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Field Theoretical Analysis of Anyonic Superconductivity

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ABSTRACT

We derive several results pertaining to anyonic superconductivity as described by a Chern–Simons field theory. (1) The renormalized Chern-Simons term at finite density is shown to vanish when the renormalized coefficient at zero density takes values $Ne^2/2\pi$. This is the field theoretical requirement to have a massless pole in the current–current correlator. We can then show that in the Chern–Simons description a system of charged anyons at zero temperature is a superconductor. This result is shown to hold to all orders in perturbation theory by generalizing a nonrenormalization theorem of the zero density case. (2) At finite temperature the renormalized Chern–Simons term does not vanish at the one–loop perturbative level. We compute the mass of this apparent "pseudo–Goldstone mode". We also exhibit evidence of critical behavior, for this same system, at a nonzero T_c. We discuss the possible implications of these perturbative results. (3) A low energy effective action for an anyonic superconductor is derived directly from Chern–Simons field theory. Several P and T violating effects occur.

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1. Introduction

It was first suggested by Laughlin[1,2] that a plasma of anyons[3] (particles with fractional statistics in two spatial dimensions) may exhibit superconductivity. Laughlin and others[1][2][4−14] have developed this idea as a possible model for high T_c copper oxide superconductors. However it is important to separate the physics of anyonic superconductivity per se from its applications to quasi–two–dimensional condensed matter systems. In this spirit one can regard anyons as fundamental particles (in $2+1$ dimensions) and study their general properties just as one does a system of bosons or fermions.

A variety of different approaches have been used to describe anyons.[15−18] It is not obvious that they are all equivalent; in fact the precise relationship between these descriptions involves subtle issues which have been largely unexplored. We will skirt such questions in this paper by anchoring our results to a particular theoretical framework: that of Chern–Simons (CS) field theory.[19−25] More precisely, we will examine a three–dimensional Euclidean field theory consisting of an abelian Chern–Simons gauge field coupled to fermions (the case of a Chern–Simons field coupled to bosons has been discussed extensively by Wen and Zee[8,13,14]). According to the results of [23], [24], and [25], this provides a field–theoretic description of free anyons. We will assume this to be true, in order to intrepret our Chern–Simons results in the language of anyons. It should be kept in mind, however, that this connection involves some unresolved subtleties.[26,27]

Our object in this paper is to verify and extend the work of Fetter, Hanna and Laughlin,[28] and of Chen, Halperin, Wilczek, and Witten.[29] These authors showed that in the random phase approximation a free gas of anyons with statistics parameter $\gamma = \pi(1-\frac{1}{N})$ where N is a large integer, has a massless pole in the current–current correlation at zero temperature which is related to superfluidity. This then implies that a charged gas of anyons would be superconducting at zero temperature. Chen et al[29] argued on general grounds that this result, i.e., the existence of a massless collective mode, should survive improvements on their approximation. We will demonstrate explicitly that this is indeed the case.

Banks and Lykken[30] studied the field theoretic realization of an anyonic system in which charged fermions in 2+1 dimensions are coupled to ordinary photons plus an additional "statistics" gauge field possessing a Chern-Simons term. They argued that superconductivity (at zero temperature) occurs if and only if the renormalized CS term of the statistics gauge field vanishes – i.e. if the quantum corrections to the bare CS term precisely cancel it.

In this paper we follow the approach of ref.[30]. We rederive in section 5 the field–theoretic criterion for anyon superconductivity by extracting the low energy effective action and by analyzing directly the condition for a massless pole in the current-current correlator. The main part of the work is described in section 2 where we calculate the renormalized CS coefficient for a relativistic finite density field theory of Chern–Simons plus fermions. The relativistic formalism (apart from the chemical potential term which, obviously, is not a Lorentz scalar) is for convenience only; our results have a well–defined nonrelativistic limit. The framework of our analysis is perturbation theory. Non–perturbative corrections to the renormalized CS term[31] as well as possible topological excitations in the form of vortices are beyond the scope of this work.[15,16] We present a detailed analysis showing that the renormalized Chern-Simons term at finite density vanishes if and only if the zero density renormalized Chern-Simons coefficient $2\pi\theta_R/e^2$ is a positive integer, N. This indicates that this system of anyons is a superfluid at zero temperature, and is a superconductor when coupled to electromagnetism.

We then show in section 3 that, at zero temperature, this result extends to all orders in perturbation theory and thus does not depend on the mean field approximation, nor on the large N limit. We do this by showing that the non-renormalization theorem of Coleman and Hill[32] can be extended to the Chern–Simons theory at finite density. We also provide topological arguments leading to the same conclusion. The physical picture is quite similar to the quantum Hall system, where an analogous topological quantization is known to occur.[33]

In section 4 we give the finite temperature extension of our analysis. We find that for $T>0$ the Chern-Simons term does *not* cancel, to one-loop order in perturbation theory. The resulting CS mass vanishes exponentially as $T\rightarrow 0$. Thus the superfluid appears to develop a thermally-activated dispersion consistent with a zero temperature phase transition. We shall discuss the implications of this in detail. On the other hand, we also find in our results indications of critical behavior at a *finite* temperature.

Section 6 is devoted to the analysis of the P and T violating Landau-Ginsburg (LG) low-energy effective action description of the anyonic superconductor. We derive the complete LG effective action (in the London limit) directly from Chern-Simons theory. Surprisingly, a phenomenologically important term of the LG theory arises from an "irrelevant" operator in the CS theory. This peculiar feature occurs due to the presence of CS terms, which are first order in derivatives. Our effective action contains several P and T violating terms obtained previously by other authors using different methods.[13,12,29,30]

The main results of this paper were summarized in a previous letter.[34] We note that Randjbar-Daemi et al[35] have recently discussed finite temperature effects using the approach of ref.[17]. We have also received new papers by Fradkin,[36] and by Panigrahi et al,[37] where very similar results to ours are obtained using different methods.

2. The renormalized Chern-Simons term

We consider a system of anyons described by a single two-component massive fermion field coupled to a CS fictitious gauge field a_μ. We introduce some finite, nonzero density of anyons by adding a chemical potential term to the action. The Euclidean path integral expression at zero temperature ($T=0$) for the partition function of the Chern-Simons theory at finite chemical potential μ is given by:

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}a_\mu exp(-S_E) \tag{2.1}$$

with

$$S_E \;=\; \int d^3x \;\left(\bar{\psi}(\not{D} \;-m)\psi \;+\; i\frac{\theta}{2}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \;-\; \mu\psi^\dagger\psi\right) \tag{2.2}$$

where $D_\mu = \partial_\mu - iea_\mu$. We shall work throughout with a nonegative chemical potential μ. We choose to work in Coulomb gauge ($\partial_i a^i=0$). We proceed by integrating out the a_0 field, which simply gives the Gauss law constraint $\delta(B - \frac{e}{\theta}\psi^\dagger\psi)$. This delta function now allows us to do the integral over a_1 and a_2 by setting

$$a_i \;=\; -\frac{e}{\theta}\epsilon_{ij}\frac{\partial^j}{\nabla^2}\psi^\dagger\psi \tag{2.3}$$

This leads to the following effective 4-fermi theory:

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\; exp[-\int d^3x \left(\bar{\psi}(\not{\partial} - m - \mu\gamma^0)\psi + \frac{ie^2}{\theta}(\bar{\psi}\gamma^i\psi)\frac{\epsilon_{ij}\partial^j}{\nabla^2}(\bar{\psi}\gamma^0\psi)\right)] \tag{2.4}$$

We use the gamma matrices $\gamma_1=\sigma_1$, $\gamma_2=\sigma_2$, and $\gamma_0=\sigma_3$ where σ_i are the Pauli spin matrices. Notice that the effect of the chemical potential is simply to replace ∂_0 by $\partial_0-\mu$. We thus define $\tilde{\partial}_\nu$ to be equal to ∂_ν unless $\nu=0$ in which case $\tilde{\partial}_0 = \partial_0-\mu$.

Our goal is to show that this theory exhibits superfluidity. More generally we would like to show that a system of charged anyons is a superconductor. In the spirit of ref. [29] we shall show that for a system of neutral anyons, the current-current correlation function has a massless pole. As discussed in ref. [30] and as will be expanded upon in section 5, a sufficient condition for such a massless pole is that the *renormalized* Chern-Simons term for this theory vanishes. In other words if the quantum corrections to the Chern-Simons coefficient precisely cancel the coefficient in the Lagrangian, we are assured that a pole is present in the current-current correlation function.

We begin by studying the fermion propagator $S(x,x')$ for this theory. The bare fermion propagator $S_0(x,x')$ is simply $1/(\tilde{\not{\partial}} - m)$. Perturbative corrections to this propagator will be computed from the Feynman rules which arise from the path integral expression above. The vertices for this theory are 4-point fermion vertices which connect a current J_i to a density J_0. Each such vertex carries a factor $-i\frac{e^2}{\theta}(\epsilon_{ij}\partial^j/\nabla^2)$. This vertex is shown in Figure 1a. It is often convenient to represent this vertex via the diagram of Figure 1b in which the standard QED vertex with a value $+ie\gamma_\mu$ is used and in which a Chern-Simons propagator is explicitly shown. This propagator (in Coulomb gauge) is nonzero only when connecting a γ_i vertex with a γ_0 vertex and then has a value $\frac{i}{\theta}(\epsilon_{ij}\partial^j/\nabla^2)$.

Our first observation when evaluating perturbative corrections to the fermion propagator is that, in the presence of a nonzero chemical potential $\mu \neq 0$, the tadpole graphs such as those of Figure 2 **do not** vanish. These tadpoles are nonvanishing since $< J_0 >= \rho_0$ is nonzero when μ is nonzero. Note that the amputated tadpole is precisely equal to the mean density ρ_0. We can thus compute the entire contribution of a single tadpole to the propagator as a function of the mean density ρ_0. Using the Feynman rules described above we find that each tadpole contributes an amount

$$-i\frac{e^2}{\theta}\gamma^i\epsilon_{ij}\frac{\partial_j}{\nabla^2}\rho_0 \tag{2.5}$$

to the fermion propagator. This single tadpole contribution can be written in the suggestive form $ie\gamma^i\mathcal{A}_i$ where

$$\mathcal{A}_i \; = \; -\frac{e}{\theta}\epsilon_{ij}\frac{\partial_j}{\nabla^2}\rho_0 \tag{2.6}$$

Notice that $\mathcal{A}_i$ is precisely the gauge potential one would obtain from a constant fictitious magnetic field $\mathcal{B} = \frac{e}{\theta}\rho_0$.

We can now compute the contribution of all the tadpoles to the Fermion propagator by summing the geometric series of Figure 2. We call the resulting object the tadpole-corrected propagator and we denote it by S_T. The result of the calculation is

$$S_T \; = \; (S_0^{-1} - ie\gamma^i\mathcal{A}_i)^{-1} \; = \; [\gamma^\mu(\partial_\mu - ie\mathcal{A}_\mu) - m - \mu\gamma^0]^{-1} \tag{2.7}$$

where we have made the definition $\mathcal{A}_0 = 0$. This leads us to the important conclusion that the tadpole-corrected propagator S_T is precisely the Green's function for a free fermion in a constant magnetic field $\mathcal{B} = \frac{e}{\theta}\rho_0$ and with chemical potential μ. We shall thus reorganize our perturbation expansion as follows. All propagators will be fully tadpole–corrected propagators. The vertices will be the same vertices as those for the basic theory, and, of course, no additional propagators are included. We shall call this "tadpole–corrected perturbation theory". We shall be able to use this reorganized expansion to prove some very powerful results about the Chern-Simons theory. In fact for many quantities of interest only one–loop effects will contribute.

To evaluate the fermion Green's function in a constant magnetic field $\mathcal{B}$ it is necessary to choose from among the many possible gauge potentials which are consistent with coulomb gauge. We choose to work in an asymmetric gauge in which $\mathcal{A}_y$=$\mathcal{B}x$, $\mathcal{A}_x$=0, $\mathcal{A}_0$=0. We have done the calculation in two ways. The first method which involves a direct evaluation of the fermion propagator is outlined below. An alternate approach using Schwinger's proper time method is described in Appendix A. We find the fermion propagator S_T by inverting the operator $\tilde{\not{D}}-m$ where $\tilde{D}_\mu = \tilde{\partial}_\mu - ie\mathcal{A}_\mu$. This is done by observing that

$$\begin{aligned} S_T &= [\tilde{\not{D}} - m]^{-1} = (\tilde{\not{D}} + m)[(\tilde{\not{D}} - m)(\tilde{\not{D}} + m)]^{-1} \\ &= (\tilde{\not{D}} + m)[\tilde{D}^2 - m^2 + e\sigma_3\mathcal{B}]^{-1} \end{aligned} \tag{2.8}$$

The propagator is thus found in two steps. We first invert the operator $Q = [\tilde{D}^2 - m^2 + e\sigma_3\mathcal{B}]$ by finding its eigenvalues and eigenfunctions. We then apply the operator $\tilde{\not{D}}+m$ to the resulting expression to obtain the propagator.

