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**Diffusive Transport Enhancement by Isolated Resonances  
and Distribution Tails Growth in Hadronic Beams \***

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# Diffusive Transport Enhancement by Isolated Resonances and Distribution Tails Growth in Hadronic Beams.

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## Abstract

The escape rates and evolution of a distribution of particles are considered for a 2-D model of transverse motion of particles in hadronic storage rings, when nonlinear resonances and external diffusion are present. Dynamic enhancement of diffusion inside separatrices can develop under a certain geometry of resonance oscillations and relatively wide resonances, leading to the fast growth of distribution tails and escape rates. The phenomenon is absent in 1-D.

## 1 Introduction

In hadronic colliders, the escape of particles to large betatron amplitudes and associated growth of distribution tails due to the small random modulations of the lattice parameters (predominantly the RF power) is an important practical issue, since it causes some problems with background levels in detectors. Experimental evidence indicates that the escape rate has an appreciable magnitude only in the presence of the beam-beam interaction. However, the present knowledge of low tune shift ( $\xi < 0.01$ ) dynamics of beam-beam interactions in hadronic colliders indicates that we cannot expect a fast escape of particles from the beam core (betatron amplitudes  $\sim 1\sigma$ ) to the tail region (amplitudes  $\sim 5\sigma$ ) to originate from the beam-beam interaction alone. Therefore, it seems apparent that the external noise and the beam-beam nonlinear dynamics "interfere" somehow to efficiently magnify their respective effects. The present paper is devoted to the description of one particular mechanism of amplification.

The most important effect of the beam-beam interaction is to drive nonlinear resonances/1,2,3/. If we consider them as isolated (an appropriate approximation at least in the absence of synchrotron modulation) we arrive at the problem of diffusive motion in the presence of a (everywhere dense) net of isolated nonlinear resonances. In most cases the resonance widths are much smaller than

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the characteristic apertures where particle loss occurs; then in 1-D the transport ability of these resonances is minimal, since their influence is confined to a small region near their separatrices.

In 2-D, resonances appear as lines in the planes of betatron energies  $I_x$  and  $I_y$  /1,2/. We can then draw the arrow of separatrix oscillations, which shows the direction of trapped particle oscillations about the resonance line. Its length  $\Delta$  is simply the width of the separatrix (or twice the maximum oscillation amplitude) and its center is the resonance line (see Fig. 1). Now consider a small kick  $\delta$  applied to a trapped particle in the direction orthogonal to the resonance line - it is clear that the center of oscillations will be displaced a distance  $\delta \cot(\alpha)$  along the resonance line. Similarly, if we introduce noise of intensity  $D$  in this direction, then the diffusion of the oscillation center along the resonance will have the intensity  $D \cot(\alpha)$ . Thus for small angles  $\alpha$  between the resonance oscillations and resonance line, diffusion is enhanced inside the separatrix. This enhancement has been termed diffusive “resonance streaming” and is well known /4/, but it does not complete the picture. Indeed, under the influence of noise the particles, besides diffusing along the resonance line, can also leave (and re-enter) the separatrix, so the overall effect of the resonance will naturally depend on the width  $\Delta$  of the resonance “stripe”, going to zero as  $\Delta$  tends to zero. For small  $\alpha$  and not small  $\Delta$ , the effects of resonance streaming lead to a strong increase in escape rate and fast growth of distribution tails even when the resonance width  $\Delta$  is much smaller than the characteristic aperture limitations. This situation is somewhat similar to the escape rate and distribution function problems in the 2-D oscillator with nonlinear resonances, damping and noise /5/, where both damping and diffusion are “renormalized” within the separatrices. In our problem, however, there is no relaxation and all quantities are time dependent. In realistic situations, the beam is small relative to the aperture during the entire storage time. In terms of distribution function evolution, this means that we are interested in region of the distribution tails. Generally, an adequate mathematical formalism to deal with the description of distribution tails is the method of weak-noise asymptotics (WNA)/6,7/. Unfortunately this method breaks down for smaller  $\Delta$  values. In this paper we develop a modified WNA which can treat arbitrarily small  $\Delta$  for asymptotically small noise intensities  $\eta$ . To build insight we shall first examine a simplified situation in Chapter 2, where the resonance is “modeled” by a stripe in the 2-D plane with a different diffusion intensity than in the remainder of the plane. The growth of distribution tails in this example can be described analytically to the larger extent. In chapter 3 we then consider the full problem in a 4-D phase space. Due to the complex geometry of this phase space, additional problems will appear. One possible resolution to these is suggested in a semi-phenomenological approach.

## 2 Demonstration diffusive example

### 2.1 Model

In this Chapter, we will consider another diffusive model that is maximally simplified to allow a clear analysis, but retains all the characteristic features of the original problem of diffusion in the presense of resonance(s). This model is a two-dimensional diffusive random walk with a coordinate-dependent diffusion coefficients. More particularly, the diffusion coefficient  $D_x$  will be

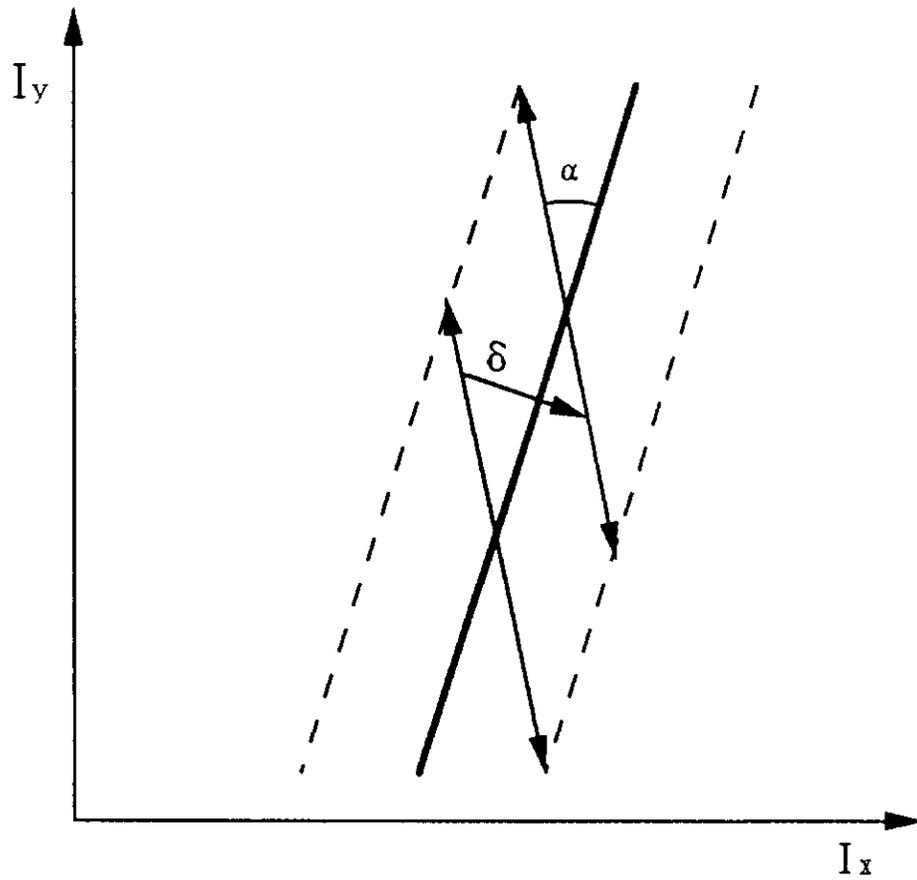


Figure 1: Displacement of resonance oscillation center by transverse kick. Thick solid line is the resonance line. Dashed lines are the separatrix.

