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2D Quantum Gravity, Multicritical Matter and Complex Matrices

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Abstract

A large N matrix model of a general complex matrix generates dynamical triangulations in which the triangles can be chequered (i.e. coloured so that neighbours are opposite colours). Gravity and the multicritical matter in such triangulations is described by the KdV-type equations found in the hermitian matrix model but *without* the doubling of degrees of freedom. However, by tuning further couplings it is possible to obtain a series of more general string equations with redoubled degrees of freedom. Other critical points give the non-perturbative behaviour seen in models of unitary matrices.

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1. Recent progress in understanding the partition function for non-perturbative two dimensional quantum gravity coupled to matter systems with central charge $c \leq 1$, equivalently non-perturbative string theory in one dimension or less[1] –[3], has generated considerable interest in matrix models. These authors used $N \times N$ hermitian matrices H as the dynamical variables and an action whose potential is just a trace over some polynomial of H: The perturbation expansion for such a model may be understood as summing over triangulations[1] of arbritrary genus two dimensional surfaces.

We have previously shown that complex matrices M (with no hermitian constraint) may be naturally interpreted[4] as summing over chequered triangulations of orientable surfaces. These are triangulations in which the polygons are divided into triangles in the obvious way, and chequered, meaning the triangles are coloured black and white in such a way that neighbouring triangles are always opposite colours (as in fig.1). The colouration is directly associated with the underlying matrices: white for M, and black for $M^\dagger$. For more details we refer the reader to ref.[4], in particular we solve there the counting problem for chequered spheres and torii, and apply these interpretations to other matrix models.

Here we will investigate the physics in such chequered triangulations. We will show that the critical points for the simplest potentials, indexed by an integer $k = 2, 3, 4 \cdots$, yield the string equations found in hermitian matrix models, except without the doubling of degrees of freedom recently exhibited by Bachas and Petropoulos[5]. This suggests that the complex matrix model is the more fundamental.

However, by tuning various couplings which are irrelevant on the sphere one can obtain a finite sequence of more general equations for *each* value of k, indexed by an integer $m = 1, 2, \cdots$, which involve redoubled degrees of freedom. We thus think of the usual k-critical behaviour as having $m = 0$. We investigate the gravity sequence (k, m)=(2,1), (2,2), (2,3), and the (k, m)=(3,1) case here. The other cases will be reported elsewhere. The (2,2) critical point is particularly interesting. We will show that it involves a pair of Painlevé I equations $\mathcal{R}_\pm = z/\kappa_\pm - 4\rho_\pm^2 - \frac{2}{3}\rho_\pm''$ as

[1] We use this term loosely. Polygonations would be a better word since in general not just 3-sided polygons are used.

in ref.[5], but the string equations are given by:

$$0=\mathcal{R}_+ \qquad \text{and} \qquad 0=\rho_-\mathcal{R}_-^2-\frac{1}{4}\mathcal{R}_-\mathcal{R}_-''+\frac{1}{8}\left(\mathcal{R}_-'\right)^2$$

We also uncover a curious form of universal relation between the orthogonal polynomial recursion relation coefficients.

There are other critical points in complex matrix models than those above. The simplest case is shown to correspond to the $k=1$ critical point found in models of unitary matrices[6][7]. Very likely multicritical generalisations of this point give the other string equations found in these references.

2. Let us briefly summarise the material from ref.[4] that will be needed here. We choose the generating function of the model to have a polynomial action in M and $M^\dagger$, and the invariance $M\to V^\dagger MW$ where V and W are arbitrary unitary matrices. This fixes the partition function to be

$$Z(g_p/\gamma)\sim\int[dM]\exp\{-\frac{N}{\gamma}\;U(M^\dagger M)\} \tag{1}$$

where U is a polynomial of the form (g_1 was set to γ in ref.[4]):

$$U(M^\dagger M)=\mathrm{tr}\{\sum_{p\geq 1}g_p(M^\dagger M)^p\}\;. \tag{2}$$

The rôle of cosmological constant will be played by γ. "$\sim$" stands for the presence of a normalisation constant which we take so that $Z=1$ when only the g_1 term is non-zero. It is necessary that this term always be non-zero since it is only then that one has a triangulation interpretation.