The eigenfunctions Φ for the operator Q, which are 2-component spinors, are found by fourier transforming in y and t:

$$\Phi(x,y,t) = e^{-i\omega t}e^{-ip_y y}\tilde{\Phi}(x,p_y,\omega) \tag{2.9}$$

The operator Q when acting on this eigenfunction gives

$$-[(\omega - i\mu)^2 - \partial_x^2 + e^2\mathcal{B}^2(x + \frac{p_y}{e\mathcal{B}})^2 + m^2 - e\mathcal{B}\sigma_3] \tag{2.10}$$

The functions $\tilde{\Phi}$ are thus eigenfunctions of a harmonic oscillator with unit mass and frequency $e\mathcal{B}$. More precisely for each normalized eigenfunction Ψ_n of the harmonic oscillator there are two eigenfunctions of Q given by

$$e^{-i\omega t}e^{-ip_y y}\begin{pmatrix}\Psi_n(x+\frac{p_y}{e\mathcal{B}})\\ 0\end{pmatrix} \quad and \quad e^{-i\omega t}e^{-ip_y y}\begin{pmatrix}0\\ \Psi_n(x+\frac{p_y}{e\mathcal{B}})\end{pmatrix} \tag{2.11}$$

with eigenvalues

$$-[(\omega-i\mu)^2+(2n+1)e\mathcal{B}\mp e\mathcal{B}] \tag{2.12}$$

respectively, where n is a nonnegative integer. If we set

$$d_n=[(\omega-i\mu)^2+2ne\mathcal{B}+m^2] \tag{2.13}$$

Then these eigenvalues are simply equal to $-d_n$ and $-d_{n+1}$ respectively.

Having found the eigenvalues we can now express the inverse of Q as

$$\begin{aligned}\frac{1}{Q}&=\frac{1}{[\tilde{D}^2-m^2+e\sigma_3\mathcal{B}]}\\ &=-\sum_{n=0}^{\infty}\int\frac{d\omega}{2\pi}\int\frac{dp_y}{2\pi}\{[\tfrac{1}{2}(\frac{1}{d_{n+1}}+\frac{1}{d_n})-\tfrac{1}{2}(\frac{1}{d_{n+1}}-\frac{1}{d_n})\sigma_3]\\ &\quad e^{-i\omega(t-t')}e^{-ip_y(y-y')}\Psi_n(x+\frac{p_y}{e\mathcal{B}})\Psi_n{}^*(x'+\frac{p_y}{e\mathcal{B}})\}\end{aligned} \tag{2.14}$$

The tadpole-corrected propagator S_T is then given by

$$S_T \;=\; [\tilde{\not{D}}-m]^{-1} \;=\; (\tilde{\not{D}}+m)Q^{-1} \tag{2.15}$$

Recall the *definition* of $\mathcal{B}$ as $\mathcal{B}=\frac{e}{\theta}\rho_0$. We have thus evaluated the propagator as a function of both ρ_0 and μ. Now the density ρ_0 depends itself on μ. Thus our next step will be to find the relationship between ρ_0 and μ. We shall do this in perturbation theory, but using the tadpole-corrected propagator S_T. We begin by computing the lowest order contribution to the mean density

ρ_0 via the diagram of Figure 3.[†] We shall see later in the paper that due to a nonrenormalization theorem the result of this lowest order calculation is, in fact, an exact result. Our strategy is to first compute ρ_0 for fixed μ and $\mathcal{B}$. Having done this we then use the fact that $\mathcal{B}$ itself depends on ρ_0 to find ρ_0 as a function of μ. (It turns out that although this last step is very illuminating, it does not play a role in the discussion of superfluidity.)

The calculation of ρ_0 proceeds as follows:

$$\begin{aligned}\rho_0 &= <\psi^\dagger(r)\psi(r)> = \; -Tr[\gamma_0 S_T(r,r)]\\ &= \sum_{n=0}^{\infty}\int\frac{d\omega}{2\pi}\int\frac{dp_y}{2\pi}|\psi_n(x+\frac{p_y}{e\mathcal{B}})|^2\,[-i(\omega-i\mu)(\frac{1}{d_{n+1}}+\frac{1}{d_n})-m(\frac{1}{d_{n+1}}-\frac{1}{d_n})]\end{aligned} \tag{2.16}$$

The integral over p_y can now be done since the functions Ψ_n are normalized eigenfunctions for the harmonic oscillator. This integral simply yields a factor $e\mathcal{B}$. Thus

$$\rho_0 \;=\; \frac{-ie\mathcal{B}}{2\pi}\sum_{n=0}^{\infty}\int\frac{d\omega}{2\pi}\,[(\omega-i\mu)(\frac{1}{d_{n+1}}+\frac{1}{d_n})-im(\frac{1}{d_{n+1}}-\frac{1}{d_n})] \tag{2.17}$$

The integrals over ω are now done using contour integral techniques. The integral $\int d\omega\{\frac{1}{d_n}\}$ is standard, and is performed by closing the contour along a semicircle of very large radius in the complex plane either above or below the real axis. The integral $\int d\omega\{\frac{(\omega-i\mu)}{d_n}\}$ can be done using a cutoff regulator. It is convergent despite initial appearances. It is evaluated by shifting the contour from the real axis to the line $-\infty+i\mu<\omega<\infty+i\mu$. The vertical parts of the contour at infinity vanish, and the resulting integral over the new contour vanishes by antisymmetry. All that remains are possible poles in the region $0<Im(\omega)<\mu$.

[†] The self energy corrections to the bare fermion propagator will be taken into account separately and will be higher order in our expansion. They will be shown, in the next section, to be irrelevant in computing the density and the renormalized Chern-Simons term.

The results are

$$\int_{-\infty}^{\infty} \frac{d\omega}{(\omega - i\mu)^2 + 2neB + m^2} = \frac{\pi}{\sqrt{2neB + m^2}} \Theta(\sqrt{2neB + m^2} - |\mu|)$$
$$\int_{-\infty}^{\infty} d\omega \frac{\omega - i\mu}{(\omega - i\mu)^2 + 2neB + m^2} = \frac{\mu}{|\mu|} \pi i \Theta(|\mu| - \sqrt{2neB + m^2}) \tag{2.18}$$

Using these results we find that

$$\begin{aligned}
\rho_0 &= \frac{eB}{4\pi} \frac{\mu}{|\mu|} \sum_{n=0}^{\infty} \left[\Theta(|\mu| - \sqrt{2neB + m^2}) + \Theta(|\mu| - \sqrt{2(n+1)eB + m^2} \right] \\
&+ \frac{eB}{4\pi} \frac{m}{|m|} \Theta(|m| - |\mu|) \\
&= \frac{eB}{2\pi} \frac{\mu}{|\mu|} \left[Int \left(\frac{\mu^2 - m^2}{2eB} \right) + \tfrac{1}{2} \right] \Theta(|\mu| - |m|) + \frac{eB}{4\pi} \frac{m}{|m|} \Theta(|m| - |\mu|)
\end{aligned} \tag{2.19}$$

where Int stands for the integer part of its argument.

Note that ρ_0 does not vanish when μ=0. This is an artifact of our regularization of the operator product $\psi^\dagger(r)\psi(r)$ using an ultraviolet cutoff. It is a consequence of the spectral asymmetry of our parity non-invariant theory and it is closely related to a similar ambiguity in the renormalization of the Chern-Simons term for this theory[19]. (It would not occur, for example, with Pauli-Villars regularization.) Nonetheless it is clearly this ρ_0 which is related to B via the relation $B = \frac{e}{\theta}\rho_0$. However, in order to get a physical picture of what is going on, it is useful to consider the "physical" density $\rho_{ph} = \rho_0(\mu) - \rho_0(\mu{=}0)$. ($\rho_{ph}$ is the expectation value of the properly renormalized density operator.)

ρ_{ph} is plotted versus μ for fixed (positive) B† when the mass $m > 0$ in Figure 4. We emphasize that this is a plot of ρ as a function of μ *for fixed* B. We have

† Although we have assumed implicitly that eB is positive, it is straightforward to show that, in general, the first term in eqn. (2.19) is proportional to $|eB|$ whereas the second term is proportional to eB. Thus for negative eB the plot in Figure 4 is shifted to the left by an amount $2|eB|$.

not yet imposed the self consistency condition which is required by the definition of $\mathcal{B}$ in terms of ρ_0. When the density is *not* an integer multiple of $e\mathcal{B}/2\pi$ the density can increase with no cost in chemical potential. Thus a new particle can be added to the system at no cost in energy. This corresponds to the filling of a Landau level. When the density reaches an integer multiple of $e\mathcal{B}/2\pi$, the level is filled and a discrete jump in chemical potential is required before the next level can be filled. The asymmetry between positive and negative μ reflects the spectral asymmetry of the theory. Evidently the two signs of μ correspond to having particles with opposite spin. We see from Figure 4 that for $\mu > 0$ one requires a chemical potential $\mu^2 = m^2 + 2e\mathcal{B}$ to begin filling the first Landau level. For $\mu < 0$ the spins point in the opposite direction. The interaction energy of the spin with the magnetic field $\mathcal{B}$ precisely cancels its orbital energy in the first Landau level leading to a zero energy mode. Thus one begins to fill the first Landau level at $\mu = -m$.

We are now ready to use the definition of $\mathcal{B} = \frac{e}{\theta}\rho_0$ to find ρ_0 as a function of μ for a given value of θ which is assumed positive. If, as we shall assume, e, $\mathcal{B}$ and θ are positive, then ρ_0 will also be positive. Let us consider the case $m > 0$ in which case $\rho_0 = \rho_{ph} + e\mathcal{B}/4\pi$. Suppose ρ_0 is fixed at some physical value (say by fixing ρ_{ph}). We see from Figure 4 that it is useful to write ρ_0 as $e\mathcal{B}/4\pi$ plus an integer number of steps of magnitude $e\mathcal{B}/2\pi$ plus some remainder.

$$\rho_0 = \frac{e\mathcal{B}}{2\pi}[N + \tfrac{1}{2} + \gamma]; \qquad 0 \leq \gamma < 1 \tag{2.20}$$

where N is an integer. Now ρ_0 is also equal to $\frac{\theta}{e}\mathcal{B}$. Thus

$$\frac{2\pi\theta}{e^2} = N + \tfrac{1}{2} + \gamma \tag{2.21}$$

Note that γ is determined entirely by θ/e^2. Thus fixing the Chern-Simons coefficient simply tells us how many Landau levels are filled and what fraction of the first unfilled level is occupied. The values of N and γ are determined entirely from the values of e and θ.

The resulting value of μ can now be determined from Figure 4. When $\gamma \neq 0$ (i.e. when there is an unfilled Landau level) the value of μ is uniquely determined to be

$$\mu^2 = m^2 + 2(N+1)\frac{2\pi\rho_0}{N+\frac{1}{2}+\gamma} = m^2 + 2(N+1)\frac{2\pi\rho_{ph}}{N+\gamma} \tag{2.22}$$

or

$$\rho_{ph} = \frac{\mu^2 - m^2}{4\pi}\frac{N+\gamma}{N+1} \tag{2.23}$$

when $\gamma \neq 0^\dagger$. On the other hand, when γ=0 and we have a filled Landau level then the value of μ is ambiguous with

$$m^2 + 2N\frac{2\pi\rho_{ph}}{N} < \mu^2 < m^2 + 2(N+1)\frac{2\pi\rho_{ph}}{N} \tag{2.24}$$

We would expect that all physical quantities will turn out to be independent of which value of μ is chosen in this range.

Notice that the condition of having precisely N filled Landau levels occurs when $2\pi\rho_{ph}/eB = N$. This then implies that $(2\pi\theta/e^2) - \frac{1}{2} = N$. Naively we might have expected the '$\frac{1}{2}$' to be missing. The reason for the presence of this term is that there is a renormalization of the Chern–Simons coefficient in this theory *at zero density* which occurs in one–loop and which is not renormalized by higher loops (see refs.[32], [19]). The renormalized Chern–Simons term at zero density has been calculated in the literature[32,19] and it will be calculated later in this section. It is given by $\theta_R = \theta - (m/|m|)(e^2/4\pi)$ when cutoff regularization is used. Although this relationship is regularization dependent, it is only the renormalized CS term which is physical. It is in fact this *renormalized* Chern–Simons term at zero density which determines the statistics of the anyons. Our condition for having N filled levels now becomes (for m>0)

$$\frac{2\pi\theta_R}{e^2} = N \tag{2.25}$$

which corresponds to a statistics parameter $\pi(1-1/N)$ where N is an integer.

† Note that this differs only slightly from the result for fermions for which $\rho = (\mu^2 - m^2)/4\pi$.

Such a statistics parameter corresponds to the horizontal parts of Figure 4 whereas a non–integer N corresponds to its vertical parts.

We now turn our attention to the calculation of the current–current correlation $< J_\mu(r) J_\nu(r') >$ in this theory, where $J_\mu = e\bar{\psi}\gamma_\mu\psi$ †. We begin by considering the (one–particle–irreducible) vacuum polarization $\Pi_{\mu\nu}$. As discussed in the previous section in this parity non–invariant theory in three dimensions, $\Pi_{\mu\nu}$ can be split up into an even and an odd part:

$$\Pi_{\mu\nu}(q) = \Pi^e_{\mu\nu}(q) + \epsilon_{\mu\nu\lambda} q^\lambda \Pi_{odd}(q)$$

where Π^e is symmetric under interchange of μ and ν. Note that gauge invariance requires the odd part of Π to have the above covariant form even at finite density for which case Lorentz invariance is lost. Our goal is to show that the *full* current–current correlation has a massless pole if and only if $2\pi\theta_R/e^2$ is an integer. As discussed previously the existence of such a pole depends entirely on the value of Π_{odd}. It will exist if $\Pi_{odd}(q{=}0){=}\theta$. This will be discussed in detail in section 5. The main point is that a massless pole exists in the current–current correlation if and only if there is a massless excitation in this theory which couples to it. If the renormalized CS term $\theta - \Pi_{odd}(q{=}0)$ vanishes then the effective action for this theory has a massless mode corresponding to the statistical photon.

