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# The Quantum Equivalence of Nambu and Polyakov String Actions

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**Abstract**

By integrating out the auxiliary metric in the Polyakov string path integral, we derive a path integral for the Nambu action complete with measure. We show how to gauge fix this Nambu form of the partition function. This involves an intermediate partial gauge fixing step. Our result is the Polyakov path integral in conformal gauge with the correct measure. The intermediate step may enjoy off–shell BRS symmetry by a generalization of the standard procedures. We show how the Teichmüller parameters arise in the Nambu formalism for general genus. These results allow us to make some observations on the physical characteristics of typical string world–sheets.

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# I. Introduction

Part of the beauty of string theory is that it is founded on a simple generalization of Einstein's postulate: that free particles follow geodesics, equivalently that their action is just the world–line length. Thus string theory is based on the Nambu action[1] which is just the world–sheet area:

$$\begin{aligned} S_N &= \frac{1}{2\pi}\int d^2\sigma\sqrt{\hat{g}} \\ \hat{g}_{ab} &= \partial_a x^\mu \partial_b x_\mu \end{aligned} \tag{1.1}$$

(where $\sigma^a(a=1,2)$ are world–sheet coordinates. $\hat{g}_{ab}$ is the induced metric: the metric the world–sheet inherits from being embedded in space–time. $x^\mu(\mu = 1,\cdots D)$ are space–time coordinates. Space–time and world–sheet will be taken to be euclidean. The string tension has been set to $\frac{1}{2\pi}$, i.e.: $\alpha' = 1$).

Actually, essentially all modern advances in the quantum theory of strings have been based on the Polyakov action,

$$S_P = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\, g^{ab}\partial_a x^\mu \partial_b x_\mu\ , \tag{1.2}$$

which introduces an auxiliary field $g_{ab}(\sigma)$ that plays the role of a world–sheet metric. The main benefit of this approach has been the ability to calculate covariantly and on arbitrary world–sheet topologies using path integral methods (first demonstrated by Polyakov [2]). While it is a straightforward exercise to prove the classical equivalence of the two formulations[2,3], this is not sufficient to imply their quantum equivalence. Indeed it is sometimes believed to be otherwise. It would be a pity if the link with world–sheet area was thus lost at the quantum level. In this paper we will show this is not so: we will show that the path integral formulations based on (1.1) or (1.2) are completely equivalent. In the process, we will demonstrate how to calculate directly with a path integral employing the Nambu action.

We were motivated to investigate this question by some recent research*[4]. We suggested a second quantized formulation of strings which is based on the Nambu action and cannot be based on the Polyakov action. The present work thus demonstrates that such a basis does not preclude our proposals, ref. [4], from being an

*I would like to thank Cliff Burgess for pressing me on this point.

appropriate generalization of first quantized treatments[2].

There are two reasons why the classically equivalent theories, using actions (1.1) or (1.2), need not be equivalent quantum mechanically. One reason is that classical theory depends only on the stationary points of the action, whereas quantum theory explores values of the action over the full function space. The other reason is that the action itself is not sufficient to define the quantum theory: In the path integral formulation, this ambiguity manifests itself in the need to define the (functional) measure. To be more specific, the Polyakov string path integral is given by

$$Z_P = \int d\mu_P(g_{ab}, x^\mu) e^{-S_P} \tag{1.3}$$

where the integration over the fields is written $d\mu_P$ to indicate that it is not just the naïve measure[5]:

$$\mathcal{D}g_{ab}\mathcal{D}x^\mu = \prod_{\underline{\sigma}} d^3g\,(\underline{\sigma})\, d^Dx\,(\underline{\sigma}) \tag{1.4}$$

and similarly for the Nambu action and path integral:

$$Z_N = \int d\mu_N(x^\mu) e^{-S_N} \ . \tag{1.5}$$

It is clear here that the measures are not just the naïve ones because these would not be invariant under (2 dimensional) diffeomorphisms. However it is also true that invariance of the measure (under all symmetries of the quantum theory) is not of itself sufficient to define the measure either.[6] In general, the measure must further be chosen to obey unitarity. This usually needs making a direct comparison with a hamiltonian formalism in a manifestly unitary gauge; and, needless to say, there are still, in general, a number of inequivalent choices depending on polarization of the phase space (i.e., choice of variables and conjugate momenta) and operator ordering. Fortunately, in the Polyakov case, a prescription for the measure exists[2,7] which has proved successful.[8] As reviewed in Section II, this boils down to the measure being the naïve one times a power of $\prod_{\underline{\sigma}} g(\underline{\sigma})$. (Invariance would have also allowed for example some power of $\prod_{\underline{\sigma}} \hat{g}(\underline{\sigma})$, $\hat{g}_{ab}$ being the induced metric; however, by comparison with the hamiltonian formalism, we know the measure must reduce to the naïve one in the gauge $g_{ab} = \delta_{ab}$ in order that (1.3) reduce to the partition function of a free field theory†).

† We are ignoring the issue of counterterms and anomalies. This is discussed later in the paper.

Note that the above implies that $Z_P$ (1.3) involves a functional integral over only undifferentiated $g_{ab}(\underline{\sigma})$ which factorizes (on a surface of any topology) into an infinite product of three dimensional integrals (using (1.4): $d^3g(\underline{\sigma}) \equiv dg_{11}(\underline{\sigma})dg_{22}(\underline{\sigma})dg_{12}(\underline{\sigma})$). As explained in Section II, it is then a simple matter (up to some subtleties) to explicitly perform the integral. The result turns out to be of the form (1.5) and provides us with an explicit expression for the measure. Up to constants it is the naïve measure times $\prod_{\underline{\sigma}} [\hat{g}(\underline{\sigma})]^{-1/2}$.

At first sight this measure seems a little peculiar: It cannot be derived from analogous methods to refs. [2,7] and it does not appear to be diffeomorphism invariant. (At least naïvely the power of $\hat{g}$ is wrong). Also, the method of derivation assumes a regularization (lattice regularization) that explicitly breaks diffeomorphism invariance. Another reason for misgivings is the very dissimilar nature of the two formulations (1.3) and (1.5). On the one hand in the critical dimension the Polyakov formalism can be gauge fixed to a free theory except, on a general topology, for some finite number of remaining degrees of freedom in the auxiliary metric (the Teichmüller parameters). Outside the critical dimension the Polyakov partition function develops an anomaly which is generally attributed to a loss of local Weyl invariance. On the other hand, it would not at first sight seem possible that the Nambu formalism could be gauge fixed to a free theory since the action (1.1) is not even bilinear in derivatives. Indeed, it has not previously to our knowledge been demonstrated that such a partition function is calculable directly. On a general topology surface, the Nambu formalism would have to be equivalent to a free theory for $x^\mu$ plus Teichmüller parameters, but it is not immediately clear where these parameters would come from, there being apparently no extra degrees of freedom in (1.5) whose remnants could yield these parameters. And, finally, we would expect some feature of the anomaly outside the critical dimension to be demonstrable, but in this case there is no analogue of local Weyl invariance.

For the above reasons, we analyze in Section III our derived form for the Nambu path integral directly and without resorting to a discretization of the world-surface. We show that by first fixing only right-handed diffeomorphisms we can derive an action which is essentially that of Siegels chiral bosons.[9] It is also equivalent to the Polyakov action in a (partial) chiral gauge. In the process we are led to a change of variables that exactly cancels the previously mentioned measure factor. We then choose a gauge for the left-handed diffeomorphisms which, on a sphere, eliminates the first gauge fixing Lagrange multiplier field. The result is precisely the Polyakov partition function with the required free field measure and standard ghosts.