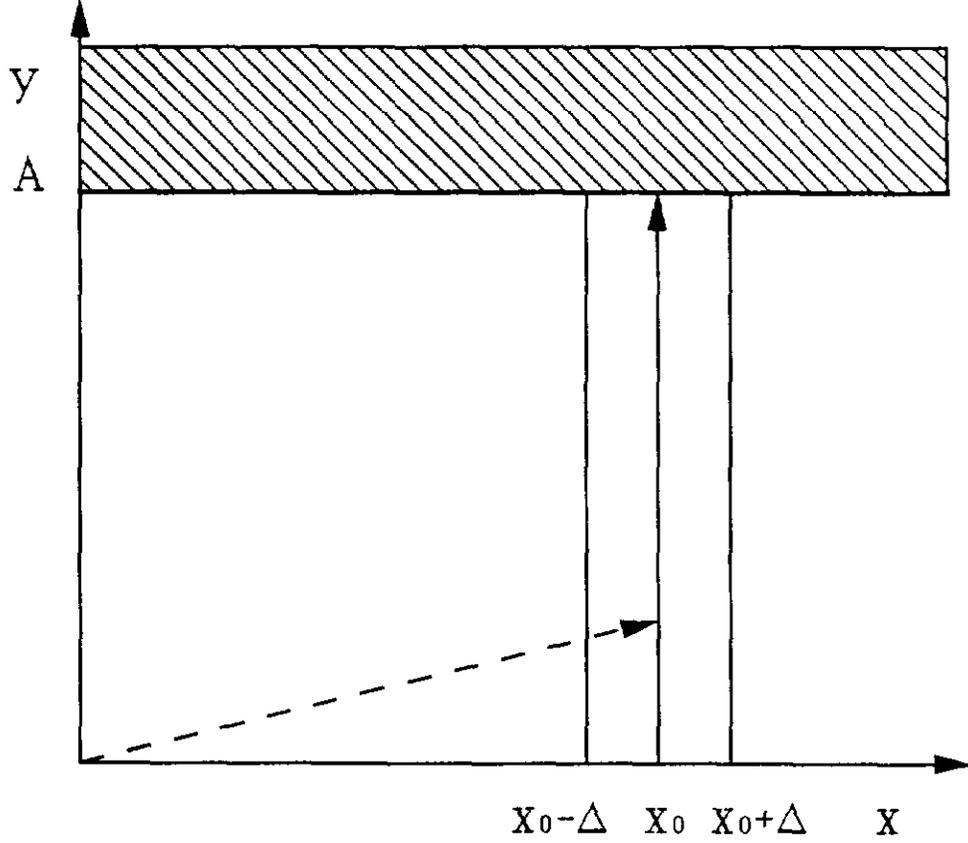


Figure 2: The stripe of a larger vertical diffusion. Shaded is absorbing boundary. The arrows show the most probable path of escape to the boundary.

constant throughout the plane, while the coefficient  $D_y$  will have a (higher) constant value  $D_{y1}$  in the stripe  $x_0 - \Delta \leq x \leq x_0 + \Delta$  and a (lower) constant value  $D_{y2}$  outside of the stripe (see Fig.2). The stochastic equations of motion are:

$$\begin{aligned}\dot{x} &= \sqrt{2D_x} \xi_x(t) \\ \dot{y} &= \sqrt{2D_y(x)} \xi_y(t)\end{aligned}\quad (1)$$

where  $\xi_x(t), \xi_y(t)$  are white noises  $\langle \xi_x(t) \xi_x(t + \tau) \rangle = \langle \xi_y(t) \xi_y(t + \tau) \rangle = \delta(\tau)$  and  $D_y(x) = D_{y1}$  if  $x_0 - \Delta \leq x \leq x_0 + \Delta$  and  $D_y(x) = D_{y2}$  otherwise. The evolution of distribution density will be governed then by the Fokker-Planck equation (FPE):

$$\frac{\partial \rho}{\partial t} = D_x \frac{\partial^2 \rho}{\partial x^2} - D_y(x) \frac{\partial^2 \rho}{\partial y^2}\quad (2)$$

The transition probability  $P(x, y; \bar{x}, \bar{y}, t)$  satisfies the same FPE (2) and is subject to the initial condition  $P(x, y; \bar{x}, \bar{y}, 0) = \delta(x - \bar{x})\delta(y - \bar{y})$ . We also will restrict the motion of particles to the positive values of  $x$  and  $y$  by making the axes  $x = 0$  and  $y = 0$  reflecting boundaries. This is equivalent to imposing the zero orthogonal component of flux conditions at these axes:  $\frac{\partial \rho}{\partial x} = 0$  for  $y = 0$  and  $\frac{\partial \rho}{\partial y} = 0$  for  $x = 0$ .

We will analyze the evolution of distribution tails and escape of particles to a distant vertical absorbing boundary  $y = A$  (see Fig.2). All the particles will be started at  $t = 0$  at  $x = 0, y = 0$ . The escape rate  $r(t)$  is the percentage of particles that are absorbed in a unit of time at the boundary  $y = A$ . It equals the integral orthogonal flux at this boundary  $J(t) = \int_0^\infty D_y(x) \frac{\partial \rho_0}{\partial y}(x, y = A) dx$  for the distribution  $\rho_0(x, y, t)$  which is subject to conditions  $\rho_0(x, y, 0) = \delta(x)\delta(y)$  and  $\rho_0(x, A, t) = 0$ . By a "distant" boundary we mean that it lies in the tail region, what is true at least if the maximum time of observation of the system  $T$  is much smaller then both diffusive time scales  $A^2/D_{y1}$  and  $A^2/D_{y2}$ . Also, under this condition the function  $\rho_0(x, y, t)$  will be much smaller than  $\rho_0(0, 0, t)$  for all times  $t < T$  in most of the plane  $x, y$  (except a small region near  $x = 0, y = 0$ ). In hadronic colliders this is a realistic approximation since the beam size is kept small relative to the aperture during the storage time.

## 2.2 Modified Weak-Noise Asymptotics

If the time  $T$  and all the parameters of the system were kept constant while diffusion intensities  $D_x, D_{y1}, D_{y2}$  were tending to zero we would have the situation where the powerful weak-noise asymptotic (WNA) method [6,7] would apply. In this asymptotics, the function  $\rho_0$  is exponentially small:

$$\rho_0(x, y, t) = Z(x, y, t) \exp\left(-\frac{\phi(x, y, t)}{\eta}\right) \quad (3)$$

( $\eta$  here is the common diffusion intensity factor  $D_x \sim \eta, D_{y1} \sim \eta, D_{y2} \sim \eta, \eta \rightarrow 0$ ). The p.d.e. for the leading exponential factor  $\phi$  has the form of the Hamilton-Jacobi equation (HJE), which is of the first order (though nonlinear) and can be solved through characteristics method. This whole approach parallels the quasiclassical approximation in quantum mechanics. In particular, the most probable paths of transition from one point to another are the only ones to contribute to an integral in the path-integral representation of  $\rho_0$ . These paths in their turn provide the minimum of an associated classical mechanical action. However the applicability of this approach is determined, among others, by the condition  $\Delta^2/D_x T \gg 1$ . Since primarily we are especially interested in the effect of narrow resonances and its dependence on the width, in this model it is highly desirable to consider the case of arbitrarily small  $\Delta$  and go beyond the limits of applicability of the standard WNA.

Physically, the condition  $\frac{\Delta^2}{D_x T} \gg 1$  means that if the most probable path of escape to absorbing boundary as well as to other points in large regions of  $(x, y)$  plane partially passes along the stripe (see below), then all the paths that are nearly as probable (giving appreciable contribution to the probability path integral) are also passing along the stripe. In other words in this case one does not need to take into account the possibility of particles "falling out" of the stripe, recrossing back, etc. while making their way along the stripe to the larger  $y$ . However, when considering small  $\Delta$  one does need to account for such possibility. We will be able to do so by taking the limit  $D_x \sim D_{y1} \sim D_{y2} \sim \eta \rightarrow 0$  and letting  $\Delta$  be arbitrary in respect to  $\eta$ . Then indeed the width  $\Delta$  is small relative to the other distances in the system and we will be able to solve the FPE applying the asymptotic form of the type (3) only in the direction along the stripe while treating the transverse direction exactly. This will provide us with the exponent of transition probability for the points

along the stripe. At the next stage, we can forget about the stripe having a width  $\Delta$  and consider it as a line, while applying the standard WNA (3) everywhere in the plane except the line. In other words, because of the variation of diffusion intensity on the small distance  $\Delta$  one needs to consider these “microscopic” scales to evaluate the renormalized “macroscopic” diffusion along the line, but after that one is left only with the “macroscopic” quantities. The “global” solution for the exponent of the function  $\rho$  in the plane  $x, y$  is constructed as in the general approach of WNA, but with the special treatment on the line. Technically it can be achieved with the variational representation of the exponent  $\phi$ . The general approach is quite similar to that of Ref./8/ for the evaluation of the tails of nonequilibrium steady-state distribution in the system with narrow resonances, damping and noise.

