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Structure of Irreducible SU(2) Parafermion Modules Derived via the Feigin-Fuchs Construction

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Abstract

We show how the Feigin-Fuchs Coulomb gas construction, with two free gaussian bosons, can be used to derive the representation theory of the $SU(2)$ parafermion models. We identify the generators of the chiral algebra within the bosonic Fock space and derive the chiral algebra of the finitely reducible models, which correspond to the $SU(2)$ and $SU(1,1)$ parafemion algebras. We focus on the $SU(2)$ case in this paper, the $SU(1,1)$ case will be discussed in a subsequent publication. Unitarity of the chiral algebra requires the parafermion Hilbert space embedding in the bosonic Fock space to be independent of two fermion zero modes. The expressions for the Virasoro highest weights of the models are doubly degenerate in the bosonic Fock space. We formulate the correlation functions of these operators in the parafemion Hilbert space, and in particular, the fusion rules for the Virasoro highest weights are derived in an elegant way. Finally, the irreducible parafermion characters are derived. We discuss the connection between our analysis and previous work on representation theory based on BRST cohomology.

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1. Introduction

The main purpose of this paper is to use the Feigin-Fuchs Coulomb Gas construction [1],[2],[3] to derive, and not just reconstruct, the representation theory of SU(2) parafermion (PF) models [4] using conformal field theory techniques.

The Coulomb gas construction, developed by Feigin and Fuchs [1],[2], and Dotsenko and Fateev [3] for the minimal conformal models [5],[6], provides in principle a powerful way of studying two dimensional conformal field theories. For instance, it presents a more practical way of computing the conformal blocks for the minimal conformal series than the bootstrap program advocated BPZ [5]. Examples of models where the chiral algebra is constructed via this method are bc ghost systems [7], Z_3 W-algebra models [8], $SU(N)$ affine Lie algebra models [9],[10],[11],[12],[13], and $SU(1,1)$ models [14],[15]. The construction of the representation theory for these models using Coulomb gas techniques is incomplete in most cases. The common element in the constructions is the use of free bosons ϕ with gaussian propagators defined on the complex plane, but with anomalous Coulomb charge conservation. The anomaly is due to the presence of a background charge Q. It signifies the coupling of these bosons to the two dimensional metric, with interaction Lagrangian $\mathcal{L}_{\text{int}} \sim \sqrt{g}QR\phi$, and with the curvature scalar R receiving support only at infinity.

In this paper we will be concerned primarily with the minimal [4] Z_N parafermions, which correspond to the GKO [16] coset $SU(2)/U(1)$, and partially with the non-minimal finitely reducible [17] parafermions, which have been shown [18] to be a subset[1]of the GKO coset $SU(1,1)/U(1)$. The representations of the minimal and non-minimal PF models are building blocks for $N=2$ superconformal field theories [19],[17]. This is because the GKO coset of the superconformal algebra by it's $U(1)$ current is either a $SU(2)$ or $SU(1,1)$ parafermion algebra. The minimal $N=2$ superconformal models with central charge $c<3$ are constructed from the $SU(2)$ PF models plus a free boson [20],[21]. The representations of the superconformal models with $c \geq 3$ are allowed representations of the $SU(1,1)$ PF algebra [17],[18]. Recall that the critical string models with $N=1$ space-time supersymmetry are built up from $N=2$ superconformal field theories [22]. In principle therefore, the Feigin-Fuchs construction of the parafermion models may be a way of providing a concise represention of the space of string compactifications with $N=1$ spacetime supersymmetry.

[1] Not all $SU(1,1)$ parafermion models are finitely reducible. The models discussed in [18] with irrational central charge are not in this class.

In practice however, there has been little progress in this direction. One problem is the lack of deductive analysis applied with the Feigin-Fuchs construction. In addition to representing the spectra of known models, one would like to be able to deduce new theories. Our focus in this paper will be to develop the calculus of the Feigin-Fuchs construction using the admittedly well studied $SU(2)$ PF model, with the clear intention of applying these techniques to the more difficult $SU(1,1)$ PF case later [23]. We resolve some of the technical problems with the $SU(2)$ PF constructions [9],[10],[11],[12],[13]. Our analysis lends itself to generalization, and application to the other Feigin-Fuchs models described above.

The Coulomb gas construction of PF models is more complicated than the $c < 1$ minimal models. In the $c < 1$ minimal models the screening operators come in Feigin-Fuchs conjugate (FF-conjugate) pairs with the sum of their charges cancelling the background charge. This does not happen in the PF models. We will have to understand the zero mode structure, the Hilbert space structure, and how the PF Hilbert space is imbedded in order to cancel the background charge and get the correlation functions right. Calculating correlation functions via the free field bosonic theory will involves working in the boson Fock space with the background charge balanced out. We will see that the parafermionic theory lives in a "smaller" Hilbert space and we will have to know how to correctly move between the two spaces.

The basis of our construction is the Fock space of two free bosons on a Lorenzian lattice. The embedding of the $SU(2)$ PF models in this basis was first given by Nemeschansky [10], and the embedding of the chiral algebra was generalized to the non-minimal case by one of us [14]. In section 2 we discuss this embedding in detail and in a unified way for both the minimal and non-minimal PF models. The starting point of our program is a natural ansatz which leads to the correct identification of ψ_1 and $\psi_1^\dagger$, the generators of the PF algebra. From this we derive compact expressions for the entire PF chiral algebra. We also derive the three chiral screening operators; two of which turn out to be fermionic. In fact, they are members of two non-commuting (η, ξ) fermion ghost systems with central charge $c = -2$ which are embedded in the boson Fock space. The lack of commutation means that states cannot in general be simultaneously diagonalized in terms of both systems.

In section 3 we discuss the issue of unitarity of the chiral algebra. We focus on the $SU(2)$ case at this point, for which the unitarity constraint for the chiral algebra is non-trivial. We find that we must require that states in the "small" PF

Hilbert space be mutually local with respect to both (η, ξ) systems and independent of both fermion zero modes $\xi_0, \tilde{\xi}_0$; then the PF chiral algebra truncates onto unitary states with non-negative conformal dimension. To calculate correlators in the PF Hilbert space, one must therefore reintroduce one of the dimension zero fermions to soak up the zero modes resulting from the integration over ξ_0 (or equivalently $\tilde{\xi}_0$) in the bosonic path integral. We need to introduce only one fermion, because when one of the zero modes is rediagonalized in the other fermion basis, it contains the other zero mode. Because this result, and hence the structure of the Hilbert space embedding into the bosonic Fock space, differs for the $SU(1,1)$ PF case, we defer further analysis of these models to a later publication [23].

The Virasoro highest weights of the PF theory can be derived systematically. We postulate that the order operators, (the highest weights of each PF module), are Virasoro primary vertex operators or screened vertex operators of the bosonic theory. The operator product expansion (OPE) of these operators with the parafermions fixes the allowed momenta, i.e. charge sectors. Then one can use the simple expressions of the parafermions derived in sec. 2 to find the lowest conformal dimension states in each PF charge sector; i.e. the Virasoro primaries and PF decendants of the theory. Thus we obtain the PF representation in the allowed charge sectors. This construction is presented in section 4. The Virasoro primaries which have negative conformal dimension decouple from the theory because their two point function is not defined in the small Hilbert space. The resulting spectrum is double the usual result; each PF/Virasoro primary with given PF charge has two alternate representations in our construction. This degeneracy is the generalization of the FF-conjugate degeneracy of the minimal $c < 1$ conformal models.

With the states at hand, and an understanding of the embedding of the PF modules in the bosonic Fock space, the correlation functions of the models can be calculated in a straightforward way. In particular we focus on the three-point functions of the PF descendants in sec. 5. We find the fusion rules by using bosonic expressions for the correlators to determine when they do not vanish. To calculate when this happens recall that correlation functions generically require the introduction of screening operators to balance the bosonic $U(1)$ charges. Screening operators in other Feigin-Fuchs constructions have previously been used to construct null states, states which are primary with respect to the chiral algebra and yet chiral algebra decendents of primaries of lower conformal dimension [24]. If the screening operators that must be added to our correlators define such a state, then the correlator must vanish. This restricts the number of screening operators that can be added and hence restricts the non-vanishing correlators. This analysis is the implementation of the notion that the truncation of the operator algebra is due to factorization onto null states [5].

Characters of the parafermion theories are constructed in sec. 6. This construction is a two step process. First, one finds characters of the reducible PF modules for each Virasoro highest weight in each charge sector by eliminating the states in the bosonic Fock space module of the highest weight which are proportional to either fermion zero mode. Secondly, one subtracts the characters of the null vectors which remain in the PF highest weight module. Null vectors are constructed using the analysis of Kato and Matsuda [24] via the screening operators. Also, we must subtract the characters of the null vectors constructed using the alternate representation of the highest weights. Our analysis gives us the parafermionic character in a form first derived in [12] but without the introduction of a BRST charge operator [25],[12].

One of our fermion systems plays an explicit role in the alternate $SU(2)$ PF construction given by Distler and Qiu [12] in which the PF Hilbert space is diagonalized in terms of (η, ξ) and a single Feigin-Fuchs boson σ. We discuss the relation of our work to this formulation in appendix A. In section 7 we show that our other non-commuting fermion system is related to the BRST charge introduced in their analysis and discuss the equivalence of both approaches. Finally in section 8 we summarize our work and discuss its implications.

Before proceeding we would like to call attention to an alternative bosonization scheme which relies on the GKO construction to find a free boson Coulomb gas representation of minimal parafermions without the introduction of background charge [26],[27],[28]. For a $SU(2)$ PF model with Z_N symmetry, this requires the introduction of $2N$ free bosons. Correlation functions and characters can be determined in this formalism, however the construction is much less elegant than the Feigin-Fuchs bosonization we will discuss in this paper.

2. Embedding of Chiral Parafermion Algebra in Boson Fock Space

We first construct the finitely reducible chiral parafermion algebra from the Fock space of two free bosons. Consider two commuting free scalar fields $\Phi_1 = \phi_1(z) + \bar{\phi}_1(\bar{z})$ and $\Phi_2 = \phi_2(z) + \bar{\phi}_2(\bar{z})$. In this section we will consider the holomorphic sector with propagators on the complex plane given by

$$\begin{aligned}\phi_1(z)\phi_1(w) &= -\ln(z-w),\\ \phi_2(z)\phi_2(w) &= +\ln(z-w).\end{aligned} \tag{2.1}$$

The currents $j_\alpha = i\partial\phi_\alpha$ generate a $U(1)\oplus U(1)$ Kac-Moody (Heisenberg) algebra. The holomorphic stress energy tensor is defined to be [10]

$$T(z) = -\frac{1}{2}\partial\phi_1(z)\partial\phi_1(z) + \frac{1}{2}\partial\phi_2(z)\partial\phi_2(z) + i\frac{Q_o}{2}\partial^2\phi_1(z), \tag{2.2}$$

where the Q_o term corresponds to background charge $-Q_o$ at the point z_∞ [3],[7]. The central charge is

$$c = 2 - 3Q_o^2 \ , \tag{2.3}$$

and the normal ordered vertex operators

$$:\exp(i\alpha\phi_1(z))\exp(i\beta\phi_2(z)): \tag{2.4}$$

have conformal dimension [3]

$$\frac{\alpha}{2}(\alpha - Q_o) - \frac{\beta^2}{2} \ . \tag{2.5}$$

We suppress the normal ordering symbols : : for vertex operators, and products of vertex operators and fields $\partial^n\phi_i$ evaluated at the same point, in expressions below. Parafermions ψ_1 and $\psi_1^\dagger$ of conformal dimension Δ_1 are defined by the operator products [4]

$$\begin{aligned}\psi_1(z)\psi_1^\dagger(w) =& (z-w)^{-2\Delta_1}\left[1 + \frac{2\Delta_1}{c}(z-w)^2 T(w) + \cdots\right] \ ,\\ T(z)\psi_1(w) =& (z-w)^{-2}\left[\Delta_1\psi_1(w) + (z-w)\partial_w\psi_1(w) + \cdots\right] \ .\end{aligned} \tag{2.6}$$