The anomaly that appears in the resulting BRS algebra outside $D = 26$ is now attributable to a loss of diffeomorphism invariance (as indeed it could have been chosen to be in the Polyakov case).

In order to better understand the intermediate partial gauge choice, we investigate BRS symmetry and gauge invariance of the action at each step. By introducing some generalization into standard BRS procedures, we show that it is possible to find an intermediate gauge fixed action that preserves the full left-handed gauge invariance while also enjoying a right-handed BRS invariance that is nilpotent off-shell. The next gauge fixing step is shown to be consistent with the right-handed BRS invariance.

In Section IV, the emergence of Teichmüller parameters, on a general genus, is demonstrated. The equivalence of Nambu and Polyakov partition functions allow us to physically interpret Teichmüller parameter dependence, and dependence of the partition function on string tension, in terms of physical qualities of the string world-sheet. We show that, with probability one, string world-sheets consist of zero area infinitely thin tubes. In Section V, we present a summary, our conclusions, and mention possible future directions.

## II. From Polyakov to Nambu‡

In this section we will perform the integration over the auxiliary metric in the Polyakov partition function (1.3), to obtain a partition function of the form (1.5). Our first step is to determine the measure in $Z_P$. We follow refs. [2,7]. The essential point is that we can define a diffeomorphism invariant measure from a diffeomorphism invariant norm on the tangent space to the fields. For example, writing the measure for a single scalar field $x(\underline{\sigma})$ as

$$d\mu(x) = \mathcal{D}x \ \mu_x \tag{2.1}$$

where $\mu_x$ is, in principle, a functional of $x(\underline{\sigma})$ and $g_{ab}(\underline{\sigma})$, and norm on the tangent space as

$$\|\delta x\|^2 = \int d^2\sigma \sqrt{g} \, \delta x^2 \ ,$$

‡Part of this section has been used as a basis for a M.Sc. thesis by Majid Al-Sarhi.[10]

we can determine $\mu_x$ from the diffeomorphism invariant condition

$$1 = \int d\mu(\delta x) e^{-||\delta x||^2} = \int \mathcal{D}(\delta x)\mu_x \, e^{-||\delta x||^2} \tag{2.2}$$

This implies

$$\mu_x = \mathcal{D}et^{\frac{1}{2}} \sqrt{g} \, . \tag{2.3}$$

Here $\mathcal{D}et$ is a functional determinant. We will drop all constants that appear multiplicatively in the measure since these contribute only to the overall normalization of the partition function. (Actually, the choice 1 in (2.2) and the dropping only of constants $c$ that appear as an infinite product: $c^\infty = \prod_{\underline{\sigma}} c$, seem to give always the right normalization in string theory.[7,8] Polchinski has argued that such infinite products may always be renormalized into the cosmological counterterm (which we introduce, see eq. (2.7)). In the course of this paper we will in fact only drop constants which are of the form of these infinite products).

It is worth noting that we can verify (2.3) directly, confirming the above trick and showing that it is convenient but not necessary: Under an infinitessimal diffeomorphism $\sigma'^a = \sigma^a - \xi^a(\underline{\sigma})$ we have

$$\begin{aligned}
\mathcal{D}x' &= \mathcal{D}x \mid \frac{\delta x'}{\delta x} \mid = \mathcal{D}x \;\; \mathcal{D}et\,(1 + \xi^a \partial_a) \\
\mathcal{D}et\sqrt{g'} &= \mathcal{D}et\,(\sqrt{g} + [\partial_a, \xi^a \sqrt{g}]) \\
&= \mathcal{D}et\sqrt{g} \;\; \mathcal{D}et^2\,(1 - \xi^a \partial_a)
\end{aligned} \tag{2.4}$$

where we have expanded the commutator in the penultimate line and assumed reality of the determinants containing $\xi$. The above expressions confirm that choice (2.1) with (2.3) is invariant.

Proceeding similarly for $g_{ab}$ we define [2,7]($c > -\frac{1}{2}$ for positivity):

$$||\delta g_{ab}||^2 = \int d^2\sigma \sqrt{g} \left( g^{aa'} g^{bb'} + c g^{ab} g^{a'b'} \right) \delta g_{ab} \delta g_{a'b'}$$

implying

$$\mu_g = \mathcal{D}et^{-\frac{3}{4}} g$$

which can readily be confirmed by a similar analysis to above. Thus altogether we have

$$Z_P = \int \mathcal{D}g_{ab} \mathcal{D}x^\mu \; \mathcal{D}et^p g \;\; e^{-S_P} \tag{2.5}$$

where $p = \frac{1}{4}(D-3)$. Actually all we will need to know is that the measure factor is $\mathcal{D}et\ g$ to some power and not the precise power.

Now replace the continuum integral $(\mathcal{D}g)$ by one over some arbitrary dense discrete lattice of points $\{\underline{\sigma}\}$, so (2.5) factorises into

$$Z_P = \int \mathcal{D}x^\mu \prod_{\underline{\sigma}} Z'_P(\underline{\sigma}) \tag{2.6}$$

$$Z'_P(\underline{\sigma}) = \int d^3g\ g^p\ \exp -\frac{\delta^2\sigma}{4\pi}\sqrt{g}\ \left[g^{ab}\hat{g}_{ab}(\underline{\sigma}) + \Lambda\right] \tag{2.7}$$

Here we have factored out the integral over $x^\mu$, which plays no further role in the analysis in this section. $\delta^2\sigma(\underline{\sigma})$ is the area element which can depend on $\underline{\sigma}$, and $\hat{g}_{ab}(\underline{\sigma})$ is the induced metric (1.1). The $\Lambda$ term is a cosmological constant counterterm to absorb any ultra violet divergences. With this regularization, this is the only possible counterterm since differentials of the metric cannot be generated; in principle certain diffeomorphism violating counterterms could be generated, but we will see that they are not needed.

Note that (2.7) is not a free integral over the elements $g_{ab}$ because $g_{ab}$ must be positive definite. This is true if and only if the eigenvalues of the matrix $\mathbf{g} = [g_{ab}]$ are strictly positive. Noting that (2.6) is invariant under $SO(2)$ rotations, we rotate to a basis in which $\hat{g}_{ab}$ is diagonal $\hat{g}_{ab} = \hat{\lambda}_a\delta_{ab}$ ($a$ not summed) and change variables to

$$\mathbf{g} \;=\; \mathbf{R}\,\mathbf{\Lambda}\,\mathbf{R}^{\mathrm{T}}$$

where $\mathbf{R}$ is the familiar $SO(2)$ rotation matrix parameterized by an angle $\theta$. $\mathbf{R} \to -\mathbf{R}$ symmetry implies the range $0 \leq \theta < \pi$. $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues $0 < \lambda_1, \lambda_2 < \infty$. The result is:

$$Z'_P = \int d\lambda_1 d\lambda_2 d\theta\ |\lambda_1 - \lambda_2|\,(\lambda_1\lambda_2)^p\ \times$$

$$\times\ \exp -\frac{\delta^2\sigma}{4\pi}\left[\cos^2\theta\left(\hat{\lambda}_1/\alpha + \alpha\hat{\lambda}_2\right) + \sin^2\theta\left(\hat{\lambda}_2/\alpha + \alpha\hat{\lambda}_1\right) + \Lambda\sqrt{\lambda_1\lambda_2}\right]$$

where $\alpha = \sqrt{\frac{\lambda_1}{\lambda_2}}$ .

