



An Exact Solution for a Thick Domain Wall in General Relativity

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ABSTRACT

We present an exact solution of Einstein equations describing a thick, static domain wall with a scalar field potential $V(\Phi) = V_0 \cos^{2(1-n)}(\Phi/f(n))$ ($0 < n < 1$). This potential becomes approximately sine-Gordon ($n \rightarrow 0$) for $f \ll m_{Pl}$. At infinity, density and pressure vanish and the space-time tends to the Minkowski vacuum on one side of the wall and to the Taub vacuum on the other side. Although the density and pressure are reflection symmetric about the center of the wall, the space-time metric has no reflection symmetry.

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1. Introduction

Hitherto, research on the gravitational effects of domain walls (i.e. plane symmetric scalar field configurations with a different value of the scalar field at $+\infty$ and $-\infty$) focused on infinitely thin walls^{1,2,3}. It has been shown that no infinitely thin, static walls with reflection symmetry exist in the framework of General Relativity. However, if the assumption of reflection symmetry is dropped, infinitely thin, static walls exist⁴. Non-static thin walls were shown to have a repulsive gravitational field that tends to Rindler space-time asymptotically. Due to a new proposal for a scenario of structure formation⁵ where domain walls with a thickness of the order of Mpc are assumed to arise after recombination, the gravitational effects of thick walls have become important^{6,7}, because such objects can only be traced via their gravitational interaction with photons and the matter accreting in their gravitational potential.

In this paper we present a solution of Einstein equations for a static, planar scalar field configuration. In a previous paper⁷ we discussed the general properties of thick domain walls and our main conclusion was that a static domain wall with positive scalar field potential cannot be reflection symmetric which implies that the vacuum space-time far away from the wall must be Rindler space⁸ on one side and Taub space⁹ on the other. The exact solution for a domain wall in this paper illustrates these properties and also demonstrates that the two vacuum states at infinity can be joined smoothly by the wall. A surprising feature of this solution is that the density and pressure distribution are symmetric about the central plane of the wall whereas the metric and therefore also the gravitational field experienced by a test particle is asymmetric.

2. The Solution

We are seeking solutions to Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \quad (2.1)$$

with the Ricci-tensor $R_{\mu\nu}$ and the energy momentum tensor for a scalar field Φ

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - g_{\mu\nu} \left[\frac{1}{2} g^{\rho\sigma} \partial_\rho \Phi \partial_\sigma \Phi - V(\Phi) \right]. \quad (2.2)$$

The solution describing a domain wall configuration shall be static and plane

symmetric, i.e. it shall admit Killing vectors

$$\partial_t, \quad \partial_x, \quad \partial_y, \quad x\partial_y - y\partial_x. \quad (2.3)$$

This allows one to take the metric of the form

$$ds^2 = e^{2A(z)} dt^2 - e^{2B(z)} dz^2 - e^{B(z)-A(z)} (dx^2 + dy^2). \quad (2.4)$$

where we have used the coordinate freedom to choose $g_{xx} = g_{yy} = e^{B-A}$. For a scalar field $\Phi(z)$ the energy momentum tensor reads:

$$\begin{aligned} T_t^t = T_x^x = T_y^y &= +\frac{1}{2} e^{-2B} \Phi'^2 + V(\Phi) \equiv \rho \\ T_z^z &= -\frac{1}{2} e^{-2B} \Phi'^2 + V(\Phi) \equiv -p \end{aligned} \quad (2.5)$$

and the Einstein-equations become:

$$\begin{aligned} G_t^t &= -e^{-2B} \left[4B'' - B'^2 - 2A'B' - 4A'' + 3A'^2 \right] / 4 = 8\pi G \rho \\ G_z^z &= -e^{-2B} \left[B'^2 + 2A'B' - 3A'^2 \right] / 4 = -8\pi G p \\ G_x^x = G_y^y &= -e^{-2B} \left[2B'' - B'^2 - 2A'B' + 2A'' + 3A'^2 \right] / 4 = 8\pi G \rho, \end{aligned} \quad (2.6)$$

where the prime denotes the derivative d/dz . The scalar field equation $\Phi^{;\mu}_{;\mu} - dV/d\Phi = 0$ simplifies to:

$$e^{-2B} \Phi'' - \frac{dV(\Phi)}{d\Phi} = 0. \quad (2.7)$$

From (2.6) one immediately finds that

$$A'' = -8\pi G e^{2B} V(\Phi) \quad (2.8)$$

$$A'' = B''/3. \quad (2.9)$$

Eqs. (2.7)- (2.9) are equivalent to the Einstein-equations (2.6) and are sufficient to determine the functions A , B and Φ for a given $V(\Phi)$.