In addition, the parafermions must have simple fusion rules [4], i.e. $[\psi_i][\psi_j] \sim [\psi_{i+j}]$. From the analysis of Dotsenko and Fateev [3], it is clear that parafermions constructed with vertex operators (2.4) with non-vanishing ϕ_1 charge α are likely to have non-trivial fusion rules, since the definition of the four-point function for such operators with $\alpha \neq 0$ will generically require insertion of screening operators. The contour choices available for the integration over the position screening operator will correspond to different conformal blocks. Therefore, to assure trivial fusion rules, we make the ansatz[2]

$$\begin{aligned}\psi_1 =& i(\alpha\partial\phi_1 + \beta\partial\phi_2)\ \mathrm{e}^{i\gamma\phi_2}\ ,\\ \psi_1^\dagger =& i(\alpha'\partial\phi_1 + \beta'\partial\phi_2)\ \mathrm{e}^{-i\gamma\phi_2}\ .\end{aligned} \tag{2.7}$$

The complex parameters $\alpha, \alpha', \beta, \beta'$, are completely determined in terms of γ by requiring that ψ_1 and $\psi_1^\dagger$ satisfy the operator products (2.6). The OPE's also fix the parameter γ to be a function of the background charge:

$$\frac{\gamma^2}{\gamma^2+1} = Q_o^2. \tag{2.8}$$

Note that γ^2 determines Δ_1

$$\Delta_1 = 1 - \gamma^2/2 \tag{2.9}$$

and that Q_o^2 determines c through eq. (2.3). Thus our ansatz implies that Δ_1 and c are not independent parameters but are related:

$$\Delta_1 = \frac{3}{2}\frac{c}{c+1}. \tag{2.10}$$

This is consistent with all the PF models obtained from $N = 2$ superconformal field theories and discussed in [17], but excludes the PF models discussed in appendix A of ref. [4]. To reproduce these models we need to consider a more general ansatz, such as adding a third boson to the construction presented here.

Finite reducibility of the algebra requires all of the operator product exponents of the theory be units of the fraction $1/N$ where N is an integer [17]. In particular, the relative monodromy of ψ_1 with $\psi_1^\dagger$ is defined by the integer b, such that the conformal dimension $\Delta_1 = 1 - b/N$. From the ansatz of eqn. (2.7) this requires $\gamma^2 = 2b/N$ and from the relation eqn. (2.8) the background charge is determined in terms of the monodromy parameters b, N

$$Q_o = \sqrt{\frac{2b}{N+2b}}\ . \tag{2.11}$$

The full expressions for the parafermions which satisfy the operator products (2.6)

[2] We can also arrive at this ansatz by demanding that we obtain Virasoro primary fields of dimension Δ_1 and then use the simplest solution which meets this criterion.

are

$$\psi_1(z) = i\sqrt{\frac{1}{2}}\left(\partial\phi_2(z) - \partial\phi_1(z)\sqrt{\frac{N+2b}{N}}\right)\exp(i\sqrt{2b/N}\phi_2(z))$$
$$\psi_1^\dagger(z) = -i\sqrt{\frac{1}{2}}\left(\partial\phi_2(z) + \partial\phi_1(z)\sqrt{\frac{N+2b}{N}}\right)\exp(-i\sqrt{2b/N}\phi_2(z)) \ . \tag{2.12}$$

Finite reducibility also restricts the form of the Virasoro primary states of eqn. (2.4). It is convenient to reparametrize these fields as

$$V_m^\ell(z) = \exp\left(\frac{-i\ell}{2b}Q_o\phi_1(z)\right)\exp\left(\frac{im}{2b}\sqrt{2b/N}\phi_2(z)\right) \ . \tag{2.13}$$

They have conformal dimension

$$\Delta_m^\ell = \frac{\ell(\ell+2b)}{4b(N+2b)} - \frac{m^2}{4bN} \ . \tag{2.14}$$

The FF-conjugate $V_m^{-\ell-2b}(z)$, and the ϕ_2 conjugate $V_{-m}^\ell(z)$ have the same conformal dimension as $V_m^\ell(z)$. Finite reducibility requires the operator product of parafermion with these vertex operators to have monodromy in units of $1/N$. This implies that m is an integer, and places no constraint on ℓ. This is easily seen by calculating the OPE of the parafermions and the V_m^ℓ, as is done in sec. 4.

Before we continue the discussion on the spectrum of the models, it is convenient to find the screening operators $\{\mathcal{J}\}$. Recall [3] that given $\mathcal{J}$, the contour integral $\oint \mathcal{J}$ can be inserted into correlation functions to balance the Coulomb charge without effecting the properties of the correlation function with respect to the chiral algebra. This requires the $\mathcal{J}$ to have conformal dimension one and operator products with $\psi_1(z)$ and $\psi_1^\dagger(z)$ which are total derivatives. Consider screening operators of the form

$$\mathcal{J} = i(A\partial\phi_1 + B\partial\phi_2)V_D^C \ , \tag{2.15}$$

with non-vanishing A or B. There are three solutions of this type, however only one is not a total derivative. (Screening operators $\mathcal{J} = \partial H$ are non-applicable since $\oint \mathcal{J}$ vanishes.) The remaining non-trivial solution is

$$J = i\partial\phi_2 V_0^{-2b} \ . \tag{2.16}$$

Since it is the screening charge which is important, J is only determined up to a total derivative term. It is also possible to find screening operators of the form V_D^C, which we denote as

$$\begin{aligned}\eta(z) =& V_N^{(N+2b)}(z) \ ,\\ \bar\eta(z) =& V_{-N}^{(N+2b)}(z) \ .\end{aligned} \tag{2.17}$$

These fields are dimension one fermions

$$\begin{aligned}\eta(z)\eta(w) &= (z-w) : \eta(z)\eta(w) := -\eta(w)\eta(z) \ ,\\ \bar\eta(z)\bar\eta(w) &= (z-w) : \bar\eta(z)\bar\eta(w) := -\bar\eta(w)\bar\eta(z) \ .\end{aligned} \tag{2.18}$$

Their charge conjugates are given by

$$\begin{aligned}\xi(z) =& V_{-N}^{-(N+2b)}(z) \ ,\\ \bar\xi(z) =& V_N^{-(N+2b)}(z) \ ,\end{aligned} \tag{2.19}$$

and these are conformal dimension zero fermions. The eqns. (2.17) and (2.19) are bosonic representations of two $c=-2$ fermion ghost systems [7] (η,ξ) and $(\bar\eta,\bar\xi)$. The representative OPE's are

$$\begin{aligned}\eta(z)\xi(w) =& (z-w)^{-1} + \cdots \ ,\\ \xi(z)\xi(w) =& (z-w)V_{-2N}^{-2(N+2b)}(w) + \cdots \ ,\\ \eta(z)\bar\xi(w) =& (z-w)^{-(1+N/b)}V_{-2N}^{0}(w) + \cdots \ ,\\ \eta(z)\bar\eta(w) =& (z-w)^{1+N/b}V_0^{2(N+2b)}(w) + \cdots \ ,\\ \xi(z)\bar\xi(w) =& (z-w)^{1+N/b}V_0^{-2(N+2b)}(w) + \cdots \ .\end{aligned} \tag{2.20}$$

The two boson Fock space can be reexpressed in terms of the (η,ξ) system and a commuting boson σ. Since the dimension zero operator V_N^N commutes with the (η,ξ) system, (the OPE's are non-singular), we define the boson σ such that

$$\begin{aligned}V_N^N(z) &= e^{a\sigma(z)} \ ,\\ \sigma(z)\sigma(w) &= -\ln(z-w) \ .\end{aligned} \tag{2.21}$$

This determines the constant a and the relation between σ and ϕ_1,ϕ_2

$$\begin{aligned}\sigma(z) =& -i\sqrt{N/2b} \ \ \phi_1(z) + i\sqrt{(N+2b)/2b} \ \ \phi_2(z) \ ,\\ a =& \sqrt{N/(N+2b)} \ .\end{aligned} \tag{2.22}$$

Since V_N^N is a dimension zero operator, the σ boson system has backgound charge $-ia$ and the stress tensor T_σ has central charge $c_\sigma = 1+3a^2$. This is the basis

Distler and Qiu [12] used for their analysis of the minimal ($b = 1$) character formulæ. Similarly, we define the boson $\bar{\sigma}$ orthogonal to the $(\bar{\eta}, \bar{\xi})$ system:

$$\begin{aligned} V^{N}_{-N}(z) = &\ e^{a\bar{\sigma}(z)} \ , \\ \bar{\sigma}(z) = &-i\sqrt{N/2b}\ \ \phi_1(z) - i\sqrt{(N+2b)/2b}\ \ \phi_2(z) \ . \end{aligned} \tag{2.23}$$

The U(1) currents of the two fermion systems are [7]

$$\begin{aligned} j(z) &= -\,\eta\xi \ , \\ \bar{j}(z) &= -\,\bar{\eta}\bar{\xi} \ . \end{aligned} \tag{2.24}$$

The charge operators j_0 and $\bar{j}_0$ count the fermion charge of the vertex operators (2.13) as

$$\begin{aligned} j_0[V^{\ell}_m] &= \frac{\ell - m}{2b} \ , \\ \bar{j}_0[V^{\ell}_m] &= \frac{\ell + m}{2b} \ . \end{aligned} \tag{2.25}$$

In particular, $j_0[\eta] = 1$, $j_0[\xi] = -1$. In appendix A, we discuss diagonalization of vertex operators with respect to the (η, ξ, σ) and the $(\bar{\eta}, \bar{\xi}, \bar{\sigma})$ systems.

Returning to the construction of the parafermion chiral algebra, we first observe that the parafermions (2.12) can be expressed as

$$\begin{aligned} \psi_1(z) &= n_{\psi} \oint \eta\ V^{-(N+2b)}_{-N+2b}(z) \ , \\ \psi^{\dagger}_1(z) &= n_{\psi^{\dagger}} \oint \bar{\eta}\ V^{-(N+2b)}_{N-2b}(z) \ . \end{aligned} \tag{2.26}$$

where the normalization coefficients $n_{\psi} = (b/N)^{\frac{1}{2}} = n_{\psi^{\dagger}}$, and the contour integrals are about the point z. In eq. (2.26), and below, we use the symbol $\oint$ as a shorthand for $\oint dz/(2\pi i)$. The parafermion chiral algebra [4] is defined by eqns. (2.6) and

$$\begin{aligned} \psi_1(z)\psi_p(0) =& z^{\Delta_1+\Delta_p-\Delta_{p+1}}\psi_{p+1}(0) + \cdots \ , \\ \psi^{\dagger}_1(z)\psi^{\dagger}_p(0) =& z^{\Delta_1+\Delta_p-\Delta_{p+1}}\psi^{\dagger}_{p+1}(0) + \cdots \ . \end{aligned} \tag{2.27}$$

These operator products characterize the action of parafermions on the identity highest weight ϕ_0 such that the states created (the ψ_p) are Virasoro highest weights. The representations of ψ_1 and $\psi^{\dagger}_1$ given by eqns. (2.26) enable us to solve for the ψ_p and $\psi^{\dagger}_p$ via the OPE's (2.27):

$$\begin{aligned} \psi_p(z) =& [n_{\psi}]^p \oint \eta\ V^{-(N+2b)}_{-N+2bp}(z) \ , \\ \psi^{\dagger}_p(z) =& [n_{\psi^{\dagger}}]^p \oint \bar{\eta}\ V^{-(N+2b)}_{N-2bp}(z) \ . \end{aligned} \tag{2.28}$$

We prove eqn. (2.28) inductively. Rewrite one of the OPE's (2.27) as

$$\psi_{p+1}(0) = \oint z^{2bp/N}\psi_1(z)\psi_p(0) \ . \tag{2.29}$$

Inserting the expressions for ψ_1 and ψ_p from (2.26) and (2.28) we find

$$\psi_{p+1} = [n_\psi]^{p+1} \oint dz_2 : \eta(z_2) : \oint dz_1 \oint dz f(z,z_1) : \eta(z_1) V_{-N+2b}^{-(N+2b)}(z) V_{-N+2bp}^{-(N+2b)}(0) : \tag{2.30}$$

where the contours C_0 are shown in figure 1, and the polynomial $f(z,z_1)$ is determinied by normal ordering the vertex operators which are at the points z, z_1 and 0 :

$$f(z,z_1) = z^{p+1} z_1^{-(p+1)} (z-z_1)^{-2} \ . \tag{2.31}$$

From figure 1, the configuration of contours can be written as a difference, $C_0 = C_1 - C_2$. The expression (2.30) vanishes for the configuration C_2 since the fermions at points z_1 and z_2 anticommute. The remaining configuration C_1 has the z_1 contour evaluated about the origin inside the z contour. If the $(z-z_1)^{-2}$ term of eqn. (2.31) is expanded about z_1, then the only part of $f(z,z_1)$ which survives both integrations over z and z_1 is $f \sim z^{-1} z_1^{-1}$. Evaluating these contours then reproduces the expression (2.28) for the case ψ_{p+1}.