Note that although the original partition function (2.5) has an infinite degeneracy associated with diffeomorphism invariance it does not cause a problem in the analysis because it only appears if we also integrate over $x^\mu(\underline{\sigma})$. However the local

Weyl invariance of the original action (1.2) does need care. A local Weyl transformation corresponds here to the scaling $\lambda_i \to \Omega\lambda_i$ and the invariance is reflected in the appearance of the ratio $\alpha^2$. Note that the measure and the counterterm are not invariant under local Weyl Transformations. Changing variables to the Weyl invariant $\alpha$ and $\beta = \sqrt{\lambda_1\lambda_2}$ allows us to factor out the non-invariant Weyl group integral (over $\beta$) which is finite for counterterm $\Lambda > 0$. (Without the counterterm, a choice of Weyl dependent variable $\beta$ other than some function of the product $\lambda_1\lambda_2$ generates diffeomorphism anomalies). It is helpful to use a remaining discrete invariance: $\alpha \to 1/\alpha$ $\theta \to \theta \pm \pi/2$, and to write $\hat{\lambda}_\pm = \hat{\lambda}_1 \pm \hat{\lambda}_2$. Dropping all constant multiplying factors (c.f. discussion below (2.3)) we obtain

$$
\begin{aligned}
Z_p' &= \int_0^{2\pi} d\phi \int_1^\infty d\alpha \left(1 - \frac{1}{\alpha^2}\right) \exp -\frac{\delta^2\sigma}{4\pi}\left[\hat{\lambda}_+\left(\alpha + \frac{1}{\alpha}\right) + \hat{\lambda}_-\left(\frac{1}{\alpha} - \alpha\right)\cos\phi\right] \\
&= \int_0^{2\pi} d\phi \left(\frac{4\pi}{\delta^2\sigma(\hat{\lambda}_+ - \hat{\lambda}_-\cos\phi)} - 2 + O\left(\delta^2\sigma\right)\right) \\
&\propto \frac{1}{\sqrt{\hat{\lambda}_1\hat{\lambda}_2}}\left(1 - \frac{1}{2\pi}\delta^2\sigma\sqrt{\hat{\lambda}_1\hat{\lambda}_2}\right) + O\left(\delta^2\sigma\right)
\end{aligned}
$$

Finally forming the product (2.6) and taking the limit $\delta^2\sigma \to 0$ we obtain $Z_P = Z_N$ with

$$
Z_N = \int \mathcal{D}x^\mu \left(\mathcal{D}et\, \frac{1}{\sqrt{\hat{g}}}\right) e^{-S_N} \tag{2.8}
$$

establishing the equivalence between the Polyakov path integral and the Nambu path integral (1.5) (with (1.1)), and determining the Nambu measure. Note that it is not trivial that the result can be expressed at all in terms of covariant quantities, or that it contains the exponential of an action. Note also that the measure is not one predictable by the methods explained at the beginning of this section (i.e., by choosing a suitable norm $||\delta x||^2$). Indeed from (2.4) it follows that (2.8) is not (naïvely) diffeomorphism invariant. Nevertheless, in the next section we will see that this measure is precisely the one required.

## III. Gauge Fixing the Nambu Path Integral

In this section we show how to gauge fix the path integral (2.8) to a tractable form; in fact, the free path integral equal to Polyakovs in conformal gauge.[3] For this it is helpful to introduce the usual complex coordinates:

$$z = \sigma^1 + i\sigma^2$$

$$\bar{z} = \sigma^1 - i\sigma^2$$

We will restrict our analysis in this section to world-sheets with the topology of a sphere. The general case will be analyzed in Section IV. In terms of these coordinates, the determinant of the induced metric takes the form

$$\hat{g} = 4\left(\hat{g}_{+-}^2 - \hat{g}_{++}\hat{g}_{--}\right) \tag{3.1}$$

where

$$\begin{aligned} \hat{g}_{++} &= \partial x \cdot \partial x \\ \hat{g}_{--} &= \bar{\partial} x \cdot \bar{\partial} x \\ \hat{g}_{+-} &= \partial x \cdot \bar{\partial} x \end{aligned} \tag{3.2}$$

(We suppress the space–time indices. We take the space–time metric to be flat for simplicity. The analysis could be readily generalised to curved metrics.) Our first observation is that choosing a gauge such that the second term vanishes in (3.1) causes the Nambu action (1.1) to collapse to the usual free action for $x$ (obtained in the gauge $g_{ab} \propto \delta_{ab}$ in (1.2)). However, the obvious choice of gauge $\hat{g}_{++} = \hat{g}_{--} = 0$ produces constraints that would seem to be intractable. Also, with some change change of variables it produces the usual ghost action, but a power of $\mathcal{D}et\ \hat{g}$ is left in the measure. Instead we will split diffeomorphisms into left and right handed parts, infinitessimally:

$$\begin{aligned} \delta_{(+)} x^\mu &= \xi^+(z, \bar{z})\, \partial x^\mu \\ \delta_{(-)} x^\mu &= \xi^- \bar{\partial} x^\mu \end{aligned} \tag{3.3}$$

and for the moment fix only right handed diffeomorphisms $\xi^-$ by the gauge $\hat{g}_{++} = 0$[§]: Under the above, the gauge changes to

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[§]A chiral gauge for the Polyakov action was considered in ref. [12], but there the fixing was completed by also setting $g_{+-} = 1$

$$\begin{aligned}\delta_{(+)}\hat{g}_{++} &= 0\\ \delta_{(-)}\hat{g}_{++} &= \hat{g}_{+-}\ \partial\xi^{-}\end{aligned}$$

Using the Fadeev–Popov procedure, one obtains from (2.8):

$$Z_N = \int \mathcal{D}x^\mu\ \mathcal{D}et^{-1/2}\hat{g}\ \mathcal{D}et\left(\hat{g}_{+-}\partial\right)\ \delta\left[\hat{g}_{++}\right] e^{-S_N} \tag{3.4}$$

Combining the determinants and using eq. (3.1), we obtain:

$$\begin{aligned}Z_N &= \int \mathcal{D}\left(x^\mu, \bar{c}, \bar{b}, \beta\right) e^{-S'}\\ S' &= \frac{1}{\pi}\int d^2\sigma\left\{\partial x\cdot\bar{\partial}x + i\beta\partial x\cdot\partial x + \bar{b}\partial\bar{c}\right\}\end{aligned} \tag{3.5}$$

Here we have raised the functional $\delta$–function constraint into the action by integrating over a Lagrange multiplier field $\beta$. $\bar{b}$ and $\bar{c}$ are the usual ghosts that appear in the conformal gauge for Polyakovs action (as follows from the identification $\xi^- \to \bar{c}$, and conformal invariance —a symmetry that survives the gauge fixing). Note that the measure factor is precisely the factor required to cancel all metric dependence from the Fadeev–Popov determinant.