### 2.3 Distribution function along the stripe

Following the program outlined above, we present the function  $\rho_0(x, y)$  in the narrow (of the order of  $\Delta$ ) vicinity of the stripe  $x_0 - \Delta \leq x \leq x_0 + \Delta$  in a “partially” asymptotic form:

$$\rho(x, y, t) = Z(x, y, t) \exp\left(-\frac{\phi(y, t)}{D_x}\right) \quad (4)$$

Exponentially strong dependence is present in (4) in  $y$  and  $t$  but not in  $x$ . We are most interested therefore in function  $\phi$ , but it turns out that the equation for  $\phi$  emerges as the compatibility condition of equation for the prefactor  $Z$  with the physical boundary conditions. Substituting (4) in (2) and singling out highest powers of  $1/D_x, 1/D_{y1}, 1/D_{y2}$  (we suppose  $D_x \sim D_{y1} \sim D_{y2} \rightarrow 0$ ), one obtains

$$-\kappa Z = D_x^2 \frac{\partial^2 Z}{\partial x^2} \quad (5)$$

where  $\kappa(y, t) = \frac{\partial \phi(y, t)}{\partial t} + \frac{D_y(x)}{D_x} \left(\frac{\partial \phi(y, t)}{\partial y}\right)^2$ . It is also convenient to introduce the notations  $\kappa_1(t)$  and  $\kappa_2(t)$  for the values of  $\kappa$  correspondingly inside and outside of the stripe. Note that though the function  $Z$  is time-dependent, the derivative  $\frac{\partial Z}{\partial t}$  does not enter the equation (5) as the corresponding term is of higher order in diffusion intensity.

The most important solution of equation (5) can be presented, if we introduce the notation  $Z_i$  for the function  $Z$  inside the stripe and  $Z_1, Z_2$  corresponding to the left and to the right from the stripe as:

$$\begin{aligned} Z_i &= A \cos\left(x_1 \frac{\sqrt{\kappa_1}}{D_x}\right) \\ Z_1 &= A_1 \exp\left(\frac{\sqrt{|\kappa_2|}}{D_x} (x_1 - \Delta)\right) \\ Z_2 &= A_1 \exp\left(-\frac{\sqrt{|\kappa_2|}}{D_x} (x_1 - \Delta)\right) \end{aligned} \quad (6)$$

where  $x_1 = x - x_0$ . This solution emerges from the requirements of: 1) symmetry relative to  $x = x_0$ , and 2) monotonically decreasing behaviour as a function of  $|x - x_0|$ . The requirement of

the symmetry is rather special in respect to obviously nonsymmetric initial conditions for  $\rho_0$  (see below). We supposed  $\kappa_1 > 0$  and  $\kappa_2 < 0$ , since only in this case can we construct the solution which would satisfy the conditions 1 and 2. Two more conditions to be imposed on the solution (6) are 3) continuity of the function  $\rho$  at the boundaries of the stripe, and 4) continuity of the orthogonal flux  $j_x = D_x \frac{\partial \rho}{\partial x}$  at the same boundaries. The condition 3 yields the relation

$$A_1 = A \cos\left(\Delta \frac{\sqrt{\kappa_1}}{D_x}\right) \quad (7)$$

The condition 4 gives one more equation:

$$A\sqrt{\kappa_1} \sin\left(\Delta \frac{\sqrt{\kappa_1}}{D_x}\right) = A_1 \sqrt{|\kappa_2|} \quad (8)$$

and together with (7) allows to find the relation between  $\frac{\partial \phi}{\partial t}$  and  $\frac{\partial \phi}{\partial y}$ :

$$\tan\left(\Delta \frac{\sqrt{\kappa_1}}{D_x}\right) = \sqrt{\frac{|\kappa_2|}{\kappa_1}} \quad (9)$$

Remember here that  $\kappa_1 = \frac{\partial \phi}{\partial t} - \frac{D_{y1}}{D_x} \left(\frac{\partial \phi}{\partial y}\right)^2$ ,  $\kappa_2 = \frac{\partial \phi}{\partial t} + \frac{D_{y2}}{D_x} \left(\frac{\partial \phi}{\partial y}\right)^2$  and that  $\kappa_1$  is positive while  $\kappa_2$  is negative.

Equation (9), though transcendental, defines  $\frac{\partial \phi}{\partial t}$  as a function of  $\frac{\partial \phi}{\partial y}$ :

$$\frac{\partial \phi}{\partial t} = G\left(\frac{\partial \phi}{\partial y}\right) \quad (10)$$

Some remarks about the derivation and the meaning of equations (9) and (10) are in order. First, one observes that the requirement of the symmetry of  $Z$  relative to  $x = x_0$  would be natural were we starting particles at  $x = x_0$ , but plays rather special role in respect to the function  $\rho_0$ . We expect such symmetric solutions to be valid in our asymptotics  $\eta \rightarrow 0$  on some section(s) of the stripe in spite of the asymmetry of the initial conditions of  $\rho_0$ . These sections are intuitively those where the particles, if their paths are retraced back in time, predominantly travel along the stripe.

Another possible class of solutions of equation (5) corresponds to the situation when the particles arrive to the section(s) of the stripe directly from the center  $x = 0, y = 0$ . At these sections, solution (9),(10) is inapplicable; but even without explicitly writing another solution one can argue that since the width of the stripe  $\Delta$  is asymptotically small, the transverse crossings of the stripe by the paths will not give any appreciable contribution to the probability path integral. The function  $\phi$  then is essentially unaffected by these sections as if the stripe were not there (see also below).

## 2.4 Exponent of transition probability along the stripe

Now we will find the solution  $\bar{\phi}$  of the pde (10) while imposing the initial condition  $\bar{\phi}(y, t) \sim \frac{(y-\bar{y})^2}{t}$  as  $t \rightarrow 0$ , for arbitrary  $\bar{y}$ . Such a solution obviously defines the exponential factor in the transition probability from point  $\bar{y}$  to point  $y$  (both inside the stripe) in time  $t$ , and will allow one to find

the full asymptotic solution for the function  $\rho_0(x, y, t)$  through a certain minimization procedure involving  $\bar{\phi}(y, t)$ .

The function  $\bar{\phi}(y, t)$  can be found through the standard characteristics method. Equation (10) is the Hamilton-Jacobi equation  $\bar{H}(y, t, p_y, p_t) = 0$  with  $p_t = \frac{\partial \phi}{\partial t}$  and  $p_y = \frac{\partial \phi}{\partial y}$ . Hamiltonian equations of motion are:

$$\begin{aligned}\frac{dp_y}{d\tau} &= 0 \\ \frac{dp_t}{d\tau} &= 0 \\ \frac{dt}{d\tau} &= 1 \\ \frac{dy}{d\tau} &= -\frac{\partial G}{\partial p_y}(p_y)\end{aligned}\tag{11}$$

Substituting the general solution of (11) in the expression for  $\phi$ :

$$\bar{\phi}(y, t_1) = \int_0^{t_1} (p_y(\tau) dy(\tau) - p_t dt(\tau))\tag{12}$$

and using the conditions  $y(t=0) = \bar{y}$ ,  $t(\tau=0) = 0$  and relation (10), one arrives at the expression

$$\bar{\phi}(y, t) = p_y(y - \bar{y}) - tG(p_y)\tag{13}$$

where  $p_y$  has to be taken from the (transcendental) relation

$$y - \bar{y} = -t \frac{dG}{dp_y}(p_y)\tag{14}$$

It is instructive to consider limiting cases of small and large stripe width  $\Delta$ . For  $\Delta \rightarrow 0$ , the solution of (9) is  $\kappa_2 = 0$ . It has a very simple physical implication: it coincides with the HJE  $\frac{\partial \phi}{\partial t} = -\left(\frac{\partial \phi}{\partial x}\right)^2 - \frac{D_{y2}}{D_x} \left(\frac{\partial \phi}{\partial y}\right)^2$  for the exponent  $\phi$  of the WNA (3) on the axis  $x = x_0$  (where  $\frac{\partial \phi}{\partial x} = 0$  from the symmetry in respect to initial condition) in the absence of any stripe (when  $D_y(x) = D_{y2}$  throughout the plane). Then, the quantity  $G$  in (10) is  $G(\frac{\partial \phi}{\partial y}) = -\frac{D_{y2}}{D_x} \left(\frac{\partial \phi}{\partial y}\right)^2$  and the function  $\bar{\phi}(y, t)$  (13) is explicitly found to be

$$\bar{\phi}(y, t) = \frac{D_x}{D_{y2}} \frac{(y - \bar{y})^2}{4t}\tag{15}$$

This is just the exponent of the usual one-dimensional distribution, spreading from the initial  $\delta$ -functional peak at  $y = \bar{y}$  under the influence of diffusion. One can also easily obtain the first-order correction to (15) in powers of  $\Delta$ .