We can integrate out the "angular" modes leaving an integral over the eigenvalues λ of $M^\dagger M$, which are non-negative:

$$Z(g_p/\gamma)\sim\int_0^\infty d^N\lambda\;\Delta^2(\lambda)\;\;\exp\{-\frac{N}{\gamma}\;U(\lambda)\} \tag{3}$$

where Δ is the Vandermonde determinant. Introducing the variable y and writing $d\mu(y)=dy\;\exp\{-N/\gamma\;U(y)\}$, we define orthogonal polynomials $P_n(y)$ and normalisation coefficients h_n and A_n so that

$$\begin{aligned} &P_n(y)=y^n+\text{lower powers}\qquad n=1,2,\cdots\\ &\int_0^\infty d\mu(y)\;P_n(y)P_m(y)=h_n\delta_{nm}\\ \text{and}\qquad&\int_0^\infty d\mu(y)\;y\,P_n^2(y)\;=A_nh_n\,. \end{aligned} \tag{4}$$

$Z = \exp(-\Gamma)$ can then be expressed in the standard way[8] in terms of h_0 and $R_m = h_m/h_{m-1}$, and the polynomials determined by the recursion relations:

$$yP_n(y) = P_{n+1}(y) + A_nP_n(y) + R_nP_{n-1}(y) \ . \tag{5}$$

By considering integrals over the measure $d\mu$ of yP_ndP_n/dy, P_ndP_n/dy and $P_{n-1}dP_n/dy$ one can derive recurrence relations for the coefficients A_n and R_n. In the large N limit the coefficients become continuous functions of n/N, so it is helpful to write $x = n/N$, $\epsilon = 1/N$, $r(x) = R_n$ and $a(x) = a_n$. To enable us to write the equations in compact form for an arbritrary action U we define orthonormal eigenkets $|x>$, operator $\hat{x}$ such that $\hat{x}|x>= x|x>$, and shift operators $S^\dagger|x>= |x+\epsilon>$ $S|x>= |x-\epsilon>$. Finally, defining the expressions

$$\Omega(x) =< x|U_y(\hat{y})|x> \ , \tag{6}$$

$$\tilde{\Omega}(x) = \gamma x - < x|S^\dagger U_y(\hat{y})|x> \tag{7}$$

where $\hat{y} = S^\dagger + Sr(\hat{x}) + a(\hat{x})$ and $U_y(\hat{y}) = \sum_{p\geq 1} pg_p\hat{y}^{p-1}$, the difference equations can be succinctly written[4]:

$$\tilde{\Omega}(x) + \tilde{\Omega}(x+\epsilon) = a(x)\Omega(x) \tag{8}$$

$$r(x)\Omega(x)\Omega(x-\epsilon) = \tilde{\Omega}^2(x) \ . \tag{9}$$

3. The spherical contribution to Γ was already analysed in ref.[4]. It is obtained by dropping all ϵ dependence in the above equations. It is entirely equivalent to the hermitian matrix model for an even potential[8], even off-criticality, up to some factors of two which may readily be adsorbed in the couplings. In this limit one sees from (8) and (9) that

$$4r(x) = a^2(x) \ . \tag{10}$$

The alternative solution $\Omega = \tilde{\Omega} = 0$, which we will call the "ghost", is ruled out by matching to small γ perturbation theory (where one recovers an interpretation in terms of triangulations): One may readily show that in the limit $\gamma \to 0$, $a(x) = r(x) = 0$ and hence $\Omega = g_1 \neq 0$. Nevertheless the ghost plays a rôle because the double scaling limit lacks this information and must be consistent with both solutions.[2]

[2] This is analogous to the simpler case of determining the sign of the sphere contribution in the Painleve I equation[1].

Substituting (10) back into (8) gives

$$\gamma x = w(a) \tag{11}$$

with
$$w(a) = 1/2 \sum_{p\geq 1} g_p (a/2)^p \frac{(2p)!}{p!(p-1)!} \ . \tag{12}$$

In ref.[4]we chose couplings $g_m^{[4]}$ so that $g_1^{[4]} = \gamma$ in order to have a direct interpretation for γ as cosmological constant. If we tune the couplings with the cosmological constant so that they are parameterized by $g_p^{[4]} = g_p(\gamma/g_1)^p$ for $p \geq 2$ and change variables $M \to \sqrt{g_1/\gamma}\, M$ in Z, we recover eqns (1)–(2). Thus by universality we can generalise so that g_1 need no longer equal γ. Now following Gross and Migdal[1] we consider the simplest polynomial actions with "k-critical" behaviour ($k = 2, 3, \cdots$). These are given by:

$$\begin{aligned} g_p &= \frac{k!\,(p-1)!}{(2p)!\,(k-p)!}(-2)^{p+1} \qquad &&\text{for} \quad p = 1, \cdots, k \\ g_p &= 0 &&\text{for} \quad p = k+1, \cdots \end{aligned} \tag{13}$$

so that, from eqn. (12):

$$w(a) = 1 - (1-a)^k \ . \tag{14}$$

Equation (11) at $x = 1$ then shows that the model goes critical as $\gamma \to 1$.

