FERMILAB-PUB-90/116-A

Origin of density fluctuations in extended inflation

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(Received 8 June 1990)

We calculate both the curvature and isocurvature density fluctuations that arise due to quantum fluctuations in a simple model of extended inflation based upon the Jordan-Brans-Dicke theory. The curvature fluctuations that arise due to quantum fluctuations in the Brans-Dicke field in general have a non-scale-invariant spectrum and an amplitude that is cosmologically acceptable and interesting without having to tune any coupling constant to a very small value. The curvature perturbations that arise due to the Higgs field are subdominant. If there are other massless fields in the theory, e.g., an axion or an ilion, then isocurvature fluctuations arise in these fields too. Production of gravitational waves and the massless particles associated with excitations of the Brans-Dicke field are also discussed. Several attempts at more realistic models of extended inflation are also analyzed. The importance of the Einstein conformal frame in calculating curvature fluctuations is emphasized. When viewed in this frame, extended inflation closely resembles slow-rollover inflation with an exponential potential, and the usual formula for the amplitude of curvature perturbations applies directly.

I. INTRODUCTION

Extended inflation is a very interesting variation on both old[1] and slow-rollover[2] inflation. In old inflation the inflaton field was the Higgs field responsible for grand-unified-theory (GUT) symmetry breaking; while in slow-rollover inflation, in order to achieve density perturbations of an acceptably small level, the inflaton field had to be a very weakly coupled gauge singlet. In extended inflation it is possible for inflation to be associated with GUT symmetry breaking, thereby once again "tying" inflation to a cosmological phase transition. Models[3] of extended inflation have been based upon alternative gravity theories where the value of the gravitational constant is determined by the value of some scalar field, the simplest theory being that of Jordan and Brans and Dicke.[4]

In extended inflation it is crucial that the field that determines the gravitational constant, which we shall refer to as the Brans-Dicke field, vary significantly during the inflationary epoch. The field that precipitates inflation, which we shall refer to as the Higgs field σ, does so because it gets hung up in a false-vacuum state—a local, but not global, minimum of its scalar potential. While the σ field is hung up, the Universe expands very rapidly—as a large power of time, but not exponentially—owing to the false-vacuum energy and the varying "gravitational constant." It is crucial that the scale factor not grow exponentially, so that the probability (per Hubble volume per Hubble time) of nucleating a bubble of true vacuum, $\epsilon(t) \sim \Gamma/H^4(t) \propto t^4$, increases with time (here Γ is the bubble nucleation rate). At the start of extended inflation ϵ is small so the σ field remains trapped in the false vacuum; when it increases to order unity the phase transition ends by the nucleation of true-vacuum bubbles. The lack of a "graceful exit" back to a radiation-dominated universe that plagued old inflation is circumvented by the variation of ϵ. Reheating is accomplished by bubble collisions and should, unlike reheating in slow-rollover inflation, be very efficient. [The requisite variation of $\epsilon(t)$ can also be accomplished if $H=$const and the vacuum transition rate Γ varies with time; this occurs in some models currently under study.[5]]

Density perturbations certainly arise as remnants of the bubbles that are nucleated during the phase transition; these perturbations have been addressed elsewhere.[6] While it is possible that the density perturbations that arise due to the bubbles are interesting, it seems uncertain: If bubble nucleation turns on rapidly, there will be very few bubbles of cosmologically interesting size; if bubble nucleation turns on slowly, there will be too many large bubbles to be consistent with the isotropy of the cosmic microwave background radiation[6] (CMBR). Unless the bubble nucleation rate is just so, it is not possible for relic bubbles to be both interesting and consistent with the isotropy of the CMBR. In any case, we will focus on the density fluctuations that arise due to quan-

tum fluctuations in the various fields in the theory during extended inflation. For comparison, in slow-rollover inflation it is these fluctuations in the inflaton field that lead to the dominant density perturbations: scale-invariant (Harrison-Zel'dovich) curvature perturbations [by scale-invariant (or Harrison-Zel'dovich) spectrum, we mean the spectrum of perturbations characterized by constant amplitude at horizon crossing: $(\delta\rho/\rho)=\text{const}$], and that also necessitate a very small coupling constant for the inflaton field.[7] Curvature perturbations arise in extended inflation, but they are not quite scale invariant (they have a power-law spectrum), and they arise due to fluctuations in the Brans-Dicke field (the field whose value controls the value of the gravitational constant). Most importantly, no dimensionless parameter needs to be set to a very small value to ensure that they are of an acceptable, or even interesting, size: The amplitude of the fluctuations is determined by the ratio of the unification scale to the Planck scale.

In this paper we compute these perturbations by a conformal transformation to the Einstein frame, the frame where the gravitational constant is constant. In this frame, extended inflation closely resembles slow-rollover inflation, with the Brans-Dicke field playing the role of the inflaton with an exponential potential:[8] For this reason, the formulas derived for curvature fluctuations and graviton production in slow-rollover inflation are directly applicable. We also address the production of massless Brans-Dicke particles and the isocurvature fluctuations that can arise if there are other massless fields in the theory, such as an axion or an ilion. Finally, we analyze density perturbations in several recent attempts to construct realistic models of extended inflation.

II. BRANS-DICKE FIELD FLUCTUATIONS

A. Some extended-inflation basics

For simplicity we consider the original La-Steinhardt model[3] of extended inflation. The theory derives from the action

$$S=\int d^4x\sqrt{-g}\left[-\frac{\mathcal{R}}{16\pi}\Phi+\frac{\omega}{16\pi}g^{\mu\nu}\frac{\partial_\mu\Phi\partial_\nu\Phi}{\Phi}+\mathcal{L}_{\text{matter}}\right]$$
$$=\int d^4x\sqrt{-g}\left[-\frac{\mathcal{R}}{8\omega}\phi^2+\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\mathcal{L}_{\text{matter}}\right], \tag{2.1}$$

where the Brans-Dicke field Φ sets the value of the "effective" gravitational constant: $G=\Phi^{-1}$. The Brans-Dicke field Φ, which has dimensions of mass squared, can be written in terms of a scalar field ϕ with dimensions of mass, curvature coupling $\xi=-1/4\omega$, and a canonical kinetic term $\Phi=2\pi\phi^2/\omega$. [We warn the reader that rewriting the action in terms of ϕ can be misleading: The kinetic term for ϕ *appears* canonical, but because of the absence of the usual $-\mathcal{R}/16\pi G_N$ term, gravity is not canonical, and by integrating by parts, derivatives of the metric tensor (from $\mathcal{R}$) may be shifted to the ϕ kinetic term.] Fluctuations in Φ are related to those in ϕ by

$$\delta\Phi=\sqrt{8\pi\Phi/\omega}\,\delta\phi\ . \tag{2.2}$$

This fact will be of some utility later.

This theory serves only as a toy model since the temperature fluctuations in the CMBR that arise due to the distribution of bubble sizes require that $\omega\lesssim 20$,[6] while precision solar-system tests of the theory require $\omega\gtrsim 500$.[9] However, this model will serve well to illustrate the salient features of the density fluctuations that arise in extended inflation.

The matter part of the Lagrangian includes the Higgs field σ and all other matter fields: $\mathcal{L}_{\text{matter}}=(\partial_\mu\sigma)^2/2-V(\sigma)+\cdots$. During extended inflation the σ field sits quietly in the false vacuum, and affects the dynamics only through the vacuum energy density that it contributes to the energy density of the Universe: $\rho_{\text{vac}}=V(\sigma=0)\equiv M^4$, where the energy scale M characterizes the energy scale of the phase transition.