The opposite asymptotics of large  $\Delta$  is easily found after observing that the argument of the tangent in (9) has to lie within the range  $(0, \frac{\pi}{2})$  for all values of  $\Delta$  to avoid unphysical negative values of the distribution function inside the stripe. The asymptotic solution of (9) for large  $\Delta$  is then found to be

$$\kappa_1 = \left(\frac{\pi D_x}{2 \Delta}\right)^2\tag{16}$$

The physical meaning is again very clear. When  $\kappa_1$  tends to zero for large  $\Delta$ , it means that the function  $\bar{\phi}(y, t)$  is the same as if the stripe were occupying the whole plane - the particles inside the stripe do not “feel” the outside region. The function  $\bar{\phi}$  (13) is

$$\bar{\phi}(y, t) = \frac{D_x (y - \bar{y})^2}{D_{y1} 4t} \quad (17)$$

The condition of applicability of the solution (17) can be found by requiring that each term in the sum  $\kappa_1 = \frac{\partial \bar{\phi}}{\partial t} - \frac{D_{y1}}{D_x} \left( \frac{\partial \bar{\phi}}{\partial y} \right)^2$  be much larger than the r.h.s. of (16), yielding:

$$\Delta \gg \pi \frac{D_x t}{|y - \bar{y}|} \sqrt{\frac{D_{y1}}{D_x}} \quad (18)$$

The quantity  $D_x t$  is the square of the r.m.s. deviation  $\Delta x(t)$  of particle in  $x$  direction over the time of observation  $t$ . Were it much smaller than  $\Delta$ , the applicability of the solution (17) would be self-obvious. However, since the distance  $y - \bar{y}$  is much larger than the r.m.s. deviation  $\Delta x$  (we suppose  $D_x \sim D_y$ ), the applicability condition (18) is less restrictive. For  $D_x \sim D_{y1}$  it can be rewritten as  $\Delta \gg \Delta x(t) \frac{\Delta x(t)}{|y - \bar{y}|}$ .

Both large and small  $\Delta$  expressions for the function  $\bar{\phi}$  (15) and (17) are monotonically decreasing functions of time. The function  $\bar{\phi}$  is actually monotonically decreasing with time  $t$  for arbitrary parameters, since the conjecture  $\frac{\partial \bar{\phi}}{\partial t}$  contradicts the condition  $\kappa_2 \geq 0$ .

## 2.5 “Global” solution

Now that the local exponent  $\bar{\phi}(y, t)$  has been found we can find the “global” solution for the exponents of distribution tails and the escape rate  $r$  on the boundary  $y = A$  for the particles initiated at  $x = 0, y = 0$ . The relationship of the rate  $r$  to the WNA (3) of the distribution  $\rho_0$  in the absence of absorbing boundaries is known from the general WNA theory [6,7]:

$$r(t) = F(t) \exp \left( \frac{R(t)}{\eta} \right) \quad (19)$$

where the exponent  $R$  is just the minimum of the exponent  $\phi(x, y, t)$  of  $\rho_0$  on the boundary:

$$R(t) = \min_x \phi(x, A, t) \quad (20)$$

The explicit expression for the exponent  $\phi(x, y, t)$  of the leading exponential term of the distribution  $\rho_0$  can be written in the standard variational form of WNA:

$$\phi(x, y, t) = \min_{\tilde{x}(t), \tilde{y}(t)} \int_0^t dt L \left( \dot{\tilde{y}}(\tau), \dot{\tilde{x}}(\tau), \tilde{y}(\tau), \tilde{x}(\tau) \right) \quad (21)$$

where the trajectories  $\tilde{y}(t), \tilde{x}(t)$  have fixed end points  $\tilde{y}(0) = 0, \tilde{x}(0) = 0, \tilde{y}(t) = y, \tilde{x}(t) = x$ . In our case however the Lagrangian  $L$  in (21) is defined as mentioned before by different expressions

for the line  $x = x_0$  and for the rest of the plane. The latter coincides with the Lagrangian of the unperturbed Hamilton-Jacobi equation:

$$\frac{\partial \phi}{\partial t} = -\left(\frac{\partial \phi}{\partial x}\right)^2 - \frac{D_{y2}}{D_x} \left(\frac{\partial \phi}{\partial y}\right)^2 \quad (22)$$

and is given by:

$$L_1 = \frac{\dot{x}^2}{2} + \frac{D_{y2}}{D_x} \frac{\dot{y}^2}{2} \quad (23)$$

On the line  $x = x_0$  the (one-dimensional) Lagrangian  $L = L_2(\dot{y}(t))$  is related to the Hamiltonian  $H = G(p_y)$  from the Hamilton-Jacobi equation (10) through the standard transformation:

$$\begin{aligned} \tilde{p}_y &= \frac{\partial L_2(\dot{y})}{\partial \dot{y}} \\ H &= \dot{y} \frac{\partial L_2}{\partial \dot{y}} - L_2 \end{aligned} \quad (24)$$

In the full minimization (21), we can allow sections of trajectories  $\tilde{x}(t), \tilde{y}(t)$  to pass by the line  $x = x_0$ , and the function  $\phi$  on these section(s) is found (up to a constant, defined by continuity) from the HJE's (9),(10). One should be reminded, that the trajectories  $\tilde{x}(t), \tilde{y}(t)$  providing the minimum in (21) are the most probable paths of arrival to the corresponding points /6,7/. After these section(s) and the values of  $\phi$  on them are found, the function  $\phi$  in the rest of the plane is the solution of the unperturbed HJE (22) with the (self-consistent) boundary conditions on the section(s). The plane  $x, y$  is decomposed into some regions, where the characteristics of the HJE (i.e. trajectories  $\tilde{x}(t), \tilde{y}(t)$ ) are coming from the center  $x = 0, y = 0$  and other regions where they come from the section(s) on the line  $x = x_0$ . The function  $\phi(x, y, t)$  is continuous, but is not differentiable (cusps are present) on the boundaries between the regions. The whole situation (including cusps) is rather similar to the WNA of the nonequilibrium steady state distributions in the oscillator with nonlinear resonances, damping and noise /5/, and more details can be found in this reference.

## 2.6 Large $\Delta$ case

In order to demonstrate the above described technique of constructing the "global" solution, let us consider the case of large  $\Delta$ , when the Hamiltonian  $G$  in (10) is  $G = -\frac{D_{y1}}{D_x} \left(\frac{\partial \phi}{\partial y}\right)^2$ . We will assume that all trajectories  $\tilde{x}, \tilde{y}$  can have at most one section on the line  $x = x_0$  (the proof of this is straightforward). The most probable paths of arrival to the points on the line  $x = x_0$  in time  $t$  consist (for  $y$  larger than a certain  $y_1$  - see below) of two sections: the straight (unperturbed) section going from  $x = 0, y = 0$  to  $x = x_0, y = y_1$  and another section going along the line as shown in Fig.2. If the time of motion on the first section is  $t_1$ , then the minimum of the unperturbed (i.e. with the Lagrangian  $L_1$  (23)) functional (21) on this section is:

$$\Delta \phi_1 = \frac{(x_0)^2}{4t_1} + \frac{D_x}{D_{y2}} \frac{y_1^2}{4t_1} \quad (25)$$