Introduce a scaling parameter δ by defining $\gamma = 1 - \mu\delta^{2k}$ and $\gamma x = 1 - z\delta^{2k}$, where μ is the renormalised cosmological constant[3] and $\delta \to 0$. The bare string coupling scales as $\epsilon = 1/N = \nu\delta^{2k+1}$ in order to give a finite spherical contribution:

$$\Gamma_0 = \frac{2}{\nu^2} \frac{\mu^{2+1/k}}{(2+1/k)(1+1/k)} \quad . \tag{15}$$

This is of course the same as observed before[1]. From eqns. (14), (11) and (10) to lowest non-trivial order in the scaling parameter:

$$\begin{aligned} a(z) &= 1 - \delta^2 \alpha(z) \\ r(z) &= \frac{1}{4}[1 - 2\delta^2 \rho(z)] \end{aligned} \tag{16}$$

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[3] a misnomer[2] for $k \geq 3$

with $\rho = \alpha = z^{1/k}$ at the spherical level. Eqn. (16) is thus the required substitution for the double scaling limit. Up to the the unusual factor of 2 in (16), ρ is the string susceptibility as follows from the equation (proved as previously[1][4]):

$$\frac{\partial^2 \Gamma}{\partial \mu^2} = \frac{2}{\nu^2}\rho(\mu) \ . \tag{17}$$

4. Actually, we will see shortly that it is not possible to derive the string equations for ρ from just the one term scaling equations (16). It is necessary to expand at least one of ρ or α in a power series in δ involving powers up to δ^{2k-2}. We will show that the following *completely* universal relation holds, when $m = 0$ up to a maximum power δ^{2k-3}:

$$\alpha = \ \rho + \delta\left(\frac{1}{2}\rho'\right) + \delta^2\left(\frac{1}{8}\rho'' + \frac{1}{2}\rho^2\right) + \delta^3\left(\frac{1}{48}\rho''' + \frac{1}{2}\rho\rho'\right) + \delta^4\Big(16\rho^3 + 4\rho\rho'' + \frac{1}{12}\rho^{(4)} + \frac{3}{32}\left(\rho'\right)^2\Big) + \delta^5\left(24\rho^2\rho' + \frac{2}{3}\rho\rho''' + \frac{1}{120}\rho^{(5)} + \frac{1}{32}\rho'\rho''\right) + \cdots \tag{18}$$

As m increases this is violated at successively lower powers. Here and elsewhere a prime denotes $-\nu d/dz$. This curious expansion is an observation we extract from our calculations for different k and m. As yet we have no deeper understanding of its meaning, or neater derivation.

We now derive the string equations for the simplest actions parameterized in (13). For completeness we start with $k = 2$. (This case was already analysed in ref.[4]). From equation (13) we find $g_1 = 4$ and $g_2 = -2/3$ and thus

$$\begin{aligned} \Omega &= \frac{4}{3}(2 + \alpha\delta^2) \\ \bar{\Omega} &= -z\delta^4 + \frac{2}{3}(2 - \rho\delta^2) \end{aligned} \tag{19}$$

so that Taylor expanding eqns. (8) and (9) to order δ^4 gives

$$0 = -\frac{2}{3}\alpha + \frac{2}{3}\rho + \frac{1}{3}\rho'\delta + \left(\frac{1}{3}\rho'' - \frac{2}{3}\alpha^2 + z\right)\delta^2 \tag{20}$$

$$0 = 4\alpha - 4\rho - 2\alpha'\delta + \left(\alpha'' + \alpha^2 - 8\alpha\rho - \rho^2 + 6z\right)\delta^2 \ . \tag{21}$$

Solving (20) for α perturbatively in δ we obtain $\alpha = \rho + \frac{1}{2}\rho'\delta + \left(\frac{1}{4}\rho'' - \rho^2 + \frac{3}{2}z\right)\delta^2 + \cdots$. Note that the relation (18) holds as advertised to order δ but is violated above this. Substituting into (21) one finds that all terms up to order δ^3 vanish and at δ^4 one obtains

$$\rho''/12 - \rho^2 + z = 0 \tag{22}$$

which under the trivial rescaling $\nu \to 2\nu$ is precisely the Painlevé I equation for 2D pure gravity found previously. However it is only one of the pair[5]; this is a consequence of the fact that α is completely determined in the scaling limit (as $\alpha = \rho$) by (18).