We thus want to show that $\Pi_{odd}(q{=}0){=}\theta$ i.e. that the renormalized Chern-Simons term at finite density vanishes.†† We do the calculation in the tadpole-improved perturbation theory described above. The idea is that in intermediate stages of the calculation we keep both μ and $B = (e/\theta)\rho_0$ as variables and only at the end we insert the appropriate value for μ which was derived above eqn.

† Note: we are defining the current as the charge current whereas the density was previously defined as the particle number density.

†† This is of course *not* true at zero density[30]. The zero density limit of this theory is extremely singular since for any finite density there are a fixed number of occupied Landau levels. This number depends only on θ but not on the density. At precisely zero density there are of course no levels occupied.

(2.24). Our first step is to evaluate the one–loop value of Π_{odd}. We shall then use a non-renormalization theorem to show that the result holds to all orders in tadpole-improved perturbation theory.

Before describing the explicit calculation of $\Pi_{odd}(q{=}0)$ we give a general argument which shows that it is equal to θ as claimed above. We show that at any order in tadpole-improved perturbation theory there is a general relation between the diagrams which contribute to $\Pi_{odd}(q{=}0)$ and those which contribute to ρ_0. We begin by considering an arbitrary diagram which contributes to ρ_0. We now take $\delta/\delta\mathcal{B}$ at fixed μ of any such diagram. This has the effect of removing precisely one tadpole insertion and replacing it by $ie\gamma^i\epsilon_{ij}\partial_j/\nabla^2$ which is applied to the graph and the result is evaluated at $q{=}0$. This is shown pictorially in Figure 5. The resulting graph is one–particle–irreducible in terms of tadpole-corrected lines since clearly all diagrams for ρ_0 are one–particle–irreducible. It is thus related to the vacuum polarization Π_{0j}. In fact

$$\frac{\delta e\rho_0}{\delta\mathcal{B}}|_\mu = i\epsilon_{ij}\frac{\partial_j}{\nabla^2}\Pi_{i0}|_{q=0} \tag{2.26}$$

Now the odd part of Π goes to zero linearly with q as q goes to zero whereas the even part of Π vanishes quadratically. Thus

$$\frac{\delta e\rho_0}{\delta\mathcal{B}}|_\mu = i\epsilon_{ij}\frac{\partial_j}{\nabla^2}\epsilon_{ik}(-i)\frac{\partial_k}{\nabla^2}\Pi_{odd}|_{q=0} = \Pi_{odd}(q=0) \tag{2.27}$$

Thus in order to evaluate Π_{odd} at $q{=}0$ all we must do is differentiate ρ_0 with respect to $\mathcal{B}$ at fixed μ. We can do this either via eqn. (2.19) or more simply from Figure 4. Note that on the vertical sections (i.e. when $(\mu^2-m^2)/2e\mathcal{B}$ is an integer) the above derivative is divergent. It is only convergent on the horizontal sections which correspond to case of completely filled Landau levels. In this case $\rho_0/\mathcal{B}$ is independent of $\mathcal{B}$. Thus $\delta\rho_0/\delta\mathcal{B} = \rho_0/\mathcal{B}$. But ρ_0 is related to $\mathcal{B}$ via the tadpole relation $\rho_0 = \frac{\theta}{e}\mathcal{B}$. Thus

$$\Pi_{odd}(q=0) = \frac{\delta e \rho_0}{\delta \mathcal{B}} = \theta \tag{2.28}$$

This is the result which was claimed above. It implies that the one loop (tadpole–improved) renormalized Chern–Simons term vanishes, and thus there is pole in the current–current correlation for this theory. It occurs if and only if the *zero density* renormalized CS coefficient θ_R is such that some integer number N of Landau levels are precisely filled. As discussed above eqn. (2.25) this occurs (for $m>0$) when $2\pi\theta_R/e^2 = N$. We also have here a hint of a possible pathology in the anyonic system when $2\pi\theta_R/e^2$ is not an integer since Π_{odd} diverges in this case. This might play a role in understanding the behaviour of anyonic systems in the presence of real magnetic fields.

In the next section we shall show that these results hold to all orders in perturbation theory. We shall do this by extending previous nonrenormalization theorems at zero density to the case of finite density. It can be seen from the above discussion that proving a nonrenormalization theorem for ρ_0 is sufficient since the corresponding result for Π_{odd} can be derived as a its consequence.

The previous result is powerful enough to be used to evaluate the renormalized Chern-Simons term at μ=0 for fixed $\mathcal{B}$. It must be emphasized that in the case μ=0, $\Pi_{odd}(q{=}0)$ is clearly *not* equal to θ since a typical value of θ requires a filling of some Landau levels which, in general cannot occur at μ=0. The result of eqn. (2.28) that $\Pi_{odd}(q{=}0) = e\delta\rho_0/\delta\mathcal{B}$ can be used together with (2.19) to see that $\Pi_{odd}(\mu{=}0, q{=}0) = e^2/4\pi$. This is a well–known result and it was used earlier in this section.

We have augmented the general result above by an explicit calculation of $\Pi_{odd}(q{=}0)$. We have done this using both the direct method and the Schwinger proper time method. The latter calculation appears in the Appendix. Here we summarize the calculation in the direct method.

The one–loop expression for $\Pi_{\mu\nu}$ is given by

$$\Pi_{\mu\nu}(x,y) = -e^2 Tr\left[\gamma_\mu S_T(x,y)\gamma_\nu S_T(y,x)\right] \tag{2.29}$$

To evaluate Π_{odd} we extract the term in eqn. (2.29) which is proportional to $\epsilon_{ij}k_0$. The basic idea is to evaluate eqn. (2.29) using equations (2.14) and (2.15). Using these equations we can write the fermion propagator as follows:

$$S_T(x,x') = \sum_n \int \frac{d\omega}{2\pi}\int \frac{dp_y}{2\pi}(\not{L} + K)e^{-i\omega(t-t')}e^{-ip_y(y-y')}\psi_n(x - \frac{p_y}{eB})\psi_n^*(x' - \frac{p_y}{eB}) \tag{2.30}$$

where

$$\begin{aligned}
K &= \frac{i}{2}\left(\frac{\omega - i\mu - im}{d_{n+1}} - \frac{\omega - i\mu + im}{d_n}\right)\\
L^0 &= \frac{-i}{2}\left(\frac{\omega - i\mu - im}{d_{n+1}} + \frac{\omega - i\mu + im}{d_n}\right)\\
L^1 &= \sqrt{\frac{eB}{2}}\left(\frac{a}{d_n} - \frac{a^\dagger}{d_{n+1}}\right)\\
L^2 &= -i\sqrt{\frac{eB}{2}}\left(\frac{a}{d_n} + \frac{a^\dagger}{d_{n+1}}\right)
\end{aligned} \tag{2.31}$$

where a and $a^\dagger$ are the raising and lowering operators for the harmonic oscillator wave functions ψ_n. Equation (2.29) can now be evaluated using the standard trace identities for the Pauli matrices. The term of interest is proportional to $\epsilon_{\mu\nu\lambda}$. We can extract Π_{odd} by calculating, for example, Π_{12} and extracting the odd part. The result is:

$$\begin{aligned}
\Pi^{12}_{odd} =& e^2 \sum_{n,r}\int\frac{d\omega}{2\pi}\int\frac{d\omega'}{2\pi}\int\frac{dp_y}{2\pi}\int\frac{dp'_y}{2\pi}e^{-i(t-t')(\omega-\omega')}e^{-i(y-y')(p_y-p'_y)}\\
& 2i(KL'_0 - K'L_0)\psi_n(x - \frac{p_y}{eB})\psi_n^*(x' - \frac{p_y}{eB})\psi_r(x' - \frac{p'_y}{eB})\psi_r^*(x - \frac{p'_y}{eB})
\end{aligned} \tag{2.32}$$

where no superscript and the superscript ' on K and L refer to whether they apply to (ω, n) or to (ω', r).

The method for extracting Π_{odd} involves first integrating over $(y-y')$ to force the external y-momentum to vanish. This yields $\delta(p_y - p'_y)$. After doing the p'_y integral, one can do the x integral which forces the external x-momentum to vanish and then yields δ_{nr}. The p_y integral can then be done and simply yields a factor eB. Note that what remains is the fourier transform of Π_{12} with respect to an external frequency $q_0 = \omega - \omega'$. The resulting expression is linear in q_0 for small q_0 and its coefficient is Π_{odd}. We find:

$$\Pi^{12}_{odd} = i\frac{eB}{\pi}e^2 q_0 \sum_{n=0}^{\infty} \int \frac{d\omega}{2\pi} \left[\frac{1}{d_n} - \frac{2(\omega - i\mu + im)(\omega - i\mu)}{d_n^2}\right] \frac{(\omega - i\mu - im)}{d_{n+1}} \tag{2.33}$$

After some manipulations we can write Π_{odd} as

$$\begin{aligned} \Pi_{odd} = & -\frac{e^2}{2\pi}\sum_{n=0}^{\infty}\int \frac{d\omega}{2\pi}\{[m(\frac{1}{d_{n+1}} - \frac{1}{d_n}) + i(\omega - i\mu)(\frac{1}{d_n} + \frac{1}{d_{n+1}})] \\ & - i4neB\frac{(\omega - i\mu)}{[(\omega - i\mu)^2 + M(n)^2]^2}\} \end{aligned} \tag{2.34}$$

where $M(n)^2 = 2neB + m^2$.

This is the same result which we obtain using Schwinger's method in Appendix A.

We now compare this expression for Π_{odd} to the previous expression for ρ_0. The term in the first line of (2.34) is identical to the similar term in ρ_0 (divided by B/e). For the term in the second line of eqn. (2.34) we perform the ω integration which results in an expression which is proportional to $\delta(\mu - M(n))$. This term vanishes whenever $\frac{\mu^2 - m^2}{2eB} \neq integer$. Therefore for any μ such that $\frac{\mu^2 - m^2}{2eB} \neq integer$, i.e. for the filled levels, we confirm explicitly the relation between Π_{odd} and ρ_0 given in eqn. (2.28).

3. Nonrenormalization theorem

We will now show that ρ, $\Pi_{odd}(0)$, and thus eqn. (2.28), are unaffected by higher order radiative corrections in tadpole–corrected perturbation theory. We do this by extending the nonrenormalization theorem of Coleman and Hill, which applies to $\Pi_{odd}(0)$ in the zero density case, to the case of finite density.

Before attempting to extend the Coleman-Hill theorem, let us briefly review it. Consider the Euclidean n-photon effective vertex, at zero density, given by summing all graphs consisting of a single fermion loop with n external photons attached. We denote this by:

$$\Gamma^{(n)}_{\mu_1\ldots\mu_n}(k_1\ldots k_n) \tag{3.1}$$

All diagrams in vacuum perturbation theory which contribute to $\Pi_{odd}(0)$ can be constructed from the $\Gamma^{(n)}$'s, by sewing together photon lines (see ref.[32] for details). One set of contributions is obtained by sewing together all but two photon lines of a $\Gamma^{(n)}$, and finding the piece of the resulting two-point function which is linear in the external momentum and antisymmetric in the vector indices. The remaining contributions are obtained by sewing together, in all possible one-photon-irreducible ways, two different $\Gamma^{(n)}$'s, such that one external photon line remains on each. These two types of contributions have the following form:

$$\begin{aligned}
&\lim_{k\to 0}\epsilon_{\mu\nu\lambda}\frac{\partial}{\partial k_\lambda}\int dk_3\cdots dk_n\Gamma^{(n)}_{\mu\nu\lambda_3\ldots\lambda_n}\left(k;-k;k_3;\ldots;-\sum_3^{n-1}k_\ell\right)\mathcal{K}_{\lambda_3\ldots\lambda_n}(k_3;\ldots;k_n)\\
&\lim_{k\to 0}\epsilon_{\mu\nu\lambda}\frac{\partial}{\partial k_\lambda}\int dk_2\cdots dl_2\cdots\Gamma^{(n)}_{\mu\rho\ldots}(k;k_2;\ldots)\Gamma^{(n)}_{\nu\lambda\ldots}(-k;l_2;\ldots)\mathcal{K}_{\rho\ldots\lambda\ldots}(k;k_2;\ldots)
\end{aligned} \tag{3.2}$$

Now for any $\Gamma^{(n)}$ gauge invariance implies

$$k^\mu\Gamma^{(n)}_{\mu\ldots}=0 \tag{3.3}$$

Differentiating this expression gives

$$\Gamma^{(n)}_{\nu\ldots} + k^{\mu}\left(\partial/\partial k^{\nu}\right)\Gamma^{(n)}_{\mu\ldots} = 0 \tag{3.4}$$

Provided that $\Gamma^{(n)}$ is analytic as $k \to 0$ this implies

$$\Gamma^{(n)}_{\ldots}(0; k_2; k_3; \ldots) = 0 \tag{3.5}$$

Furthermore, if $n > 2$, so that k_1 and k_2 are independent variables, then

$$\Gamma^{(n)}_{\ldots}(k_1; k_2; \ldots) = \mathcal{O}(k_1 k_2) \tag{3.6}$$

as k_1, $k_2 \to 0$. These relations imply that all contributions to $\Pi_{odd}(0)$ of two-loop and higher order vanish. This is the Coleman-Hill nonrenormalization theorem.