The increment of  $\phi$  on the second section can be taken, for the case of large  $\Delta$  we are considering, from (17):

$$\Delta\phi_2 = \frac{D_x (y - y_1)^2}{D_{y1} 4(t - t_1)} \quad (26)$$

The time of motion on the second section is  $t - t_1$ , since the full time should be  $t$ . Minimizing now the sum  $\phi = \Delta\phi_1 - \Delta\phi_2$  by both  $t_1$  and  $y_1$ , we find:

$$\begin{aligned} \phi(x_0, y, t) = & \frac{1}{4t} \left[ x_0 \left( \sqrt{1 - \frac{D_{y1} D_{y2}}{D_{y1}^2 - D_{y2}^2}} - \frac{D_{y2}}{\sqrt{D_{y1}^2 - D_{y2}^2}} \right) \right. \\ & \left. + y \sqrt{\frac{D_x}{D_{y1}}} \right]^2 \end{aligned} \quad (27)$$

and the quantity  $y_1$  independent of  $t$ :

$$y_1 = x_0 \frac{D_{y2} \sqrt{D_{y1}}}{\sqrt{D_x (D_{y1}^2 - D_{y2}^2)}} \quad (28)$$

To find the function  $\phi(x, y, t)$  in the entire plane, one needs to solve the unperturbed HJE (24) subject to the boundary condition (27) on the line  $x = x_0, y > y_1$  and to the requirement of the characteristics of this equation to start from  $x = 0, y = 0$ . It is easy to see that the time dependence of the solution is “purely diffusive”  $\phi(x, y, t) = \varphi(x, y)/t$  (note that this is true only for large  $\Delta$ ). Substituting this dependence in (27) and solving the resulting equation with the boundary conditions on the line  $x = x_0, y > y_1$  by characteristics method, one can explicitly find the solution  $\varphi_1$  in the part of the plane. This region I is defined by the condition that  $\varphi_1(x, y)$  is smaller than the unperturbed function  $\varphi_0(x, y) = x_0^2/4 + \frac{D_x}{D_{y2}} \frac{y^2}{4}$ . Then  $\varphi = \varphi_1$  in region I and  $\varphi = \varphi_0$  in the rest of the plane (region II). The function  $\varphi(x, y)$  is not differentiable on the boundaries between the regions (has cusps). The qualitative sketch of the contours of the function  $\phi$  is shown in Fig. 3. In the same graph, the characteristics, which are the most probable path of arrival to each point, are shown.

The quantity  $\Delta\phi$  (27) for  $y = A$  is the exponent  $R(t)$  (20) of the escape rate to the boundary  $y = A$  if the boundary is larger than a certain  $y = y_2$  (see Fig.3). The most probable path of escape to the boundary partially goes along the stripe as is shown in Fig.1.

### 3 Full system: nonlinear resonances and noise

#### 3.1 Local FPE

Our primary system of consideration is the two-dimensional Hamiltonian oscillator with external noise:

$$\begin{aligned} \dot{\vec{x}} &= \vec{p} \\ \dot{\vec{p}} &= -\frac{\partial U(\vec{x}, t)}{\partial \vec{x}} + \sqrt{2\eta} \tilde{\xi}(t) \end{aligned} \quad (29)$$

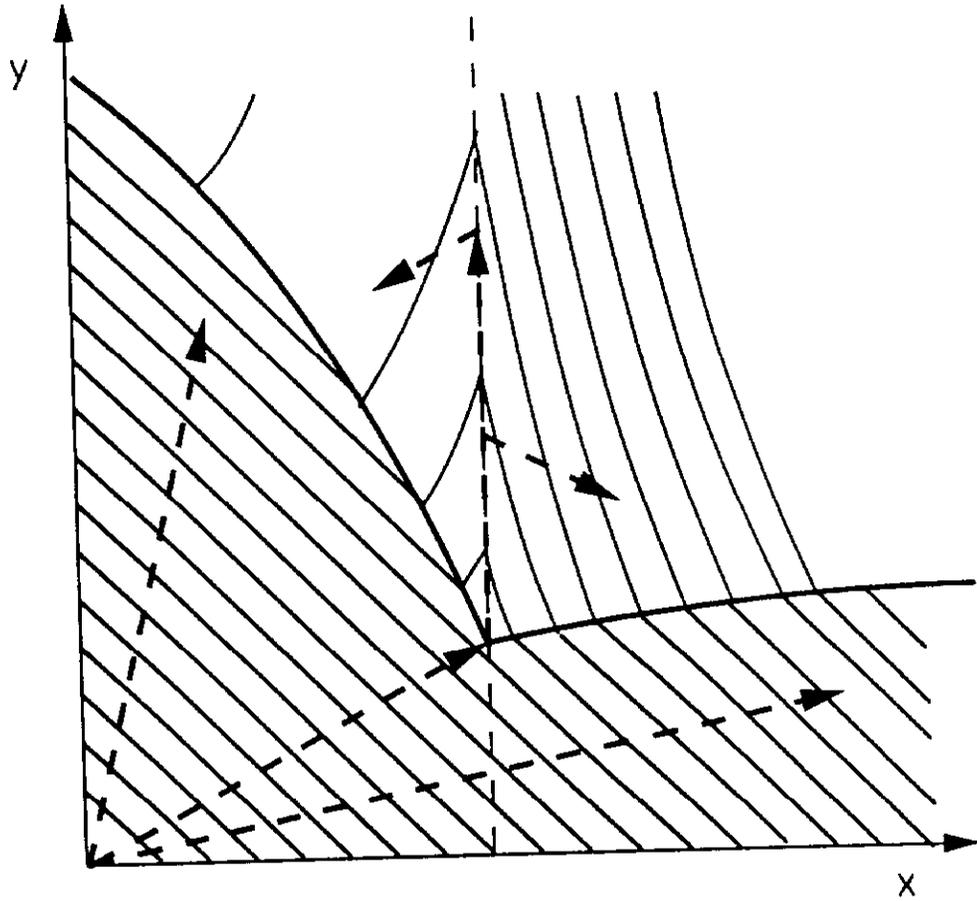


Figure 3: The qualitative graph of the contours of function  $\varphi(x, y)$ . Dashed lines show the most probable paths from the starting point  $x = 0, y = 0$  to the points in the different regions.

where  $\eta$  is the diffusion intensity.  $\xi_i(t)$  here is the white-noise vector process  $(\xi_i(t)\xi_k(t-\tau)) = \delta_{ik}\delta(\tau)$ . We suppose that the potential  $U$  consists of an unperturbed time-dependent part  $U_0(\vec{x})$  corresponding to exactly integrable motion and a small perturbation  $U = U_0 + \epsilon\delta U(\vec{x}, t)$ , time-periodic with frequency  $\Omega$ . The Hamiltonian of the system (without noise) can be presented therefore in action-angle variables of the unperturbed system:

$$H = H_0(\vec{I}) - \epsilon \sum_{\vec{l}, n} V_{\vec{l}n}(\vec{I}) \cos(\vec{l} \cdot \vec{\theta} - n\Omega t) \quad (30)$$

where  $H_0 = \vec{p}^2/2 - U_0(\vec{x})$  and the perturbation was expanded in Fourier series in both  $\theta$  and  $t$ . Each harmonic  $V_{\vec{l}n}$  excites a nonlinear resonance on the line  $l_x\nu_x(\vec{I}) - l_y\nu_y(\vec{I}) - n\Omega = 0$ , where  $\vec{\nu} = \partial H_0 / \partial \vec{I} / 9/$ . The amplitude of oscillations of  $\vec{I}$  at the separatrix defines the ‘‘resonance width’’  $\Delta I$  in  $\vec{I}$  space and is proportional to  $\sqrt{\epsilon}$  (see below). The resonant Hamiltonian can be obtained by dropping all the nonresonant harmonics, introducing new (canonical) variables

$$\begin{aligned} I_1 &= \frac{I_y}{l_y} \\ I_2 &= -I_x + \frac{l_x}{l_y} I_y \\ \psi_1 &= l_x\theta_x - l_y\theta_y - n\Omega t \\ \psi_2 &= -\theta_x \end{aligned} \quad (31)$$

and expanding the Hamiltonian  $H_0(I)$  to second order in deviations in  $I_1$  from the center  $I_{10}(I_2)$  /9/. The result will be:

$$H = \lambda \frac{p_1^2}{2} - \epsilon V_{\vec{m}} \cos(\psi_1) \quad (32)$$

where  $\lambda = \frac{\partial(l_x\nu_x - l_y\nu_y)}{\partial I_1}$ , and  $\vec{m} = (\vec{l}, n)$ . The resonance width, which is the amplitude of oscillations of  $p_1$  on the separatrix of pendulum (32), is

$$p_{1r} = 2 \left| \frac{\epsilon V_{\vec{m}}}{\lambda} \right|^{1/2} \quad (33)$$