3. Unitarity of the Chiral Algebra and Fermion Zero Modes

The conformal dimension for ψ_p and $\psi_p^\dagger$, given by

$$\Delta_p = p(N-bp)/N \tag{3.1}$$

is unbounded from below for positive b, as p becomes large. For a unitary identity module, these negative dimension states must decouple from the physical spectrum. To determine the constraints required for this decoupling we consider parafermion correlation functions.

Denote the correlator of operators $O_1, O_2, \ldots,$ in the bosonic theory with the presence of the background charge placed at the point at infinity via the insertion of the vertex operator $V_0^{2b}(z_\infty)$, as $\langle O_1, O_2, \ldots \rangle_{2b}$. It is clear that the two point correlator $\langle \psi_p^\dagger(z)\psi_p(w)\rangle_{2b}$, with the parafermion operators defined by eqns. (2.28) vanishes by a lack of ϕ_1 charge conservation. We therefore need to construct the correct conjugates for the parafermions ψ_p, $\psi_p^\dagger$ in the presence of background charge.

Consider the non-vanishing two point correlator

$$\begin{aligned} c_p =&\langle V_{-N+2bp}^{-(N+2b)}(z) V_{N-2bp}^{N}(0)\rangle_{2b} \\ =&z^{-2\Delta_p} \ . \end{aligned} \tag{3.2}$$

Introduce into the correlation function the factor $1 = \oint \eta(w')\xi(w)$ on the complex plane at radius between $|z|$ and infinity. Then deform the contour so that $\oint \eta(w')$ winds about the operators at the points z and 0 in the correlator. The operator products of $\eta(w')$ with the background charge vertex operator V_0^{2b} and the vertex operator V_{N-2bp}^{N} are both non-singular and mutually local; therefore the only non-vanishing winding of the contour integral is about the operator $V_{-N+2bp}^{-(N+2b)}$. Using the definition of ψ_p given by eqn. (2.28), we see that the correlator (3.2) is

$$c_p = n_{\psi}^{-p} \langle \xi(w) \psi_p(z) V_{N-2pb}^{N}(0)\rangle_{2b} \ . \tag{3.3}$$

To interpret the non-vanishing correlators, note that the parafermions given by eqns. (2.28) are independent of both zero modes, ξ_0 and $\tilde{\xi}_0$. For instance, the operator ψ_p is clearly independent of ξ_0 because the contour integral $\oint \eta = \eta_0$ projects out the ξ_0 dependence. Also, by explicitly diagonalizing ψ_p with respect to the $(\tilde{\eta}, \tilde{\xi}, \tilde{\sigma})$ system we find it independent of the zero mode $\tilde{\xi}_0$. Therefore the correlator c_p describes a mapping from the Hilbert space of parafermion operators, which does not include the zero mode ξ_0, to the Fock space of the bosonic theory. The operator $\xi(w)$ inserted into the correlator c_p soaks up the path integral over the zero mode.

At this point the analysis depends upon the value of the parameter b; in this paper we consider $b \geq 1$. In this case we must find a trunction of the parafermion algebra because the parafermion dimension Δ_p is unbounded from below. The natural ansatz is to require that the parafermion Hilbert space be independent of both zero modes ξ_0 and $\tilde{\xi}_0$. Again consider the correlator c_p given by eqn. (3.3). The $\tilde{j}$ charge of V_{N-2bp}^{N} is given by (see eqn. (2.25)) $\tilde{j}_0[V_{N-2bp}^{N}] = (N - bp)/b$. If $K = N/b$ is integral, then this operator can be rediagonalized in the $(\tilde{\eta}, \tilde{\xi}, \tilde{\sigma})$ basis. In this case, and if $p \leq N$ the $\tilde{j}$ charge is a positive integer and the operator is independent of $\tilde{\xi}_0$, (see appendix A). This is precisely the truncation required for non-negative conformal dimensions Δ_p ! In the correlator c_p both zero modes are soaked up by the operator $\xi(w)$. This is clear since the $\tilde{\xi}$ charge of ξ is $\tilde{j}_0[\xi] = -K - 1$. The expression for $\xi(w)$ in the $(\tilde{\eta}, \tilde{\xi}, \tilde{\sigma})$ basis is $\xi = \exp(\sqrt{K(K+2)}\ \tilde{\sigma})\ \tilde{\xi}\partial\tilde{\xi} \cdots \partial^K \tilde{\xi}$. The analogous expression holds for $\tilde{\xi}$ in terms of (η, ξ, σ).

We denote the holomorphic subspace of states in the boson Fock space as $\mathcal{H}$. Consider the subspace $\mathcal{H}_{\text{local}} \subset \mathcal{H}$ which contains states which are relatively local with respect to the (η, ξ) and $(\tilde{\eta}, \tilde{\xi})$ systems. States in this subspace can be rediagonalized with respect to the fermion systems as we have previously discussed. We define the small Hilbert space $\mathcal{H}_{\mathrm{small}} \subset \mathcal{H}_{\mathrm{local}}$ to be the restriction of states in $\mathcal{H}_{\mathrm{local}}$ to only those states which are independent of both fermion zero modes ξ_0 and $\tilde{\xi}_0$. When diagonalized in the (η, ξ, σ) basis, states in $\mathcal{H}_{\mathrm{small}}$ are independent of ξ_0, and when rediagonalized in the $(\tilde{\eta}, \tilde{\xi}, \tilde{\sigma})$ basis, they are independent of $\tilde{\xi}_0$. We can decompose the relatively local Hilbert space $\mathcal{H}_{\mathrm{local}}$ as

$$\mathcal{H}_{\mathrm{local}} = \mathcal{H}_{\mathrm{small}} \oplus \xi_0 \mathcal{H}_1 \oplus \tilde{\xi}_0 \mathcal{H}_2 \ . \tag{3.4}$$

Since states are in general not simultaneously diagonalizable in both fermion bases, there will be an overlap of states in the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We will construct the embedding of the irreducible parafermion modules into the small Hilbert space in sec. 6.

The condition that ψ_p and its conjugate V^N_{N-2bp} be in $\mathcal{H}_{\mathrm{local}}$ requires N/b to be an integer. The chiral algebra states ψ_p for $p > N$ decouple from other states in the $\mathcal{H}_{\mathrm{small}}$ because, by construction, there are no combination of states in the small Hilbert space which can be combined via fusion to form the conjugate field V^N_{N-2bp}, which for $p > N$ is proportional to $\tilde{\xi}_0$.

This analysis is for the case $b \geq 1$. The vertex operators we have discussed depend only upon the combination N/b, which is an integer. The models which can be constructed in this framework with $b > 1$ are in fact equivalent to the $b = 1$ models. Keeping N/b fixed, there are extra states in the $b > 1$ models without counterparts in the $b = 1$. However these are projected out of the spectrum since they are not in $\mathcal{H}_{\mathrm{local}}$. This is the subset of $b \geq 1$ models found to have associative parafermion four-point correlators [17]. We see that the Coulomb gas contruction and independence of both zero modes requires truncation of the $b \geq 1$ models to this subset.

For the parafermions $\psi^\dagger_p$ the non-vanishing two point correlator is

$$c^\dagger_p = n^{-p}_{\psi^\dagger} \langle \tilde{\xi}(w) \psi^\dagger_p(z) V^N_{-N+2pb}(0) \rangle_{2b} \ . \tag{3.5}$$

The identity operator ϕ_0 of the small Hilbert space has the non-vanishing two-point correlator

$$\begin{aligned} c_0 =& \langle \xi(w) \phi_0(z) V^N_N(0) \rangle_{2b} \ , \\ =& \langle \tilde{\xi}(w) \phi_0(z) V^N_{-N}(0) \rangle_{2b} \ , \\ =& 1 \ . \end{aligned} \tag{3.6}$$

The dimension zero operators V_N^N and V_{-N}^N are the conjugates of the identity operator in the small Hilbert space. The operators V_{N-2p}^N and V_{-N+2p}^N which appear in the correlators c_p and $c_p^\dagger$ above are, up to normalization, $\psi_p^\dagger$ and ψ_p acting on the identity conjugates

$$\begin{aligned} V_{N-2p}^N &\propto \oint \psi_p^\dagger(w) V_N^N(z) \ , \\ V_{-N+2p}^N &\propto \oint \psi_p(w) V_{-N}^N(z) \ . \end{aligned} \tag{3.7}$$

Note that the vertex operator V_{N-2p}^N is the conjugate of both $\psi_p^\dagger$ and ψ_{N-p} and we must identify $\psi_p^\dagger$ with ψ_{N-p}. We generalize this identification in the next section.

4. Parafermion Descendants and Virasoro Highest Weights for $b = 1$

In this section we consider the $b = 1$ models, which correspond to the unitary $SU(2)/U(1)$ GKO models [4],[20]. The irreducible modules are obtained from a finite set of highest weight primary conformal fields ϕ_q which satisfy the finite reducibility constraints. Define the monodromy parameters ω_q and $\omega_q^\dagger$ via the operator products between parafermions and PF/Virasoro highest weights

$$\begin{aligned} \psi_1(z)\phi_q(0) &= z^{-\omega_q} \tilde{\phi}_{q+2} \ , \\ \psi_1^\dagger(z)\phi_q(0) &= z^{-\omega_q^\dagger} \tilde{\phi}_{q-2}^\dagger \ . \end{aligned} \tag{4.1}$$

The independent parameters $\omega_q, \omega_q^\dagger \in R$ satisfy the finite reducibility constraints,

$$\begin{aligned} \omega_q &= + \, q/N \quad \bmod 1 \ , \\ \omega_q^\dagger &= - \, q/N \quad \bmod 1 \ , \\ & q \in \text{integers} \end{aligned} \tag{4.2}$$

and also satisfy the constraint that the ϕ_q are highest weights with repect to the PF algebra. This requires the states $\tilde{\phi}_{q+2}$ and $\tilde{\phi}_{q-2}^\dagger$ of eqns. (4.1) to have conformal dimension greater than or equal to the conformal dimension of the PF/Virasoro highest weight ϕ_q. This is equivalent to the constaints

$$\begin{aligned} \omega_q &\le 1 - 1/N \ , \\ \omega_q^\dagger &\le 1 - 1/N \ . \end{aligned} \tag{4.3}$$

We postulate that the only PF/Virasoro primary fields are vertex operators V_m^ℓ defined by eqn. (2.13), (up to application of screening charges). The OPE's of these operators with the parafermions (2.12) are easily evaluated, for general b, within the free field theory.

$$\begin{aligned}\psi_1(z)V^{\ell}_m(0) = \frac{n_\psi}{2b}\, z^{-m/N-1}\Bigg[&(\ell-m)\\ &+z\left(\sqrt{\frac{2b}{N}}\,(\ell-m+N)i\partial\phi_2(0)-\sqrt{2b(N+2b)}\,i\partial\phi_1(0)\right)\\ &+\mathcal{O}(z^2)\Bigg]V^{\ell}_{m+2b}(0)\ ,\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}\psi_1^\dagger(z)V^{\ell}_m(0) = \frac{n_{\psi^\dagger}}{2b}\, z^{+m/N-1}\Bigg[&(\ell+m)\\ &-z\left(\sqrt{\frac{2b}{N}}\,(\ell+m+N)i\partial\phi_2(0)+\sqrt{2b(N+2b)}\,i\partial\phi_1(0)\right)\\ &+\mathcal{O}(z^2)\Bigg]V^{\ell}_{m-2b}(0)\ .\end{aligned}\tag{4.5}$$

It is easy to see that for $b=1$ there do not exist any vertex operators that satisfy the highest weight constraints (4.2) and (4.3) unless $m=\ell$ or $m=-\ell$. In either case, (4.2) and (4.3) imply $\ell=0,1,2,\ldots,N-1$.