For $k = 3$ one obtains $\Omega = 6 - 4a + \frac{4}{5}[a^2 + r + r_+]$ and $\tilde{\Omega} = \gamma x + 4r - \frac{4}{5}[ra + ra_-]$ where subscripts $+/-$ mean arguments $x \pm \epsilon$. Substituting into eqn. (8) and expanding to order δ^6 one obtains

$$\begin{aligned} 0 =& 4\alpha - 4\rho - 2\rho'\delta + \left(\frac{1}{2}\alpha'' - \rho'' + 4\alpha^2 - 6\rho\alpha\right)\delta^2 - \left(\frac{1}{3}\rho''' + 3\rho'\alpha\right)\delta^3 \\ &+ \left(\frac{1}{24}\alpha'''' - \frac{1}{12}\rho'''' + 2\alpha^3 - \frac{3}{2}\rho''\alpha - \rho\alpha'' - \rho'\alpha' - 5z\right)\delta^4 \end{aligned} \tag{23}$$

Solving for α one finds that α is given by the series (18) up to δ^3, but with the δ^4 contribution $(-\frac{1}{192}\rho'''' - \frac{3}{4}\rho^3 + \frac{1}{4}(\rho')^2 + \frac{7}{16}\rho\rho'' + \frac{5}{4}z)\delta^4$. Expanding (9) and substituting this series one finds (eventually) that all terms of order less than δ^6 disappear and one is left with

$$-\frac{1}{160}\rho'''' + \frac{1}{8}(\rho')^2 + \frac{1}{4}\rho\rho'' - \rho^3 + z = 0\ . \tag{24}$$

Under the trivial rescaling $\nu \to 2\nu$, this is one of the pair of $k = 3$ multicritical string equations obtained previously[1][5].

For $k = 4$ one proceeds similarly. From now on the derivations become rather long but are handled admirably by the algebraic computing program FORM. The solution for α obtained from (8) consists of all the terms shown in (18) and at order δ^6:

$$\frac{35}{32}z - \frac{15}{32}\rho^4 + \frac{47}{64}\rho^2\rho'' - \frac{19}{768}\rho\rho^{(4)} + \frac{53}{64}\rho\,(\rho')^2 + \frac{47}{92160}\rho^{(6)} - \frac{21}{512}\,(\rho'')^2 - \frac{13}{256}\rho'\rho'''\ .$$

Substituting this into the expanded eqn.(9) one finds at order δ^8, under the same rescaling $\nu \to 2\nu$ as previously, the $k = 4$ multicritical equation derived from hermitian matrix models (c.f. Gross and Migdal[1]).

We conclude that the k−critical behaviour given by the simplest actions U_k (as parameterized by eqn. (13)) gives the square root of the partition function for hermitian matrix models with even potentials[1][5]. The $\nu \to 2\nu$ rescaling and eqn. (17) imply that, in the scaling limit, the vacuum energy Γ for the complex matrix is half the contribution from a $2N \times 2N$ hermitian matrix. This fact together with observations in ref.[4] may be clues to a more direct derivation.

5. We now generalise the actions U_k by adding terms that are irrelevant on the sphere. Thus for example, in this model the most general polynomial action of order 8 with $k = 2$ critical behaviour at the spherical level is:

$$U = k_2 U_2 + k_3 U_3 + k_4 U_4 \tag{25}$$

where the k_i are parameters such that $k_2 > 0$, $\sum_i k_i \neq 0$, and $\sum_i i k_i \neq 0$. (The last condition ensures that the quadratic term coefficient is non-zero). By rescaling γ we can set $\sum_i k_i = 1$, then eqn.(14) implies[4]

$$w(a) = 1 - k_2(1-a)^2 - k_3(1-a)^3 - k_4(1-a)^4 \ , \tag{26}$$

which implies the same scaling relations as before. Eliminating k_4 and solving for α in terms of ρ using (8) we obtain:

$$\alpha = \rho + 1/2\rho'\delta - \left\{ \frac{210z + 4(24 - 59k_2 - 3k_3)\rho^2 + (24 + 11k_2 - 3k_3)\rho''}{4(48 - 13k_2 - 6k_3)} \right\} \delta^2 \tag{27}$$

provided that the denominator does not vanish. Substituting this into (9) yields at order δ^4:

$$\rho''/12 - \rho^2 + z/k_2 = 0 \ . \tag{28}$$

which is the same equation as (22) under the rescaling of the (non-universal) string coupling $\nu \to \nu k_2^{-1/4}$, and consequently $\rho \to \rho k_2^{-1/2}$ (to preserve (17)). We have thus demonstrated universality for generic values of the couplings.

However, we must investigate the special case $6k_3 = 48 - 13k_2$. This is the first new critical point $m = 1$. In this case one finds that (9) vanishes to order δ^5 and (8) to order δ^3 without imposing any relation on α and ρ. The reason can be traced to

[4] w is a linear functional of the action U.

Ω and $\tilde{\Omega}$. Their constant terms (order δ^0) which are generally non-zero vanish in this case. Indeed on the sphere they now read:

$$\frac{5}{2}\Omega = \delta^2(16-k_2)(\rho-\alpha) + \delta^4\left\{4(6-k_2)\alpha\rho + (k_2-16)\alpha^2\right\} - 4/3\delta^6(6-k_2)\alpha^3 \ .$$
$$5\tilde{\Omega} = \delta^2(16-k_2)(\rho-\alpha) + \delta^4\left\{2(16-k_2)\alpha\rho - 2(6-k_2)(\alpha^2+\rho^2) - 5z\right\} + 4\delta^6(6-k_2)\alpha^2\rho \ . \tag{29}$$

In general, at the m^{th} critical point Ω and $\tilde{\Omega}$ are of order δ^{2m}. It is convenient to define $\rho_\pm = \rho \pm \alpha$. In terms of these, one finds at the next orders in (8) and (9) resectively:

$$(20z - 5k_2\rho_+^2 + 5/6k_2\rho_+'') - (k_2-16)(3\rho_-^2 - 1/2\rho_-'') = 0 \tag{30a}$$
$$(20z - 5k_2\rho_+^2 + 5/6k_2\rho_+'')\rho_- + (k_2-16)(\rho_-^3 - 1/2\rho_-\rho_-'' + 1/2(\rho_-')^2) = 0 \tag{30b}$$

Eliminating the first expression between the two equations gives, *provided* $k_2 \neq 16$:

$$4\rho_-^3 - \rho_-\rho_-'' + 1/2(\rho_-')^2 = 0 \ .$$

By writing $\rho_- = \chi^2$ this equation may be solved exactly. The result is given in terms of Jacobi elliptic functions (cn is the cosine-amplitude):

$$\rho_- = \frac{1}{4}\nu^2 A^2\left(1+\mathrm{cn}[A(z-B)]\right) / \left(1-\mathrm{cn}[A(z-B)]\right) \tag{31}$$

where A and B are integration constants. As such, for $A \neq 0$, it is doubly periodic in z with an infinite lattice of double poles, and does not have an asymptotic expansion in z. For $A=0$ however we have $\rho_- = \nu^2/(z-B)^2$. Substituting this back into (30b) one finds the second term vanishes[5], and the eqn. reads:

$$\frac{1}{12}\left(\rho_+''/2\right) - (\rho_+/2)^2 + z/k_2 = 0 \ . \tag{32}$$

Thus from eqn. (28) we see that we recover the previous solution for ρ but with the correction $-\frac{1}{2}\nu^2/(z-B)^2$. Defining the string coupling $g_s = \nu k_2^{1/4}/(2\mu^{5/4})$ we obtain half the asymptotic expansion from hermitian models as before, but minus $\ln\mu$:

$$\Gamma = \frac{2}{15}g_s^{-2} - 47/48\ln\mu - \frac{7}{5}(g_s/24)^2 - \cdots \tag{33}$$

[5] For $A \neq 0$ the second term is a constant, adsorbed by shifting z.