Significantly the above already coincides with the Polyakov partition function: in the partial gauge $g_{++} = 0$.¶ To show this statement, identify

$$\sqrt{g}g^{++} \equiv 4i\beta \tag{3.6}$$

and note that the gauge choice implies $\sqrt{g}g^{+-} = 2$ and $g^{--} = 0$. This in turn implies the only non–vanishing component of the anti–ghost $b_{ab}$ is $b_{--} = \bar{b}$ and hence (after some algebra) the ghost Lagrangian for the Polyakov partition function simplifies to:

$$\sqrt{g}\ b_{ad}\nabla^a c^d = \bar{b}\partial\bar{c}$$

Not surprisingly, the BRS and left handed diffeomorphism symmetries also coincide: they can be determined uniquely from the requirements that they obey the correct

¶The integral over the Weyl mode has been factored out as implied by the Weyl invariant identification (3.6).

algebra (for symmetries involving BRS this occurs, as usual, only on-shell) and that the action be invariant. We will go back and analyze the symmetries in detail, but first let us show how to completely fix the gauge. The left-handed diffeomorphisms (c.f. (3.3)) take the following form on the new fields:

$$\begin{aligned} \delta_{(+)}\beta &= \xi^+\partial\beta - \partial\xi^+\beta + i\bar{\partial}\xi^+ \\ \delta_{(+)}\bar{c} &= \xi^+\partial\bar{c} \\ \delta_{(+)}\bar{b} &= \xi^+\partial\bar{b} \end{aligned} \tag{3.7}$$

In particular, this means that (on a sphere) we can choose the gauge $\beta = 0$. From the point of view of the identification with the Polyakov action (3.6), this is not surprising, but referring to our gauge fixing procedure (3.4-5) we see that this means we can gauge away our first gauge fixing constraint which we would otherwise not know how to handle! Fixing the gauge $-i\beta = 0$ and introducing the Fadeev–Popov ghosts (by using (3.7)), one obtains:

$$\begin{aligned} Z_N &= \int \mathcal{D}\left(x^\mu, b, c, \bar{b}, \bar{c}\right) e^{-S} \\ S &= \frac{1}{\pi}\int d^2\sigma \left\{\partial x.\bar{\partial}x + b\bar{\partial}c + \bar{b}\partial\bar{c}\right\} , \end{aligned} \tag{3.8}$$

in other words, precisely the Polyakov action in conformal gauge.

The rest of this section is devoted to a careful analysis of the symmetries involved in this two-step gauge fixing procedure. We do this to ensure that no crucial symmetries are violated in the procedure. In particular, we show that the first gauge fixing step can be introduced such as to both leave the left-handed local invariance undisturbed and transform the right-handed invariance into an off-shell-nilpotent BRS invariance. We then show that the next gauge fixing step leaves the right-handed BRS invariance undisturbed. At the end of the section we discuss the quantum anomalies that appear in the procedure; they are just the usual anomalies that appear when $D \neq 26$.

Returning to the splitting of diffeomorphisms (3.3) and gauge choice $\hat{g}_{++} = 0$, let us make some preliminary comments. First of all, in order to do separate diffeomorphisms (3.3) we must complexify, i.e., allow $x$ to take on complex values, but we preserve the degrees of freedom by integrating only over $x$ and not $\bar{x}$. We would not have had to do this in Minkowski signature: this is just the usual effect of treating chirality in euclidean space. Compare, e.g., the euclidean action and gauge transformation for chiral bosons[9] with eqs. (3.5) and (3.7). Secondly, it is straightforward to see, by Riemanns theorem (a coordinate choice can always be made over the whole sphere such as to diagonalize the metric), that it is always possible to choose the above gauge.

Note that the algebra of diffeomorphisms yields two subalgebras, a left-handed one:

$$\begin{aligned} \left[\delta^1_{(+)}, \delta^2_{(+)}\right] &= \delta^{12}_{(+)} \\ \text{where} \qquad \delta^i_{(+)} &= \xi^+_i \partial \qquad i = 1,2 \\ \text{and} \qquad \delta^{12}_{(+)} &= (\xi^+_1 \partial \xi^+_2 - \xi^+_2 \partial \xi^+_1)\partial \ , \end{aligned} \tag{3.9}$$

and similarly a right-handed subalgebra for $\delta_{(-)}$. But they do not decouple (not normal subalgebras):||

$$\left[\delta_{(+)}, \delta_{(-)}\right] = \left(\xi^+ \partial \xi^-\right) \bar{\partial} - \left(\xi^- \bar{\partial} \xi^+\right) \partial \ . \tag{3.10}$$

Fixing the gauge $\hat{g}_{++} = 0$ using standard methods of BRST[11] we specialize to

$$\begin{aligned} \delta_{(-)} x^\mu &= \bar{\epsilon}\, \bar{\delta} x^\mu \\ \bar{\delta} x^\mu &= \bar{c}\, \bar{\partial} x^\mu \end{aligned} \tag{3.11}$$

where $\bar{c}$ is a ghost, $\bar{\delta}$ is the BRS operator and $\bar{\epsilon}$ is a constant Grassmann parameter. Derive

$$\bar{\delta} \bar{c} = \bar{c}\, \bar{\partial} \bar{c} \tag{3.12}$$

by nilpotency of $\bar{\delta}$ ($\bar{\delta}^2 = 0$) and introduce an antighost $\bar{c}^+_-$ that transforms into the Lagrange multiplier field $\beta^+_-$ by

$$\bar{\delta} \bar{c}^+_- = i \beta^+_- \tag{3.13}$$

$$\bar{\delta} \beta^+_- = 0 \ . \tag{3.14}$$

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||except of course in the special case where the diffeomorphisms are conformal $\bar{\partial}\xi^+ = \partial \xi^- = 0$.

(The indices refer to the fields required conformal weights: conformal symmetry will survive the gauge fixing. $\bar{c}$ has conformal weight $\bar{h} = -1$). The ghost and gauge-fixing lagrangians are then given together as $\mathcal{L}_{(-)} = \bar{\delta}(\bar{c}^{+}_{-}\partial x \cdot \partial x)$.

The (off-shell) BRS invariance is automatic from nilpotency of $\bar{\delta}$. However, we must modify this procedure. The reason is that (one can show) the above action cannot respect the remaining $\xi^{+}$ invariance off-shell. This is a consequence of the fact that the two algebras are coupled (3.10). (If one simply ignores this problem, the end result is eq. (3.5). But we solve the problem here to ensure that we are not performing some sleight of hand). Actually, we must ensure that the action is invariant under finite left-handed diffeomorphisms. This is equivalent (for those connected to the identity) to ensuring that the transformation of the fields respect the subalgebra (3.9). And since the actions invariances must be right-handed BRS and left-handed diffeomorphisms, the commutator of the two must close, i.e., we require

$$\left[\delta_{(+)}, \bar{\epsilon}\bar{\delta}\right] = \delta'_{(+)} + a\bar{\epsilon}\bar{\delta}$$

where $a$ is a number. Equation (3.10) (with $\xi^{-} = \bar{\epsilon}\bar{c}$ as in (3.11)) shows that this equation is not true: the induced $\delta_{(-)}$ transformation is not of the form of BRS—$\delta_{(-)} = \bar{\epsilon}\bar{\delta}$. However, if we define $\bar{c}$ to transform as

$$\delta_{(+)}\bar{c} = \xi^{+}\partial\bar{c} \tag{3.15}$$

then one finds

$$\left[\delta_{(+)}, \bar{\epsilon}\bar{\delta}\right] x^{\mu} = \bar{\epsilon}\ \bar{c}\ \bar{\partial}\xi^{+}\partial x^{\mu}$$

which is of the required form (with $a = 0$ and $\xi^{+\prime} = \bar{\epsilon}\ \bar{c}\ \bar{\partial}\xi^{+}$). It is equivalent and convenient to define a BRS transformation of the gauge parameter

$$\bar{\delta}\xi^{+} = \bar{c}\bar{\partial}\xi^{+} \tag{3.16}$$

The two symmetries then commute when applied to $x^{\mu}$; this may also be verified for $\bar{c}$:

$$\left[\delta_{(+)}, \bar{\delta}\right] x^{\mu} = \left[\delta_{(+)}, \bar{\delta}\right] \bar{c} = 0 \tag{3.17a}$$