To see this let us first consider $m\geq 0$. The constraint on ω_q of eqn. (4.3) requires $\ell=m$ and $m<N$, as is seen by comparison of the explicit OPE eqn. (4.4) with the highest weight definition eqn. (4.1). The constraint $\omega^\dagger$ of eqn. (4.3) requires $m\geq 0$. These solutions are consistent with the OPE of eqn. (4.5). Now consider $m<0$. The constraints on $\omega^\dagger$ given by eqns. (4.3) and (4.5) require $\ell=-m$ and $m<N$. These solutions are consistent with the OPE of eqn. (4.4). The PF/Virasoro highest weights are labeled as

$$\begin{aligned}\phi_0 =&V^0_0\ ,\\ \phi_q =&V^q_q\ ,\\ \phi'_q =&V^{N-q}_{-(N-q)}\ ,\end{aligned}\tag{4.6}$$

where $1\leq q\leq N-1$.

In sec. 3 we discussed how unitarity of the bosonized PF theory required the parafermions and the PF Hilbert space to be independent of the zero modes ξ_0 ans $\tilde{\xi}_0$. Note that the highest weight states we derived above are also independent of both fermion zero modes. It is at first troubling that the FF-conjugate of a PF/Virasoro highest weight state is not a highest weight. However the criterion of zero mode independence provides the solution to this problem. By construction, the vertex operators ϕ_q and ϕ'_q have the same properties with respect to the chiral algebra, (i.e. the same conformal dimension and PF charge). We claim this is the Feigin-Fuchs double degeneracy in the bosonic construction of PF/Virasoro highest weights. FF-conjugation takes the highest weight state V_ℓ^ℓ to $V_\ell^{-\ell-2}$. However if the original vertex operator is independent of both zero modes, it is clear that the conjugate is not. (The fermionic charges $\tilde{j}_0$ and j_0 of V_ℓ^ℓ are positive or zero, whereas those of the conjugate are strictly negative.) To move back to the small Hilbert space we integrate $V_\ell^{-\ell-2}$ by the screening operator η or $\tilde{\eta}$. The easiest way to see which screening operator one should use is to demand that the new state be in the list of highest weight states (4.6). Since

$$\begin{aligned}\phi_q =& V_q^q = \oint \eta V_{-(N-q)}^{-(N-q)-2} = \oint \eta (\phi'_q)^{FF} \ , \\ \phi'_q =& V_{-(N-q)}^{N-q} = \oint \tilde{\eta} V_q^{-q-2} = \oint \tilde{\eta}(\phi_q)^{FF} \ ,\end{aligned} \tag{4.7}$$

we identify $\phi_q \sim \phi'_q$ in the PF Hilbert space.

The state ϕ_q is the holomorphic part of the order operator σ_q of the parafermion theory. Consider the action of the parafermions on the highest weights eqns. (4.6). In section 2, we studied this problem for the identity highest weight ϕ_0. A similar analysis for the other highest weights follows by use of the explicit expressions for the parafermions given by eqn. (2.26). We use the same notation as Lykken [17] to label the PF decendents which are Virasoro highest weights.

$$\begin{aligned}\phi_p^q(z) &= A_{(-N+q-1+2p)/N} \cdots A_{(-N+q+1)/N}\phi_q \ , \\ \tilde{\phi}_p^{N-q}(z) &= A^\dagger_{(-q-1+2p)/N} \cdots A^\dagger_{(-q+1)/N}\phi_q \ .\end{aligned} \tag{4.8}$$

$$\begin{aligned}\phi'^q_p(z) &= A_{(-N+q-1+2p)/N} \cdots A_{(-N+q+1)/N}\phi'_q \ , \\ \tilde{\phi}'^{N-q}_p(z) &= A^\dagger_{(-q-1+2p)/N} \cdots A^\dagger_{(-q+1)/N}\phi'_q \ .\end{aligned} \tag{4.9}$$

The operators A and $A^\dagger$ are the modes of the parafermions ψ_1 and $\psi_1^\dagger$, in the appropriate charge sector, defined in the usual way via contour integration [4]. The states ϕ_p^q and $\tilde{\phi}_p^{N-q}$ for $p = 0, 1, 2, \ldots$ are parafermion decendents of $\phi_q \equiv \phi_0^q = \tilde{\phi}_0^{N-q}$. The states ϕ'^q_p and $\tilde{\phi}'^{N-q}_p$ are parafermion decendents of $\phi'_q \equiv \phi'^q_0 = \tilde{\phi}'^{N-q}_0$ By applying the bosonized form of ψ_1 amd $\psi_1^\dagger$ given in (2.26), to (4.6) we find, (up to normalization constants),

$$\begin{aligned}\phi_p^q(z) &= \oint \eta V_{q-N+2p}^{q-(N+2)}(z) \ , \\ \tilde{\phi}_p^{N-q}(z) &= V_{q-2p}^q(z) \quad p \le q \ ,\end{aligned} \tag{4.10}$$

$$\begin{aligned} \phi'^q_p(z) &= V^{N-q}_{q-N+2p}(z) \quad p \leq N-q \ , \\ \bar{\phi}'^{N-q}_p(z) &= \oint \bar{\eta} V^{(N-q)-(N+2)}_{q-2p}(z) \ . \end{aligned} \tag{4.11}$$

It is clear that these equations are the correct result. They have the correct dimension and their bosonic ϕ_1 and ϕ_2 charges agree with (4.8), (4.9), (2.26), and (4.6). (For instance, ϕ^q_p has ϕ_2-charge $q - N + 2p + N$, where the last N is the ϕ_2-charge of η.) Note that the parafermions are $\phi^0_p = \psi_p$ and $\bar{\phi}'^0_p = \psi^\dagger_p$, and their conjugates are $\bar{\phi}^0_p = V^N_{N-2p}$ and $\phi'^0_p = V^N_{-N+2p}$. Also note that under the identification in (4.10) and (4.11), $\phi^q_0 = \bar{\phi}^{N-q}_0$ and $\phi'^q_0 = \bar{\phi}'^{N-q}_0$, as can be checked by explicit calculation.

The truncation of the index p of the states $\bar{\phi}^{N-q}_p$ and ϕ'^{N-q}_p can be seen in two ways. The first way is by explicit calculation using the OPE given by eqn. (4.5). The second way is by observing that the parafermion highest weights eqn. (4.6) are independent of both fermion zero modes, as are the parafermions. Therefore any parafermion decendents must also be independent of the zero modes. The fermion charges of $\bar{\phi}^{N-q}_p$ are $j_0 = p$ and $\bar{j}_0 = q - p$. Therefore if $p > q$, the state is not independent of $\bar{\xi}_o$, which is a contradiction.

We now show that there is a similar truncation for the ϕ^q_p and $\bar{\phi}'^q_p$ fields. The conformal dimension h^p_q of the fields ϕ^q_p , $\bar{\phi}^q_p$, ϕ'^q_p and $\bar{\phi}'^q_p$ is given by (2.14), with $\ell = N - q$ and $m = \ell - 2p$:

$$h^p_q = \Delta^{N-q}_{N-q-2p} = \frac{q(N-q)}{2N(N+2)} + \frac{p(N-q-p)}{N}, \tag{4.12}$$

The dimension of the $\bar{\phi}^q_p$, ϕ'^q_p fields is bounded by zero by the truncation described above. However, for large enough p these dimensions become negative for the ϕ^q_p, $\bar{\phi}'^q_p$ fields. Therefore, for unitary parafermion modules these states must decouple from the spectrum. This truncation is understood by considering the non-vanishing two-point function for ϕ^q_p

$$\langle \xi(w) \phi^q_p(z_1) \bar{\phi}^q_p(0) \rangle_{2b} = z^{-2h^q_p} \ . \tag{4.13}$$

The conjugate field to ϕ^q_p is therefore $\bar{\phi}^q_p$, which is independent of the fermion zero modes if $p \leq N - q$. Restricting to only those bosonic vertex operators in the small Hilbert space enforces this truncation. For $p > N - q$ there do not exist operators in the small Hilbert space that fuse onto $\bar{\phi}^q_p$. This assures that the ϕ^q_p with negative conformal dimensions decouple from the small Hilbert space.

In summary, we have

$$
\begin{aligned}
\phi^q_p(z) &= \oint \eta V^{q-(N+2)}_{q-N+2p}(z) \ , \\
\tilde{\phi}^q_p(z) &= V^{N-q}_{N-q-2p}(z) \ , \\
\phi'^q_p(z) &= V^{N-q}_{q-N+2p}(z) \ , \\
\tilde{\phi}'^q_p(z) &= \oint \tilde{\eta} V^{q-(N+2)}_{N-q-2p}(z) \ , \\
&\text{with } p \leq N-q \ .
\end{aligned}
\tag{4.14}
$$

We see that $p \leq N-q$ corresponds to the restrictions given in ref. [4] on the representation of the PF algebra.

There exist two Z_2 degeneracies in this parametrization. First, this parametrization is by construction redundant,

$$
\phi'^q_p(z) = \tilde{\phi}^q_{N-q-p}(z) \ . \tag{4.15}
$$

This redundancy requires us to identify the fields

$$
\phi^q_p(z) = \tilde{\phi}'^q_{N-q-p}(z) \ , \tag{4.16}
$$

because of the double degeneracy in the definition of the conjugate of $\tilde{\phi}^q_p$. For example, instead of defining the conjugate as in eqn. (4.13), one can insert the fermion $\tilde{\xi}(w)$ into the correlator to soak up the fermion zero modes. This leads to the field $\tilde{\phi}^q_{N-q-p}$ as the conjugate. If one classifies the PF Hilbert space $\mathcal{H}_{PF}$ by its L and M charge, $L = N-q$, $M = L-2p$, these identifications are the $\mathcal{H}^{L,M}_{PF} = \mathcal{H}^{N-L,N+M}_{PF}$ symmetry [29].

The second Z_2 symmetry is a degeneracy of states due to FF-conjugation symmetry. Our case is more subtle than the $c<1$ minimal models. FF-conjugation takes the vertex operator V^{ℓ}_m to $V^{-\ell-2}_m$. By looking at the fermionic charges j_0 and $\tilde{j}_0$ of V^{ℓ}_m, if the original vertex operator is independent of both zero modes, then its FF-conjugate is not. To define a conjugate in the small Hilbert space, we integrate $V^{-\ell-2}_m$ by the screening operator η or $\tilde{\eta}$. The choice of which screening operator we use is dictated, (up to the first Z_2 degeneracy), by the constraint that the resulting field be in the list given by eqn. (4.14). This leads to the identifications

$$
\begin{aligned}
\phi^q_p &= \phi'^q_p \ , \\
\tilde{\phi}^q_p &= \tilde{\phi}'^q_p \ .
\end{aligned}
\tag{4.17}
$$

5. Fusion Rules of $b=1$ Virasoro Highest Weights

To further understand the mechanism responsible for truncation of non-unitary states, we calculate the three-point functions of the Virasoro highest weights. More precisely, we will determine when the three-point functions do not vanish. This corresponds to the fusion rules for the states in the PF representation. Although the formalism developed allows us to calculate the four-point function, the calculation is not necessary to understand the truncation of the non-unitary states.

The PF/Virasoro highest weights of eqn. (4.6) are independent of both fermion zero modes ξ_0 ans $\bar{\xi}_0$. Since this is also the case for the parafermions, the entire parafermion module is in the small Hilbert space described in section 3. To calculate a PF correlator we must move to the boson Fock space which is where the path integral in terms of free bosons is defined. To do this we add $\xi(w)$ or $\bar{\xi}(w)$ into the correlator to soak up the zero modes. Also, to satisfy charge conservation in the bosonic theory, we can add powers of the screening operator $S=\oint J(y)$, where $J(y)$ is given by eqn. (2.16). This screening charge is also independent of both fermion zero modes (see appendix A).