The equation of motion for Φ and the Friedmann equation are

$$\ddot{\Phi}+3H\dot{\Phi}-\frac{1}{a^2}\nabla^2\Phi=\frac{8\pi}{2\omega+3}(\rho-3p)\ ,$$
$$H^2\equiv\frac{\dot{a}^2}{a^2}=\frac{8\pi\rho}{3\Phi}+\frac{\omega}{6}\frac{\dot{\Phi}^2}{\Phi^2}-H\frac{\dot{\Phi}}{\Phi}\ . \tag{2.3}$$

During extended inflation $\rho\simeq\rho_{\text{vac}}\equiv M^4$, $p\simeq-\rho_{\text{vac}}$, and the scale factor a and Φ evolve as

$$a(t)=a_0(1+Bt)^{\omega+1/2}\Longrightarrow a_0(Bt)^{\omega+1/2}\quad(\text{for } Bt\gg 1)\ ,$$
$$\Phi(t)=\Phi_0(1+Bt)^2\Longrightarrow\Phi_0B^2t^2\quad(\text{for } Bt\gg 1)\ , \tag{2.4}$$

where B is defined in terms of ω, M, and the value of Φ at the start of inflation ($\equiv\Phi_0$) by

$$B\Phi_0^{1/2}=\frac{1}{q\omega}M^2\ ,\quad q=\left[\frac{(6\omega+5)(2\omega+3)}{32\pi\omega^2}\right]^{1/2}\ . \tag{2.5}$$

Eq. (2.4) implies that during inflation, the expansion rate is time dependent:

$$H\equiv\frac{\dot{a}}{a}=\frac{(\omega+\frac{1}{2})B}{1+Bt}\Longrightarrow\frac{\omega+\frac{1}{2}}{t}\quad(\text{for } Bt\gg 1)\ . \tag{2.6}$$

Since there is little variation in Φ during the matter or radiation-dominated regimes, the value of Φ at the end of inflation is approximately equal to its value today, the Planck mass squared:

$$\Phi_e\simeq G_N^{-1}\equiv m_{\text{Pl}}^2\simeq\Phi_0B^2t_e^2=\frac{t_e^2M^4}{q^2\omega^2}\Longrightarrow t_e\equiv q\omega\frac{m_{\text{Pl}}}{M^2}\ , \tag{2.7}$$

where the time t_e corresponds to the end of extended inflation.

Around this slightly ill-defined time the σ field makes the transition to the true vacuum through the rapid nucleation of Coleman–De Luccia bubbles, and bubble collisions reheat the Universe to a temperature of the order M. The quantity q is a dimensionless constant of order unity and for $\omega\gg 1$, $q\to\sqrt{3/8\pi}$. Throughout we shall reserve G_N for the present value of the gravitational con-

stant, and the Planck mass $m_{\rm Pl} \equiv G_N^{-1/2} = 1.22 \times 10^{19}$ GeV. (Since the value of Φ at the end of inflation will affect our results for the amplitude of the density perturbations, one should keep in mind the possibility that in a more realistic model Φ might evolve significantly after inflation, in which case Φ_e would be very different from $m_{\rm Pl}^2$. We will consider models where this occurs in Sec. IV.)

B. Production of fluctuations

The physical wavelength of a linear perturbation grows with the scale factor of the Universe: $\lambda_{\rm phys} \propto a(t)$. During inflation, a given perturbation begins sub-Hubble sized and then crosses outside the "horizon" [i.e., $\lambda_{\rm phys} \simeq H(t)^{-1}$]; later, during the matter- or radiation-dominated epoch, it crosses back inside the horizon ($\lambda_{\rm phys} \simeq H^{-1}$ again). We will need to know the time t that a fluctuation of present physical wavelength λ crossed outside the horizon during extended inflation; in terms of this time, λ is given by

$$\lambda = \frac{M}{2.75K} \frac{a(t_e)}{a(t)} H^{-1}(t) , \tag{2.8}$$

where the reheat temperature is assumed to be M and $a(t_0)/a(t_e) = M/2.75$ K (i.e., we assume that the Universe evolved adiabatically after inflation and ignore the variation in g_*, so that $T \propto a^{-1}$). Writing $\lambda = \lambda_{\rm Mpc}$ Mpc $\simeq \lambda_{\rm Mpc} 10^{38}$ GeV^{-1} and taking $a(t_e)/a(t) \simeq (t_e/t)^{\omega+1/2}$ it follows that

$$\begin{aligned} \lambda_{\rm Mpc} &= 10^{-25} \frac{q m_{\rm Pl}}{M} (t_e/t)^{\omega-1/2} , \\ \frac{t_e}{t} &= 10^{25/(\omega-1/2)} \left[\frac{M}{q m_{\rm Pl}} \right]^{1/(\omega-1/2)} \lambda_{\rm Mpc}^{1/(\omega-1/2)} . \end{aligned} \tag{2.9}$$

It is interesting to exhibit the effective value of the gravitational coupling G as a function of epoch when the fluctuation of wavelength λ "went outside the horizon":

$$\frac{G}{G_N} = \frac{m_{\rm Pl}^2}{\Phi} = \left[\frac{t_e}{t} \right]^2 = \left[10^{25} \frac{M}{q m_{\rm Pl}} \lambda_{\rm Mpc} \right]^{2/(\omega-1/2)} , \tag{2.10}$$

for $\omega = 10$ and $M = 10^{14}$ GeV, $G/G_N \simeq 10^4 \lambda_{\rm Mpc}^{0.21}$. In addition, since the bubble nucleation rate per Hubble volume $\epsilon(t) = \Gamma/H^4 \simeq (t/t_e)^4$, we can express $\epsilon(t)$ in terms of the scale that is leaving the horizon at time t instead of t:

$$\epsilon(t) \propto \lambda_{\rm Mpc}^{-4/(\omega-1/2)} . \tag{2.11}$$

As bubble nucleation "switches on," say ϵ increases from 0.1 to 1, a range of scales cross outside the horizon: From the relation above we see that the logarithmic interval of scales $[\Delta(\ln\lambda)]$ that cross outside the horizon as bubble nucleation commences is proportional to $(\omega - \frac{1}{2})/4$. This implies that the range of bubble sizes expected varies exponentially with ω, and one can easily appreciate why there is an upper bound to ω from bubble nucleation.

Now let us compute the horizon-crossing amplitude of a fluctuation in the Brans-Dicke field (i.e., when it crosses outside the horizon during extended inflation). We can *estimate* its amplitude by setting the fluctuation amplitude in the equivalent field ϕ equal to the value of $H/2\pi$ at the epoch of horizon crossing:

$$\frac{\delta\Phi_\lambda}{\Phi} \simeq \omega^{-1/2} 10^{50/(\omega-1/2)} \times q \left[\frac{M}{q m_{\rm Pl}} \right]^{(2\omega+1)/(\omega-1/2)} \lambda_{\rm Mpc}^{2/(\omega-1/2)} . \tag{2.12}$$

(Since ϕ is only minimally coupled in the limit that $\omega \gg 1$, $\delta\phi = H/2\pi$ is only technically correct in this limit.) We see that the size of the fluctuation can be large, just like the value of $m_{\rm Pl}^2/\Phi$, and for the same reason: During extended inflation, Φ can be very small compared to its present value. Moreover, we see that in the limit of exponential inflation, i.e., $\omega \gg 1$, the spectrum of fluctuations becomes "flat", that is, independent of λ, as one would expect. Finally, the amplitude of the Φ fluctuations decreases to zero as $\omega \to \infty$ (in the limit of $\omega \to \infty$, the Brans-Dicke field Φ freezes out and the theory becomes general relativity).