To ensure invariance we now require

$$\left[\delta_{(+)}, \bar{\delta}\right] \beta^{+}_{-} = 0 \tag{3.17b}$$

$$\left[\delta_{(+)},\bar{\delta}\right]\bar{c}_-^+ = 0\ . \tag{3.17c}$$

Also, the left-handed transformation of the Lagrange multiplier is determined by requiring the $\beta_-^+\partial x.\partial x$ term to be invariant (up to a total derivative). The transformation is just that implied by its tensor indices:

$$\delta_{(+)}\beta_-^+ = \Delta\beta_-^+ \tag{3.18a}$$

where

$$\Delta = \xi^+\partial - \left(\partial\xi^+\right)\ . \tag{3.18b}$$

Now it is clear that (3.14), (3.17b) and (3.18) are incompatible. Thus we generalize the BRS transformation (3.13) to

$$\bar{\delta}\bar{c}_-^+ = \bar{Q}\bar{c}_-^+ + i\beta_-^+$$

$$\bar{Q} = A\bar{c}\bar{\partial} + B\left(\bar{\partial}\bar{c}\right)$$

where we have added the most general possible homogeneous BRS transformation compatible with conformal invariance ($A$ and $B$ are real numbers). We assume

$$\delta_{(+)}\bar{c}_-^+ = \Delta_{\bar{c}}\bar{c}_-^+$$

where $\Delta_{\bar{c}}$ is a linear differential operator containing $\xi^+$. Requiring $\bar{\delta}^2$ to vanish on $\bar{c}_-^+$ yields:

$$\bar{\delta}\beta_-^+ = \bar{Q}\beta_-^+$$

$$\bar{\delta}(\bar{Q}) = \bar{Q}^2 \tag{3.19a}$$

which is sufficient to ensure $\bar{\delta}^2\beta_-^+ = 0$. Eq. (3.17c) implies

$$\Delta_{\bar{c}} = \Delta$$

$$\bar{\delta}(\Delta) - \delta(\bar{Q}) = [\bar{Q},\Delta] \tag{3.19b}$$

which is sufficient to ensure (3.17b). Solving equations (3.19) determines $A = -B = 1$ uniquely. Thus we have

$$\begin{aligned}
\bar{\delta}\bar{c}_-^+ &= \bar{c}\,\bar{\partial}\bar{c}_-^+ - \bar{\partial}\bar{c}\,\bar{c}_-^+ + i\beta_-^+ \\
\bar{\delta}\beta_-^+ &= \bar{c}\,\bar{\partial}\beta_-^+ - \bar{\partial}\bar{c}\,\beta_-^+
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\delta_{(+)}\bar{c}^{+}_{-} &= \xi^{+}\partial\bar{c}^{+}_{-} - \partial\xi^{+}\bar{c}^{+}_{-} \\ \delta_{(+)}\beta^{+}_{-} &= \xi^{+}\partial\beta^{+}_{-} - \partial\xi^{+}\beta^{+}_{-}\end{aligned} \tag{3.21}$$

Finally note that transformations (3.15) and (3.21) obey the subalgebra (3.9) automatically since they are of the form of tensor transformations in one dimension. Now, using the fact that $\delta_{(+)}$ and $\bar{\delta}$ commute on the fields (3.17) and the $\delta_{(+)}\bar{c}^{+}_{-}$ transformation is that implied by its indices, it immediately follows that the ghost plus gauge fixing action

$$\begin{aligned}S_{(-)} &= \frac{1}{2\pi}\int d^2\sigma\, \bar{\delta}\left(\bar{c}^{+}_{-}\partial x.\partial x\right) \\ &= \frac{1}{2\pi}\int d^2\sigma \left\{ i\beta^{+}_{-}\partial x.\partial x + \bar{c}\bar{\partial}\bar{c}^{+}_{-}\partial x.\partial x - (\bar{\partial}\bar{c})\bar{c}^{+}_{-}\partial x.\partial x \right. \\ &\quad \left. -\bar{c}^{+}_{-}\bar{\partial}(\partial x.\partial x)\,\bar{c} - 2\bar{c}^{+}_{-}\partial x.\bar{\partial}x\,\partial\bar{c}\right\}\end{aligned} \tag{3.22}$$

is invariant under an off-shell nilpotent right-handed BRS and finite left-handed gauge transformations. Our gauge-fixed partition function is thus (using (1.1) and (2.8)):

$$Z_N = \int \mathcal{D}\left(x^{\mu}, \bar{c}, \bar{c}^{+}_{-}, \beta^{+}_{-}\right) \mathcal{D}et^{-\frac{1}{2}}\hat{g}\; e^{-(S_N + S_{(-)})}$$

This is equal to our intermediate result (3.5) as may be shown by using the functional $\delta$-function obtained by integrating over $\beta^{+}_{-}$ to simplify the action (using also (3.1)), changing variables to

$$\bar{b} = -\bar{c}^{+}_{-}\partial x.\bar{\partial}x \tag{3.23}$$

which, recalling that a change of Grassmann variables generates an inverse jacobian (Berezinian) and using (3.1), can be seen to cancel the measure factor, and finally re-expressing the functional $\delta$-function as a term in the action ($\beta^{+}_{-} = 2\beta$). The change of variables (3.23) implies (using (3.20-21):

$$\delta_{(+)}\bar{b} = \xi^{+}\partial\bar{b} - \bar{\partial}\xi^{+}\partial x.\partial x(\partial x.\bar{\partial}x)^{-1}\bar{b}$$

$$\bar{\delta}\bar{b} = \bar{c}\,\bar{\partial}\bar{b} - 2i\beta\partial x.\bar{\partial}x + \partial\bar{c}\,\bar{\partial}x.\bar{\partial}x(\partial x.\bar{\partial}x)^{-1}\bar{b}$$

However, the use of the constraint $\partial x.\partial x = 0$ to simplify the actions (3.22) and (1.1) deforms the $\beta$ and $\bar{b}$ transformations to:

$$\begin{aligned} \delta_{(+)}\bar{b} &= \xi^+\partial\bar{b} \\ \delta_{(+)}\beta &= \xi^+\partial\beta - \partial\xi^+\beta + i\bar{\partial}\xi^+ \\ \bar{\delta}\bar{b} &= -\left\{\bar{\partial}\bar{b}\bar{c} + 2i\beta\partial x.\bar{\partial}x + 2\bar{b}\bar{\partial}\bar{c} + \bar{\partial}x.\bar{\partial}x\right\} \\ \bar{\delta}\beta &= \bar{\partial}(\bar{c}\beta) \end{aligned} \tag{3.24}$$