Since we do not know any systematic way of solving eqs. (2.7) - (2.9) we try an ansatz for $B(z)$

$$B = -n \ln \cosh(z - z_0) - \ln K \quad (2.10)$$

($n, z_0, K = \text{const}$) which turns out to yield a reasonable scalar field potential $V(\Phi)$ and a density ρ and pressure p that vanish for $|z| \rightarrow \infty$. With $B(z)$ given by (2.10), $A(z)$ and $V(z)$ can be calculated by (2.9) and (2.8), respectively. The scalar field equation (2.7) (or equivalently one of eqs. (2.6)) yields $\Phi(z)$ and by eliminating z from $\Phi(z)$ and $V(z)$ one gets V as a function of Φ . Carrying out all these steps one finally obtains the following solution to eqs. (2.7) - (2.9):

$$e^{2A} = \frac{1}{\cosh^{2n/3}(z - z_0)} e^{-4n(z-z_0)/3} \quad (2.11)$$

$$e^{2B} = \frac{1}{K^2} \frac{1}{\cosh^{2n}(z - z_0)} \quad (2.12)$$

$$e^{B-A} = \frac{1}{K} \frac{1}{\cosh^{2n/3}(z - z_0)} e^{2n(z-z_0)/3} \quad (2.13)$$

$$\Phi - \Phi_0 = f \arcsin \left[\tanh(z - z_0) \right] \quad (2.14)$$

$$V(\Phi) = V_0 \left[\cos \left((\Phi - \Phi_0)/f \right) \right]^{2(1-n)} \quad (2.15)$$

$$f \equiv \left[\frac{n(1-n)}{12\pi G} \right]^{1/2}, \quad V_0 \equiv \frac{nK^2}{24\pi G}, \quad 0 < n < 1 \quad (2.16)$$

K, n, z_0, Φ_0 are constants. Note that we have already eliminated all integration constants that are associated with a mere rescaling of the coordinates. (This leaves us with dimensionless coordinates.) The only physically meaningful constants are then n and K . n determines the energy scale f of the scalar field as well as the power of the cos in the potential and K determines the amplitude of the potential.

The energy density ρ , the pressure p perpendicular to the wall and V as a function of z are given by:

$$\rho = (2 - n) V_0 \frac{1}{\cosh^{2(1-n)}(z - z_0)} \quad (2.17)$$

$$p = -n V_0 \frac{1}{\cosh^{2(1-n)}(z - z_0)} \quad (2.18)$$

$$V(z) = V_0 \frac{1}{\cosh^{2(1-n)}(z - z_0)} \quad (2.19)$$

The density has a single maximum at z_0 and tends to zero for $|z| \rightarrow \infty$. p is negative and has a single minimum at z_0 which agrees with the predictions that were derived from general properties of the Einstein equations for planar scalar fields in a previous paper⁷. Note that the scalar field probes only a half-period of the cos-potential, i.e. the cos in (2.15) is positive for finite z and goes to zero for $|z| \rightarrow \infty$. The width Δ of the density peak is the proper distance between some points \tilde{z} and $-\tilde{z}$ where the density is ρ_{max}/e , i.e.

$$\Delta = \int_{-\tilde{z}}^{\tilde{z}} e^B dz \quad (2.20)$$

Since $e^B \propto 1/K$ and the density maximum $\rho_{max} = nK^2(2 - n)/(24\pi G)$, the width of the peak Δ is inversely proportional to the square root of the density maximum.

Eqs. (2.17) - (2.19) show that the matter distribution is reflection symmetric about the plane $z = z_0$ whereas the metric component e^{2A} ((2.11)) is asymmetric. This also agrees with the claim in⁷ that no planar, static scalar field with reflection symmetry exists in curved space.

Next, we consider the asymptotic form of the solution. ρ , p , V vanish for $|z| \rightarrow \infty$ and the asymptotic form of the metric is

$$ds^2 = e^{-2n(z-z_0)} dt^2 - \frac{1}{K^2} e^{-2n(z-z_0)} dz^2 - (dx^2 + dy^2) \quad \text{for } z \rightarrow +\infty \quad (2.21)$$

and

$$ds^2 = e^{2n(z-z_0)/3} dt^2 - \frac{1}{K^2} e^{-2n(z-z_0)} dz^2 - e^{-4n(z-z_0)/3} (dx^2 + dy^2) \quad \text{for } z \rightarrow -\infty \quad (2.22)$$

