a point p of the manifold on P, and z is a point of X just above the corresponding point p', then $d(w, z)$ is probably about κl. But, recall that $X(M)$ was a Poisson distribution of points in M, therefore, with probability $e^{-\rho V[w,z]}$, the set $[w, z]$ will contain no points of X other than x and y, and so $d(w, z)$ will be equal to 2. Moreover, we can take infinitely many pairs (w_i, z_i), such that $d(w_i, z_i)$ is approximately the manifold-distance between x and y, and such that any two distinct $[w_i, z_i]$ contain no point of X in common, other than x and y. The events "$d(w_i, z_i) = 2$" are then independent, and each has probability bounded away from 0 independently of i. Hence almost surely one of the events occurs, and so the minimum of the $d(w_i, z_i)$ is almost surely 2.

There are various ways to get around this problem, but none are particularly natural. In our opinion, it is more appropriate to return to the question of how one actually measures distance. One can either use standard rods and clocks, or standard clocks and light beams. It is the latter approach which is clearly more adaptable to our (causal) setup. That is, as a standard inertial observer, we measure times and distances by sending out light rays and measuring the time elapsed before they are returned. This means that we need to define the distance between a point and a given geodesic.

We define a *geodesic* in $X(M)$ to be a chain C such that, for every pair of points w and z in C, the length of the section of C between w and z is equal to $d(w, z)$. A point $x \in X$ is *related* to C if there are points w and z on C with $w \leq x \leq z$.

Now, if C is a geodesic and x is related to C, let $l(x)$ be the highest point on C which is below x, and $u(x)$ be the lowest point of C which is above x. Then we define $d(x, C) = d(l(x), u(x))/2$. Evidently this is approximately equal to a fixed constant times the manifold-distance between x and the point of C half-way between $l(x)$ and $u(x)$.

If we have two geodesics, there is now a natural way to define the speed of one geodesic with respect to the other, however, our 'velocity' only has meaning in the sense of an average distance travelled over a certain length of time. Clearly the smaller the time interval, the less reliable this 'velocity': it seems that our model does not incorporate the idea of an instantaneous velocity—at least not in any normal sense.

By this process, we have now set up the basic ingredients of special relativity for the causal set. To summarise: we have taken the causal structure of a discrete poset representing a spacetime, and we have shown how to define distance on that causal set. We use a definition analogous to the continuum case, and show that our definition does indeed correspond with the continuous metric in the continuum limit. We have also explored the question of measurement of dimension for the set. In a manifold there is a clear definition of dimension via the dimension of the tangent space at a point. However, the poset is neither a vector space, nor is it locally equivalent to one. It is therefore quite important that we have established that a working definition of dimension can be constructed. It is also amusing that this definition depends upon the scale of measurement.

It may seem that these definitions are merely stating the obvious, however, that is only because one is still thinking in terms of the poset as being embedded in an underlying manifold. This is precisely the situation we were trying to avoid. We have been exploring definitions which are expressible only in terms of the poset itself, without any reference to an underlying manifold. The problem with abstracting spacelike distance is an excellent example of a situation in which what is obvious for a manifold is quite incorrect for a poset.

If one believes in a fundamentally discrete spacetime, then one must know what properties this discrete set has, and how to measure these properties. What we have done is shown how to measure the basic physical properties of a discrete spacetime, and the extent to which they are measurable. It may or may not be possible to construct a dynamical theory on top of this structure, but we hope that at least we have provided a starting point.

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