Consider the correlator in the parafermionic theory

$$C^{q_1}{}_{q_2 q_3} = \langle \phi^{q_1}_{p_1}(z_1)\ \tilde{\phi}^{q_2}_{p_2}(z_2)\ \tilde{\phi}^{q_3}_{p_3}(z_3) \rangle \ . \tag{5.1}$$

We suppress the p_j labels on $C^{q_1}{}_{q_2 q_3}$ to avoid cluttering the notation. The indices are raised and lowered by the metric tensor $\delta^q_{q'}$ given by the two point function in the PF theory

$$\langle \phi^q_p(z) \tilde{\phi}^{q'}_{p'}(w) \rangle = \delta^q_{q'} (z-w)^{-2h^q_p} \ . \tag{5.2}$$

Again we have suppressed the p and p' index in the metric tensor. By $SL(2,C)$ invariance [5] the correlator is given by

$$C^{q_1}{}_{q_2 q_3} = c^{q_1}{}_{q_2 q_3} \prod_{\substack{i<j \\ k \neq i,j}} (z_i - z_j)^{[h_i + h_j - h_k]} \ , \tag{5.3}$$

where $c^{q_1}{}_{q_2 q_3} \in R$ are the operator product coefficient. Hence the fusion rules are

$$\tilde{\phi}^{q_2}_{p_2} \times \tilde{\phi}^{q_3}_{p_3} = \sum_{q_1} c^{q_1}{}_{q_2 q_3} \tilde{\phi}^{q_1}_{p_1} \ . \tag{5.4}$$

To find the appropriate representation of $C^{q_1}{}_{q_2 q_3}$ in the bosonic Fock space, we use the bosonic form of ϕ^q_p and $\tilde{\phi}^q_p$ given in (4.14), and write down the correlator in the boson Fock space by adding a ξ and screening operators.

$$C^{q_1}{}_{q_2q_3} \propto \langle \xi(w) \left[\prod_{i=1}^{t} S\right] \oint_{z_1} dz'\eta(z') V^{q_1-(N+2)}_{q_1-N+2p_1}(z_1)\, V^{N-q_2}_{N-q_2-2p_2}(z_2)\, V^{N-q_3}_{N-q_3-2p_3}(z_3)\, \rangle_{2b} \,. \tag{5.5}$$

Deform the contour integral over z' about the other vertex operators. It is clear that the OPE of η with V^{q}_{q-2p} is non-singular for $p \geq 0$. The OPE of η with the screening operator J is singular but once we perform the $\oint dz'$ integration we have a total derivative. Since $S = \oint J$ we get no contribution. Therefore the only non-vanishing contribution is $\oint \eta\xi(w) = 1$. It is convenient to define $L_j = N - q_j$, $M_j = L_j - 2p_j$, $j = 1, 2, 3$:

$$C^{q_1}_{q_2q_3} \propto \langle \left[\prod_{i=1}^{t} S\right] V^{-(L_1+2)}_{-M_1}(z_1)\, V^{L_2}_{M_2}(z_2)\, V^{L_3}_{M_3}(z_3)\, \rangle_{2b} \tag{5.6}$$

Taking into account the background charge, the bosonic charge constraints are

$$\begin{aligned} -L_1 + L_2 + L_3 &= 2t \,, \\ -M_1 + M_2 + M_3 &= 0 \,. \end{aligned} \tag{5.7}$$

The correlator eqn. (5.5) is not specified until the contours of integration in the definition of the screening operators are defined. For a three point function, there is only one linearly independent set of contours to choose. There are two ways of seeing this fact. One way is to choose the points $z_1 = 1$, $z_2 = 0$ and take the limit $z_3 \to \infty$ in eqn. (5.3). We write the correlator as $C^{q_1}{}_{q_2q_3} = \lim_{z_3\to\infty} c^{q_1}{}_{q_2q_3} z_3^{2\Delta_3}$. Therefore when the points z_1, z_2, z_3 are taken to these limits, the contour integrals over the screening operators reduce to integrals either over the inverval $[0, 1]$ or the interval $[1, \infty]$, (see ref. [3]). The integration over $[0, \infty]$ vanishes since it is up to a constant the integral about all three points, which can be contracted to zero. Therefore, all of the contours can be defined to be over the interval $[0, 1]$.

The second way to see that there is only one linearly independent choice of contours is to construct a homology basis of loops for a configuration of N^{th} root branch cuts with λ branch points. One result of this construction [30],[31] is the fact that independent of the monodromy $1/N$, there are $\lambda - 2$ linearly independent closed contours for points on the sphere. We can write the correlator above as

$$\text{corr} = \prod_{i=1}^{t} \oint_i f(y_1, \ldots y_t; z_j) \tag{5.8}$$

where we have labeled the points where the screening operators are inserted at y_i. For each contour integral over y_i, there are $\lambda - 2 = t$ choices of linearly independent closed contours. However, the mutual monodromy between the screening operators is given by $J(y_j)J(y_k) = e^{2\pi i2/(N+2)}J(y_k)J(y_j)$ as is easily seen by calculating their OPE. As a result, if two variables y_i are assigned the same contour, then the correlator vanishes. Since there are precisely as many linearly independent contours as points y_i to assign, and since the screening operators $J(y_i)$ are all the same, there is only one independent set of contours to choose.

Consider a subset of s contours, $s \leq t$ such that the integrals are well defined about one of the vertex operators at the points z_i of eqn. (5.5). We define these local states as Ω_i^s. If it is possible to define these states, then the correlator must vanish. Either the Ω_i^s will vanish identically, or they will correspond to null vectors, i.e. states which are simultaneously chiral algebra decendents and highest weights. The correlator of such null vectors with the other Virasoro primaries vanishes [5].

Suppose it is possible to define the state $\Omega_1^s = [\prod_{i=1}^s S]\, V_{-M_1}^{-(L_1+2)}(z_1)$. This state is well defined if the monodromy of each variable y_i with respect to z_1 is local. The condition is derived by dragging the operator $J(y_i)$ about z_1. Since the $J(y_k)$ are mutually non-local, each other $J(y_k)$ must be dragged about z_1 to restore the original $J(y_i)J(y_k)$ monodromy. The condition for a well defined state is

$$\tfrac{1}{2}s(s-1)\frac{2}{N+2} + s\frac{L_1+2}{N+2} - s = sI_s \ . \tag{5.9}$$

If the integer I_s is non-negative, the state Ω_1^s vanishes, if I_s is negative then it is a null vector [8],[24]. Thus whenever we can close the contour, we get a vanishing correlator. To have a non-vanishing correlator, the number of screening operator insertions t must be strictly less than the smallest value of s which satisfies eqn. (5.9) for any I_s. In the present case, that occurs when $I_s = 0$ and we find that the smallest value of s is $s_{\min} = N + 1 - L_1$. Thus $t \leq N - L_1$. The charge constraint (5.7) eliminates t in terms of of the L_1. Hence $L_1 \leq 2N - L_2 - L_3$. Since $t \geq 0$ we also get $L_1 \leq L_2 + L_3$. Putting this together we have

$$L_1 \leq \min\left[2N - L_2 - L_3, L_2 + L_3\right] \ . \tag{5.10}$$

Now consider the possible existence of the state $\Omega_2^s = [\prod_{i=1}^s S]\, V_{M_2}^{L_2}$. The condition for a well defined state is now

$$\tfrac{1}{2}s(s-1)\frac{2}{N+2} - s\frac{L_2}{N+2} - s = sI_s \ . \tag{5.11}$$

The smallest integer s for which this state is well defined corresponds to $I_s = -1$ and gives $s_{\min} = L_2 + 1$. Therefore $t \leq L_2$. In the same way the state Ω_3^s gives the constraint $t \leq L_3$. Putting these two constraints together with (5.7) yields

$$L_1 \geq |L_2 - L_3| \ . \tag{5.12}$$

Note also that eqn. (5.7) implies that

$$\begin{aligned} L_1 &= L_2 + L_3 \bmod 2 \ , \\ M_1 &= M_2 + M_3 \quad . \end{aligned} \tag{5.13}$$

Equations (5.10),(5.12) and (5.13) represent the non-trivial fusion rules for the fields in the correlator $C^{N-L_1}{}_{N-L_2,N-L_3}$. Note that the primary fields G^j_m in the $\widehat{su}(2)_N$ affine Kac-Moody theory [32] can be identified with the $\tilde{\phi}^q_p$ of the parafermion theory in the following way [4]:

$$G^j_m(z) = \tilde{\phi}^{N-2j}_{j-m}(z) \exp\left(im\sqrt{2/N}\ \phi_3(z)\right) \ , \tag{5.14}$$

where $\phi_3(z)$ is a free massless boson with the same signature as $\phi_1(z)$ but no background charge. We see that the $C^{N-L_1}{}_{N-L_2,N-L_3}$ correspond to the fusion rules of the $j_i = L_i/2$ and $m_i = M_i/2$ Kac-Moody primary fields [33]:

$$G^{L_2/2}_{M_2/2} \times G^{L_3/2}_{M_3/2} = \sum_{L_1} c^{N-L_1}{}_{N-L_2,N-L_3} G^{L_1/2}_{M_1/2} \ . \tag{5.15}$$

Similarly, we can calculate the fusion rules for the "primed" fields by examining the correlator $C'_{q_1}{}^{q_2 q_3} = \langle \tilde{\phi}'^{q_1}_{p_1}\ \phi'^{q_2}_{p_2}\ \phi'^{q_3}_{p_3} \rangle$. To represent this correlator in the bosonic theory, we must insert $\tilde{\xi}(w)$ into the correlator. Due to the global bosonic $\phi_2 \to -\phi_2$ symmetry of the theory, the calculation completely parallels that given above and the fusion rules are the same as above.

Now consider the "conjugate" case for the three point function

$$\begin{aligned} &C^{q_1 q_2}{}_{q_3} = \langle \phi^{q_1}_{p_1}(z_1)\ \phi^{q_2}_{p_2}(z_2)\ \tilde{\phi}^{q_3}_{p_3}(z_3) \rangle \\ &\propto \langle \xi(w) \left[\prod_{i=1}^{t} [S]\right] \oint_{z_1} dz' \eta(z') V^{-(L_1+2)}_{-M_1}(z_1)\ \oint_{z_2} dz'' \eta(z'') V^{-(L_2+2)}_{-M_2}(z_2)\ V^{L_3}_{M_3}(z_3)\ \rangle_{2b} \end{aligned} \tag{5.16}$$

Deform the contour integral over z'' about the other vertex operators. Again the only non-vanishing contribution is $\oint \eta\xi(w) = 1$. We deform the contour integral over z' so that it surrounds all of the vertex operators, including the screening operators.

$$C^{q_1 q_2}{}_{q_3} \propto \langle \oint_{\text{all}} dz' \eta(z') \left[\prod_{i=1}^{t} S\right] V^{-(L_1+2)}_{-M_1}(z_1)\ V^{-(L_2+2)}_{-M_2}(z_2)\ V^{L_3}_{M_3}(z_3)\ \rangle_{2b} \tag{5.17}$$

Again, we attempt to construct local states Ω_i^s and then require $t < s_{min}$. After some algebra, we find the constraints

$$
\begin{aligned}
N - M_1 =& M_2 - M_3 \ , \\
N - L_1 =& L_2 + L_3 \bmod 2 \ , \\
|L_2 - L_3| \leq N - L_1 \leq& \min\left[2N - L_2 - L_3, L_2 + L_3\right] \ .
\end{aligned}
\tag{5.18}
$$

This denotes the non-trivial constraints for non-vanishing operator product coefficient $C^{N-L_1,N-L_2}{}_{N-L_3}$. Similarly, the fusion rules for the correlator $C'_{q_1q_2}{}^{q_3}$ are calculated in the bosonic theory by adding $\tilde{\xi}(w)$ to the correlator and the constraints on the operator product coefficients are also given by eqns. (5.18).