The change of sign in the torus contribution implies, by the arguments of Gross and Migdal[1], that the physics is non-unitary. Non-perturbatively, the partition function Z vanishes linearly as $\mu \to B$. This much can be deduced from the string equations. An analysis (e.g. along the lines of David[9]) is necessary to determine the values of A and B the matrix integral chooses. If the matrix model chooses $A \neq 0$ it seems reasonable to conclude that a (k,m)=$(2,1)$ large N limit does not exist. If the matrix model chooses $A = 0$ and $B = \infty$, then $\alpha = \rho$ exactly and (32) collapses to the (k,m)=(2,0) universality class.

6. The analysis above was only valid if $k_2 \neq 16$. The case $k_2 = 16$ is much more interesting. This gives the (2,2) critical point. From eqns. (29) we see that this corresponds to vanishing linear terms in Ω and $\tilde{\Omega}$. To investigate universality at this point we must generalise at least to a $k_5 U_5$ term in (25). Eliminating k_5 by $\sum_i k_i = 1$, we find that the $m=1$ critical point occurs at $k_4 = 10 - 55k_2/16 - 17k_3/8$ and the $m=2$ point further requires $k_3 = 80/3 - 10k_2/3$. The ghost solution now becomes important. (For the $m=1$ case the ghost merely reproduces the real solution). One finds on the sphere that

$$\begin{aligned} \Omega &= \left((8 - k_2/4)\rho_-^2 - k_2/4\rho_+^2\right)\delta^4 + \cdots \\ \tilde{\Omega} &= \left((4 - k_2/8)\rho_-^2 + k_2/8\rho_+^2 - z\right)\delta^4 + \cdots \end{aligned} \tag{34}$$

Thus solving the ghost in the scaling limit gives:

$$\kappa_+\rho_+^2 = z/4 \qquad \text{and} \qquad \kappa_-\rho_-^2 = z/4$$

where $\kappa_+ = k_2/16$. In this case $\kappa_- = 2 - \kappa_+$, but it is unrelated to κ_+ in general. (For a non-zero k_6 coefficient one finds the same κ_+ but $\kappa_- = 24/7 - 27k_2/112 - 3k_3/56$). We will see that the string equation is consistent with both this behaviour and the true solution $\rho = \alpha = \sqrt{z/k_2}$.

One finds that the first non-vanishing contribution to (8) is at order δ^4:

$$-2z/k_2 + \rho_+^2/2 - \rho_+''/12 = 0 \tag{35}$$

while (9) is order δ^8 but has (35) as a factor. Thus one must write $\rho_+ = f + \zeta_1(z)\delta + \zeta_2(z)\delta^2 - \cdots$, where f is a solution of (35) and solve for the coefficients ζ. One finds at order δ^5 in (8), the following equation for ζ_1:

$$\frac{1}{2}\nu + k_2 \left\{ -\frac{1}{48}\rho_-''' + \frac{1}{4}\rho_-' f + \zeta_1 f - \frac{1}{12}\zeta_1'' \right\} = 0 \ .$$

(We use (35) to eliminate all differentials on f higher than the first). The δ^9 term in (9) vanishes if ζ_1 is a solution of this equation, thus it is necessary now to solve for ζ_2. From the δ^6 term in (8) we obtain a second order differential equation for ζ_2 containing ρ_-, f and ζ_1, which is too long to write. Assuming that ζ_2 satisfies this equation we find that the next order (δ^{10}) in (9) does not vanish identically but gives:

$$0 = \rho_- \mathcal{R}_-^2 - \frac{1}{4}\mathcal{R}_- \mathcal{R}_-'' + \frac{1}{8}\left(\mathcal{R}_-'\right)^2 \tag{36a}$$

where
$$\mathcal{R}_-(z) = z - \kappa_- \left(4\rho_-^2 - \frac{2}{3}\rho_-''\right) \tag{36b}$$

This equation together with (35):

$$\mathcal{R}_+ = z - \kappa_+ \left(4\rho_+^2 - \frac{2}{3}\rho_+''\right) = 0$$

form the string equations for the (2,2) critical point. Remarkably all reference to the subleading functions ζ has disappeared. Note that the ghost is responsible for the specific solution $\mathcal{R}_- = \mathcal{R}_+ = 0$ which is the system of two Painlevé equations from hermitian matrix non-even potentials[5]. However the true solution has asymptotic expansion:

$$\begin{aligned} 2\rho_- = &-\frac{1}{4}\nu^2/z^2 - \frac{225}{32}\nu^6\kappa_-/z^7 - \frac{690675}{1204}\nu^{10}\kappa_-^2/z^{12} - \cdots \\ &- \text{coeff.}\sqrt{z}\left(\nu^2/z^{5/2}\right)^{2n+1}\kappa_-^n - \cdots \\ 2\rho_+ = &\sqrt{z/\kappa_+} - \frac{1}{24}\nu^2/z^2 - \cdots - \text{coeff.}\sqrt{z}\left(\nu^2/z^{5/2}\right)^n \kappa_+^{(n-1)/2} - \cdots \end{aligned}$$

the latter expansion being the standard one[1].