We now want to extend these arguments to the case of finite fermion density. We thus define a finite density Euclidean n-photon effective vertex, given by summing all graphs consisting of a single *tadpole-corrected* fermion loop with n external photons attached. Order by order in *tadpole-corrected* perturbation theory, the structure of the graphs which contribute to $\Pi_{odd}(0)$ is identical to the zero-density case. Furthermore, we can apply the same construction to obtain all the graphs contributing to ρ. These are obtained by sewing together all but one external photon line of a $\Gamma^{(n)}$, and have the form:

$$\int dk_2 \cdots dk_n \Gamma^{(n)}_{\mu\lambda_2\ldots\lambda_n}\left(0; k_2; \ldots; -\sum_{2}^{n-1} k_\ell\right) \mathcal{K}_{\lambda_2\ldots\lambda_n}(k_2; \ldots; k_n) \tag{3.7}$$

Thus, to prove the desired nonrenormalization theorem for ρ and $\Pi_{odd}(0)$, it suffices to show that, for k_1,$k_2 \to 0$:

$$\begin{aligned} \Gamma^{(n)}_{\ldots}(k_1 \ldots) &= \mathcal{O}(k_1), \qquad & n > 1 \\ \Gamma^{(n)}_{\ldots}(k_1, k_2, \ldots) &= \mathcal{O}(k_1 k_2), \qquad & n > 2 \end{aligned} \tag{3.8}$$

By gauge invariance and the argument presented above, these relations are true provided that $k \to 0$ is in the region of analyticity of the $\Gamma^{(n)}$.

We prove the nonrenormalization theorem therefore by demonstrating the analyticity of the $\Gamma^{(n)}$ as $k^2 \to 0$ in the Euclidean region. This is obvious for the zero density system, since the physical (Minkowski) threshold for fermion-antifermion pairs begins at $k^2 = 4m^2$. At finite density, however, one must also worry about the production of fermion-hole pairs. In our case, since the $\Gamma^{(n)}$ are defined in tadpole–corrected perturbation theory, this corresponds to a (Minkowski) photon being absorbed by a fermion in a Landau level, causing a transition to an unoccupied state. The Landau levels allow continuous values of momentum but are discretely spaced in energy (with spacing $e\mathcal{B}/m$ in the non-relativistic limit). Therefore, when we have N completely filled Landau levels, physical singularities are absent for (Minkowski) $k_0 < e\mathcal{B}/m$. Thus as we approach $k^2 \to 0$ from the Euclidean region the $\Gamma^{(n)}$ are analytic, and the nonrenormalization theorem holds precisely for $\theta = Ne^2/2\pi$.

Note that for other values of θ we obtain no definite conclusions; this is similar to the $m = 0$ case of the zero density system. For self-consistency, we should also note that the Goldstone pole, which is the end result of this analysis, does not appear in the individual 1PI diagrams of the $\Gamma^{(n)}$.

To understand *why* the nonrenormalization theorem works, it is useful to study the topological properties of ρ (and, by extension, $\Pi_{odd}(0)$). By topological we mean dependent only on the asymptotic behavior of the effective background gauge field. The physical content of the finite density nonrenormalization theorem is that the spatially averaged mean density ρ is insensitive to local perturbations of the background mean field. We would like to understand in detail why this is true.

The first step is to relate ρ to quantities which measure the spectra of operators in a *nonconstant* background field. Recall that ρ is defined as

$$\begin{aligned}\rho &= -Tr[\gamma_0 S_T(x,x)] \\ &= -Tr\left[\gamma_0(S_0^{-1} - ie\gamma^i \mathcal{A}_i)^{-1}\right]\end{aligned} \tag{3.9}$$

Let us, for simplicity, assume static gauge fields with arbitrary spatial variation.

Then we can write:

$$\rho = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} Tr\left[\gamma_0 (\not{D} - m)^{-1}\right] \tag{3.10}$$

where

$$\not{D} = -i\gamma_0(\omega - i\mu) + \gamma^i(\partial_i - ie\mathcal{A}_i) \tag{3.11}$$

Applying eq.(2.8), we get

$$\rho = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} Tr\left[\gamma_0(\not{D} + m)[D^2 - m^2 + e\mathcal{B}\gamma_0]^{-1}\right] \tag{3.12}$$

Since the $\gamma^i D_i$ part of the integrand will not survive the trace, we can rewrite this as

$$\rho = i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} Tr\left[\gamma_0[D^2 - m^2 + e\mathcal{B}\gamma_0]^{-1}[\gamma_0(\omega - i\mu) + im]\right] \tag{3.13}$$

Although D^2 and $\mathcal{B}$ do not commute for nonconstant $\mathcal{B}$, one can easily derive the following identity:

$$[D^2 - m^2 + e\mathcal{B}\gamma_0]^{-1} = [1 - Ge\mathcal{B}Ge\mathcal{B}]^{-1}(1 - Ge\mathcal{B}\gamma_0)G \tag{3.14}$$

where

$$G = (D^2 - m^2)^{-1}$$

This identity allows us to perform the gamma matrix traces in our expression for ρ. The result is:

$$\rho = 2i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} Tr\left[[1 - Ge\mathcal{B}Ge\mathcal{B}]^{-1}[(\omega - i\mu) - imGe\mathcal{B}]G\right] \tag{3.15}$$

With a few additional manipulations, one finds that this is equivalent to

$$\rho = i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} Tr \left[(\omega - i\mu) \left(\frac{1}{D_+^2} + \frac{1}{D_-^2} \right) + im \left(\frac{1}{D_+^2} - \frac{1}{D_-^2} \right) \right] \tag{3.16}$$

where

$$D_\pm^2 = D^2 - m^2 \pm e\mathcal{B}$$

Of course, for constant $\mathcal{B}$, this expression immediately reduces to eqn. (2.17).

To see the topological nature of ρ let us first consider the $\mu \to 0$ case. In this limit only the second term of our expression –the one proportional to m– survives. Define

$$\begin{aligned} H^\pm &= -D_i^2 \mp e\mathcal{B} \\ E &= \omega^2 + m^2 \end{aligned} \tag{3.17}$$

Then we can write

$$\rho = \frac{m}{2\pi} \int_{m^2}^{\infty} dE \frac{1}{\sqrt{E - m^2}} Tr \left[\frac{1}{H^+ + E} - \frac{1}{H^- + E} \right] \tag{3.18}$$

Provided that $\mathcal{B}$ goes to a nonzero constant value at spatial infinity, the spectrum of $H^\pm$ consists entirely of discrete bound eigenvalues with no continuum. The analytic behavior of the (regulated) trace as a function of complex E consists of simple poles on the negative real axis. In addition the integrand has a branch cut, which we take to lie on the real axis to the right of the branch point $E{=}m^2$. Since the trace is analytic at $E{=}m^2$, and since the integrand is $\mathcal{O}(E^{-3/2})$ as $|E| \to \infty$, we can rewrite eqn. (3.18) as a contour integral:

$$\rho = \frac{m}{4\pi} \oint_C dz \frac{1}{\sqrt{z - m^2}} Tr \left[\frac{1}{H^+ + z} - \frac{1}{H^- + z} \right] \tag{3.19}$$

where the contour encloses the entire complex plane except for the branch cut and an infinitesimal disk around the branch point $z = m^2$. Now note that $H^\pm$

are related to the two-dimensional massless Euclidean Dirac operator. Writing:

$$i \not{D} = \begin{bmatrix} 0 & L \\ L^\dagger & 0 \end{bmatrix} \tag{3.20}$$

we have:

$$H^- = LL^\dagger, \quad H^+ = L^\dagger L \tag{3.21}$$

Thus the nonzero eigenvalues of $H^\pm$ are paired, and their residues cancel in the evaluation of the contour integral. We are left with the zero mode contribution:

$$\rho = -\frac{m}{2|m|}\Delta \tag{3.22}$$

where[36]

$$\begin{aligned} \Delta &= dim\ Ker(L^\dagger L) - dim\ Ker(LL^\dagger) \\ &= dim\ Ker(L) - dim\ Ker(L^\dagger) \end{aligned} \tag{3.23}$$

Thus we have found that ρ is proportional to the *index* of the operator L.

Returning now to the general case of finite density, we must consider the contribution to ρ from the first term in eqn. (3.16). This is given by

$$-\frac{i}{4\pi} \oint_{C'} dz Tr \left[\frac{1}{H^+ + z} + \frac{1}{H^- + z} \right] \tag{3.24}$$

where we have changed variables:

$$(\omega - i\mu)^2 + m^2 \to z$$

The closed contour C' crosses the real axis at $+\infty$ and at $-(\mu^2 - m^2)$. We observe that, for any $\mathcal{B}$ field configuration which goes asymptotically to a nonzero constant, the spectra of $H^\pm$ are purely discrete in z. Furthermore the integrand contains no branch cuts. The contour integral picks up the residues of discretely spaced poles. This discrete spacing is determined by the asymptotic behavior of the bound state eigenfunctions; thus the evaluation of the contour integral is insensitive to local perturbations in the background field.

Analogous results for $\mu>0$ apply to the second term of (3.16). It should be noted that our analysis of the contour integrals in eqns.(3.19) and (3.24) assumes $m^2>0$ and that $z = -(\mu^2-m^2)$ does not coincide with any poles. These are, of course, the same restrictions that appeared in our discussion of the Coleman-Hill theorem.

4. Finite temperature behaviour

So far we discussed the anyonic system at zero temperature. The more interesting question is, obviously, the behaviour of the system at a finite temperature T. The passage to finite temperature does not alter the fact that the condition for the massless mode in the current–current correlator is the vanishing of θ_R. Hence, we have to repeat the calculation of the ρ and Π_{odd}, which are now also function of the temperature, and check whether we still have $\Pi_{odd} = \frac{e\rho}{\mathcal{B}}$. We shall only be calculate the one–loop contributions to these quantities (in tadpole–improved perturbation theory). Our proofs of the nonrenormalization theorem do not extend to the finite temperature case and it is nearly certain that the theorem fails to hold.

Technically, the standard procedure to pass to the finite temperature calculation in Eulidean space involves the compactification of the (Euclidean) time direction into the range $0 \leq t \leq \beta = \frac{1}{T}$ and the imposition of antiperiodic boundary conditions (in time) for fermions and periodic boundary conditions for bosons. For our calculation this implies replacing the integral over frequencies ω with a sum over discrete 'Matsubara' frequencies $\omega_l = \frac{2\pi}{\beta}(l+\frac{1}{2})$, l being an integer. In particular we can evaluate the mean density ρ_0 at finite temperature by using equation (2.17) but, instead of integrating over ω, we sum over the discrete frequencies. This results in the expression:

$$\rho_0 \; = \; \frac{-ie\mathcal{B}}{2\pi\beta}\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}[(\omega_l - i\mu)(\frac{1}{d_{n+1}(l)}+\frac{1}{d_n(l)})-im(\frac{1}{d_{n+1}(l)}-\frac{1}{d_n(l)})] \quad (4.1)$$

where

$$d_n(l) = (\omega_l - i\mu)^2 + M^2(n) = (\omega_l - i\mu)^2 + 2neB + m^2$$

The frequency sums $\sum G(l)$ can be done exactly by evaluating contour integrals of the form

$$\frac{1}{2\pi i}\oint \frac{cos\pi x}{sin\pi x} G(x) \tag{4.2}$$

where the integral is over a contour which surrounds the real axis. When evaluating ρ_0 we get the following answers for the sums:

$$\begin{aligned} \frac{2\pi}{\beta}\sum_l \frac{1}{d_n(l)} &= -\frac{\pi}{2M}\left[tgh[\frac{\beta}{2}(\mu - M)] - tgh[\frac{\beta}{2}(\mu + M)]\right] \\ \frac{2\pi}{\beta}\sum_l \frac{(\omega - i\mu)}{d_n(l)} &= \frac{\pi i}{2}\left[tgh[\frac{\beta}{2}(\mu - M)] + tgh[\frac{\beta}{2}(\mu + M)]\right] \end{aligned} \tag{4.3}$$

Inserting these results into the expression (4.1) for ρ_0 we get the following expression for the density at finite temperature:

$$\rho_o = \frac{eB}{4\pi}\{\sum_{n=0}^{\infty}\left[tgh[\frac{\beta}{2}(\mu + M(n))] + tgh[\frac{\beta}{2}(\mu - M(n))]\right] - \frac{m}{|m|}tgh[\frac{\beta}{2}(\mu - |m|)]]\} \tag{4.4}$$

Note that in the limit $\beta \to \infty$ this expression reduces to the zero temperature result given in eqn.(2.19). At nonzero temperature the density is no longer a step function as the chemical potential is varied. The steps are smoothed out as is shown for specific values of the parameters in Figure 6. This result is, of course well known from the theory of the Quantum Hall Effect.

We now compute the renormalized CS term at finite temperature. The simplest way to do this is to use eqn.(2.28) which is valid at finite temperature. Recall that the renormalized Chern–Simons term is proportional to $d(\rho/B)/d(B)$. In the zero temperature case ρ was (piecewise) proportional to eB and thus the renomalized Chern–Simons term vanished. At finite temperature we see from eqn. (4.4) that this is no longer the case. In fact ρ/eB is a monotonic function of eB.