(2.21) is a Rindler space-time⁸, i.e. Minkowski space-time in an accelerated

frame and (2.22) is the Taub vacuum⁹ which after a transformation of the z coordinate $z \rightarrow \hat{z} = e^{-4n(z-z_0)/3}(3/4n)$ and a rescaling $(\hat{t}, \hat{x}, \hat{y}) = 2\sqrt{n/3}(t, x, y)$ can be cast into the form

$$ds^2 = \hat{z}^{-1/2}(d\hat{t}^2 - d\hat{z}^2) - \hat{z}(d\hat{x}^2 + d\hat{y}^2) . \quad (2.23)$$

Thus, the vacuum space-time far away from the wall is different on the two sides of the wall although the matter variables exhibit a reflection symmetry. It was shown that this asymmetry is a general feature of static, thick domain walls but it was not clear whether a continuous connection between these different vacua is possible. This example shows that the Minkowski vacuum and the Taub vacuum can be joined smoothly by a scalar field.

Finally, we briefly reiterate the properties of trajectories of test particles moving perpendicular to the wall, as already discussed for general domain walls in⁷. The first integrals of the geodesic equations for a test particle on a curve $x^\mu = (t(\tau), z(\tau), 0, 0)$ (τ is an affine parameter along the geodesic, a dot denotes differentiation with respect to τ) are:

$$\dot{t} = \tilde{E}e^{-2A} \quad (2.24)$$

$$\dot{z}^2 = e^{-2B} \left[\tilde{E}^2 e^{-2A} - \mu^2 \right] \quad (2.25)$$

where \tilde{E} is the energy constant associated with the Killing vector ∂_t and $\mu^2 = 1, 0$ for massive and massless particles, respectively. The acceleration of the particle measured by an observer that remains at a constant distance from the wall is given by

$$\ddot{z} = e^{-2B} \left[\frac{2}{3}n\tilde{E}^2 e^{-2A} \left(1 + 2 \tanh(z - z_0) \right) - \mu^2 n \tanh(z - z_0) \right] \quad (2.26)$$

$A(z)$ is a monotonic function and $\dot{z}^2 \geq 0$ implies by eq. (2.25) that massive particles ($\mu^2 = 1$) can only move in the region $z \geq z_T$, where z_T is the single turning point

$$\tilde{E}^2 - e^{2A(z_T)} = 0 . \quad (2.27)$$

Thus, any massive particle coming from the Minkowski vacuum and moving towards the wall bounces at $z = z_T$ and is repelled back into the Minkowski vacuum. This means that any test particle is accelerated towards the Minkowski

side. Massive particles on the Taub side are attracted by the wall. For photons the possible trajectories are quite different: from (2.26) follows that massless particles ($\mu^2 = 0$) moving perpendicular to the wall feel a repulsive force on both sides of the wall since $\ddot{z} > 0$ for $\tanh(z - z_0) > -1/2$ and $\ddot{z} < 0$ for $\tanh(z - z_0) < -1/2$. However, they can penetrate the wall freely without any turning point. The interesting feature is that the equilibrium point where $\ddot{z} = 0 \Leftrightarrow \tanh(z - z_0) = -1/2$ does not coincide with the density maximum at z_0 .

Since domain walls are supposed to have a cosmological relevance in a model for structure formation⁵ we finally give the choice of parameters n, K that emulates the corresponding parameters in⁵. f is the energy scale of the scalar field which is assumed to be $\approx 10^{15} GeV$. This gives a value for $n \approx 10^{-8}$. Note that there are two possible values for n corresponding to this energy scale of Φ since eq. (2.16) has two solutions for a given f . But the other value of n would be close to one which gives by eq. (2.15) a potential that is almost a constant. For n close to zero the potential is approximately $\propto \cos^2(\Phi/f)$. The second free parameter in $V(\Phi)$, V_0 corresponds to m_ν^4 in⁵, where $m_\nu \approx 10^{-2} eV$ is the neutrino mass. Note that the pressure p is non-zero, but much smaller than V and ρ for $f \ll m_{Pl}$, whereas for a wall in Minkowski space it is always zero. Thus it is entirely due to gravitational effects.

3. Summary

We have shown that there exist solutions of Einstein equations describing a static planar domain wall with finite thickness. The density and pressure tend to zero at infinity and the scalar field has a kink-like distribution attaining different values at $z \rightarrow +\infty$ and $z \rightarrow -\infty$. A salient property is that the vacuum space-time far away from the wall is different on the two sides of the wall inspite of a reflection symmetric density distribution.

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