C. Evolution of Brans-Dicke-field fluctuations

Before we go on to analyze the curvature (density) perturbations let us first consider the evolution of the fluctuations in the Brans-Dicke field itself (the isocurvature perturbations). During extended inflation Φ grows as t^2; using Eq. (2.3), it is simple to show that super-horizon-sized fluctuations in Φ grow as t. Thus $\delta\Phi_\lambda/\Phi$ decreases as t^{-1} during extended inflation. During the radiation- and matter-dominated epochs that follow extended inflation the value of Φ remains roughly constant. Likewise, it is simple to show that super-horizon-sized fluctuations in Φ also remain approximately constant. Once a fluctuation in Φ reenters the horizon, it follows from Eq. (2.3) that its amplitude decreases as $a^{-1}(t)$. For fluctuations that reenter the horizon during the present matter-dominated epoch ($\lambda \gtrsim 13$ Mpc), the decrease in their amplitude until today is given by $a(t_H) \simeq 10^{-6} \lambda_{\rm Mpc}^2$, where t_H is the time when the fluctuation crossed back inside the horizon and the value of the scaler factor today is $a_0 = 1$. For fluctuations that cross back inside the horizon during the radiation-dominated epoch ($\lambda \lesssim 13$ Mpc), the decrease in amplitude is $a(t_H) \simeq 10^{-5} \lambda_{\rm Mpc}$.

Using all of these facts we can compute the present amplitude of the fluctuations in the Brans-Dicke field. For fluctuations of present wavelength less than about 13 Mpc,

$$\frac{\delta\Phi_\lambda}{\Phi} = \omega^{-1/2} 10^{-5+25/(\omega-1/2)} \times q \left[\frac{M}{q m_{\rm Pl}} \right]^{2\omega/(\omega-1/2)} \lambda_{\rm Mpc}^{(\omega+1/2)/(\omega-1/2)} , \tag{2.13}$$

while, for fluctuations of present wavelength greater than about 13 Mpc,

$$\frac{\delta\Phi_\lambda}{\Phi}=\omega^{-1/2}10^{-6+25/(\omega-1/2)}\times q\left[\frac{M}{qm_{\rm Pl}}\right]^{2\omega/(\omega-1/2)}\lambda_{\rm Mpc}^{2\omega/(\omega-1/2)}\ . \qquad (2.14)$$

Again, we see that for the interesting values of M and ω the fluctuations are of interesting amplitude; e.g., for $\omega=10$ and $M=10^{14}$ GeV,

$$\frac{\delta\Phi_\lambda}{\Phi}=3\times10^{-14}\lambda_{\rm Mpc}^{1.1}\ \ (\lambda\lesssim 13\ \text{Mpc})\ ,$$
$$\frac{\delta\Phi_\lambda}{\Phi}=3\times10^{-15}\lambda_{\rm Mpc}^{2.1}\ \ (\lambda\gtrsim 13\ \text{Mpc})\ . \qquad (2.15)$$

On scales less than that of the present horizon, $\lambda\lesssim 3000$ Mpc, the fluctuations in Φ correspond to massless Φ particles; while on the largest scales, $\lambda\gtrsim 3000$ Mpc, they correspond to spatial fluctuations in the gravitational constant. The consequences of such fluctuations in the Brans-Dicke field, and any resulting constraints, remain to be discussed. Brans-Dicke field fluctuations should have numerous effects, including contributing energy density, causing temperature fluctuations in the CMBR, affecting the timing of the millisecond pulsar, and possibly affecting various precision solar-system tests of general relativity. However, because the model we are considering is truly a toy model which most certainly needs modification—perhaps making the Brans-Dicke field massive, or even massive and unstable—we will not consider the (isocurvature) fluctuations in Φ any further.

The fluctuations in the Brans-Dicke field that arise during inflation also correspond to energy-density fluctuations and thus give rise to fluctuations in the spacetime geometry (curvature fluctuations) as a given mode crosses outsize the horizon during inflation. We consider curvature fluctuations next.

D. Curvature fluctuations

Since the effective source of Newtonian gravity is proportional to $G\rho$ and $G\rho\propto\rho/\Phi$, one might expect that fluctuations in Φ give rise to density fluctuations of a similar amplitude. As we shall see this is essentially correct. While it is tempting to try to analyze the production of curvature fluctuations in the frame of Eq. (2.1), known as the *Jordan conformal frame,* because the effective gravitational constant is varying [for interesting models the value of G today is several orders of magnitude smaller than it is during inflation, cf. Eq. (2.10)] and because the fluctuating field, the Brans-Dicke field Φ, is not minimally coupled, such a procedure is suspect.

The surest way to analyze curvature fluctuations is to work in a conformally rescaled frame where the gravitational part of the action takes the usual Einstein-Hilbert form. This frame is known as the *Einstein conformal frame.* The rescaling to the Einstein conformal frame is accomplished by the conformal transformation

$$\bar g_{\mu\nu}=\Omega^{-2}(t)g_{\mu\nu}\ ,\quad \Omega^2=m_{\rm Pl}^2/\Phi\ ,\quad \Psi=\Psi_0\ln(\Phi/m_{\rm Pl}^2)\ , \qquad (2.16)$$

where $\Psi_0^2=(2\omega+3)m_{\rm Pl}^2/16\pi$. In the Einstein frame the action is given by

$$\bar S=\int d^4x\sqrt{-\bar g}\left[-\frac{\bar{\mathcal{R}}}{16\pi G_N}+\tfrac{1}{2}\bar g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi+\exp(-\Psi/\Psi_0)\bar g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma-\exp(-2\Psi/\Psi_0)M^4\right]\ , \qquad (2.17)$$

where an overbar indicates the value of a quantity in the Einstein frame and as usual $G_N=m_{\rm Pl}^{-2}$ is the present value of the gravitational constant. We will assume that the Higgs field σ is anchored in the false vacuum so that its kinetic term can be neglected; the only effect of the σ field is to contribute a false-vacuum energy whose value in the Einstein frame is $\exp(-2\Psi/\Psi_0)M^4$. Note too that at late times, $t\gg t_e$, when $\Phi\simeq m_{\rm Pl}^2$, the conformal factor $\Omega\to 1$, so that the Jordan and Einstein frames become equivalent. (Since Φ grows with time, the conformal factor decreases monotonically to 1.) During extended inflation it is simple to show that[10]

$$\bar a(\bar t)=\bar a_0(1+C\bar t)^{\omega/2+3/4}\ ,$$
$$\bar H=\frac{C(\omega/2+3/4)}{1+C\bar t}\longrightarrow\frac{\omega/2+3/4}{\bar t}\quad(\text{for } C\bar t\gg 1)\ , \qquad (2.18)$$
$$1+C\bar t=(1+Bt)^2\ ,$$

where $2B/C=\sqrt{\Phi_0/m_{\rm Pl}^2}$. These facts will be of use shortly.

In the Einstein frame the Brans-Dicke field Ψ takes on the appearance of a minimally coupled scalar field with a potential $V(\Psi)=M^4\exp(-2\Psi/\Psi_0)$. The equation of motion for Ψ is familiar:

$$\ddot\Psi+3\bar H\dot\Psi-\frac{1}{\bar a^2}\nabla^2\Psi+\frac{dV(\Psi)}{d\Psi}=0\ . \qquad (2.19)$$

Assuming that the Φ field is homogeneous, its evolution is just that of "slow roller": $d\Psi/d\bar t\simeq-(dV/d\Psi)/3\bar H$. (It is simple to show that $\ddot\Psi/\bar H\dot\Psi\sim\omega^{-1}$, which for $\omega\gg 1$ justifies the slow-roll approximation.) *That is, when extended inflation is viewed from the Einstein frame, it resembles slow-rollover inflation off an exponential potential, with the rescaled Brans-Dicke field Ψ playing the role of the inflaton.*

Because Ψ behaves just like an inflaton field and because the gravitational part of the action is just of general relativity, we can compute the curvature fluctuations that result from quantum fluctuations in Ψ by taking advantage of the machinery developed for slow-rollover inflation.[7] When a given scale λ crosses back inside the horizon after extended inflation (denoted by "hor") the amplitude of the fluctuation on that scale is given by

$$\left[\frac{\delta\rho}{\rho}\right]_{\rm hor}\simeq\frac{\bar H^2}{d\Psi/d\bar t}\simeq-\frac{3\bar H^3}{dV/d\Psi}\ , \qquad (2.20)$$

where the quantities on the right side of Eq. (2.20) are to be evaluated when the scale crossed outside the horizon during inflation. Moreover, well after extended inflation

the Jordan and the Einstein frames coincide so that the curvature fluctuations in both frames are the same. That is, the fluctuation amplitude in the Jordan frame, which is what are interested in, is equal to that computed in the Einstein frame, where the amplitude is most easily and unambiguously computed.