We have derived the correct fusion rules [4],[33],[29] for the Virasoro primary fields ϕ_p^q, $\tilde{\phi}_p^q$, ϕ'^q_p, $\tilde{\phi}'^q_p$, $0 \leq p \leq N - q$ by analyzing the non-normalized three-point functions. They are given by

$$
\begin{aligned}
\tilde{\phi}_{p_2}^{q_2} \times \tilde{\phi}_{p_3}^{q_3} &= \sum_{q_1} C^{q_1}{}_{q_2q_3} \tilde{\phi}_{p_1}^{q_1} \ , \\
p_1 &= p_2 + p_3 + (q_2 + q_3 - N - q_1)/2 \ , \\
q_1 &= q_2 + q_3 - N \bmod 2 \ , \\
|N - q_2 - q_3| \leq q_1 &\leq N - |q_2 - q_3| \ , \\
\\
\phi_{p_2}^{q_2} \times \tilde{\phi}_{p_3}^{q_3} &= \sum_{q_1} C^{q_1q_2}{}_{q_3} \tilde{\phi}_{p_1}^{q_1} \ , \\
p_1 &= -p_2 + p_3 + (-q_2 + q_3 - q_1)/2 \ , \\
q_1 &= -q_2 + q_3 \bmod 2 \ , \\
|q_2 - q_3| \leq q_1 &\leq N - |N - q_2 - q_3| \ .
\end{aligned}
\tag{5.19}
$$

It is clear that the conformal blocks, the holomorphic part of four point functions, can be easily calculated and expressed in terms of contour integrals over rational polynomials.

6. Minimal $b = 1$ Parafermion Characters

The characters of the minimal parafermion models (the string functions) are well understood [34],[29] and frequently used as building blocks of string model building. In this section we will use the concepts developed in the previous sections to obtain these string functions. The small Hilbert space of the parafermion modules is independent of both fermion zero modes ξ_0 $\tilde{\xi}_0$. It is therefore necessary to include only the states of the full bosonic Fock space which are not proportional to these modes. It is also necessary to subtract the states in the modules of any null vectors of the parafermion theory. We construct the irreducible PF character for the module $\{\tilde{\phi}_p^q\}_{\rm irr}$ of the Virasoro highest weights $\tilde{\phi}_p^q$ in each PF charge sector. The construction of the PF character for $\{\phi_p^q\}_{\rm irr}$ is similar.

Consider the state $\tilde{\phi}_p^q = V_M^L$, where $L = N - q$ and $M = N - q - 2p$, with conformal dimension $h_p^q = \Delta_M^L$ given by eqn. (4.12). The character of the boson module $[V_M^L]$ is

$$\chi_M^L = \frac{q^{\Delta_M^L - c/24}}{[\varphi(q)]^2} , \tag{6.1}$$

where $q = e^{i\pi\tau}$, $\tau \in C$, and $\varphi(q) = \prod_{n=1}^{\infty}(1 - q^n)$. A power series expansion of χ_M^L with respect to q about $q = 0$ is the sum of states in the Hilbert space, where each state is weighted by it's "Boltzman factor" of $e^{i\pi L_0 \tau}$. The factor $q^{-c/24}$, where c is the central charge (2.3), represents the vacuum energy contribution, and the factor $[\varphi(q)]^{-2}$ is the contribution of the bosonic oscillators. Following Feigin and Fuchs [2], we assume that for the values of c given by (2.3) the only null vectors in the bosonic theory are parafermion null vectors.

Let $\{V_M^L\}$ be the submodule of $[V_M^L]$ which contains only decendents of V_M^L restricted to the small Hilbert space $\mathcal{H}_{\rm small}$, and denote $\widehat{\chi}_M^L$ as its character.[3]By construction, the sum of this character and similar characters shifted by the charges of the zero modes must be the boson character χ_M^L

$$\chi_M^L = \widehat{\chi}_M^L + (\xi_0)\widehat{\chi}_{M+N}^{L+(N+2)} + (\tilde{\xi}_0)\widehat{\chi}_{M-N}^{L+(N+2)} , \tag{6.2}$$

where $(\tilde{\xi}_0), (\xi_0)$ denote the charges of the zero modes. The conformal dimensions Δ_{M+N}^{L+N+2} and Δ_{M-N}^{L+N+2} of the "second level" vertex operators are greater than Δ_M^L. (If their conformal dimensions were less than Δ_M^L then they would not be included in eqn. (6.2).) Equation (6.2) begins a recursion relation for the subcharacter $\widehat{\chi}_M^L$. The next level for the recursion relation is generated by

$$\begin{aligned}\chi_{M+N}^{L+N+2} + \chi_{M-N}^{L+(N+2)} =& \widehat{\chi}_{M+N}^{L+N+2} + \widehat{\chi}_{M-N}^{L+(N+2)} \\ &+ (\xi_0)\widehat{\chi}_{M+2N}^{L+2(N+2)} + (\tilde{\xi}_0)\widehat{\chi}_{M-2N}^{L+2(N+2)} + ((\xi_0) + (\tilde{\xi}_0))\widehat{\chi}_M^{L+2(N+2)} .\end{aligned} \tag{6.3}$$

[3] There are states in $\mathcal{H}_{\rm small}$ which decouple because their conjugates are not in $\mathcal{H}_{\rm small}$. We examined some of these in sections 3 and 4. They are in charge sectors which are not counted. More precisely, we are calculating the reducible character $\widehat{\chi}_M^L$ which counts states in $\mathcal{H}_{\rm small}$ whose conjugates are also in $\mathcal{H}_{\rm small}$.

The last term $\widehat{\chi}_M^{L+2(N+2)}$ is counted once and denotes an intersection of the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ defined by eqn. (3.4). Again, note that the conformal dimensions of the third level, $\Delta_{M+2N}^{L+2(N+2)}$, $\Delta_{M-2N}^{L+2(N+2)}$, $\Delta_M^{L+2(N+2)}$ are greater than the dimensions of the second level. The character $\widehat{\chi}_M^L$ is found recursively by continuing this process

$$\widehat{\chi}_M^L = \sum_{r,s=0}^{\infty} (-1)^{r+s} \chi_{M+(s-r)N}^{L+(r+s)(N+2)} \ . \tag{6.4}$$

This expression is obtained by adding and subtracting alternating levels, accounting for intersections of modules, and including only characters at level $X+1$ with higher conformal dimension of their highest weights than at level X. This is shown in figure 2.

This is not the end of the analysis for the irreducible character because of the existence of PF null vectors in the modules $\{\tilde{\phi}_p^q\}$. These null vectors are the remaining null vectors of the PF module. We have already taken into account most of the null vectors in the theory. In particular, there are "boundary states", such as $V_{\pm(L+1)}^L$, which are not in $\mathcal{H}_{\rm small}$. They were eliminated by the construction of $\{\tilde{\phi}_p^q\}$. The null vectors we need to indentify below are only those which exist in $\mathcal{H}_{\rm small}$. An explicit construction of null vectors for the minimal conformal series was first given by Feigin and Fuchs [2]. The construction below follows the later analysis of Kato and Matsuda [24], which we apply to the $SU(2)$ parafermion case. We have discussed in section 2 the screening operators $\eta, \bar{\eta}$, and J given by eqns. (2.17) and (2.16). We can use them to construct null vectors in $\mathcal{H}_{\rm small}$ since their screening charges are independent of both zero modes. Consider first the use of the $\eta, \bar{\eta}$ to construct null states in the module $\{\tilde{\phi}_r^q\}$. These are given by the states

$$\begin{aligned}
\tilde{\Omega}^{(\eta)L}_M(0) &= \oint_0 dz \eta(z) V_{M-N}^{L-(N+2)}(0) \ , \\
\tilde{\Omega}^{(\bar{\eta})L}_M(0) &= \oint_0 dz \bar{\eta}(z) V_{M+N}^{L-(N+2)}(0), \\
\tilde{\Omega}^{(\eta\bar{\eta})L}_M(0) &= \oint dz_i \eta(z_1)\bar{\eta}(z_2) V_M^{L-2(N+2)}(0) \ .
\end{aligned} \tag{6.5}$$

Multiple applications of the fermions vanish by the anti-commutation property. The states in (6.5) are by construction highest weights in the module of V_M^L, however we will now show that they vanish identically. To see this consider $\tilde{\Omega}^{(\eta)L}_M$; for it not to vanish, the operator product of η with the vertex operator at $z=0$ must have singularity z^{ω}, where ω is a negative integer. One finds $\omega = p-1$, so that the null vector is non-trivial only for $p=0$ in which case it is equal to the original highest weight state $\tilde{\phi}_0^q$. Similarly $\tilde{\Omega}^{(\tilde{\eta})L}_M$ is identically vanishing or trivial. The third state $\tilde{\Omega}^{(\eta\tilde{\eta})L}_M$ is identically zero because, for the highest weights in consideration, the monodromy of the variable z_1 about the origin is a positive integer. When that is the case, the contour integration gives zero. It is useful to state the values of L and M required for non-vanishing null states

$$\begin{aligned} \tilde{\Omega}^{(\eta)L}_M &\neq 0 \quad (L-M) \leq 0 \;, \\ \tilde{\Omega}^{(\tilde{\eta})L}_M &\neq 0 \quad (L+M) \leq 0 \;, \\ \tilde{\Omega}^{(\eta\tilde{\eta})L}_M &\neq 0 \quad (L \pm M) \leq 0 \;. \end{aligned} \tag{6.6}$$

The non-trivial null vectors therefore require the use of the screening operator J. Consider null vectors of the form

$$\tilde{\Omega}^{(r)L}_M = \left[\prod_{i=1}^{r} S\right] V_M^{L+2r} \;, \tag{6.7}$$

where S is a contour integral over J. By construction these null vectors are independent of both fermion zero modes $\xi_0, \tilde{\xi}_0$. These states are non-vanishing if

$$\tfrac{1}{2}r(r-1)\frac{2}{N+2} - r\frac{L+2r}{N+2} - r = -r(n_1+1) \;, \tag{6.8}$$

where n_1 is a positive integer. Represent each null vector of this type by the vertex operator used to construct it, (i.e. represent $\tilde{\Omega}^{(r)L}_M$ by V_M^{L+2r}). Then we have found null vectors represented by $V_M^{n_1 2(N+2)-(L+2)}$. Each of these null vectors has a degenerate module. The construction described above can be used to find the null vectors of the null vector modules by replacing $L \to 2n_1(N+2)-(L+2)$. These new null vectors are represented by the vertex operator $V_M^{(n_1+n_2)2(N+2)+L}$. Similarly each of these null vectors has a degenerate module, however the form of this next level of null vectors is equivalent to the first set of null vectors.

Now consider the embedding of the null vector modules. The null vector with lowest conformal dimension is the $n_1=1$ case of the first set and represented by the vertex operator $V_M^{2(N+2)-(L+2)}$. The first null vector of its module is the $n_2=0$ case of the second set and is represented by $V_M^{2(N+2)+L}$. The full embedding diagram for the null vector in the two series above is given in figure 3. For each null vector represented by $V_M^{L'}$, one can construct the states $\tilde{\Omega}^{\eta L'}_M$, $\tilde{\Omega}^{\tilde{\eta} L'}_M$, and $\tilde{\Omega}^{\eta\tilde{\eta} L'}_M$. However, by the constraints given by eqn. (6.6), since $L' > L$ these states vanish identically.