There appears to be no further critical phenomena for k=2 beyond this point except for the possibility of setting κ_-=0 . In that case the above string equations still apply. Nevertheless the critical behaviour is different: The ρ_- expansion is truncated to the torus term, and the possible non-perturbative physics collapses to that of the (2,0) point. We call this the (2,3) critical point. It does not quite satisfy the characterization below (29): The ghost solution on the sphere changes to $\rho_+^2 \sim \rho_-^3$ in Ω (higher powers of ρ_- are accessible) and substituting into $\tilde{\Omega}$ one finds agreement with (35).

7. We now briefly discuss the (k,m)=(3,1) point. The details of the derivation differ from the previous points, but the result is similar to the (2,1) case. For general values of the couplings irrelevant on the sphere one confirms universality of (24). However when the δ^0 terms vanish in Ω and $\tilde{\Omega}$ one obtains the $m=1$ critical point. Order by order in δ one obtains pairs of simultaneous second order differential equations for $\rho_- = f + \zeta_1\delta + \zeta_2\delta^2 + \cdots$. For example for f one finds $f'' - 6f^2 = 0$ from (8), and $-2f^3 + ff'' - (f')^2 = 0$ from (9). Solving these one finds $\rho_- = f = \nu^2/(z-B)^2$ in the continuum limit, for some integration constant B. Unlike the (2,1) case there is no constant A. Solving the next orders gives $\zeta_1 = -\frac{1}{4}\rho_+' + C\nu^3/(z-B)^3$ for some integration constant C, while ζ_2 can only be solved implicitly. (This expansion agrees with the first two terms in (18)). All reference to ζ_2 and C then disappears and one is left with the (3,0) equation for ρ_+ (up to a rescaling of ν and shift in z). This is analogous to (2,1) case.

Finally we note that for the simplest action U_2 another critical point, with $k=1$, was observed in ref.[4]. We find that it is governed to all genus by the $k=1$ mKdV equation of ref.[6][7]. Translating from ref.[4], the critical point is at $\gamma = -3$. Thus we set $\gamma = -3(1-\mu\delta^2)$ and $x = (1-z\delta^2)/(1-\mu\delta^2)$. We find $a = 3(1-\alpha\delta^2)$ and $r = \frac{9}{4}(1-2\rho\delta^2)$, with $\rho = z/4$ and $\alpha = 0$ on the sphere. With the above scaling laws, equations (8) and (9) give $\alpha = z/2 - \rho$ in the scaling limit and the differential equation:

$$0 = \chi(z + 4\,\chi^2) - \chi''$$

where $$\rho = z/2 + \chi^2$$

This can be expanded and shown to agree with the sphere and torus analysis in ref.[4]. Up to rescaling, and the non-universal linear term in ρ, the above is as found in refs.[6][7]. Interestingly enough this solution implies that $\Omega = \tilde{\Omega} = 0$ on the sphere so in this case the ghost is the real solution. The ghost evades our previous objections because this critical point is accessed on the second sheet of the sphere solution: $a(x) = 1 + \sqrt{1-\gamma x}$, as shown in ref.[4]. It is thus disconnected from perturbation theory in γ and does not have a triangulation interpretation. The mKdV equations arise in the strong coupling phase of unitary matrix models[7]. As such they do not have a direct string interpretation either[10], since the correct perturbation expansion yields integrals over products of unitary matrices.

We stress that the critical points (k, m) however have equally valid interpretations in terms of triangulations as those from hermitian models. We have only scratched the surface of these new phenomena. It is obviously of interest to delineate the critical series further, especially the cases $k = m$ which yield new string equations, and to understand how they fit into the KdV interpretation and the newly discovered Virasoro algebra[11].

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Fig. 1: A portion of a chequered triangulation.