Remembering that $\bar{H}^2=8\pi V/3m_{\rm Pl}^2$ and $dV(\Psi)/d\Psi=-2V/\Psi_0$, it is simple to evaluate Eq. (2.20) for $(\delta\rho/\rho)_{\rm hor}$:

$$\left[\frac{\delta\rho}{\rho}\right]_{\rm hor}\simeq 4\pi\left[\frac{2\omega+3}{6}\right]^{1/2}\left[\frac{M}{m_{\rm Pl}}\right]^2\frac{m_{\rm Pl}^2}{\Phi}$$
$$\simeq 10^{50/(\omega-1/2)}4\pi\left[\frac{2\omega+3}{6}\right]^{1/2}q^{-2/(\omega-1/2)}$$
$$\times\left[\frac{M}{m_{\rm Pl}}\right]^{(2\omega+1)/(\omega-1/2)}\lambda_{\rm Mpc}^{2/(\omega-1/2)}\ . \qquad (2.21)$$

Up to a factor of order ω this is precisely the same as the fluctuation amplitude in Φ, cf. Eq. (2.12).[11]

This expression diverges in the large ω limit because the effective potential appearing in Eq. (2.17) becomes flat, $dV/d\Psi\to 0$. In the Jordan frame, this theory reduces to standard Einstein gravity in the same limit. Once again, the fluctuation amplitude diverges because the scalar field ϕ does not roll if $\rho_{\rm vac}\equiv M^4$ is a constant.

Implicit in computing $(\delta\rho/\rho)_{\rm hor}$ was the assumption that quantum fluctuations in the field Ψ are given by $\bar{H}/2\pi$. Since Ψ in minimally coupled and has a canonical kinetic term this is the correct thing to do.[12] What would have been the outcome if we had worked in the Jordan frame and assumed that $\delta\phi=H/2\pi$? The fluctuations in Φ are computed from those in ϕ by Eq. (2.1):

$$\frac{\delta\Phi}{\sqrt{\Phi}}=\left[\frac{8\pi}{\omega}\right]^{1/2}\delta\phi=\left[\frac{2}{\omega\pi}\right]^{1/2}H\ . \qquad (2.22)$$

From this and the fact that $\delta\Psi=\Psi_0(\delta\Phi/\Phi)$, which follows from the definition of Ψ, we find that

$$\delta\Psi=\left[\frac{2\omega+3}{2\omega}\right]^{1/2}\left[\frac{2\omega+1}{2\omega+3}\right]\frac{\bar{H}}{2\pi}\ . \qquad (2.23)$$

Thus, only in the limit $\omega\gg 1$ is the result in the Jordan frame assuming $\delta\phi=H/2\pi$ consistent with the result in the Einstein frame assuming $\delta\Psi=\bar{H}/2\pi$. This is simple to understand: The fluctuation amplitude of $H/2\pi$ applies only to a minimally coupled, massless scalar field with canonical kinetic term. In the Einstein conformal frame Ψ is a minimally coupled scalar field with canonical kinetic term, and because its potential is very flat it is effectively massless. Thus $\delta\Psi=\bar{H}/2\pi$ applies. In the Jordan conformal frame the field ϕ is only minimally coupled in the $\omega\to\infty$ limit, and thus $\delta\phi=H/2\pi$ only technically applies for $\omega\to\infty$: This is the limit in which the two methods for estimating $\delta\Psi$ agree.

Even if ω is not large it is still possible to compute the quantum fluctuations, $\delta\phi$, in the Jordan frame. In this case there is a subtle, yet important, correction to $\delta\phi$ that arises from the interaction of ϕ with the curvature scalar. In order to construct a Hamiltonian formalism and the corresponding quantum theory, one must transform all second-order derivatives in the $-\mathcal{R}\phi^2/8\omega$ term of Eq. (2.1) into first-order derivatives through an integration by parts. Consequently, there are terms containing derivatives in ϕ beyond the usual kinetic term, $\partial_\mu\phi\partial^\mu\phi/2$. The equal-time commutation relationship $[\phi(t,x),\partial\phi(t,x')/\partial t]=ia^{-3}(t)\delta^3(x-x')$, which determines the amplitude of the quantum fluctuations, is then replaced by a more complicated expression (see, e.g., Salopek, Bond, and Bardeen, Ref. 8). Thus, it is much easier, although not necessary, to work in the Einstein conformal frame where one has a standard kinetic term for the field whose fluctuations lead to density perturbations.

Note that the power-law spectrum of curvature fluctuations that arise due to quantum fluctuations in Φ, given by Eq. (2.21), becomes flatter as ω becomes large. The amplitude of these fluctuations is very interesting: for $\omega=10$ and $M=10^{14}$ GeV,

$$\left[\frac{\delta\rho}{\rho}\right]_{\rm hor}\simeq 4\times 10^{-4}\lambda_{\rm Mpc}^{0.21}\ . \qquad (2.24)$$

The associated temperature fluctuations on large angular scales, $\theta\sim 1°$ to $180°$, corresponding to scales $\lambda\sim 100$ to 10 000 Mpc, are given by[13]

$$\left[\frac{\delta T}{T}\right]_{\theta\gtrsim 1°}\simeq\frac{1}{15}\frac{\bar{H}^2}{d\Psi/d\bar{t}}$$
$$\simeq 10^{50/(\omega-1/2)}\left[\frac{2\omega+3}{6}\right]^{1/2}q^{-2/(\omega-1/2)}\left[\frac{M}{m_{\rm Pl}}\right]^{(2\omega+1)/(\omega-1/2)}10^{4/(\omega-1/2)}(\Omega_0 h)^{-2/(\omega-1/2)}\left[\frac{\theta}{1}\right]^{2/(\omega-1/2)}\ , \qquad (2.25)$$

where we use the fact that the comoving scale λ corresponds to an angular size of $\theta=34.4$ arcsec $(\Omega_0 h)\lambda_{\rm Mpc}$ at recombination. For $\omega=10$ and $M=10^{14}$ GeV, the temperature fluctuations are certainly too large to be consistent with the current limit to the quadrupole anisotropy, $\delta T/T\lesssim 3\times 10^{-5}$. Increasing ω or decreasing M slightly can remedy this problem, while still predicting fluctuations of an interesting size on smaller scales. That bubble nucleation occur rapidly enough so that there are not too many large bubbles requires that ω must be less

than about 20.[6] This fact together with the desire to associate M with a scale of order the GUT scale seems to imply that the fluctuations will be both of an interesting magnitude and *not* exactly scale invariant. The fact that the amplitude of the density perturbations increases with scale may be of some importance in that it boosts the fluctuation amplitude on large scales. (According to some observers, a scale-invariant spectrum lacks sufficient power on large scales to be consistent with the observed large-scale structure—large-scale streaming motions, the cluster-cluster correlation function, and the large voids seen in the Harvard Smithsonian Center for Astrophysics redshift survey.)

Again, we remind the reader that the model considered is truly a toy model which certainly requires modification. However, since the key feature of extended inflation is significant variation in the gravitational constant during inflation, one might expect that this simple toy model with $\omega \lesssim 20$ would at least mimic features of a more realistic model of extended inflation. In Sec. IV we will analyze curvature fluctuations in several attempts at more realistic models.