As discussed in section 4, there exists a Z_2 degeneracy in the bosonic Fock space for each Virasoro primary of the PF theory. This is the FF-conjugate symmetry of the $SU(2)$ PF theory, $\tilde{\phi}_p^q = \tilde{\phi}'^q_p$. So we must also consider the construction of null vectors using the "primed" representation of the field. We define

$$\tilde{\Omega}'^{(r)L}_{M} = \oint dz\tilde{\eta}(z) \left[\prod_{i=1}^{r} S\right] V_M^{L-(N+2)+2r} . \tag{6.9}$$

The non-vanishing null vectors are represented by the vertex operator $V_M^{L+2n'_1(N+2)}$. The next level of null vectors is represented by $V_M^{(n'_1+n'_2)2(N+2)-(L+2)}$. We see therefore that these null vectors of the "primed" representation are contained in the series given by fig. 3. More specifically, they are the Feigin-Fuchs Z_2 symmetry null vectors in $\{V_M^L\}$. This result differs significantly from the unitary minimal conformal series case (case IIIb of ref. [2].) In the minimal conformal series case, the first null vector of the "Feigen-Fuchs conjugate" representation of the highest weights is not in the module of the first null vector of the original representation. This generates the "ladder" embedding diagram of the minimal conformal modules. The source of the difference in our case is the extra $-r$ term on the left side of eqn. (6.8). This comes from the operator product of the $\partial\phi_2$ terms in the definition of the screening operator J with the highest weight vertex operators in the construction of the null vectors.

The correct character of the module $\{\tilde{\phi}_p^q\}_{\rm irr}$ is obtained by subtraction of the modules for each of the null vectors in the diagram fig. 3. Since all of the null vectors are in the module of the first null vector represented by $V_M^{2(N+2)-(L+2)}$, we subtract only its module. The character (string function) is given by

$$\tilde{c}_p^q = \hat{\chi}_M^L - \hat{\chi}_M^{2(N+2)-(L+2)} , \tag{6.10}$$

which can also be written as

$$\begin{aligned}\tilde{c}_p^q =& q^{\Delta_M^L - c/24} \left[\varphi(q)\right]^{-2} \sum_{r,s=0}^{\infty} (-1)^{r+s} q^{\frac{1}{2}s(s+1)+\frac{1}{2}r(r+1)+sr(N+1)} \\ &\left[q^{\frac{1}{2}r(L+M)+\frac{1}{2}s(L-M)} - q^{N+1-L+\frac{1}{2}r(2N+2-L+M)+\frac{1}{2}s(2N+2-L-M)}\right] .\end{aligned} \tag{6.11}$$

This expression was first given by Distler and Qiu [12], however our derivation differs significantly from theirs. Since $\tilde{\phi}_p^q \sim \tilde{\phi}'^q_p$, $\tilde{c}_p^q = \tilde{c}'^q_p$. Similarly $c_p^q = c'^q_p$. Finally, the global bosonic ϕ_2 charge symmetry $m \leftrightarrow -m$ implies $\tilde{c}_p^q = c'^q_p$ and $c_p^q = \tilde{c}'^q_p$.

7. Interpretation as BRST Cohomology

We now consider the relationship between our analysis and the BRST cohomology [25] analysis of the $SU(2)$ PF theory given by Distler and Qui [12]. Their representation of the PF operators uses the (η, ξ, σ) basis introduced in sec. 2 and discussed further in appendix A. In their analysis, the physical PF Hilbert space is assummed to be in the submodule $\widehat{\mathcal{H}}$ of the boson Fock space $\mathcal{H}$ which is independent of the ξ zero mode. The boson Fock space has the decomposition $\mathcal{H} = \widehat{\mathcal{H}} \oplus \xi_0 \widehat{\mathcal{H}}$, and a BRST operator

$$\mathcal{Q} = \oint \eta(z) \prod_{n=1}^{N+2} \oint J(w_n) \ , \tag{7.1}$$

with the property $\mathcal{Q}^2 = 0$, maps states from $\widehat{\mathcal{H}}$ to $\widehat{\mathcal{H}}$. By calculating the cohomology $\mathrm{ker}\mathcal{Q}/\mathrm{im}\mathcal{Q}$, they found the characters of the PF theory.

The Hilbert space $\widehat{\mathcal{H}}$ of [12] is related to $\mathcal{H}_{\mathrm{small}}$ defined in section 3 by $\widehat{\mathcal{H}} = \mathcal{H}_{\mathrm{small}} + \tilde{\xi}_0 \mathcal{H}_{\mathrm{small}}$. In our analysis the reducible PF modules consist of the states in $\mathcal{H}$ which are independent of both zero modes, and whose conjugates are also independent of both zero modes. These modules are a subset of $\mathcal{H}_{\mathrm{small}}$. Restricting to states in the kernel of $\mathcal{Q}$ corresponds to these reducible modules. In addition to this, moding out by the image of $\mathcal{Q}$ is equivalent to subtracting the PF null vector module from $\{\phi^q_p\}$.

We begin by showing that moding out by states in the image of $\mathcal{Q}$ is equivalent subtracting the first PF null vector module. Recall that in sec. 6 we calculated the character of the PF module $\{\tilde{\phi}^q_r\}_{\mathrm{irr}}$. Alternatively, we could have directly calculated the character of $\{\phi^q_r\}_{\mathrm{irr}}$. Consider the null vectors of the module $\{\phi^q_r\}$, given by

$$\Omega^{(r)N-L}_{\ \ N-M} = \oint dz \eta(z) \left[\prod_{i=1}^{r} S \right] V^{2r-(L+2)}_{-M} \ , \tag{7.2}$$

where $L = N - q$, $M = L - 2r$. Following the analysis of sec. 6, the first series of null vectors is given by $r = n_1(N+2) - 1 + (L+2)$, and represented by the vertex operator $V^{L+2n_1(N+2)}_{-M}$. The case $n_1 = 0$ corresponds to $r = L+1$ and the "null vector" in this case is actually the original highest weight ϕ^q_r. The second series of null vectors is represented by the vertex operators $V^{(n_1+n_2)2(N+2)-(L+2)}_{-M}$. All null vectors are in these series. In particular, the lowest dimension null vector is in the second series and is given by

$$\Omega^{(N+2)\ N-L}_{\ \ \ \ \ \ N-M} = \mathcal{Q} V^{2(N+2)-(L+2)}_{-M} \ . \tag{7.3}$$

Namely, the first null vector is in $\mathrm{im}\mathcal{Q}$. The other PF null vectors are in $\{\Omega^{(N+2)\ N-L}_{\ \ \ \ \ \ N-M}\}$ and hence also in $\mathrm{im}\mathcal{Q}$.

The BRST operator is not unique. The cohomology of the conjugate BRST operator

$$\tilde{Q} = \oint \tilde{\eta}(z) \prod_{n=1}^{N+2} \oint J(w_n) \tag{7.4}$$

generates the irreducible module of $\tilde{\phi}^q_r = \tilde{\phi}'^q_r$ as we have shown in section 6.

We have shown that the PF null vectors of $\{\phi^q_p\}$ are in im$Q$. This together with the fact that our PF character agrees with [12], establishes the equivalence of our reducible PF module to ker$Q$. Although we do not prove this directly, we present an argument which shows one of the relationships between our analysis and ker$Q$.

Consider the vertex operator V^ℓ_m in ker$Q$. It is in $\mathcal{H}_{\rm small}$ when its fermion charges j_0 and $\bar{j}_0$ are non-negative integers. This requires $\ell \pm m \geq 0$, and $\ell = m$ mod 2 (see eqn (2.25)). We now show that if $\ell \leq N$ then the conjugate of V^ℓ_m is also in $\mathcal{H}_{\rm small}$. This agrees with the analysis of ref. [12], where $V^\ell_m \in \ker Q$ only if $\ell \leq N$.

Following the analysis of sections 3 and 4, we find the conjugate of V^ℓ_m in $\mathcal{H}_{\rm small}$ by introducing a factor $1 = \oint \eta(w')\xi(w)$ into the non-vanishing two point correlator.

$$z^{-2\Delta^\ell_m} = \langle V^\ell_m(z) V^{-(\ell+2)}_{-m}(0)\rangle_{2b} = \langle \xi(w) V^\ell_m(z) \oint \eta(w') V^{-(\ell+2)}_{-m}(0)\rangle_{2b} \ . \tag{7.5}$$

There are actually two possible representations of the conjugate, given by $\oint \eta V^{-(\ell+2)}_{-m}$ and $\oint \tilde{\eta} V^{-(\ell+2)}_{-m}$, which require insertion of either $\xi(w)$ or $\tilde{\xi}(w)$ to balance the bosonic charges and soak up fermion zero modes. These states are in $\mathcal{H}_{\rm small}$ (independent of both fermion zero modes) if

$$S^\ell_m = \oint \tilde{\eta} \oint \eta V^{-(\ell+2)}_{-m} \tag{7.6}$$

vanishes. For instance, if $\oint \eta V^{-(\ell+2)}_{-m}(0)$ is proportional to $\tilde{\xi}(0)$, then multiplication by $\tilde{\eta}(w')$, and integration over w' will be non-vanishing because of the $\tilde{\eta}(w')\tilde{\xi}(0)$ OPE. The state S^ℓ_m vanishes if the monodromy of either η and $\tilde{\eta}$ about the origin is a non-negative integer

$$2N - (\ell \pm m) \geq 0 \ . \tag{7.7}$$

Hence if $\ell \pm m \leq 2N$, the representations of the conjugate are in $\mathcal{H}_{\rm small}$. Since $\ell \pm m \geq 0$, the bound $\ell \pm m \leq 2N$ is saturated when $m = \ell$, and we have $\ell \leq N$. Note that we cannot define S^ℓ_m itself to be the conjugate of V^ℓ_m; insertion of both $1 = \oint \eta(w')\xi(w)$ and $1 = \oint \tilde{\eta}(y')\tilde{\xi}(y)$ does not lead to the definition of a local conjugate state because $\oint \tilde{\eta}(w')\xi(w) \neq 0$.

Note that both Q and $\tilde{\xi}_0$ have the same dimension and bosonic ϕ_1 and ϕ_2 charge. This is not unexpected given that the restriction of physical states to the ker$Q$ is equivalent to the restriction of physical states to $\mathcal{H}_{\rm small}$. Our zero mode $\tilde{\xi}_0$ plays the opposite role of Q on the states in $\widehat{\mathcal{H}}$; state which are annihilated when multiplied by $\tilde{\xi}_0$ are not in the ker$Q$.

8. Discussion and Conclusions

We have developed the Feigin-Fuchs Coulomb gas construction to the point where it can be used to derive, versus reconstruct, the structure of unitary irreducible representations. One of the keys to the analysis was the correct treatment of fermion zero modes. We developed our treatment of the zero modes by demanding that all negative dimension states decouple from the theory. This led us to describe the physical PF Hilbert space as the subset of the two boson Hilbert space which was mutually local with respect to both the (η, ξ) and $(\tilde{\eta}, \tilde{\xi})$ system and independent of both the ξ_0 and $\tilde{\xi}_0$ zero mode. We discussed the connection between this Hilbert space and the BRST constraint that all physical state be in the kernel of the BRST charge Q. Recall that in our approach physical states and their conjugates must be in the small Hilbert space if the chiral algebra is to be unitary. Therefore there exists an explicit connection between unitarity and the BRST constraint $\phi_{\rm phys} \in \ker Q$.

We have shown that our treatment reproduces the correct fusion rules and character formula. One advantage of our approach is that less assumptions are required to define the representation theory. In particular, we derived the PF highest weights, and used unitarity as the criterion to justify construction of the space $\mathcal{H}_{\rm small}$. The irreducible characters were found by subtracting the modules of all possible null vectors (constructed from screening operators), from the reducible PF characters. This leads naturally to the identification of a BRST operator.

Our goal was to determine how well the Feigen-Fuchs construction could derive representation theory. A greater test will be to apply the formalism to models for which the representation theory is not well understood, such as the $SU(1,1)$ PF models [23]. We have previously discussed the connection between these models and $N = 1$ spacetime SUSY. However, it is also of interest to find conformal field theories (exact string solutions) corresponding to string propagation on a non-flat space-time manifold with Lorentzian signature, such as de Sitter space, which is the coset manifold $= SO(4,1)/SO(3,1)$. It is natural to use the GKO constructions G/H, with G or both G and H as non-compact Kac-Moody algebras, to try to understand such models. The simplest of these constructions are the $SU(1,1)$ PF models. It is not obvious that the GKO construction G/H should be identified with the coset manifold G/H. The correct identification between CFT's and target manifolds requires a matching of the physical states in perturbation theory. To find the representation theory of these models, the Feigen-Fuchs construction will clearly be a valuable tool.