The relation $\Pi_{odd} = e\rho_0/\mathcal{B}$ is thus never valid, and the renormalized CS term is nonzero for any finite temperature. In fact by differentiating eqn.(4.4) we find:

$$\begin{aligned}\theta_R(\mu, T) &= -\, e\mathcal{B}e^2 \frac{d(\rho/e\mathcal{B})}{d(e\mathcal{B})} \\ &= -\frac{\alpha e\mathcal{B}}{2}\beta \sum_{n=0}^{\infty} \frac{n}{M(n)} \left[tgh^2[\frac{\beta}{2}(\mu + M(n))] - tgh^2[\frac{\beta}{2}(\mu - M(n))] \right]\end{aligned} \tag{4.5}$$

where $\alpha = e^2/4\pi$. Note that the term inside the sum is positive for all values of n as long as μ is nonzero. Thus θ_R is nonzero for any nonzero value of μ. Note that it is not necessary to use eqn.(2.28) to evaluated the renormalized Chern–Simons term. We have, in fact, evaluated it explicitly as a further check on the calculation. The steps leading to eqn. (2.34) can be repeated and one obtains precisely the same equation but with the integral over ω replaced with the sum over ω_l. Recall that the cancellation of the bare and one–loop CS term was a result of the vanishing of the term in the second line of eqn.(2.34) for the filled Landau levels -i.e., $\frac{\mu^2 - m^2}{2e\mathcal{B}} \neq integer$. It is straightforward to see that at finite temperature this term does not vanish.

It is important to emphasize that this result, namely the presence of a nonzero renormalized Chern–Simons term at finite temperature, has only been demonstrated in the one–loop approximation. Although it is difficult to imagine that higher order perturbative effects would force the mass of this mode to vanish, it is, in principle, possible. More reasonably, nonperturbative effects may generate a massless mode in the current–current correlation even at finite temperature. It is well known that although long range order is not possible at finite temperature in 2+1 dimensions it is still possible for a massless mode to be present as occurs for a Kosterlitz–Thouless transition.[39] These ideas have been used quite widely in analyzing the finite temperature behaviour of anyonic systems.[40] . We can certainly not rule out such behaviour but we must point out that *in perturbation theory* a mass is present and superfluidity is lost.

In order to obtain an estimate of the size of the renormalized Chern-Simons term and of the resulting mass of the "pseudo–Goldstone mode" we evaluate θ_R in the low temperature limit. (The precise limit will be described below.) We shall specialize to the case of most interest for which the T=0, μ=0 Chern-Simons coefficient is an integer, N, i.e. for which N Landau levels are filled. We assume a density ρ of anyons. Keeping in mind the distinction between ρ_{phys} and ρ_0 which was discussed in section 2, we then require a field eB=$2\pi\rho/N$. If we then assume that the temperature is sufficiently low so that $\beta(M_N - M_{N-1})) >> 1$ (recall that $M_N^2 = 2NeB + m^2$) we can compute the sums in both eqn. (4.4) and eqn. (4.5) since only one term in each sum contributes significantly. We can express the result in terms of the renormalized value of N, N_{ren}=$2\pi\theta_R/e^2$ as

$$N_{ren} = \frac{2\pi\rho}{N}\beta\left(\frac{N-1}{M_{N-1}} + \frac{N}{M_N}\right) exp\left(-\frac{\beta}{2}(M_N - M_{N-1})\right) \tag{4.6}$$

In the non-relativistic limit $\rho << m^2$ this becomes

$$N_{ren} = \frac{2\pi\rho}{mN}\beta(2N-1)exp\left(\frac{-\beta\pi\rho}{mN}\right) \tag{4.7}$$

We see that for integer N and for small temperature, the renormalized Chern-Simons term is exponentially suppressed compared to its unrenormalized value.

The mass of the "pseudo–Goldstone" mode is given by:

$$m_{PG} = \frac{\theta_R}{\Pi_e} = (\frac{2\beta}{N})(\frac{\pi\rho}{mN})^2(2N-1)exp\left(\frac{-\beta\pi\rho}{mN}\right) \tag{4.8}$$

where for Π_e we have used the estimate derived in section 6. We can get a rough idea of the order of magnitude of this mass by putting in some possible numbers for the mass and density such as may occur in high T_c superconductors. Choosing the density ρ to be $10^{14}cm^{-2}$ and the mass m to be the electron mass and a temperature T of 100^oK we find that m_{PG} is approximately 5×10^{-6} ev. This corresponds to a distance scale of roughly 5 cm. This estimate is of course extremely crude since there are large uncertainties in the exponent.

The value of N_{ren} for non integer values of N (which is simply defined as $2\pi\theta/e^2$ with θ the $T{=}\mu{=}0$ value of the Chern-Simons term) also displays some interesting behaviour. At zero temperature it is infinite as can be seen from equations (2.19) and (2.28). At finite temperature it is finite, though much larger than the value for integer N (at small temperatures.) We plot N_{ren} as a function of N for various values of β and ρ in Figure 7. The exponential supression of N_{ren} for integer N is evident in the figures. What is perhaps most interesting from these figures is that even at the one loop level there is critical behaviour of N_{ren} as a function of temperature. Note that for any small temperature, at some point near any integer value of N, dN_{ren}/dN vanishes. For any such integer value of N there is a critical temperature $T_c(N)$ above which this derivative is nonzero. It is plausible that even if there were some nonperturbative mechanism which would restore the massless mode at finite temperature or restore superconductivity through coupling between layers, the above critical behaviour may signal an end to such behaviour and might thus be related to the phase transition to a normal state.

In summarizing this section we once again point out that since we have no control of nonperturbative effects, we cannot argue convincingly that superfluidity is lost at any finite temperature. What we have, however, shown is that one of the main steps in the argument for superfluidity at zero temperature, namely the presence in the RPA approximation of a massless pole in the current–current correlation, is lost at finite temperature.

5. The criterion for anyonic superconductivity

The field–theoretic criterion for superconductivity of a system of charged anyons at zero temperature was derived by Banks and Lykken[30] using an effective low energy approach. Here, following the same approach, we rederive this criterion in more detail and argue that it applies also to systems with non-zero temperature. We also show that the same criterion emerges by analyzing diagramatically the condition for a massless pole in the current-current correlator.

We begin again with a single two-component massive fermion field coupled to the fictitious CS gauge field a_μ, now in a Minkowski space-time.

$$\mathcal{L}_a = i\bar{\psi}(\not{D} \; - m)\psi + \frac{\theta}{2}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \tag{5.1}$$

By integrating over the fermionic degrees of freedom and those of the fictitious gauge field above a certain cutoff one gets for the low energy effective action

$$\mathcal{L}_{eff} = -\frac{1}{4}\Pi_e f^{\mu\nu}f_{\mu\nu} + \tfrac{1}{2}\theta_R\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \tag{5.2}$$

where $\Pi_e{=}\Pi_e(k^2{=}0)$ and $\Pi_o{=}\Pi_o(k^2{=}0)$ are the parity even and odd parts of the vacuum polarization

$$\Pi_{\mu\nu}(k^2) = \Pi_e(k^2)(k_\mu k_\nu - g_{\mu\nu}k^2) - \Pi_o(k^2)\epsilon_{\mu\nu\lambda}k^\lambda \tag{5.3}$$

at zero four momentum, and $\theta_R{=}\theta{-}\Pi_o$ is the renormalized CS coefficient. The quantum system which corresponds to (5.2) is equivalent[19] to that of one polarization of a massive ($m = \frac{\theta_R}{\Pi_e}$) boson with spin $\frac{\theta_R}{|\theta_R|}$ for the case of $\theta_R{\neq}0$, and to that of a massless scalar for $\theta_R{=}0$. Due to the equation of motion $J^\mu{=}\theta\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$, a massless fictitious gauge field implies a massless pole in the current–current correlator. Thus *a sufficient condition for the existence of a massless pole is the vanishing of* θ_R.

The next stage is to examine whether the system admits superconductivity when coupled to photons. Electromagnetism is introduced to the system via the Lagrangian

$$\mathcal{L} = \mathcal{L}_a - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e_A J_\mu A^\mu \tag{5.4}$$

with $J_\mu = -\bar{\psi}\gamma_\mu\psi$ the fermionic current, and A_μ, e_A the electromagnetic gauge field and coupling constant. The corresponding low energy effective action is given now by[30]:

$$\begin{aligned}\mathcal{L}_{eff} = & -\frac{1}{4}(1+\Pi_e) F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\Pi_o \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \\ & + \tfrac{1}{2}\theta_R \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{1}{4}\Pi_e f^{\mu\nu} f_{\mu\nu} \\ & - \tfrac{1}{2}\Pi_e F_{\mu\nu} f^{\mu\nu} - \tfrac{1}{2}\Pi_o \epsilon^{\mu\nu\lambda}(A_\mu \partial_\nu a_\lambda + a_\mu \partial_\nu A_\lambda)\end{aligned} \tag{5.5}$$

In the case of interest, when θ_R=0, we can rewrite this effective action in a Landau-Ginzburg (LG) form by considering the path integral

$$\mathcal{Z} = \int \mathcal{D} a_\mu \mathcal{D} A_\mu e^{i\mathcal{L}_{eff}} \tag{5.6}$$

and changing variables from a^λ to the dual of $f_{\mu\nu}$, ${}^*f^\lambda = \frac{1}{2}\epsilon^{\mu\nu\lambda} f_{\mu\nu}$. This should be accompanied by imposing the Bianchi identity on ${}^*f^\lambda$ via a Lagrange multiplier $\tilde{\varphi}$:

$$\mathcal{Z} = \int \mathcal{D}\, {}^*f_\lambda \mathcal{D}\tilde{\varphi} \mathcal{D} A_\mu e^{i[\mathcal{L}_{eff}(A_\mu,\, {}^*f_\lambda) + \tilde{\varphi}\partial_\alpha\, {}^*f^\alpha]} \tag{5.7}$$

Gaussian integration over ${}^*f^\lambda$ leads to the LG effective Lagrangian

$$\begin{aligned}\mathcal{L}_{eff}(A_\mu, \varphi) &= \frac{1}{2\Pi_e}(\partial_\alpha \tilde{\varphi} + \theta A_\alpha + \Pi_e\, {}^*F_\alpha)^2 = \tfrac{1}{2}(\partial_\alpha \varphi + C A_\alpha + a\, {}^*F_\alpha)^2 \\ &= \tfrac{1}{2}(\partial_\alpha\varphi + CA_\alpha)^2 + a(\partial_\alpha \varphi + CA_\alpha)^* F^\alpha + \frac{1}{4}\Pi_e F_{\mu\nu}F^{\mu\nu}\end{aligned} \tag{5.8}$$

where $\varphi = \frac{\tilde{\varphi}}{\sqrt{\Pi_e}}$ (assuming that Π_e does not vanish) , $C = \frac{\theta}{\sqrt{\Pi_e}}$, a=$\sqrt{\Pi_e}$ and ${}^*F$ is the dual to F.

Obviously one has to add to (5.8) the first line of eqn. (5.5). This effective action, which is invariant under $\varphi \to \varphi - C\Lambda$, $A_\mu \to A_\mu + \partial_\mu \Lambda$, is the Stuckelberg form of the Higgs mechanism. We thus see that a massless fictitious gauge field indeed leads to a photon mass term as well as to a (P,T violating) supercurrent. A discussion of the low energy action and the associated current is presented in section 6.

To incorporate a fixed non-zero density of anyons one has to add a chemical potential term $\mu\psi^\dagger\psi$ to the Euclidean version of the Lagrangian given in eqn. (5.2). By introducing a chemical potential term we break Lorentz invariance. This system is described in the low energy region by an effective action which is required to be invariant under the group of rotations rather then the full Lorentz group. Hence, instead of eqn. (5.2) one gets

$$\mathcal{L}_{eff} = -\frac{1}{4}\Pi_e^B f_{ij} f^{ij} - \frac{1}{4}\Pi_e^E f_{0j} f^{0j} + \tfrac{1}{2}\theta_R \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \tag{5.9}$$

where Π_e^B and Π_e^E, the coefficients of the magnetic and electric terms, are related to the non-relativistic parity–even vacuum polarization

$$\Pi^e_{\mu\nu}(k^2) = \Pi_e(k^2)(k_\mu k_\nu - g_{\mu\nu}k^2) + \Pi'_e(k^2)(k^i k^j - \delta^{ij}\overrightarrow{k^2})\delta^i_\mu \delta^j_\nu \tag{5.10}$$

via $\Pi_e^E = \Pi_e(\overrightarrow{k^2}=0, k^0=0)$, $\Pi_e^B = \Pi_e(\overrightarrow{k^2}=0, k^0=0) + \Pi'_e(\overrightarrow{k^2}=0, k^0=0)$. (We shall see in section 6 that this expression is not quite complete and that another tensor structure constructed out of the dual of **k** is also possible.) Notice, however, that there is no difference between the form of the CS term here and the one in the relativistic $\mu=0$ case. There is no gauge invariant way to split the CS term. This is an an outcome of the fact that the CS term is a topological (metric independent) term. It is straightforward to show that the same proof of the equivalence at $\theta_R=0$ to the theory of a massless scalar applies also in the non-relativistic case. Another explicit derivation of this result is given in the appendix of ref. [30]. Turning on the electromagnetic interactions and following the same steps that led to eqn. (5.8) we now get for $\theta_R=0$

$$\begin{aligned}\mathcal{L}_{eff}(A_\mu,\varphi) =& \tfrac{1}{2}(\partial_0\varphi + CA_0)^2 - \frac{v^2}{2}(\partial_i\varphi + CA_i)^2 \\ &+ a(\partial_0\varphi + CA_0)\ {}^*F^0 + b(\partial_i\varphi + CA_i)\ {}^*F^i\end{aligned} \tag{5.11}$$

where now $v = \sqrt{\frac{\Pi^B_b}{\Pi^B_e}}$, $C = \frac{\theta}{\sqrt{\Pi^B_e}}$, $a{=}\sqrt{\Pi^B_e}$ and $b{=}a$. Notice, however, that this effective action is not the most general gauge and rotation invariant action of a non-relativistic real scalar coupled to an abelian gauge field in 2+1 dimensions. The most general case allows $a{\neq}b$ as was found in ref. [29] where, for particular coefficients C, v, a and b, this LG action reproduced the current–current correlation function derived from the RPA method. In section 6 we show how this splitting may occur in the present formulation, and we relate, at the one–loop level, the even part of the zero–momentum vacuum–polarization to the corresponding parameters of the action.