Finally, we again emphasize that curvature fluctuations are most directly and unambiguously addressed in the Einstein frame. In the Einstein frame the gravitational part of the action is just that of general relativity, and the redefined Brans-Dicke field Ψ precisely resembles a minimally coupled inflation filed with canonical kinetic energy term. Because extended inflation, as viewed in the Einstein frame, directly corresponds to slow-rollover inflation, the usual formulas for computing curvature fluctuations[7] apply. On the other hand, in the Jordan frame the gravitational part of the action is not that of general relativity, the effective gravitational constant varies by several orders of magnitude from the time of inflation until the present, and the Brans-Dicke field Φ is not minimally coupled. By working carefully in the Jordan frame one can get the correct result.[11,14]

(We should remark that working in the Einstein frame is not without its disadvantages: All the scalar fields have nonstandard kinetic terms; the complexity has been shifted to them, so to speak. However, during inflation the matter fields play a very passive role, and the trade-off of simplicity in the gravitational sector for complexity in the matter sector is worthwhile. Ater inflation, when the matter fields play a more significant role, it makes more sense to work in the Jordan frame, where the matter fields take on the standard form.)

III. FLUCTUATIONS IN OTHER FIELDS

A. Higgs field σ

During most of extended inflation the Higgs field plays a very passive role, quietly resting in the false-vacuum state, $\sigma=0$. At the end of extended inflation the σ field tunnels to the true vacuum; density fluctuations will certainly arise from the nucleation and thermalization of bubbles. Here we are interested in the perturbations that might arise due to quantum fluctuations in the σ field long before the end of extended inflation. However, we will not find them. Quantum fluctuations in the Higgs field are highly suppressed for a very simple reason: The mass of the σ field $m_\sigma^2 = V''(0) \sim M^2$ is much larger than the Gibbons-Hawking temperature $T_{\rm GW} = H/2\pi$. Very roughly, $m_\sigma/(H/2\pi) \sim \sqrt{3\pi/2}(1/\sqrt{G}M)$ where $G^{-1/2} < m_{\rm Pl}$ during inflation. As we have seen in the previous section, M must be significantly less than $m_{\rm Pl}$ to ensure that the Brans-Dicke fluctuations are acceptably small. Moreover, $G^{-1/2}$ must be much less than M during inflation, otherwise quantum gravitational corrections, which have not been taken into account, would be important. Thus, the mass of the Higgs field is several orders of magnitude larger than the Gibbons-Hawking temperature, and so fluctuations in the σ field are highly suppressed. Note that we have addressed the fluctuations in the σ field in the Jordan frame, as in this frame the kinetic term for σ is canonical. Were we to carefully address the fluctuations in the σ field in the Einstein frame by using a redefined field Σ which has a canonical kinetic term, we would find that $m_\Sigma/(\bar{H}/2\pi) \simeq m_\sigma/(H/2\pi)$.

B. Other massless fields

Any nearly massless scalar field, i.e., $m^2 \ll H^2$, in the theory will have fluctuations of order $H/2\pi$ imprinted upon it on all scales, as they cross outside the horizon. In the case that the energy density contributed by that field is subdominant, i.e., much smaller than that of the Higgs field, these fluctuations will not contribute significantly to the curvature fluctuations, but instead give rise to isocurvature fluctuations. This occurs in much the same way it does in slow-rollover inflation.[15] As a simple and interesting example, we will treat isocurvature axion fluctuations.

To analyze these fluctuations it is most appropriate to work in the Jordan frame, where matter fields have their usual kinetic and potential terms, but where the gravitational constant is varying. [In the Einstein frame kinetic terms in the matter Lagrangian are rescaled by factors of $\Omega^2 = \exp(-\Psi/\Psi_0)$ and potential terms by factors of $\Omega^4 = \exp(-2\Psi/\Psi_0)$.] Since the fluctuations we are interested in do not involve the gravitational degrees of freedom, the variation of G is only of interest in so far as it affects the expansion rate H. As noted earlier, a slow variation in the expansion rate H only changes the amplitude of quantum fluctuations in a massless scalar field by order $\dot{H}/H$.[12]

Consider a complex scalar field ϕ that carries PQ charge and undergoes spontaneous symmetry breaking, after which ϕ obtains a vacuum expectation value $\langle\phi\rangle = f_a \exp(-i\theta)$, where $f_a = \langle|\phi|\rangle$ is the vacuum expectation value that breaks Peccei-Quinn (PQ) symmetry. The axion degree of freedom is θ. Suppose that PQ-symmetry breaking occurs before, or early on, during inflation. Since θ is massless, no particular value is energetically favored during inflation. Later, around a temperature of 1 GeV, instanton effects become important, and θ develops a potential of depth about m_π^4 and minimum at $\theta=0$. Within the inflationary region, θ will take on some arbitrary value $\theta_1 \neq 0$. The misalignment of θ_1 with the eventual minimum of the axion potential

leads to coherent axion production, with the number density of axions produced being proportional to θ_1^2.[16] Fluctuations in θ will lead to fluctuations in the number of axions produced and correspond to isocurvature axion perturbations: $(\delta n_a/n_a)\simeq 2(\delta\theta/\theta_1)$.

Quantum fluctuations in ϕ give rise to quantum fluctuations in θ: $\delta\theta_\lambda \simeq H/f_a$, where H is value of the Hubble parameter when the scale λ crossed outside the horizon. During extended inflation $H\propto(\omega+\frac{1}{2})/t$, and we have previously related t to λ, cf. Eq. (2.9). Bringing this all together, we find that the spectrum of isocurvature axion perturbations is given by

$$\left[\frac{\delta n_a}{n_a}\right]_\lambda \simeq \frac{1}{\theta_1}\frac{M^2}{f_a m_{\rm Pl}}\frac{t_e}{t} \simeq 10^{25/(\omega-1/2)}\theta_1^{-1}\frac{M}{f_a}\left[\frac{M}{q m_{\rm Pl}}\right]^{(\omega+1/2)/(\omega-1/2)} \times \lambda_{\rm Mpc}^{1/(\omega-1/2)} . \tag{3.1}$$

When the Universe is matter dominated and a given scale has crossed back inside the horizon, these isocurvature perturbations will give rise to density perturbations of the same amplitude. In the limit that $\omega \gg 1$, the spectrum of isocurvature axion perturbations given by Eq. (3.1) is identical to that in slow-rollover inflation.[15] For M/f_a, $\theta_1\sim 1$, $\omega=10$, and $M=10^{14}$ GeV the amplitude of fluctuations is

$$\left[\frac{\delta n_a}{n_a}\right]_\lambda \simeq 3\times 10^{-6}\lambda_{\rm Mpc}^{0.11} , \tag{3.2}$$

which is definitely cosmologically interesting.

Any field that could develop isocurvature fluctuations in slow-rollover inflation can also do so in extended inflation. A second interesting example is provided by the "ilion" field, which in a particular model of baryogenesis gives rise to the baryon asymmetry.[17] In this case case ilion fluctuations result in isocurvature baryon-number fluctuations.[18] In general, in slow-rollover inflation the spectrum of isocurvature fluctuations was scale invariant; in extended inflation they will have some scale dependence because the Hubble parameter is not constant during inflation (inflation is power law rather than exponential).

C. Graviton perturbations

To analyze gravitational-wave perturbations (the transverse, traceless tensor metric perturbations) it is most appropriate to work in the Einstein frame, as the results derived or slow-rollover inflation[19] are directly applicable. As mentioned previously, long after extended inflation the Jordan and Einstein frames coincide so that the results we derive for tensor fluctuations in $\bar{g}_{\mu\nu}$ at late times are identical to those in $g_{\mu\nu}$—the ones we are interested in.