Appendix A. Mapping Between Different Coulomb Gas Representations

In this appendix we discuss the relationship between the different bosonizations in the literature [9],[10],[11],[12],[13],[14]. We will see that the relationship is basically a change of basis in the two boson space. Though the bosonization presented in [9],[10],[11],[12],[13] is only for the minimal case, in this appendix we give a unified treatment of both the $SU(2)$ and $SU(1,1)$ parafermion models.

In the Distler and Qiu basis the parafermionic energy momentum tensor is given by

$$T(z) = T_\sigma(z) + T_{\eta\xi}(z) \tag{A.1}$$

where

$$\begin{aligned} T_\sigma(z) &= -\frac{1}{2}(\partial_z\sigma(z))^2 + \frac{a}{2}\partial_z^2\sigma(z) \\ T_{\eta\xi}(z) &= -\frac{1}{2}(\partial_z\chi(z))^2 + \left(\frac{-i}{2}\right)\partial_z^2\chi(z) \;=\; -\eta(z)\partial_z\xi(z) \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} a &\equiv iq_0 = \sqrt{N/(N+2b)} \\ \eta(z) &\equiv e^{i\chi(z)} \qquad \dim\eta = 1 \\ \xi(z) &\equiv e^{-i\chi(z)} \qquad \dim\chi = 0 \end{aligned} \tag{A.3}$$

The $\eta\xi$ OPE is

$$\eta(z)\xi(w) = (z-w)^{-1} + \text{finite parts} \tag{A.4}$$

In order that the energy momentum tensor in (A.1) agree with the one in (2.2), we must have

$$\begin{aligned} \phi_1(z) &= -i\sqrt{N/2b}\;\;\sigma(z) - \sqrt{(N+2b)/2b}\;\;\chi(z) \\ \phi_2(z) &= \pm\left(-i\sqrt{(N+2b)/2b}\;\;\sigma(z) - \sqrt{N/2b}\;\;\chi(z)\right) \end{aligned} \tag{A.5}$$

Inverting this relationship we get

$$
\begin{aligned}
\sigma(z) &= -\, i\sqrt{N/2b}\ \ \phi_1(z) \pm i\sqrt{(N+2b)/2b}\ \ \phi_2(z) \\
\chi(z) &= -\, \sqrt{(N+2b)/2b}\ \ \phi_1(z) \pm \sqrt{N/2b}\ \ \phi_2(z)
\end{aligned}
\tag{A.6}
$$

Taking the $(+)$ sign gives the relationship between σ, χ and ϕ_1, ϕ_2 whereas taking the $(-)$ sign gives the relationship between $\tilde{\sigma}$, $\tilde{\chi}$ and ϕ_1, ϕ_2. The $\pm$ in the above equations reflects the $\phi_2 \to -\phi_2$ symmetry that exists in the energy momentum tensor. From the field $\tilde{\chi}$ one gets the $\tilde{\eta}$ screening operator and $\tilde{\xi}$ zero mode in the same way the χ gave the η screening operator and ξ zero mode;

$$
\begin{aligned}
\tilde{\eta}(z) &\equiv e^{i\tilde{\chi}(z)} \qquad \dim \tilde{\eta} = 1\ , \\
\tilde{\xi}(z) &\equiv e^{-i\tilde{\chi}(z)} \qquad \dim \tilde{\chi} = 0\ , \\
\tilde{\eta}(z)\tilde{\xi}(w) &= (z-w)^{-1} + \text{finite parts}\ .
\end{aligned}
\tag{A.7}
$$

By induction we can also show that

$$
\begin{aligned}
\eta(z)\partial_z\eta(z)\ldots\partial_z^{n-1}\eta(z) &\propto\ e^{in\chi(z)} \\
\xi(z)\partial_z\xi(z)\ldots\partial_z^{n-1}\xi(z) &\propto\ e^{-in\chi(z)} \\
\partial_z\xi(z)\ldots\partial_z^{n}\xi(z) &\propto \partial_z^n(\ e^{-in\chi(z)})
\end{aligned}
\tag{A.8}
$$

And similarly for the tilded basis. Using (A.8) we have that the vertex operators defined in (2.13) become

$$
\begin{aligned}
V_m^\ell(z) &=\ \exp\left((m/a - \ell a)\,\sigma(z)/2\right)\ \exp\left(i\,(\ell - m)\,\chi(z)/2\right)\ , \\
&=\ \exp\left((-m/a - \ell a)\,\tilde{\sigma}(z)/2\right)\ \exp\left(i\,(\ell + m)\,\tilde{\chi}(z)/2\right)\ .
\end{aligned}
\tag{A.9}
$$

Thus we see that when the j ($\bar{j}$) charge is non-negative, the vertex operator is independent of the ξ ($\tilde{\xi}$) zero mode. In the minimal case when $b=1$, $(\ell \pm m)$ is a positive even integer for the states in the PF representation and we have

$$
\begin{aligned}
V_m^\ell(z) &\propto\ \exp\left((m/a - \ell a)\,\sigma(z)/2\right)\eta(z)\partial_z\eta(z)\ldots\partial_z^{(\ell-m)/2}\eta(z)\ , \\
&\propto\ \exp\left((-m/a - \ell a)\,\tilde{\sigma}(z)/2\right)\tilde{\eta}(z)\partial_z\tilde{\eta}(z)\ldots\partial_z^{(\ell+m)/2}\tilde{\eta}(z)\ .
\end{aligned}
\tag{A.10}
$$

One can also show that in the η, ξ basis the expression for the parafermions become

$$
\begin{aligned}
\psi_1(z) =& i\sqrt{\frac{b}{N}}\ \partial_z\chi(z)\exp\left(\sigma(z)/a - i\chi(z)\right) = -\sqrt{\frac{b}{N}}\ \partial_z\xi(z)\mathrm{e}^{\sigma(z)/a} \\
\psi_1^\dagger(z) =& -\left(\sqrt{\frac{N+2b}{b}}\ \partial_z\sigma(z) - i\frac{N+b}{b}\sqrt{\frac{b}{N}}\partial_z\chi(z)\right)\exp\left(-\sigma(z)/a + \chi(z)/2\right) \\
=& -\sqrt{\frac{b}{N}}\left(\frac{N}{b}\eta(z)\partial_z(\mathrm{e}^{-\sigma(z)/a}) + \frac{N+b}{b}\partial_z\eta(z)\right)\mathrm{e}^{-\sigma(z)/a}
\end{aligned}
\tag{A.11}
$$

When written in terms of the tilded basis the above expressions are the same except that the role of ψ_1 and $\psi_1^\dagger$ is switched.

There are three screening operators in our Coulomb gas representation. They are η, $\tilde{\eta}$, and J. Equation (2.16) defines J. In terms of the (η,ξ), $(\tilde{\eta},\tilde{\xi})$ bases it is

$$
\begin{aligned}
J =& \sqrt{\frac{N}{2}}\left(\frac{1}{a^2}\left(\partial \mathrm{e}^{a\sigma}\right)\xi + \mathrm{e}^{a\sigma}\partial\xi\right)\ , \\
=& -\sqrt{\frac{N}{2}}\left(\frac{1}{a^2}\left(\partial \mathrm{e}^{a\tilde{\sigma}}\right)\tilde{\xi} + \mathrm{e}^{a\tilde{\sigma}}\partial\tilde{\xi}\right)\ .
\end{aligned}
\tag{A.12}
$$

Note that up to a total derivative, J is equal to the screening operator presented in [12]. We also have that

$$
\begin{aligned}
J \propto& \ \mathrm{e}^{a\sigma}\partial\xi + \text{total derivative}\ , \\
\propto& \ \mathrm{e}^{a\tilde{\sigma}}\partial\tilde{\xi} + \text{total derivative}\ .
\end{aligned}
\tag{A.13}
$$

The relevent quantity is the screening charge $S = \oint J$, and (A.13) shows that S is independent of both zero modes.

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Figure Captions

Fig. 1: Manipulation of the contour configuration C_0 into $C_1 - C_2$, leads to straightforward evaluation of ψ_{p+1}.

Fig. 2: Graphical solution for the character $\widehat{\chi}^L_M$ is obtained by adding and subtracting alternating rows on the diagram.

Fig. 3: Embedding diagram of null vectors in $\{V^L_M\}$.

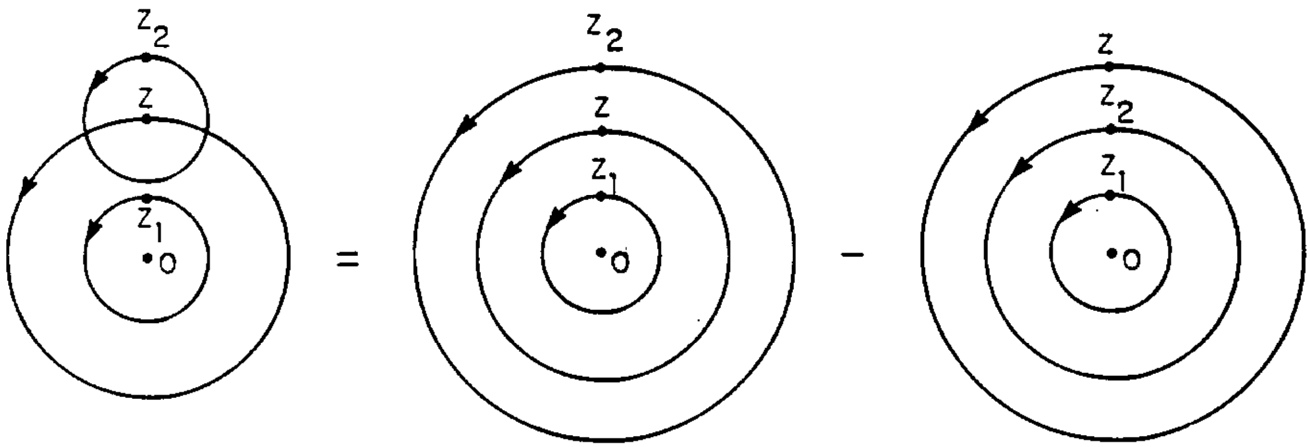

Figure 1

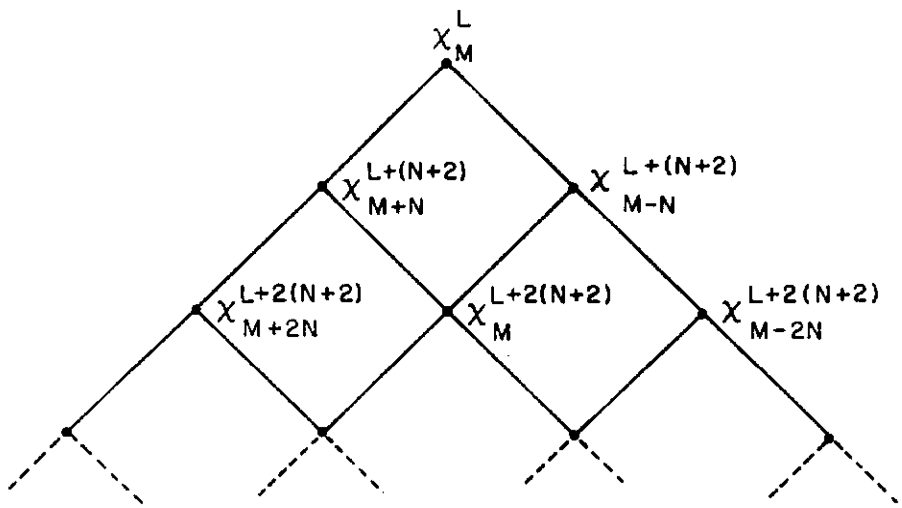

Figure 2

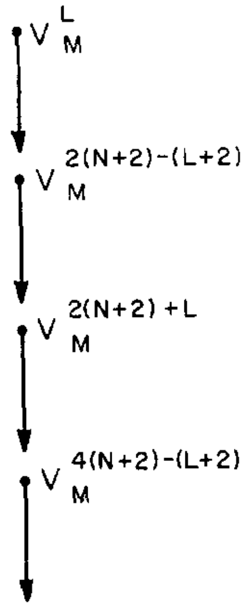

Figure 3