Finally, we will naturally be interested in considering the anyonic system at finite temperature. Technically, as was shown in section 4, the passage to non-zero temperature in the system is achieved by taking a compactified imaginary time direction with radius of $\beta = \frac{1}{T}$ which, in momentum space, involves a summation over the Matsubara frequencies. These changes do not affect the criterion for superconductivity.

As an alternate to the above discussion we investigate the condition for a massless pole in the current–current correlator $< J_\mu(r)J_\nu(r') >$ diagramatically. We denote by $K_{\mu\nu}$ the fourier transform of the correlator and express it in the perturbation expansion shown in Figure 8. It is clear that K is related to Π by summing a geometric series:

$$K \ = \Pi \left[\frac{1}{1 - C\Pi}\right]$$

where C represents the Chern-Simons propagator, which was discussed in Section 2. The condition for the presence of a massless pole in K is that the determinant $det(1-C\Pi)$ vanishes when the momentum k vanishes. Using the expression (5.10)for $\Pi_e^{\mu\nu}$ near $k = 0$ we can now explicitly evaluate the determinant $det(1-C\Pi)$. The result is

$$det(1 - C\Pi)(k = 0) \propto (1 - \frac{\Pi_{odd}}{\theta})^2 \tag{5.12}$$

We thus see explicitly that a massless pole is present only if $\Pi_{odd}(k=0)=\theta$. This is precisely the condition that the renormalized Chern-Simons term at finite density vanish.

When coupled to electromagnetism this system is a superconductor. This can be easily seen diagramatically by treating the current–current correlation K evaluated above, as the leading order one–particle–irreducible vacuum polarization for the electromagnetic photon. The photon propagator is then estimated by summing the geometric series shown in Figure 9. In Feynman gauge the photon propagator is given by $(k^2-\tilde{K})^{-1}$ where $\tilde{K}$ is the coefficient of $g_{\mu\nu}$ in K. Clearly if K has a massless pole and thus $\tilde{K} \sim 1/k^2$ as $k^2 \to 0$, the photon propagator will have no massless pole, and will in fact have a pole at a nonzero value of k^2. This leads to the Meissner effect.

A similar argument for the duality between the fictitious and electromagnetic gauge fields was presented by Wen and Zee[14]. For $\theta_R \neq 0$ the Green's function of a_μ behaves (in a relativistic formulation) as $< a_\mu(k)a_\nu(-k) > \sim \frac{(g_{\mu\nu}-k_\mu k_\nu/k^2)}{k^2-m^2}$ where m is given by θ_R/Π_e. This two point function induces a current–current correlator which in the $k \to 0$ limit behaves like $< J_\mu(k)J_\nu(-k) > \sim \frac{(g_{\mu\nu}k^2-k_\mu k_\nu)}{m^2}$. When coupled to photons this leads to an effective Maxwell term for the photon without a mass term. This argument completes the picture of dual behaviour between the fictitious and electromagnetic gauge fields.

6. Low energy effective action

The low energy effective action of a system of charged anyons was derived in the last section. For the finite density, non-relativistic case one has:

$$
\begin{aligned}
\mathcal{L}_{eff}(A_\mu,\varphi) = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\theta}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \\
& \tfrac{1}{2}(\partial_0\varphi + CA_0)^2 - \frac{v^2}{2}(\partial_i\varphi + CA_i)^2 \\
& + a(\partial_0\varphi + CA_0)\ {}^*F^0 + b(\partial_i\varphi + CA_i)\ {}^*F^i
\end{aligned}
\tag{6.1}
$$

where the parameters v, C a and b are functions of the magnetic and electric even parts of the vacuum-polarization as follows: $v = \sqrt{\frac{\Pi^B}{\Pi^E}}$, $C = \frac{\theta}{\sqrt{\Pi^B}}$, $a = \sqrt{\Pi^B}$ and b=a. Note the presence of the electromagnetic Chern-Simons term which is absent in ref.[29]. To generate a splitting between a and b one has to go back to equation (5.9) and add to it an additional $B\partial_i E^i$-like term, namely: $\frac{1}{2}\Pi_N\epsilon^{ij}f_{ij}\partial_k f^{0k}$. Π_N is the coefficient[30] of $q^0(\epsilon^{ik}q^j + \epsilon^{jk}q^i)q^k$ in the non-relativistic expression for Π^{ij}. In spite of the fact that this gauge and rotational invariant term is higher in derivatives it conspires with the mixed CS term to give a contribution to the term whose coefficient is a in (6.1). With such a term, and its mixed f with F analog, one gets: $a = \sqrt{\Pi^B}[1 + \frac{\Pi_N\theta}{\Pi_B\Pi_B}]$ and $b = \sqrt{\Pi^B}$.

Eqn. (6.1) is a description of the low lying collective excitations exhibiting the Stuckelberg form of the Higgs mechanism. This action was discussed in refs. [29] and [42]. In the absence of an electromagnetic field it describes a sound wave which turns into the longitudinal component of the massive photon. The "anyonic" origin of this action appears via the P and T violating terms which are proportional to a and b. If we set aside the F^2 term then the current and the density are given by:

$$
\begin{aligned}
J_i = & \frac{-\delta\mathcal{L}_{eff}}{\delta A_i} = -v^2C(\partial_i\varphi + CA_i) + a\epsilon_{ij}\partial^j(\partial_0\varphi + CA_0) - b\epsilon_{ij}\partial_0(\partial^j\varphi + CA^j) \\
J_0 = & \frac{\delta\mathcal{L}_{eff}}{\delta A_0} = C(\partial_0\varphi + CA_0) + \tfrac{1}{2}aC\epsilon^{ij}F_{ij}
\end{aligned}
\tag{6.2}
$$

The first term of the current is the standard London supercurrent and the additional terms emerge from the P and T violating terms of the action[42].

We can estimate the values of the parameters v, C, a, and b by a one–loop dimensional analysis. This gives:

$$\begin{aligned} v^2 &= \frac{\Pi^B}{\Pi^E} \sim \frac{\rho}{m^2} \qquad & C &= \frac{\theta}{\sqrt{\Pi^B}} \sim e\sqrt{m} \\ b &= \sqrt{\Pi^B} \sim \frac{eN}{\sqrt{m}} \qquad & a &= \sqrt{\Pi^B} + \frac{\theta \Pi_N}{\sqrt{\Pi^B \Pi^E}} \sim \frac{eN}{\sqrt{m}} \end{aligned} \tag{6.3}$$

We can compute these parameters at the one–loop level from the even part of the vacuum polarization. (These calculations will be presented in a future publication.) Note that, for the even part of the vacuum polarization, the nonrenormalization theorem does not apply and there are higher loop corrections.

It is interesting to compare our effective action to those of refs.[13], [12], [29], and [30]. We obtain all of the P and T violating terms discussed in these papers. Unlike ref.[29] we find that the coefficient of $A_i^* F^i$ is nonzero: it equals $\theta/2$. It will be important to resolve this discrepancy since this term has experimental consequences which are potentially observable.[13]

7. Summary and discussion

In this work we have presented a field theoretical analysis of anyonic superconductivity in the framework of CS gauge theory. By developing a tadpole–improved perturbation theory, in which fermions propagate in a constant background magnetic field, we obtain a simple physical picture of the anyon gas analogous to the Integer Quantum Hall (IQH) system. A crucial difference, however, is that the quantum constraints of CS theory relate the fermion density to the "statistics" mean field. This allows the fermi gas with exactly filled Landau levels to be compressible, resulting in a Goldstone mode and superfluidity.[36] We have seen that the dynamics of this phenomenon are entirely determined by a one–loop CS graph, with no higher order corrections (at zero temperature). This is, *a priori*, a remarkable result. It becomes rather intuitive, however, in the IQH mean field picture with the Landau gap and purely discrete spectrum. It is amusing to note that the well–known insensitivity of the IQH quantization to impurities[33] may be considered an "experimental confirmation" of our extended Coleman-Hill theorem!

Our results, extending those of Fetter et al and of Chen et al, show rather conclusively that zero temperature anyonic superconductivity *does exist* in a rigorous theoretical framework.

There remain the open questions of whether anyons provide a mechanism for high T_c superconductivity, and whether CuO superconductors utilize such a mechanism. Our CS formalism is not well-adapted to addressing these questions. As we have seen, our finite temperature perturbative results give indications both of a zero temperature phase transition (giving a pseudo-Goldstone mass) and of critical behavior at a finite $T_c \sim eB/m$. Previous results and arguments[14,15,18] [43] indicate that the true behavior is of Kosterlitz-Thouless type; however, since we exhibit neither vortices nor a local order parameter we have not addressed this possibility directly. The less attractive scenario of a zero temperature phase transition should be considered seriously, even though our evidence for it is less than compelling. It is quite possible that the resolution of this issue will involve subtleties of the relationship between our CS description and other descriptions of the anyon gas.

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APPENDIX A

Fermion propagator, density and Π_o using Schwinger's proper time integral

The tadpole–corrected fermion propagator was shown in section 2 to be equivalent to that of a fermion in a constant fictitious magnetic field which is defined via the density: $\mathcal{B}=e\rho/\theta$. We now calculate this propagator using Schwinger's proper time integral:

$$S_T(x,x') = < x| \int_0^\infty ds e^{-Hs}(\not{D} \ + m)|x' > \tag{A.1}$$

where $H=-(\not{D} \ \ \not{D} \ )=D^2+e\mathcal{B}\sigma_3$. We differ from the original calculation[44] in using a Euclidean metric and working in three dimensions. The matrix element of the operator $U(s)=e^{-Hs}$ is thus given[44] by

$$\begin{aligned} < x|U(s)|x' > = &\frac{C(x,x')}{s^{3/2}} e^{-L(-is)} e^{-\frac{es\mathcal{B}\sigma_3}{2}} \\ &\times e^{-\frac{1}{4}(x-x')_\mu [eFcot(eFs)]^{\mu\nu}(x-x')_\nu} \end{aligned} \tag{A.2}$$

where $C(x-x') = \frac{1}{8\pi^{3/2}} e^{i\int_{x'}^{x} d\xi^\mu A_\mu(\xi)}$ and $e^{-L(-is)} = \frac{e\mathcal{B}s}{sinh(e\mathcal{B}s)}$. Expanding the various factors in (A.2) we get the following expression for the tadpole–corrected fermion propagator

$$\begin{aligned} S_T(x,x') = &\frac{e\mathcal{B}}{8\pi^{3/2}} e^{i\int_{x'}^{x} d\xi^\mu A_\mu(\xi)} \int_0^\infty ds \frac{e^{-m^2 s}}{s^{1/2} sinh(e\mathcal{B}s)} e^{-\frac{(t-t')^2}{4s}} \\ &\times e^{-\frac{1}{4}e\mathcal{B}cot(e\mathcal{B}s)(x-x')_i(x-x')^i} \times [G_1 + G_0\sigma_3 + G^i\sigma_i] \end{aligned} \tag{A.3}$$

where G_1, G_0 and G^i are given by

$$\begin{aligned} G_1 &= mcosh(e\mathcal{B}s) + \frac{1}{2s}(t-t')sinh(e\mathcal{B}s) \\ G_0 &= -msinh(e\mathcal{B}s) - \frac{1}{2s}(t-t')cosh(e\mathcal{B}s) \\ G^i &= -\frac{(x^i - x'^i)}{2sinh(e\mathcal{B}s)} \end{aligned} \tag{A.4}$$

Note that here we supressed the chemical potential. In the following computations before integrating over the frequency we first analytically continue $\omega \rightarrow \omega - i\mu$.

The next task is to calculate the fermion density. This is achieved by substituting $S_T(x-x')$ from (A.4) into

$$\rho_0 = -Tr[\gamma_0 S_T(x,x)] \tag{A.5}$$

We have to integrate over $d^2(x-x')\delta(x_i-x_i')$ and then over $d(t-t')\delta(t-t')$. The last delta function is written as $\delta(t-t')=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega(t-t')}$. After the first integration we find

$$\begin{aligned}\rho &= -\frac{eB}{8\pi^{3/2}}\int d(t-t')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega(t-t')}\int ds\frac{e^{-ms^2}}{s^{1/2}} \\ &\times e^{\frac{-(t-t')^2}{4s}}[-m-\frac{1}{2s}(t-t')(coth(eBs)] \\ &= \frac{2eB}{4\pi^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-s(m^2+\omega^2)}[m+i\omega coth(eBs)]\end{aligned} \tag{A.6}$$

If we now write

$$coth(eBs) = \frac{1+e^{-2eBs}}{1-e^{-2eBs}} = (1+e^{-2eBs})\sum_{n=0}^{\infty}e^{-2neBs} \tag{A.7},$$

substitute it into (A.6) and replace ω with $\omega - i\mu$ we find exactly the same expression (2.17) as found using the method described in section 2.