The dimensionless amplitude of a gravitational-wave perturbation as it crosses outside the horizon during extended inflation is

$$\bar{h}_\lambda \simeq \frac{\bar{H}}{m_{\rm Pl}} , \tag{3.3}$$

where $\bar{H}$ is to be evaluated at horizon crossing during extended inflation. Once the mode is outside the horizon its amplitude remains constant until it reenters the horizon after extended inflation. It is a simple matter to evaluate Eq. (3.3) for the amplitude of the tensor-metric perturbation h_λ at post-extended-inflation horizon crossing:[20]

$$h_\lambda \simeq \left[\frac{8\pi}{3}\right]^{1/2}\left[\frac{M}{m_{\rm Pl}}\right]^2\frac{m_{\rm Pl}^2}{\Phi} \simeq \left[\frac{8\pi}{3}\right]^{1/2} 10^{50/(\omega-1/2)} q^{-2/(\omega-1/2)}\left[\frac{M}{m_{\rm Pl}}\right]^{(2\omega+1)/(\omega-1/2)} \times \lambda_{\rm Mpc}^{2/(\omega-1/2)} . \tag{3.4}$$

For $\omega=10$ and $M=10^{14}$ GeV we find that

$$h_\lambda \simeq 5\times 10^{-6}\lambda_{\rm Mpc}^{0.21} . \tag{3.5}$$

The gravitational-wave mode just reentering the horizon today, $\lambda\sim 3000$ Mpc, leads to a quadrupole anisotropy in the CMBR of amplitude $\delta T/T\sim h_\lambda$, which for the parameters above correspond to $\delta T/T\sim 3\times 10^{-5}$— very close to the current upper limits to the quadrupole anisotropy.

At post-extended-inflation horizon crossing the ratio of energy density in the gravitational-wave mode just crossing inside the horizon to that of the total energy density is given by

$$\frac{\lambda d\rho_{\rm GW}/d\lambda}{\rho_{\rm tot}} \simeq \frac{4}{3\pi}\left[\frac{\bar{H}}{m_{\rm Pl}}\right]^2 , \tag{3.6}$$

for the mode that is just crossing inside the horizon today ($\lambda\sim 3000$ Mpc) this is

$$\Omega_{\lambda\sim 3000\ {\rm Mpc}} = \frac{\lambda d\rho_{\rm GW}/d\lambda}{\rho_{\rm crit}} \simeq 10^{114/(\omega-1/2)} q^{-4/(\omega-1/2)} \times \left[\frac{M}{m_{\rm Pl}}\right]^{2(2\omega+1)/(\omega-1/2)} . \tag{3.7}$$

For the parameters above, $\Omega_{\lambda\sim 3000\ {\rm Mpc}}\sim 10^{-10}$.

It is straightforward to compute the spectrum of relic gravitational waves today;[19] they extend from $\lambda\sim 10^{-25}(m_{\rm Pl}/M)$ Mpc, the mode that reentered the horizon just after reheating, to $\lambda\sim 3000$ Mpc, the mode that is just reentering the horizon today. The fraction of critical density contributed today varies with λ as

$$\begin{aligned} &\Omega_\lambda \propto \lambda^{2+4/(\omega-1/2)} , \quad 13 \lesssim \lambda \lesssim 3000\ {\rm Mpc} , \\ &\Omega_\lambda \propto \lambda^{4/(\omega-1/2)} , \quad 10^{-25}(m_{\rm Pl}/M) \lesssim \lambda \lesssim 13\ {\rm Mpc} , \end{aligned} \tag{3.8}$$

and can by normalized by the result above for $\Omega_{\lambda\sim 3000\ {\rm Mpc}}$. In the limit that $\omega \gg 1$ this is the same spectrum as that predicted in slow-rollover inflation.

In addition to the relic gravitational waves that arise as

quantum fluctuations, there is an additional source due to bubble collisions, which is considered elsewhere.[21]

IV. OTHER MODELS OF EXTENDED INFLATION

Were it not for the fact that precision solar-system tests of the Brans-Dicke theory require ω to be greater than about 500, Brans-Dicke theory with $\omega \sim 10$ and $M \sim 10^{14}$ GeV would provide a very elegant and viable model of extended inflation. The rub is that for $\omega \lesssim 500$, the effective gravitational constant today, which varies as $G^{-1} = \Phi \propto 4 \ln t / 3(2\omega + 3)$, is changing too rapidly to be consistent with the most stringent solar-system limits to $\dot{G}$. If, after extended inflation, there were some mechanism to prevent the time variation of Φ, e.g., a potential of the general form $\lambda(\Phi - m_{\rm Pl}^2)^2$, the above difficulty could be circumvented. In many theories, including superstrings and other theories that involve higher dimensions, a field such as the Brans-Dicke field arises, and is known as the "dilaton." There are a variety of reasons for wanting and expecting the dilaton field to acquire a mass, and extended inflation provides yet another. For the sake of a simple model, imagine that the Brans-Dicke field does acquire a mass, in the form of an additional (potential) term in the Lagrangian:

$$\mathcal{L} \to \mathcal{L} - \frac{\lambda\omega}{16\pi}(\Phi - m_{\rm Pl}^2)^2 . \tag{4.1}$$

Such a potential for Φ would both "anchor" Φ, thereby preventing the gravitational constant from varying, and provide a mass for the Brans-Dicke field, $m_\Phi^2 = \lambda m_{\rm Pl}^2$. As we shall now discuss, in so doing it would not adversely affect extended inflation provided that $m_\Phi \lesssim (16\pi/\omega)^{1/2} M^2 / m_{\rm Pl} \sim 10^9$ GeV (for $M = 10^{14}$ GeV).

The addition of such a term to the Lagrangian density of the theory modifies the equations of motion for $a(t)$ and Φ; in the Jordan conformal frame they become

$$\begin{aligned} \ddot{\Phi} + 3H\dot{\Phi} - \frac{1}{a^2}\nabla^2\Phi &= \frac{8\pi}{2\omega+3}(\rho - 3p) + \frac{2\lambda\omega}{2\omega+3} m_{\rm Pl}^2 (m_{\rm Pl}^2 - \Phi) , \\ H^2 &= \frac{8\pi\rho}{3\Phi} + \frac{\omega}{6}\frac{\dot{\Phi}^2}{\Phi} - H\frac{\dot{\Phi}}{\Phi} + \frac{\lambda\omega}{6\Phi}(\Phi - m_{\rm Pl}^2)^2 . \end{aligned} \tag{4.2}$$

During extended inflation $\rho \simeq M^4$, $p = -\rho$, and the vacuum energy associated with the σfield being in the false vacuum controls the right-hand sides of both these equations. If the potential for Φ is not to interfere with the implementation of extended inflation, then the additional terms on the right-hand sides of Eqs. (4.2) must be subdominant; this requires that

$$\begin{aligned} \frac{\omega}{16\pi}\frac{m_\Phi^2 m_{\rm Pl}^2}{M^4}\left[1 - \frac{\Phi}{m_{\rm Pl}^2}\right] &\ll 1 , \\ \frac{\omega}{16\pi}\frac{m_\Phi^2 m_{\rm Pl}^2}{M^4}\left[1 - \frac{\Phi}{m_{\rm Pl}^2}\right]^2 &\ll 1 . \end{aligned} \tag{4.3}$$

Both of these conditions are met during extended inflation provided that $m_\Phi \lesssim (16\pi/\omega)^{1/2} M^2 / m_{\rm Pl}$. Likewise, if these conditions are met, the new terms in the equations of motion for $\bar{a}(\bar{t})$ and Ψ are subdominant. In addition, conditions (4.3) also guarantee that $\Psi \ll \bar{H}/2\pi$ (and equivalently that $m_\Phi \ll H/2\pi$); therefore, during extended inflation the Brans-Dicke field Ψ still behaves like a massless scalar field (mass much less than the Gibbons-Hawking temperature) and $\delta\Psi = \bar{H}/2\pi$, implying that our previous calculation for the curvature fluctuations is applicable here.