Note that $\beta$ no longer transforms homogeneously under $\delta_{(+)}$ as a consequence of simplifying (1.1). The altered $\bar{\delta}$ transformations are determined uniquely by requiring the action $S'$ still be invariant. It is straightforward to show that the new $\delta_{(+)}$ transformations still satisfy the subalgebra (3.9) (and that the action $S'$ is invariant). However, the BRS transformation is nilpotent on $\bar{b}$ now only on-shell:

$$\bar{\delta}^2\bar{b} = -2i\beta\;\bar{\partial}x.\bar{\partial}x\;\partial\bar{c} \tag{3.25}$$

(Of course, this effect is standard in BRS and does not destroy its power. It follows from the fact that we have used the $\beta$ eqs. of motion). We find also that the modified transformations (3.24) commute only on-shell:

$$\begin{aligned} \left[\delta_{(+)},\bar{\delta}\right]\bar{b} &= -2\bar{\partial}\xi^+\left(\bar{b}\partial\bar{c} + i\beta\partial x.\partial x\right) \\ \left[\delta_{(+)},\bar{\delta}\right]\beta &= 2\bar{\partial}\xi^+\partial\bar{c}\,\beta \end{aligned} \tag{3.26}$$

Note that the $\delta_{(+)}$ transformations (3.15), (3.24) are as quoted in equation (3.7).

Now let us gauge fix left-handed diffeomorphisms. The new term in the $\beta$ transformation allows us to gauge away the first gauge fixing constraint (in (3.5)). Thus specialize to $\xi^+ = \epsilon c, \delta_{(+)} = \epsilon\delta$ where $c$ is our left-handed ghost, $\epsilon$ is a constant Grassmann parameter and $\delta$ is our left-handed BRS operator:

$$\begin{aligned} \delta x^\mu &= c\partial x^\mu \\ \delta^2 = 0 \;\Rightarrow\; \delta c &= c\partial c \end{aligned} \tag{3.27}$$

The BRS transformations of the other fields $\bar{c}, \bar{b}, \beta$ follow from the replacement $\xi^+ = \epsilon c$ (and (3.7)), and off-shell nilpotency on these fields is readily confirmed. That $\delta$ and $\bar{\delta}$

anticommute (up to equations of motion (3.26)) follows from the commutator relations $[\bar{\delta}, \delta_{(+)}]$ except for

$$\left\{\delta, \bar{\delta}\right\} c = 0 \quad \text{and} \quad \bar{\delta}^2 c = 0$$

which follow from $\{\delta, \bar{\delta}\}\bar{c} = 0$ and $\delta^2 \bar{c} = 0$ by symmetry. Now introduce antighost $b$ and Lagrange multiplier field $\gamma$ in standard fashion:

$$\delta b = \gamma, \quad \delta\gamma = 0 \tag{3.28}$$

and write the left–handed ghost plus gauge fixing action:

$$S_{(+)} = \frac{i}{\pi} \int d^2\sigma \delta \left[b\beta\right] = \frac{1}{\pi} \int d^2\sigma \left\{i\gamma\beta - b\left(ic\,\partial\beta - i\partial c\,\beta - \bar{\partial} c\right)\right\} \tag{3.29}$$

This is automatically invariant under left–handed BRS: $\delta$. It will be invariant under $\bar{\delta}$ if we choose the transformations on the new fields so that

$$\begin{aligned} \bar{\delta}(b\beta) &= \text{a total derivative} \\ \text{and} \qquad \{\bar{\delta}, \delta\}b &= 0\,. \end{aligned} \tag{3.30}$$

The first condition determines:

$$\bar{\delta} b = \bar{c}\,\bar{\partial} b \tag{3.31}$$

which (e.g.: by symmetry with $\delta\bar{b}$) satisfies $\bar{\delta}^2 = 0$. And the second condition gives:

$$\bar{\delta}\gamma = -c\,\partial\bar{c}\,\bar{\partial} b + \bar{c}\,\bar{\partial}\gamma$$

One may readily check that $\bar{\delta}^2\gamma = \{\delta, \bar{\delta}\}\gamma = 0$, for example:

$$\{\delta, \bar{\delta}\}\gamma = -\delta^2(\bar{\delta} b) = 0$$

follows from (3.30) and (3.28). Thus we have confirmed that we may gauge fix left–handed diffeomorphisms by setting $\beta = 0$ while leaving right–handed BRS undisturbed. Finally, including the new action (3.29) and variables into the partition function (3.5) and integrating over $\gamma$ gives the result (3.8), the Polyakov action in conformal gauge. The BRS transformations become:

$$\begin{aligned}
\bar{\delta}x^\mu &= \bar{c}\bar{\partial}x^\mu \\
\bar{\delta}\bar{c} &= \bar{c}\bar{\partial}\bar{c} \\
\bar{\delta}\bar{b} &= -\left\{\bar{\partial}x.\bar{\partial}x + 2\bar{b}\bar{\partial}\bar{c} + \bar{\partial}\bar{b}\bar{c}\right\} \\
\bar{\delta}c &= \bar{c}\bar{\partial}c \\
\bar{\delta}b &= \bar{c}\bar{\partial}b
\end{aligned} \tag{3.32}$$

for right-handed BRS, from eqs. (3.11, 12, 24), setting $\beta = 0$ consistently ($\bar{\delta}\beta = 0$). And nilpotency ($\bar{\delta}^2 = 0$) off-shell follows from (3.25). The left-handed BRS transformations are just those of (3.32) with all barred terms exchanged with unbarred. These are just the previously stated transformations ((3.27) and below) except for $\delta b$ which is modified by integrating out $\gamma$. (It is determined uniquely from $\delta S = 0$, thus clearly by symmetry of (3.8) it is as given in (3.32) by exchanging barred with unbarred). Thus $\delta^2 = 0$ holds on all fields, and $\{\delta, \bar{\delta}\} = 0$ holds except possibly on $b$ and $\bar{b}$ where we find

$$\left\{\delta, \bar{\delta}\right\}\bar{b} = -2\bar{\partial}c\left[\bar{b}\partial\bar{c} + \partial x.\bar{\partial}x\right]$$

and by symmetry $\{\delta, \bar{\delta}\}b$, which vanish only on-shell. Hence the total BRS charge $\delta + \bar{\delta}$ is nilpotent on-shell (which is all that is required). Note that equations (3.32) are the usual BRS transformations. ($\delta\bar{c}$ and $\bar{\delta}c$ are often set to zero which is true on shell, but off shell they follow from nilpotency of the full BRS operator).

We have shown how to gauge fix the Nambu path integral (2.8) to a free theory (on a sphere) and verified the quantum equivalence to the Polyakov formalism up to anomalies. Now that we have fully gauge fixed we may check to see whether our gauge fixing steps were anomalous. Of course, because (3.8, 3.32) are just the usual gauged fixed action and BRS symmetries, one finds the well-known anomalies in the left and right moving BRS symmetries which disappear if and only if $D = 26$.[3] Naturally, our interpretation of these BRS anomalies is that the diffeomorphism invariance of the Nambu action becomes anomalous on quantization unless $D = 26$. As is well known, in gauge fixing the Polyakov action, it is possible to regard the BRS anomalies by a suitable addition of local counterterms which depend on the auxiliary metric,[3] as a loss of local Weyl invariance; Clearly such a choice is not available to us here.