The third step is to calculate the odd part of the vacuum polarization Π_o. We substitute (A.3) into eqn. (2.29) and just as in section 2 we extract the term proportional to $\epsilon_{ij}\omega$ The resulting term is:

$$\Pi_o^{ij}(q) = \frac{2ie^2(eB)^2}{8^3\pi^6}\epsilon_{ij}\int d^3p\int d^3x\int d^3x' e^{ipx+(p-q)x'}\int_0^\infty \frac{ds_1}{\sqrt{s_1}}\int_0^\infty \frac{ds_1}{\sqrt{s_2}}$$
$$\times \frac{e^{m^2(s_1^2+s_2^2)}}{sinh(eBs_1)sinh(eBs_2)} e^{\frac{-t^2}{4s_1}} e^{\frac{-t'^2}{4s_2}} \times e^{[-\frac{1}{4}eB[coth(eBs_1)x^2+coth(eBs_2)x'^2]]}$$
$$[(\frac{tt'}{4s_1s_2} - m^2)sinh[eB(s_1-s_2)] - m(\frac{t}{2s_1} - \frac{t'}{2s_2})cosh[eB(s_1-s_2)]]. \tag{A.8}$$

Now doing the x and x' integration, and then the p integration, substituting $k^ik_i = 0$ we get:

$$\Pi_o^{ij}(0,q_0) = \frac{2ie^2(eB)}{16\pi^3}\epsilon_{ij}\int \omega\int dt\int dt' e^{ipx+(p-q)x'}\int_0^\infty \frac{ds_1}{\sqrt{s_1}}\int_0^\infty \frac{ds_1}{\sqrt{s_2}}$$
$$\times \frac{e^{m^2(s_1^2+s_2^2)}}{sinh(eBs_1)sinh(eBs_2)} e^{\frac{-t^2}{4s_1}} e^{\frac{-t'^2}{4s_2}}$$
$$[(\frac{tt'}{4s_1s_2} - m^2)sinh[eB(s_1-s_2)] - m(\frac{t}{2s_1} - \frac{t'}{2s_2})cosh[eB(s_1-s_2)]]. \tag{A.9}$$

After integrating over t and t' this reduces to

$$\Pi_o^{ij}(0,q_0) = \frac{ie^2(eB)}{4\pi^2}\epsilon_{ij}\int d\omega\int_0^\infty \frac{ds_1}{\sqrt{s_1}}\int_0^\infty \frac{ds_1}{\sqrt{s_2}}$$
$$\times \frac{e^{m^2(s_1^2+s_2^2)}}{sinh(eBs_1)sinh(eBs_2)} e^{-\omega^2 s_1} e^{-(\omega-q_0)^2 s_2}$$
$$[[-(\omega(\omega-q_0) - m^2)sinh[eB(s_1-s_2)] - imq_0cosh[eB(s_1-s_2)]]\,. \tag{A.10}$$

Replacing hyperbolic trignometric functions with sums and integrating over the proper time variables s_1 and s_2 one gets the same result as given in eqn. (2.34).

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Figure Captions

Fig 1: Two representations of the Feynman diagrams for the Chern-Simons theory.

Fig 2: Tadpole contributions to the fermion propagator. S_F represents the full fermion propagator.

Fig 3: One loop diagram for the density ρ_0 in tadpole-improved perturbation theory.

Fig 4: The physical density $\rho_{ph} = \rho(\mu) - \rho(0)$ is plotted versus $\mu^2 \times sign(\mu)$ at fixed B. Here $sign(\mu)$ is the sign of μ.

Fig 5: Diagramatic representation of the result that $\delta e\rho_0/\delta B|_\mu = \Pi_{odd}(q=0)$. Only the simplest class of diagrams are shown.

Fig 6: Numerical results for the physical density as a function of μ for various values of the inverse temperature β.

Fig 7: The renormalized Chern-Simons coefficient $N_{ren} = 2\pi\theta_{ren}/e^2$ is plotted versus its unrenormalized value (by which we mean its zero density renormalized value) for various temperatures and densities.

Fig 8: Diagramatic expansion for the full current-current correlator for the pure Chern-Simons theory in terms of the one-particle- irreducible graphs.

Fig 9: Diagramatic expansion for the electromagnetic photon propagator in terms of the current-current correlator K of the pure Chern-Simons theory. Electromagnetic corrections to the vacuum polarization are not shown.