There is one new and potentially worrisome wrinkle associated with the mass term introduced for the Φ field: In general, extended inflation need not end when the value of Φ is precisely equal to $m_{\rm Pl}^2$, because the end of inflation is determined by the tunneling of the σ field to the true vacuum. Thus, after reheating, Φ may be left oscillating about the minimum of its potential. These coherent Φ-field oscillations behave just like nonrelativistic matter and will come to dominate the mass density of the Universe long before the Universe is supposed to become matter dominated (at a temperature of about 10 eV). To be more specific, if these oscillations come to dominate the energy density of the Universe when the temperature is $T_* \gg 10$ eV, then the Universe reaches a temperature of 3 K at the age of 10 Gyr $(T_*/10\ {\rm eV})^{-1/2}$. (A similar problem was encountered in slow-rollover inflation with the Polonyi field.[22])

The cure for this dread disease is simple: The Brans-Dicke field must be unstable and decay. This is not difficult to arrange for a field of such large mass. A more thorough discussion of a model of extended inflation where the Brans-Dicke field acquires a mass and how extended inflation fits into a realistic particle-physics model will be given in Ref. 23. In addition, if the value of Φ at the end of inflation ($\equiv \Phi_e$) is not equal to $m_{\rm Pl}^2$, then the formula for $(\delta\rho/\rho)_{\rm hor}$ must be modified.

To begin, Eq. (2.7) which defines t_e becomes

$$m_{\rm Pl}^2 \left[\frac{\Phi_e}{m_{\rm Pl}^2}\right] = \Phi_0 B^2 t_e^2 = \frac{t_e^2 M^4}{q^2\omega^2} \implies t_e = \frac{\sqrt{\Phi_e}}{m_{\rm Pl}} q\omega \frac{m_{\rm Pl}}{M^2} , \tag{4.4}$$

where $\sqrt{\Phi_e}/m_{\rm Pl}$ is not necessarily equal to 1. Now Eq. (2.9) relating t_e/t to $M/m_{\rm Pl}$, $\lambda_{\rm Mpc}$, and ω becomes

$$\frac{t_e}{t} = 10^{25/(\omega - 1/2)} \left[\frac{m_{\rm Pl}}{\sqrt{\Phi_e}}\right]^{1/(\omega - 1/2)} \times \left[\frac{M}{q m_{\rm Pl}}\right]^{1/(\omega - 1/2)} \lambda_{\rm Mpc}^{1/(\omega - 1/2)} , \tag{4.5}$$

which is the same as Eq. (2.9) except for the factor of $\sqrt{\Phi_e}/m_{\rm Pl}$. Equation (2.21) for the amplitude of the curvature fluctuations becomes

$$\left[\frac{\delta\rho}{\rho}\right]_{\rm hor} \simeq 4\pi\left[\frac{2\omega+3}{6}\right]^{1/2}\left[\frac{M}{m_{\rm Pl}}\right]^2\frac{\Phi_e}{\Phi}$$
$$\simeq 10^{50/(\omega-1/2)}4\pi\left[\frac{2\omega+3}{6}\right]^{1/2}\left[\frac{\sqrt{\Phi_e}}{m_{\rm Pl}}\right]^{-(2\omega+1)/(\omega-1/2)} q^{-2/(\omega-1/2)}\left[\frac{M}{m_{\rm Pl}}\right]^{(2\omega+1)/(\omega-1/2)}\lambda_{\rm Mpc}^{2/(\omega-1/2)} . \quad (4.6)$$

The amplitude of the fluctuations is *changed* by a factor of $(m_{\rm Pl}^2/\Phi_e)^{(\omega+1/2)/(\omega-1/2)}$. The amplitude of graviton perturbations is changed by the same factor, while isocurvature perturbations are changed by a factor of $(m_{\rm Pl}^2/\Phi_e)^{(\omega+1/2)/2(\omega-1/2)}$. If the value of Φ_e is greater than $m_{\rm Pl}^2$, then all the fluctuation amplitudes are smaller; if the value of Φ_e is less than $m_{\rm Pl}^2$, then all the fluctuation amplitudes are larger.

Steinhardt and Accetta[3] have proposed a different approach in constructing a realistic model of extended inflation, which they call hyperextended inflation. They start with the action

$$S=\int d^4x\sqrt{-g}\left[-\frac{M^2\mathcal{R}}{16\pi}-\frac{\mathcal{R}\Phi}{16\pi}-\frac{\beta\mathcal{R}\Phi^2}{16\pi M^2}-\cdots+\frac{\omega}{16\pi}\frac{(\partial_\mu\Phi)^2}{\Phi}+\mathcal{L}_{\rm matter}\right], \quad (4.7)$$

where β is a dimensionless constant, M is an energy scale less than the Planck mass (the GUT scale?), and Φ is, as before, the Brans-Dicke field. The key modification is to include higher-order terms in the coupling between the Brans-Dicke field Φ and the curvature. (We have adopted a slightly different notation than theirs; in their notation ω is a function of Φ which is then expanded in powers of Φ.)

In general, the conformal transformation to the Einstein frame is quite complicated because of the three different terms involving Φ and $\mathcal{R}$. However, the analysis can be simplified by considering regimes where one of the three terms dominates. Those regimes are (a) $\Phi \lesssim M^2$, where the first term dominates, (b) $M^2 \lesssim \Phi \lesssim M^2/\beta$, where the second term dominates, and (c) $\Phi \gtrsim M^2/\beta$, where the third term dominates. In regime (a), the theory is a rescaled version of general relativity where $G=M^{-2}$. Since a crucial feature of extended inflation is the time variation of the gravitational constant it is pointless to address regime (a). In regime (b), the theory resembles the Brans-Dicke theory, and in regime (c), the theory also has a time-varying gravitational constant. We can exclude the possibility that we are in regime (b) today, since if we were, we would again be confronted by the problems of a simple Brans-Dicke model. Today then, the effective gravitational theory must be described by regime (c), in which case we can read off the gravitational constant $G_N=M^2/\beta\Phi_\infty^2$ and deduce the present value of Φ: $\Phi_\infty=Mm_{\rm Pl}/\sqrt{\beta}=m_{\rm Pl}^2(M/m_{\rm Pl}\sqrt{\beta})$. If $M<\sqrt{\beta}\,m_{\rm Pl}$, the value of Φ today is less than the Planck mass squared, which will be the case unless $\beta \ll 1$. The value of Φ today can be much greater than $m_{\rm Pl}^2$, provided that $\beta \ll 1$. We will return to the significance of this point shortly.

There are several possible scenarios for hyperextended inflation. First, that the period of inflation relevant for us, the last 60 or so e-folds, occurred during regime (b), in which case the analysis of the previous sections applies since the effective action during inflation is just that of a Brans-Dicke theory. There is one important difference however: the final value of the Φ during phase (b), denoted by Φ_e, may not be equal to $m_{\rm Pl}^2$. We have just treated this possibility above, in the model where the Brans-Dicke field is given a mass; the amplitude of fluctuations is this circumstance is given by Eq. (4.6). If Φ_e is less than $m_{\rm Pl}^2$, the amplitude of the curvature perturbations is *increased* relative to the original La-Steinhardt model[3] by a factor of about $(m_{\rm Pl}^2/\Phi_e)$. Since the largest possible value of Φ_e is M^2/β, the increase in the amplitude of curvature perturbations is *at least* a factor of $(\sqrt{\beta}M/m_{\rm Pl})^{2(\omega+1)/(\omega-1/2)}$, which nearly cancels a similar factor of $M/m_{\rm Pl}$ in Eq. (4.6):

$$\left[\frac{\delta\rho}{\rho}\right]_{\rm hor} \gtrsim 10^{50/(\omega-1/2)}4\pi\left[\frac{2\omega+3}{6}\right]^{1/2} q^{-2/(\omega-1/2)} \times\beta^{(\omega+1)/(\omega-1/2)}\lambda_{\rm Mpc}^{2/(\omega-1/2)} . \quad (4.8)$$

In this case it is difficult to achieve curvature perturbations of an acceptable magnitude. If $\beta \ll 1$, so that $\Phi_e \gtrsim m_{\rm Pl}^2$, the amplitude of the density perturbations is smaller than the original La-Steinhardt model,[3] and it is quite possible to achieve density perturbations of an appropriate size.