Now let us consider the effect of noise. First, instead of stochastic equations of motion, we will use the equivalent language of distribution functions. The evolution of the distribution of particles, corresponding to the primary equations of motion (29) is governed by the FPE /6/:

$$\frac{\partial \rho}{\partial t} + \vec{p} \frac{\partial \rho}{\partial \vec{x}} - \frac{\partial(U_0 + \epsilon\delta U)}{\partial \vec{x}} \frac{\partial \rho}{\partial \vec{p}} = \eta \frac{\partial^2 \rho}{\partial \vec{p}^2} \quad (34)$$

We will be constructing the solution of the FPE (34) under a set of limitations on the parameters. First, we will suppose that the Hamiltonian part of dynamics can be well described in terms of isolated nonlinear resonances, so that the resonances do not overlap. This is true under the condition  $(\epsilon |V_{\vec{m}} \lambda|)^{1/2} \gg \nu$  ( $\nu$  here is one of the components of  $\vec{\nu}$ ) /9/. Then, we will be interested only in the

tails of distribution, which means that the characteristic energies  $E = H_0$  and times of observation  $T$  should satisfy the condition  $E/\eta T \gg 1$ . Also, it will be supposed that the diffusion in our system is a slow process relative to both the unperturbed motion (time scales  $\tau_1 \sim 1/\nu$ ) and the resonant oscillations (time scales  $\tau_2 \sim (\epsilon V_{\bar{m}} \lambda)^{-1/2}$ ). The last condition is more restrictive and can be obtained by requiring that the r.m.s. time  $\tau$  required to shift the particle by diffusion to the distance equal to the resonance width (33)  $\tau \sim p_{1r}^2/\eta$  is much larger than  $\tau_2$ :

$$(\epsilon V_{\bar{m}}!)^{3/2} \gg \lambda^{1/2} \eta \quad (35)$$

This inequality, as well as the previous one holds for small enough noise intensity  $\eta$ .

When the diffusion is slow it is natural to assume that the distribution will smear along the Hamiltonian trajectories before it will undergo any appreciable changes under the influence of diffusion, so that the distribution after a short time will be constant along these trajectories. This assumption is the basis of so called "thermal averaging" technique /10/ used to describe the evolution in the 1-D version of our system with damping. In Ref./5/, such averagings were performed as an intermediate stage of describing 2-D systems (with damping). Following /5/ we will now carry out the thermal averaging in two steps. First, transform the FPE (34) to the unperturbed action-angle variables  $\vec{I}, \vec{\theta}$  and (supposing the distribution  $\rho$  depends only on  $\psi_1$  and not  $\psi_2$ ) average the FPE over both "fast" phases  $\vec{\theta}$  keeping "slow" phase  $\psi_1$  constant (see/5/ for more details). This will yield

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \epsilon V_{\bar{m}} \sin(\psi_1) (l_x \frac{\partial \rho}{\partial I_x} + l_y \frac{\partial \rho}{\partial I_y}) + (l_x \nu_x - l_y \nu_y - n \Omega) \frac{\partial \rho}{\partial \psi_1} = \\ = \eta \frac{\partial}{\partial I_k} G_{0kl} \frac{\partial \rho}{\partial I_l} + \eta R_2 \frac{\partial \rho}{\partial \psi_1} - \eta R_3 \frac{\partial^2 \rho}{\partial \psi_1^2} + \eta R_{4k} \frac{\partial^2 \rho}{\partial I_k \partial \psi_1} \end{aligned} \quad (36)$$

Here, only one (resonance) harmonic of Fourier expansion (30) was retained. The thermal averaged diffusion tensor  $G_{0kl}(I_x, I_y)$  is

$$G_{0kl} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_x d\theta_y \frac{\partial I_k(\vec{x}, \vec{p})}{\partial p_i} \frac{\partial I_l(\vec{x}, \vec{p})}{\partial p_i} \quad (37)$$

where summation over the repeated indices is implied. The quantities  $R_2$  through  $R_4$  in (36) are other averages of the type (37) and are not given explicitly since the corresponding terms will drop out in subsequent transformations. In accelerator problems, the whole formalism is somewhat simplified since the linear betatron part of the Hamiltonian always dominates the zero Fourier harmonic of the nonlinear part, coming either from beam-beam interaction or sextupole or octupole terms. The actions  $I_x, I_y$  then are just the unperturbed linear betatron energies  $I_x = p_x^2/2 + \nu_{x0}^2 x^2/2$ ,  $I_y = p_y^2/2 + \nu_{y0}^2 y^2/2$ . The tensor  $G_{0kl}$  then is diagonal:

$$G_{0kl} = \frac{I_k}{\nu_{k0}} \delta_{kl} \quad (38)$$

Note that the FPE (36) is local and is applicable only in the vicinity of the chosen resonance since all nonresonant harmonics of perturbation  $U$  were dropped.

It is more convenient now to use the same local variables  $p_1, I_2, \psi_1$  as used in Hamiltonian (32). The FPE in these variables is

$$\frac{\partial \rho}{\partial t} + \lambda p_1 \frac{\partial \rho}{\partial \psi_1} + \epsilon V_{\bar{m}} \sin \psi_1 \frac{\partial \rho}{\partial p_1} = \eta \left( Q_{11} \frac{\partial^2 \rho}{\partial p_1^2} - 2Q_{21} \frac{\partial}{\partial p_1} \left( \frac{\partial}{\partial I_2} - \kappa \frac{\partial}{\partial p} \right) \rho + Q_{22} \left( \frac{\partial}{\partial I_2} - \kappa \frac{\partial}{\partial p} \right)^2 \rho \right) - \eta P \quad (39)$$

where the diffusion tensor in  $(I_1, I_2)$  space is

$$\begin{aligned} Q_{11} &= \frac{I_1}{l_1 \nu_y} \\ Q_{21} &= \frac{l_x I_1}{l_y \nu_y} \\ Q_{22} &= \frac{(-I_2 + l_x I_1)}{\nu_x} + \frac{l_x^2}{l_y \nu_y} I_1 \end{aligned} \quad (40)$$

and  $\kappa = \frac{\partial I_{10}}{\partial I_2}$  ( $I_{10}(I_2)$  is the resonance line). The quantity  $P$  consists of the terms which are proportional either to the first derivative of  $\rho$  with respect to  $p_1, I_2$  or to the second derivative at least one of which is with respect to  $\psi_1$ . This quantity will be shown to be safely dropped to the precision of consideration.

### 3.2 Modified weak-noise asymptotics

Now we will introduce the WNA only in the direction along the resonance line  $p_1 = 0$  in the same manner as in the illustrative example of Chapter 2. The method was proposed in /8/ to describe analogous systems with damping.