## IV. General Genus and Typical Strings

So far we have only discussed the Nambu formulation for a sphere. Now let us discuss the differences that arise on a general topology. Let us assume for simplicity a closed oriented string theory. (Actually since non–orientable or open strings do not have separate chiral degrees of freedom it is not immediately clear how to generalize to these cases). Thus we consider gauge fixing the Nambu partition function on a Riemann surface with some non–zero genus. Recall first that (for the Polyakov partition function) it is no longer possible to pick the gauge $g_{ab} \rightarrow \Omega(\underline{\sigma})\delta_{ab}$ over the whole surface. (We are considering only the diffeomorphism group and ignoring the local scale degree of freedom $\Omega(\underline{\sigma})$ which is left unfixed). Instead, global restrictions leave one with an integral over some finite number of degrees of freedom: $g_{ab} \rightarrow \Omega(\underline{\sigma})\bar{g}_{ab}(\underline{\sigma};\tau)$ , $\bar{g}_{ab}(\tau)$ being some coset representative for the diffeomorphism orbits, $\tau$ being the Teichmüller parameters (strictly moduli) which label the cosets uniquely (and range over a fundamental region determined by the modular group**). It is intuitively clear how the Nambu partition function will also yield Teichmüller parameters (we will be specific later): It is no longer possible in general to choose one specific gauge for all metrics $\hat{g}_{ab}$ e.g., $\hat{g}_{++} = 0$ (c.f. below (3.3)). Instead one must integrate over a more general choice of gauge introducing some function of the Teichmüller parameters. From our discussion around (3.6) we would expect the result to coincide with the Polyakov partition function in the more general gauge. Then again it is clearly not possible in general to fix $\beta = 0$ completely (directly from (3.7) or by the identification (3.6)) introducing some further dependence on the moduli. Thus the Teichmüller parameters will arise in general partly from an integration over right–handed gauge fixing choices and partly from some remaining degrees of freedom in the gauge fixing parameter $\beta$.

Now let us introduce a specific family of gauges which in fact extracts all the Teichmüller parameter dependence at the first gauge fixing step. Every metric $\hat{g}_{ab}$ is diffeomorphic to some coset representative $\Omega(\underline{\sigma})\bar{g}_{ab}(\underline{\sigma};\tau)$. For each coset representative we can make a further change of coordinates (in the action) which diagonalizes the metric $\bar{g}_{ab}(\underline{\sigma};\tau) \rightarrow \bar{\Omega}(\underline{\sigma}';\tau)\delta_{ab}$ at the expense of altering the boundary conditions to ones that now are a function of Teichmüller parameters. This is guaranteed at any genus by the uniformization theorem.[13]

**The group of diffeomorphisms modulo those connected to the identity.

(For clarity's sake, let us detail the genus one contribution: the torus. In order to perform the path integral, we fix on some choice of coordinates, for example, we may choose to represent the torus by a square $0 \leq \sigma^1, \sigma^2 \leq 1$ with opposite sides identified.[7] We then integrate over all fields periodic in $\sigma^1, \sigma^2$ ($\sigma^1 \sim \sigma^1 + 1, \sigma^2 \sim \sigma^2 + 1$). Every metric is diffeomorphic to some unique coset representative $\Omega \bar{g}_{ab}$ where $\bar{g}_{ab}$ is constant:[7]

$$[\bar{g}_{ab}] = \begin{bmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{bmatrix} .$$

and $\tau = \tau_1 + i\tau_2$ is the (complex) Teichmüller parameter (modulus) restricted to the fundamental region $\{\tau|\ |\tau| \geq 1, \tau_2 > 0, |\tau_1| \leq \frac{1}{2}\}$. The above clearly displays the remaining metric degrees of freedom to be integrated over. One can make a change of coordinates in the action to $(z', \bar{z}')$ so that $\bar{g}_{ab}$ becomes diagonal and the boundary conditions become some (fixed) function of the Teichmüller parameters; the popular choice is such that the torus is represented by a parallelogram with corners $z' = 0, 1, \tau, 1 + \tau$, and opposite sides identified. At genus $g > 1$, the cosets are labelled by $3g - 3$ complex Teichmüller parameters and, by the uniformization theorem, one can change coordinates so that the metric is diagonal and the Riemann surface is represented by a polygon (in the upper half plane plus point at $\infty$) with sides identified in some specified way and corners given by functions of the Teichmüller parameters.)

Let $z'(\tau), \bar{z}'(\tau)$ be some specific change of coordinates that diagonalizes $\bar{g}_{ab}(\underline{\sigma}; \tau)$. Let

$$\zeta^a(\underline{\sigma}; \tau) = \frac{\partial \sigma^a}{\partial z'(\tau)}$$
$$\bar{\zeta}^a(\underline{\sigma}; \tau) = \frac{\partial \sigma^a}{\partial \bar{z}'(\tau)} ,$$

then fixing the gauge by setting $\hat{g}_{ab}(\underline{\sigma})$ to $\Omega(\underline{\sigma})\bar{g}_{ab}(\underline{\sigma}; \tau)$ for some $\Omega$ and some $\tau$ is equivalent to requiring

$$\begin{aligned} \zeta^a(\tau)\zeta^b(\tau)\hat{g}_{ab}(\underline{\sigma}) &= \hat{g}'_{++}(z', \bar{z}') = 0 \\ \text{and} \qquad \bar{\zeta}^a\bar{\zeta}^b\hat{g}_{ab} &= \hat{g}'_{--} = 0 \end{aligned} \tag{4.1}$$

for some $\tau$ (as may be seen by inverting the coordinate transformation). Our first gauge fixing is accomplished by inserting into the path integral (according to the usual Fadeev–Popov trick):

$$\text{constant} \;= \int [d\tau] \int d\mu\left(\xi^{-}\right) \mathcal{D}et\left(\frac{\delta\left(\zeta^a\zeta^b\hat{g}_{ab}\right)}{\partial\xi^{-}}\right) \delta\left[\zeta^a\zeta^b\hat{g}_{ab}\right]$$

Here $[d\tau]$ is the modular invariant measure for the Teichmüller parameters and the integral is over some fundamental region. (Modular invariance fixes the measure uniquely up to some constant). $d\mu(\xi^-)$ is the group measure for right–handed diffeomorphisms where left/right–handedness are referred to the $z'(\tau)$ coordinate system, so that, for example, a general infinitessimal diffeomorphism is given by:

$$\delta x^\mu(\underline{\sigma}) = \xi^+(\underline{\sigma})\,\zeta^a\partial_a x^\mu + \xi^-\bar{\zeta}^a\partial_a x^\mu \;.$$

Viewing our manipulations in the $z'(\tau)$ coordinate system, we are choosing the family of gauges $\hat{g}'_{++}(z'(\tau),\bar{z}'(\tau)) = 0$ which is clearly possible from the discussion about (4.1). The result, again viewed in the primed coordinate system, is of the form (3.5) except the partition function is also integrated over $\tau$. The dependence on $\tau$ appears implicitly through the choice of boundary conditions (position of corners of the polygon). Now we can choose the second gauge fixing to be $\beta^{(\prime)} = 0$, as before. This is a globally allowed choice of gauge for the right–handed diffeomorphisms because the first gauge fixing produces a partition function which is equivalent (through identification (3.6)) to the Polyakov partition function in the partial gauge $g'_{++} = 0$ (defined as in (4.1)) while this second gauge fixing is the same as setting $g'_{--} = 0$ and completes the gauge choice (4.1). Thus the final result is the same as that obtained for the Polyakov partition function: the standard partition function (3.8) defined with some standard choice of $\tau$ dependent boundary conditions and integrated over a fundamental region for $\tau$ using the modular invariant measure.