Now consider regime (c). In this regime the action is effectively given by

$$S=\int d^4x\sqrt{-g}\times\left[-\frac{\beta\mathcal{R}\Phi^2}{16\pi M^2}+\frac{\omega}{16\pi}g^{\mu\nu}\frac{\partial_\mu\Phi\partial_\nu\Phi}{\Phi}+\mathcal{L}_{\rm matter}\right] . \quad (4.9)$$

By means of the conformal transformation

$$g_{\mu\nu}\to\Omega^2\bar{g}_{\mu\nu} , \quad \Omega^2=M^2m_{\rm Pl}^2/\beta\Phi^2 , \quad (4.10)$$

the theory can be written as

$$\bar{S}=\int d^4x\sqrt{-\bar{g}}\left[-\frac{\bar{\mathcal{R}}}{16\pi G}+\tfrac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi-V(\Psi)\right], \quad (4.11)$$

where $V(\Psi)=(M^4/\beta^2m_{\rm Pl}^4)\exp(-2\Psi/\Psi_0)V(\sigma)$, $\Phi G=\exp(\Psi/2\Psi_0)$, and the new definition of Ψ_0 is $\Psi_0^2=3m_{\rm Pl}^2/16\pi$. In rescaling the theory, we have assumed that $\omega \ll 6$, in which case the kinetic term for Ψ

that arises from the original kinetic term for Φ, cf. Eq. (4.7), is negligible compared to that which arises from the conformal transformation involving $\mathcal{R}$. The evolution of the rescaled scale factor $\bar{a}$ and Φ are easy to analyze in the Einstein conformal frame:[24]

$$\bar{a} \propto \bar{t}^{3/4}\ , \quad \Phi \propto \bar{t}^{1/2}\ . \tag{4.12}$$

In the Jordan frame it follows that

$$a \propto t^{1/2}\ , \quad \Phi \propto a^2 \propto t\ . \tag{4.13}$$

In this regime there is no inflation. However, Φ does evolve, as it must to reach its final value, $\Phi = M m_{\rm Pl}/\sqrt{\beta}$. Because of the large density perturbations that may arise during inflationary regime (b) unless $\beta \ll 1$, care must be taken in implementing this version of hyperextended inflation.

There is one last possibility, that the inflation relevant to us occurred during regime (c), which requires that $\omega \gtrsim 6$. In this case, the transformation to the Einstein conformal frame is not a simple one. If we treat Φ as slowly varying, the transformation above is valid with the change

$$\Psi_0^2 = \frac{3m_{\rm Pl}^2}{16\pi}\left[1 + \frac{\omega}{6}\frac{M^2}{\beta\Phi}\right]\ . \tag{4.14}$$

In this case

$$\bar{a} \propto \bar{t}^{\,m}\ , \quad m = \frac{3}{4} + \frac{\omega}{8}\frac{M^2}{\beta\Phi}\ , \quad \Phi \propto \bar{t}^{1/2}\ . \tag{4.15}$$

Early on, when $\Phi \lesssim \omega M^2/2\beta$, inflation occurs as $m \geq 1$; as Φ increases to the value $\omega M^2/2\beta$, $m \to 1$ and superluminal expansion ceases. In terms of the evolution of Φ, the epoch of inflation lasts from $\Phi = M^2/\beta$ to $\Phi = \omega M^2/2\beta$. During inflation,

$$a \propto t^{2m-1}\ , \quad \Phi \propto a^{1/(2m-1)}\ . \tag{4.16}$$

Using the usual formula for the amplitude of the curvature fluctuations, $(\delta\rho/\rho)_{\rm hor} \simeq \bar{H}^2/\dot{\Psi}$, we find that

$$\left[\frac{\delta\rho}{\rho}\right]_{\rm hor} \simeq 8\pi\sqrt{m/6}\,10^{25/(m-1)} \times \left[\frac{M}{m_{\rm Pl}}\right]^{(2m-1)/(m-1)} \lambda_{\rm Mpc}^{1/(m-1)}\ . \tag{4.17}$$

Since m varies during inflation, we cannot immediately evaluate this expression.

During inflation, that is, while $M^2/\beta \lesssim \Phi \lesssim \omega M^2/2\beta$,

$$d\ln a = (2m-1)d\ln\Phi = \left[\frac{1}{2} + \frac{\omega M^2}{4\beta\Phi}\right] d\ln\Phi\ . \tag{4.18}$$

Integrating this expression, we find that the total number of e-folds in the scale factor $a(t)$ during inflation is $(\omega/2)^2\exp(\omega/4 - 1/2)$; we immediately see that in order to achieve the 60 or so e-folds of inflation necessary, ω must be excess of about 240. Further, it is straightforward to compute the value of m around the time that the cosmologically interesting scales went outside the horizon ($N = 60$ or so e-folds before the end of inflation): $m \simeq N/2 \sim 30$. Returning to Eq. (25) we find, provided that $\omega \gtrsim 240$,

$$\left[\frac{\delta\rho}{\rho}\right]_{\rm hor} \simeq 4\pi\sqrt{N/3}\,10^{50/N} \times \left[\frac{M}{m_{\rm Pl}}\right]^{2(N-1)/(N-2)} \lambda_{\rm Mpc}^{2/(N-2)}\ . \tag{4.19}$$

Since $m \simeq N/2 \gg 1$, this expression is very nearly independent of N and $\lambda_{\rm Mpc}$: $(\delta\rho/\rho)_{\rm hor} \simeq 300(M/m_{\rm Pl})^2$. Fluctuations of a cosmologically interesting amplitude can be attained for $M \sim 10^{15}$ GeV or so, provided that $\omega \gtrsim 240$. Whether or not this leads to an overproduction of large bubbles remains to be seen.

V. CONCLUDING REMARKS

In slow-rollover inflation the dominant curvature fluctuations arise due to quantum fluctuations in the inflaton field. In extended inflation, there is no inflaton field *per se;* the Higgs field σ, whose vacuum expectation value is the order parameter for the phase transition, plays a very passive role until it makes its transition to the true vacuum at the end of inflation, thereby reheating the Universe. It is the quantum fluctuations in the Brans-Dicke field that give rise to the dominant curvature fluctuations (aside from those associated with bubbles). These curvature fluctuations are most directly addressed in the Einstein conformal frame: In the Einstein conformal frame extended inflation resembles slow-rollover inflation with the Brans-Dicke field playing the role of the inflaton, with an exponential potential; the calculation of curvature fluctuations is precisely the same as in slow-rollover inflation. Unlike slow-rollover inflation, these fluctuations are typically not scale invariant and, even more important, it is not necessary to tune any parameter to a very small value to ensure that they have an acceptable amplitude: The requisite amplitude, $(\delta\rho/\rho)_{\rm hor} \sim 10^{-5}$, is obtained provided that the scale M (GUT scale?) is of order 10^{-5} that of the Planck scale. In principle, the density perturbations that arise from bubbles can also be important; however, that seems to require that a parameter, in the Brans-Dicke example ω, be tuned to be just so.

Just as in slow-rollover inflation, isocurvature fluctuations can arise in any massless field present in the theory, e.g., the axion or the ilion. Such isocurvature fluctuations are of a similar magnitude as they are in slow-rollover inflation, but they too typically have a non-scale-invariant spectrum. Because ordinary matter fields have canonical kinetic terms in the Jordan frame, these fluctuations are most appropriately computed in the Jordan frame.

While we have only analyzed density fluctuations in extended inflation for the simplest toy model and a couple

of attempts at a more realistic model, there is some hope that these models will serve to illustrate the general features that one can expect in a viable model of extended inflation; the reason to expect that this is true is that the key feature of extended inflation is significant variation in the gravitational constant, which occurs in the toy model analyzed here for $\omega \lesssim 20$.

ACKNOWLEDGMENTS

This work was supported in part by the Department of Energy (at Chicago and at Fermilab) and by NASA through Grant No. NAGW-1340 (at Fermilab). We thank M. Kamionkowski and P. J. Steinhardt for valuable comments.

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