The solution of the FPE (39) in the limit  $\eta \rightarrow 0$  and for arbitrary  $\epsilon |V_{\bar{m}}|/\eta$  has the following functional form:

$$\rho(p_1, I_2, \psi_1, t, \eta) = Z(p_1, I_2, \psi_1, t, \eta) \exp \left( -\frac{\phi(I_2, t)}{\eta} \right) \quad (41)$$

where  $\frac{1}{Z} \frac{\partial Z}{\partial I_2} \rightarrow const$ ,  $\frac{1}{Z} \frac{\partial Z}{\partial t} \rightarrow const$  for  $\eta \rightarrow 0$ . Substituting (41) in (39) and singling out highest degrees of  $1/\eta$ , we arrive to:

$$\begin{aligned} \lambda p_1 \frac{\partial Z}{\partial \psi_1} + \epsilon V_{\bar{m}} \sin \psi_1 \frac{\partial \rho}{\partial p_1} &= \frac{1}{\eta} (Q_{22} q^2 + \frac{\partial \phi}{\partial t}) Z - 2(Q_{21} q - \kappa q Q_{22}) \frac{\partial Z}{\partial p} + \\ &\eta (Q_{11} + Q_{22} \kappa^2 - 2\kappa Q_{21}) \frac{\partial^2 Z}{\partial p_1^2} + \eta F_1 \frac{\partial^2 Z}{\partial p_1 \partial \psi_1} + F_2 q \frac{\partial Z}{\partial \psi_1} \end{aligned} \quad (42)$$

where  $q = \frac{\partial \phi}{\partial I_2}$ . Note that similar to Chapter 2, the derivative  $\frac{\partial Z}{\partial t}$  does not enter the equation (42). The quantities  $F_1, F_2$  originate from the last term in (39) and will drop out in subsequent

calculations. Introducing the notations

$$\begin{aligned}
a(I_2, t) &= Q_{22} \left( \frac{\partial \phi}{\partial I_2} \right)^2 - \frac{\partial \phi}{\partial t} \\
b(I_2, t) &= 2(\kappa Q_{22} - Q_{21}) \frac{\partial \phi}{\partial I_2} \\
c(I_2) &= Q_{11} + Q_{22} \kappa^2 - 2Q_{21} \kappa
\end{aligned} \tag{43}$$

equation (42) can be rewritten as

$$\tilde{L}_H Z = \left( \frac{a}{\eta} - b \frac{\partial}{\partial p} + c \eta \frac{\partial^2}{\partial p^2} \right) Z + \eta F_1 \frac{\partial^2 Z}{\partial p_1 \partial \psi_1} + F_2 q \frac{\partial Z}{\partial \psi_1} \tag{44}$$

where  $\tilde{L}_H$  is the Liouville operator  $\tilde{L}_H = \frac{\partial H}{\partial p_1} \frac{\partial}{\partial \psi_1} - \frac{\partial H}{\partial \psi_1} \frac{\partial}{\partial p_1}$  of the resonant Hamiltonian (32). Utilizing the condition (35) of diffusion slowness relative to the resonant dynamics now we can perform the second stage of thermal averaging, i.e. average equation (44) along the trajectories of the Hamiltonian  $H$  (32). The procedure is the same as in /5/ - we suppose that the function  $Z$  depends on  $p_1$  and  $\psi_1$  only through the action  $J(H)$  for the Hamiltonian  $H$  (32) and average the equation (44) over time. The resulting equation is

$$\frac{a}{\eta} Z + \frac{\partial}{\partial J} \left( bF + c \eta G(J) \frac{\partial}{\partial J} \right) Z = 0 \tag{45}$$

where

$$\begin{aligned}
F &= \left\langle \frac{\partial J(p_1, \psi_1)}{\partial p_1} \right\rangle \\
G(J) &= \left\langle \left( \frac{\partial J(p_1, \psi_1)}{\partial p_1} \right)^2 \right\rangle
\end{aligned} \tag{46}$$

The symbol  $\langle \dots \rangle$  in (46) implies the averaging over time along the trajectories of the Hamiltonian  $H$ . The quantity  $F$  can be shown to be independent on  $J$ . Notice also that the last two terms in (44) vanished under the averaging.

### 3.3 Phenomenological approach

Physically, equation (45) is very much alike the equation (5) of our model example. Therefore, it is clear that together with the “physical” boundary conditions it uniquely defines the relation between  $\frac{\partial \phi}{\partial t}$  and  $\frac{\partial \phi}{\partial I_2}$  (analogous to the example of Chapter 2). Technically however the equation (45) is intractable since the quantities  $F$  and  $G$  are expressed through elliptic integrals /10/. One way of handling this problem is the phenomenological simplification of functions  $F$  and  $G$  as suggested in /8/. In this approach, we substitute the exact trajectories of the pendulum (32) by “simplified” trajectories shown in Fig.4, and perform the averagings in (44) along these trajectories. Equations (44),(45) are formally the same as those of Ref./8/, while the difference is in the definition of

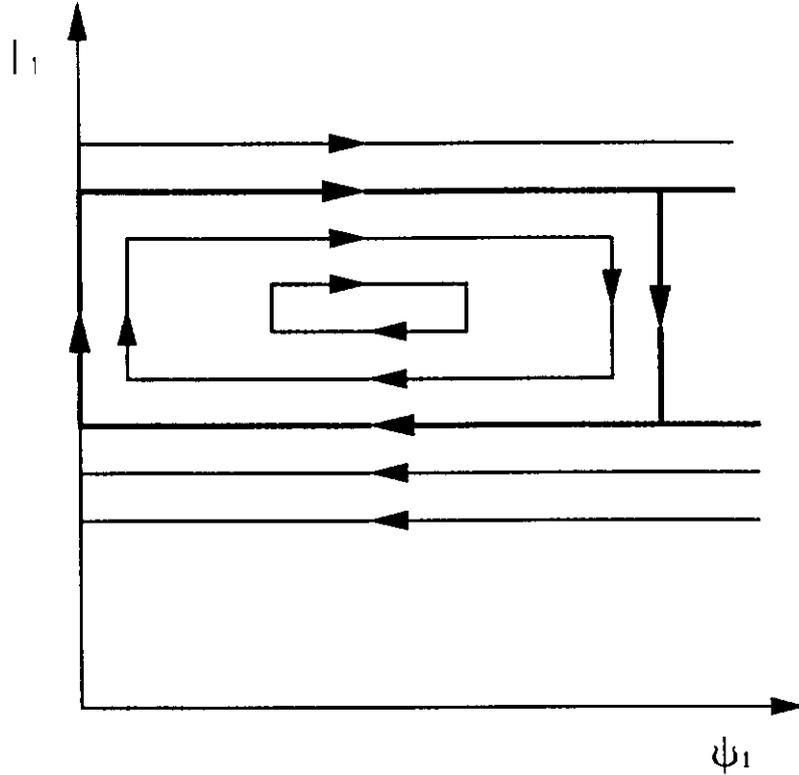


Figure 4: “Simplified” trajectories approximating pendulum trajectories of nonlinear resonance

coefficients  $a$  and  $b$  (43). Therefore we can use all the intermediate calculations from Ref./8/. The resulting one-dimensional HJE, defining the relation between  $\frac{\partial\phi}{\partial t}$  and  $\frac{\partial\phi}{\partial I_2}$  analogously to the HJE(9) is equation (32) of Ref./8/:

$$\left[ \frac{2}{ac}(b^2 - 4ac) \right]^{\frac{1}{2}} = \tan \left[ \left( \frac{a}{c} \right)^{\frac{1}{2}} \frac{k p_{1r}}{\eta} \right] \quad (47)$$

where  $k$  is the phenomenological constant of the order of unity. Equation (47) indeed is very much similar to the equation (9), so that most of the discussion of Chapter 2 applies also in this case. In particular, for the (large resonance width  $p_{1r}$ )/(small noise  $\eta$ ) regime the solution of equation (47) is  $a = 0$ , or

$$\frac{\partial\phi}{\partial t} + Q_{22} \left( \frac{\partial\phi}{\partial I_2} \right)^2 = 0 \quad (48)$$

This equation has a very clear meaning - the quantity  $Q_{22}$  is the component of the primary diffusion tensor  $G_0$  (37) in the direction along the resonance line  $I_1 = I_{10}$  when transformed to the new vector basis (direction  $I_1 = I_{10}$ ), (direction  $I_2 = \text{const}$ ). Therefore the diffusion inside the resonance evolves indeed as a one-dimensional process with geometrically clear transformation of diffusion rate. The condition of applicability of the solution (48) can be obtained similarly to that of Chapter 2, yielding

$$\frac{p_{1r}}{\eta} \gg \frac{\sqrt{cQ_{22}t}}{|I_{21} - I_{20}|} \quad (49)$$

where  $t$  is the time allowed for the transition from  $I_2 = I_{20}$  to  $I_2 = I_{21}$ . This condition will hold for small enough noise intensity  $\eta$ . It is interesting to note also that for very narrow resonances the requirement for the smallness of noise intensity (35) necessary to apply the thermal averaging is more restrictive than the condition (49).