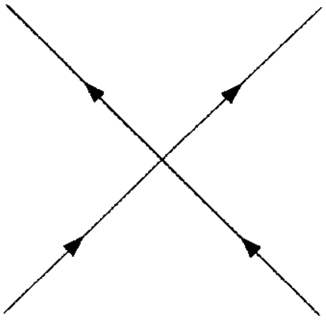


Figure 1 a

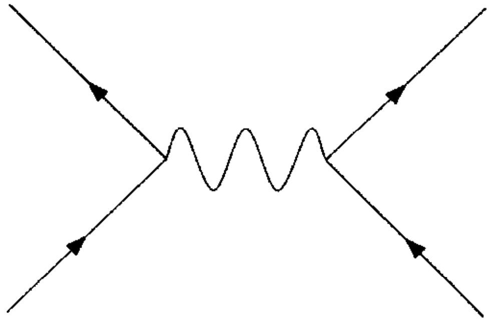


Figure 1 b

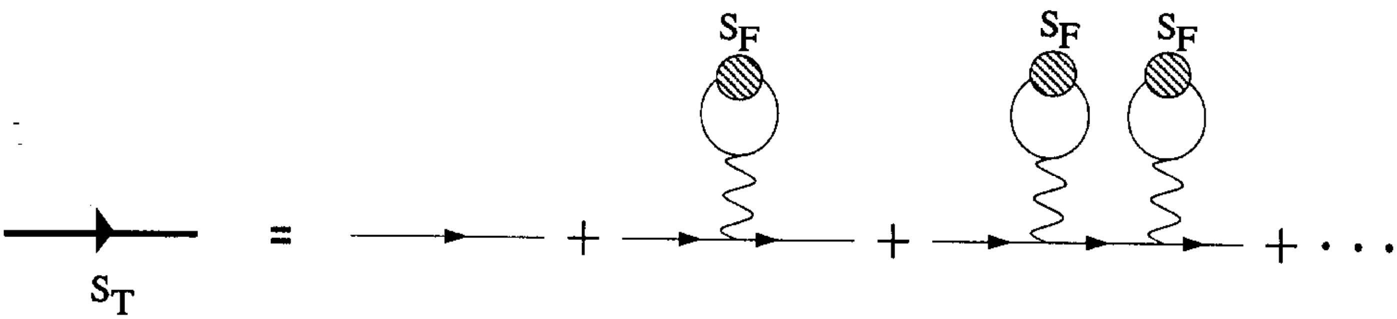


Figure 2

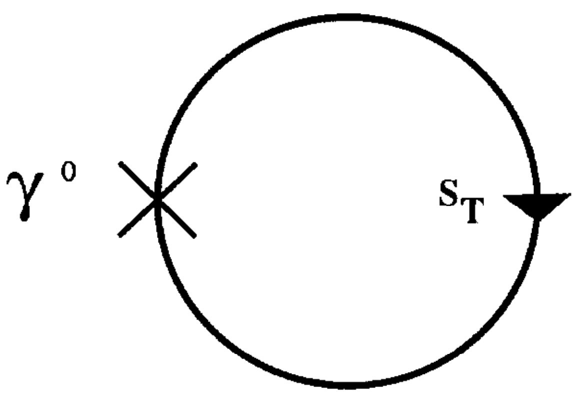


Figure 3

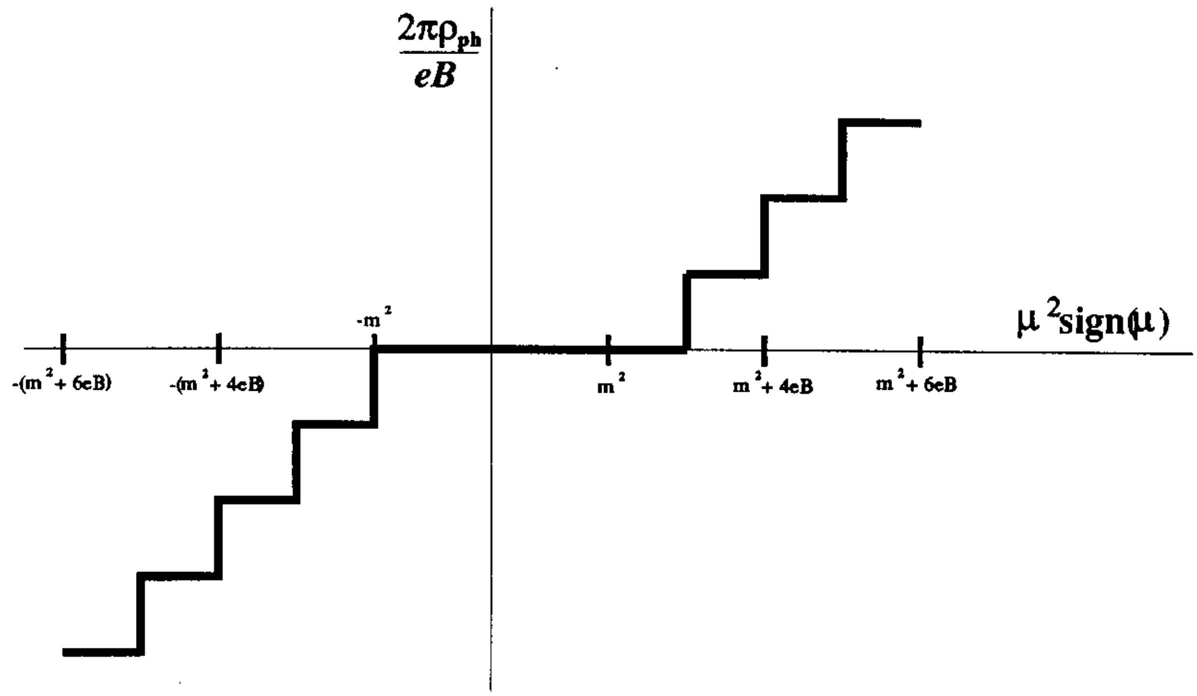


Figure 4

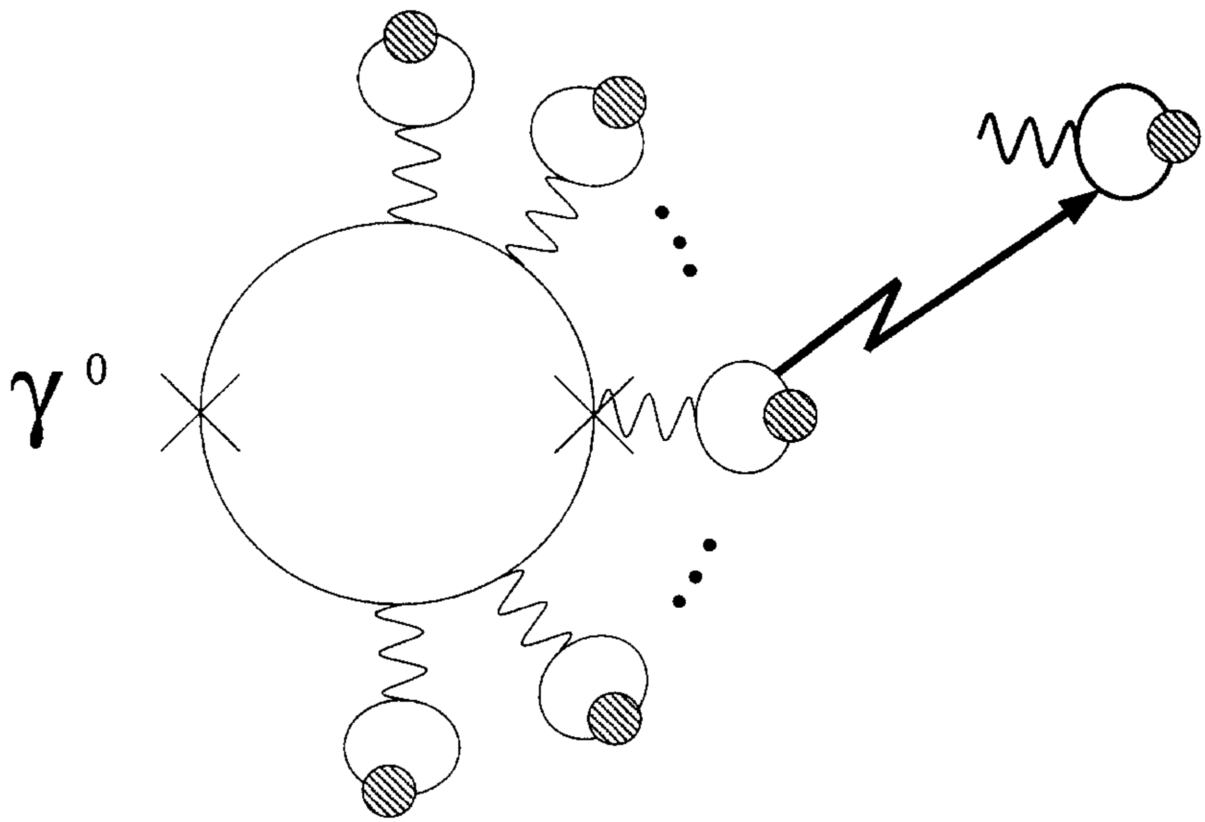


Figure 5

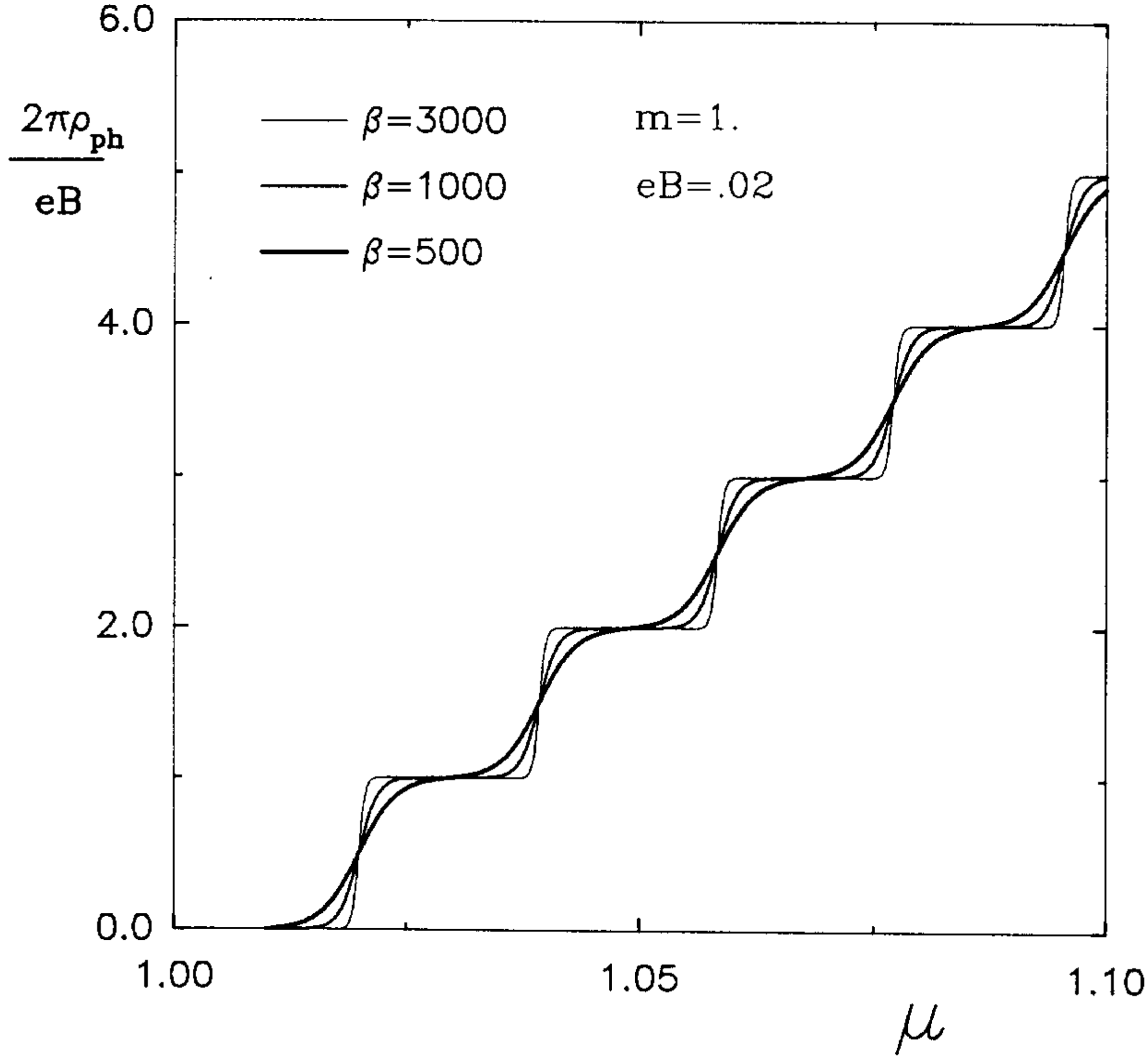
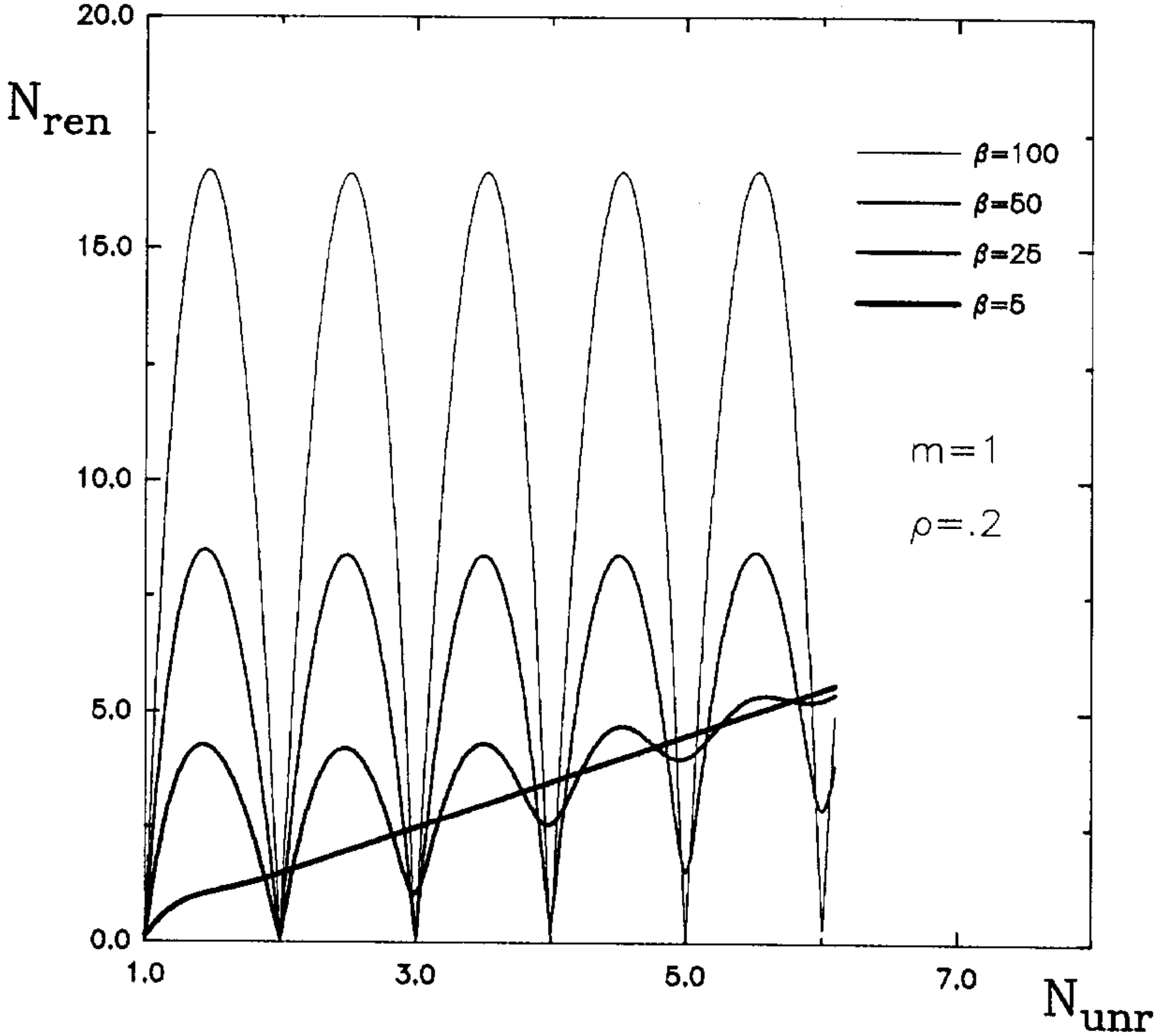


Figure 6



20.0
15.0
10.0
5.0
0.0
1.0
3.0
5.0
7.0
N_{ren}
N_{unr}
β=100
β=50
β=25
β=5
m=1
ρ=.2

Figure 7a

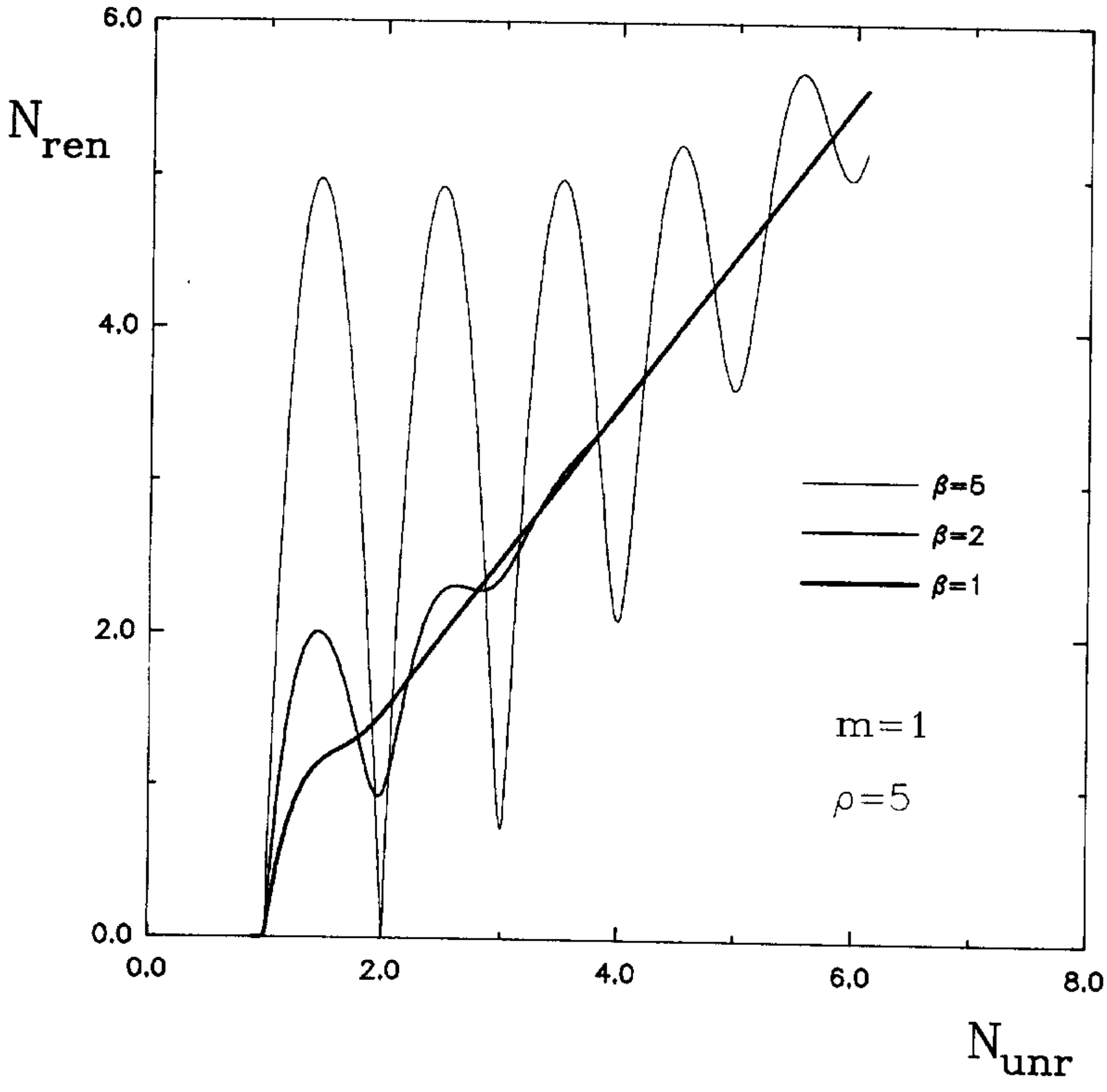


Figure 7b

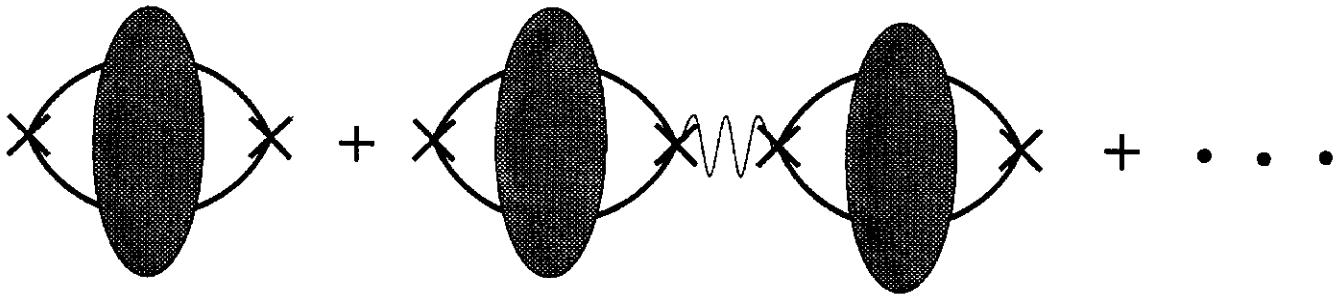


Figure 8

Figure 9