We have completed our demonstration that the string quantum theory based on the physical area action (1.1) (i.e., the actual area of the world–sheet configuration as measured in space–time) and defined by the partition function (2.8) is quantum equivalent to the Polyakov partition function. It is amusing to note that various results derived using the latter partition function now receive a direct physical interpretation. For example, the moduli (Teichmüller parameters restricted to the fundamental region) now characterize geometric properties of the world–sheet configuration in space–time (rather than the auxiliary field of the Polyakov formalism), while the integrand—in the integral over moduli—is a measure of the probability (for the euclidean path integral, amplitude for Minkowski space) of obtaining configurations with these values of the moduli. Thus at genus one the fact that the integrand is non-zero everywhere and diverges only for $\tau_2 \to \infty$†† (due to the tachyon) implies that no range of moduli leads to world-sheets that cannot be embedded in 26 euclidean dimensions while tori for which one direction is infinitely longer than the other are infinitely more probable than any other configuration. (The first conclusion is expected from geometry: Whitney's isometric embedding theorem guarantees the existence of an embedding $(x^\mu(\underline{\sigma}))$ for each choice of metric $\hat{g}_{ab}(\underline{\sigma})$ in four or greater euclidean dimensions. It would thus be interesting if sense could be made of the Nambu formalism in small numbers of dimensions). The fact that the partition function (at any genus) has string tension dependence:

$$Z \propto T^{13}$$

(by dimensions, also explicitly [7]) allows one to deduce the probability measure as a function of physical area $A$ of the configuration. Writing the Nambu partition function so as to explicitly display the string tension dependence:

$$Z_N(T) = \int d\mu_N(x^\mu) e^{-TA[x]}$$

where $A[x]$ is the area ($2\pi S_N$ as given in (1.1)), and substituting

$$1 = \int dA d\gamma/2\pi \exp i\gamma(A - A[x])$$

one finds the probability measure for a given area $P(A)$:

$$Z_N = \int dA\ P(A)$$

$$P(A) \propto e^{-TA} \left(\frac{d}{dA}\right)^{13} \delta(A)$$

a distribution at $A = 0$.

Taken together, the above facts imply that a typical string world-sheet configuration (at least for genus one and presumably in general) is constructed from infinitely thin tubes of zero area.

††This follows for example from the explicit result [3,7] and properties of the discriminant [14].

## V. Conclusions

In this paper we have constructed and investigated a partition function for first quantized strings based on the physical area (Nambu) action (1.1). In Section II we constructed the Nambu partition function by integrating out the auxiliary metric from the Polyakov partition function. We used (implicitly) a lattice regularization. As was noted there, it is not guaranteed that the result contains the inverse exponential of the Nambu action, and that the other factors may be interpreted as the measure (2.8). We note that by construction, within the lattice regularization, the result of Section II proves equivalence between the Polyakov partition function and this Nambu partition function for any space–time dimension and, in fact, for a variety of backgrounds: the analysis is unaltered if one introduces a general spacetime metric, nor is it altered if one couples in a general antisymmetric tensor field $B_{\mu\nu}(x)$ (since this does not involve the auxiliary metric) or a constant dilaton (since this couples only to the Euler number). However, since lattice regularization is not diffeomorphism invariant, the Nambu partition function can develop an anomaly in this symmetry.

The result we obtained for the measure, in Section II, is puzzling because it does not appear to be diffeomorphism invariant. Nevertheless, in Section III we showed that this is the measure required so that on gauge fixing the Nambu partition function, the Polyakov partition function in conformal gauge with free measure (and the usual ghosts) is obtained. Our strategy for gauge fixing the Nambu path integral was first to fix right–handed diffeomorphisms. Our choice of gauge eliminated the square root in the Nambu action and produced a partition function that can already be identified with a partially gauge fixed Polyakov partition function by suitable reinterpretation of the gauge fixing Lagrange multiplier. We completed the gauge fixing by choosing a gauge that eliminated the Lagrange multiplier.

The remainder of Section III was devoted to a careful discussion of the BRS and gauge symmetries involved in this two step procedure, checking that they are preserved when required. Once the partition function was fully gauge fixed, we were able to investigate quantum anomalies in the gauge symmetries (diffeomorphisms). There is just the usual anomaly that requires $D = 26$ for flat spacetime. Outside the critical dimension the loss of gauge invariance invalidates our gauge fixing procedure. It would be very interesting if progress could be made in understanding this anomaly in the Nambu formalism more explicitly. The extension of this gauge fixing procedure to the previously mentioned backgrounds is straightforward and is valid providing they satisfy the usual conformal invariance conditions. The extension to non-constant dilaton backgrounds and other backgrounds is an interesting question not addressed by this work. Once gauge fixed, however, the couplings (and vertex operators) will clearly be those for the Polyakov formalism.

In Section IV, we generalized our analysis to general genus introducing a family of gauges for the first gauge fixing and thereby exposing the Teichmüller parameters. The final result is again equivalent to the Polyakov partition function in conformal gauge. We noted that the generalization of this gauge fixing procedure to open or non-orientable strings is not so clear.

The appearance of exactly the same results from this Nambu formulation (as have been obtained from the Polyakov formulation) in the critical dimension allows us to give a physical interpretation of the vacuum diagrams' dependence on moduli and string tension in terms of typical world-sheets (i.e., qualities of the embeddings that appear almost always in the euclidean path integral). We showed that typical world-sheets consist of infinitely thin tubes of zero area. This conclusion is in accord with expectations for a partition function where the probability of a given configuration is exponentially suppressed by the area of the world-sheet. Indeed intuitively one expects typically infinitely thin and infinitely long tubular world-sheets with zero area since these configurations are undamped, but explore the maximum configuration space.

Finally, let us note that the methods described in this paper may have applications elsewhere. It would be interesting to consider the supersymmetrization of this work (presumably via the Green-Schwarz action [3,15]). But perhaps most intriguing is the possibility that these methods might shed light on quantum theories of membranes (and higher dimensional analogues) controlled by a volume action (of the form (1.1) but with $d^2\sigma$ replaced by $d^n\sigma$ where $n = 3$ or higher). Similar to the Polyakov action one can create a classically equivalent action containing an auxiliary metric which is bilinear in derivatives. Such theories prove very difficult to quantize (see however ref. [16]). Since the metric involves $\frac{1}{2}n(n+1) \geq 6$ degrees of freedom, it would seem unlikely that an extension of the analysis of Section II to these cases would show that the auxiliary metric actions are quantum equivalent to the volume actions (based on (1.1)). On the other hand, progress might be made on a covariant gauge fixing of the volume action since for example the partial gauge choice

$$\sqrt{\hat{g}} - \frac{1}{n}\,\partial_a x^\mu \partial_a x_\mu = 0$$

turns the square root into a free lagrangian and leaves $(n-1)$ further gauge choices to use to eliminate, or form manageable, constraints.

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