It should be born in mind that the solution (47) is not unique in the same way as solution (9) of the *model example*. Indeed, both solutions were constructed subject to certain “physical” conditions, implicitly based on the conjecture that the inside-separatrix (inside-stripe) trajectories are more probable than the outside-separatrix (outside-stripe) paths. Generally this need not always be the case and another “competitive” solution corresponds in the precision of consideration simply to (locally) ignoring the resonance (see /5/ for details in the similar situation in the steady-state problem in the system with damping).

### 3.4 Global solution.

After finding the equation (47) for the variation of the exponent of distribution function along the resonance, we can forget about the resonance as a structure in  $I_x, I_y, \psi_1$  space and treat it as a line in  $I_x, I_y$  plane. The distribution function at this stage can be presented in a standard WNA form both in the entire plane  $I_x, I_y$  and on the line. The equation for the exponent  $\phi$  in the plane is the unperturbed HJE:

$$\frac{\partial \phi}{\partial t} + \frac{I_x}{\nu_{x0}} \left( \frac{\partial \phi}{\partial I_x} \right)^2 + \frac{I_y}{\nu_{y0}} \left( \frac{\partial \phi}{\partial I_y} \right)^2 = 0 \quad (50)$$

which is obtained by substituting the WNA form in the FPA (36) with  $V_{\tilde{m}} = 0$  and supposing  $\frac{\partial \rho}{\partial \psi_1} = 0$  (tensor  $G_{okl}$  is taken from (38). Along the line, function  $\phi$  has to satisfy the one-dimensional HJE (47). The function  $\phi$  also has to be continuous.

The construction of the global solution  $\phi$  can be done basing on the variational representation (21). The Lagrangian  $L$  is the unperturbed one everywhere except on the resonance line, where it corresponds to the one-dimensional HJE (47). The plane  $I_x, I_y$  will be divided into characteristic regions, qualitatively looking as in our model example shown in Fig.3.

The general algorithm of constructing the “global” solution can be formulated along the same lines as that of Ref./5/ for the systems with damping and applies to the case of arbitrary number of resonance lines or infinite resonance webs, generic in nearly-integrable systems. When performing the “global” minimization for  $\phi$  over all possible paths we can allow the paths either to cross the resonance lines or to pass along them. The second case will correspond to the solution (47) while the first one to the resonance-ignoring solution /5,8/. Which solution will locally take over, i.e. whether it is “profitable” for the particles to travel along the resonance at given point, depends on all the other points on the resonance line(s) and is therefore a nonlocal problem. The only local condition which can be derived is the lower bound of “profitability”. It can be obtained by requiring that it be “easier” for the particles to go along the resonance line when the resonance is present than going in the same direction in the absence of resonance. Because of the time-dependent character of the problem this condition is more complicated than the similar one of time-independent steady-state (see Appendix B of Ref./5/). In our case, one has to compare two different HJE’s and which one

is more “profitable” depends in general on the time allowed for transition.

Let us obtain the “lower bound” condition for the case of a large resonance width, when the effective one-dimensional HJE along the resonance line (47) reduces to a simple form (48). The “competing” HJE along the direction of the resonance line in the absence of resonance can be obtained by inserting the trajectory  $p_1 = 0$  and varying  $I_2(t)$  in the variational representation of the type (21):

$$\phi(p_1, I_2, t) = \int_0^t dt \left[ b_{11} \dot{p}_1^2 + 2b_{21} \dot{p}_1 \dot{I}_2 + b_{22} \dot{I}_2^2 \right] \quad (51)$$

where the symmetric tensor  $b_{\mu\nu}$  is the inverse to the diffusion tensor  $T_{\mu\nu}$  in coordinates  $p_1, I_2$ . The latter one, if to assign index 1 to  $p_1$  and 2 to  $I_2$ , is seen from the r.h.s. of equation (39) to be

$$\begin{aligned} T_{11} &= Q_{11} - 2\kappa Q_{21} + \kappa^2 Q_{22} \\ T_{21} &= Q_{21} - Q_{22}\kappa \\ T_{22} &= Q_{22} \end{aligned} \quad (52)$$

The resulting HJE is

$$\frac{\partial \phi}{\partial t} + \frac{Q_{22}Q_{11} - Q_{21}^2}{Q_{11} - 2\kappa Q_{21} + \kappa^2 Q_{22}} \left( \frac{\partial \phi}{\partial I_2} \right)^2 = 0 \quad (53)$$

Thus, both HJE’s (48) and (53) correspond to simple one-dimensional diffusion and the coefficients before  $\left( \frac{\partial \phi}{\partial I_2} \right)^2$  are the diffusion intensities (divided by  $\eta$ ). Therefore the “lower bound” condition can be obtained by requiring that the coefficient in (48) is larger than that in (53):

$$Q_{22} > \frac{Q_{22}Q_{11} - Q_{21}^2}{Q_{11} - 2\kappa Q_{21} + \kappa^2 Q_{22}} \quad (54)$$

If the condition (54) is not fulfilled, the diffusion inside the resonance is not enhanced and the resonance will not have any appreciable effect on the distribution tails and escape rates.

The “global” solution for  $\phi$  taking into account the net of resonance lines (which is infinite and everywhere dense for generic near-integrable systems (30)) can be constructed from the “global” minimization principle along the same lines as in the theory of time-independent steady-states in the systems with damping /5/. A slight modification will be that when varying the paths of arrival to a certain point, and having some sections of it going by the resonance lines, the times of starting points on those sections also have to be varied.

## 4 Discussion and Conclusions

We described the mechanism of diffusion enhancement by two-dimensional resonances with external noise, leading for wide enough resonances to the enhanced rate of transport of particles from the beam core to large amplitudes and hence an increased speed of growth of the tails of distribution. The basic scenario of diffusion enhancement inside the separatrices due to the small angle between the resonance line and the resonance oscillations direction was previously described in the literature

/4/. However, that calculation of “renormalized” diffusion intensity inside the separatrix does not provide the essential information about the “macroscopic” transport rate along the resonance and its overall effect on the distribution function. The basic dynamic process to take into account to evaluate the “macroscopic” transport rate is the diffusion of particles in transverse to the resonance line direction, so that because of the different longitudinal diffusion intensities inside and outside of the separatrix, the transverse diffusion modulates the longitudinal one. This makes the effective one-dimensional random walk along the resonance line a more complex stochastic process, in fact not even describable by any sort of diffusion process.

The present paper was devoted to the description of the physics of phenomenon, and left aside the more technical (though very important) question under what conditions can it manifest itself in the real hadronic colliders. Few general observations about this aspect of the problem can be made however without a substantial effort.

The necessary condition of resonance-induced enhancement is the condition (54), requiring smallness of angle between the resonance line and resonant oscillations direction. The question then is when this small angle can appear. The important point is that it depends only on nonlinear tune shifts  $\delta\nu_x$ ,  $\delta\nu_y$  dependencies on betatron amplitudes  $A_x$ ,  $A_y$ , and not on the harmonic amplitudes (defining the resonance width). The tune shifts are created by both the multipole components of magnetic fields and the nonlinear beam-beam interaction field. Since the hadronic beams are usually round, the beam-beam interaction is symmetric and preliminary numerical evidence is that the resonant oscillations are always nearly orthogonal to the resonance line. Thus it does not look likely that the phenomenon can manifest itself in the absence of multipole components. Superimposing the latter on the top of beam-beam interactions can however change the situation.

Consider now the effect of the multipoles in the absence of beam-beam force. The lowest order multipole tune-shifts come either from the first-order perturbation term of the octupole component or the second-order one of the sextupole component and have the same functional forms:

$$\begin{aligned}\delta\nu_x &= C_1 A_x^2 + C_2 A_y^2 \\ \delta\nu_y &= C_2 A_x^2 + C_3 A_y^2\end{aligned}\quad (55)$$

where  $A_x, A_y$  are the betatron amplitudes and the coefficients  $C_1, C_2, C_3$  are the integrals of the multipole amplitudes along the ring and can vary in respect to each other in the wide range. It is easy to see that the resonance line is straight in the action variables  $J_x = A_x^2$ ,  $J_y = A_y^2$  and that the angle between this line and the resonant oscillations can be varied arbitrarily by varying the constants  $C_1, C_2, C_3$ . Thus the phenomenon of the resonance-enhanced diffusion can be more easily observed in the absence of beam-beam interaction and is conceivable when both beam-beam interaction and lattice nonlinearities affect the tune shifts.

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