Fermi National Accelerator Laboratory

FERMILAB–PUB–89/174-T
August, 1989

Higher Integrals of Motion in a Perturbed k=1 SU(2) Wess-Zumino-Witten Theory

Ken-ichiro Kobayashi[1]
Department of Physics, College of Liberal Arts and Sciences
Kyoto University, Kyoto 606, Japan

Tsuneo Uematsu[2]
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL 60510, USA

Abstract

We investigate higher integrals of motion in the k=1 SU(2) Wess-Zumino-Witten(WZW) model perturbed by a certain relevant operator. While the perturbed system is a special case of a sine-Gordon theory, it is shown to the lowest order in perturbation theory that there exist extra conserved currents due to the SU(2) symmetry in the original WZW model.

---

[1]JSPS Fellow, address after September 1, 1989: Lyman Laboratory of Physics, Harvard University Cambridge, MA 02138, USA

[2]On leave of absence from Department of Physics, College of Liberal Arts and Sciences Kyoto University, Kyoto 606, Japan

In recent years, there has been much progress in two-dimensional conformal field theories(CFT's) [1,2] in the context of studying string compactifications and also statistical models of critical phenomena.

The critical points of statistical systems described by these conformal field theories correspond to the renormalization-group fixed-points in a larger set of 2-dimensional field theories. It has now become increasing interest to study 2-dimensional field theories away from critical points i.e. off-critical behavior of conformal field theories [3,4,5]. It is also suggested that integrable lattice models carry an infinite dimensional algebraic structure characteristic to CFT even away from critical points [6]. Since an infinite number of conserved quantities are important to solve the theories exactly, there has been numerous works on the study of the conserved currents in the conformal field theories away from criticality by adding some relevant perturbations to the original theories [7,15,16,17]. Actually, it has been known that there exist higher integrals of motion in perturbed conformal field theories such as minimal models of CFT, W-algebras, and sine-Gordon system [8,9,10,11].

In ref.[10], Sasaki and Yamanaka gave a general prescription for obtaining the higher integrals of motion in the quantum sine-Gordon system. Recently Eguchi and Yang [11] have studied the deformation of the Virasoro minimum models by the (1,3) operator which leads to the sine-Gordon theory and clarified the connection between the result of ref. [10] and those of Zamolodchikov's [7,8,9]. They also argued that the perturbed CFT's based on the coset construction are described by the Toda field theory.

In this paper we investigate SU(2) Wess-Zumino-Witten(WZW) model [12,13,14] with level $k = 1$ and therefore with the central charge $c = 1$ perturbed by a certain relevant operator. We will show that for a special value of the β in the sine-Gordon theory, we get more higher integrals of motion than obtained by Sasaki and Yamanaka [10].

Let us consider SU(2) WZW model with $k = 1$ i.e. $c = 1$ case which is realized by a free boson ϕ compactified on a circle of radius r [18,19]. The action for this system is given by

$$S = \frac{1}{2\pi}\int d^2z \partial\phi\bar{\partial}\phi \tag{1}$$

where $\phi(z,\bar{z}) = \frac{1}{2}[\phi(z) + \bar{\phi}(\bar{z})]$.

We now take $r = 1/\sqrt{2}$ in which case there exists affine $SU(2)\times SU(2)$ symmetry. The $SU(2)\times SU(2)$ generators in our convention are given in terms of ϕ and $\bar{\phi}$ as:

$$\begin{aligned} J^+(z) &=: e^{i\sqrt{2}\phi(z)}: \qquad & \bar{J}^+(\bar{z}) &=: e^{i\sqrt{2}\bar{\phi}(\bar{z})}: \\ J^-(z) &=: e^{-i\sqrt{2}\phi(z)}: \qquad & \bar{J}^-(\bar{z}) &=: e^{-i\sqrt{2}\bar{\phi}(\bar{z})}: \\ J^3(z) &= \frac{i}{\sqrt{2}}\partial\phi(z) \qquad & \bar{J}^3(\bar{z}) &= \frac{i}{\sqrt{2}}\bar{\partial}\bar{\phi}(\bar{z}) \end{aligned} \tag{2}$$

where : : means the usual normal-ordered product. The operators $J^{\pm}, J^3$ satisfy the following operator algebras:

$$\begin{aligned} J^+(z)J^-(w) &\sim (z-w)^{-2} : e^{i\sqrt{2}\phi(z) - i\sqrt{2}\phi(w)} : \\ &\sim \frac{1}{(z-w)^2} + \frac{2J^3(w)}{(z-w)} \\ J^3(z)J^{\pm}(w) &\sim \frac{\pm J^{\pm}(w)}{(z-w)} \end{aligned} \tag{3}$$

where we have used the following formula:

$$: e^{i\alpha\phi(z)} :: e^{i\beta\phi(z)} : \quad \sim (z-w)^{\alpha\beta} : e^{i\alpha\phi(z) + i\beta\phi(w)} : \tag{4}$$

Here we normalize the holomorphic scalar field ϕ as:

$$< \phi(z)\phi(w) >= -ln(z-w) \ ,$$

and we take a similar normalization for $\bar{\phi}$.

We now add to the action of this theory a relevant perturbation term:

$$\lambda \int \Phi(w,\bar{w}) d^2w \quad . \tag{6}$$

In order to investigate higher-order integrals of motion we shall adopt the argument by Zamolodchikov based on perturbation theory[7,8,9]. Correlation functions in the presence of perturbation are given by

$$< X >_\lambda = \frac{1}{Z_\lambda} \sum_{n=0}^{\infty} \int \cdots \int \frac{(-\lambda)^n}{n!} < X\Phi(y_1)\cdots\Phi(y_n) >_0 d^2y_1 \cdots d^2y_n \tag{7}$$

with $Z_\lambda =< \mathbf{1} >_\lambda$.

By taking X to be $\partial_{\bar{z}} A(z,\bar{z})$, in the lowest order of perturbation theory we get

$$\partial_{\bar{z}} A(z,\bar{z}) = \lambda \partial_{\bar{z}} \int d^2 w [A(z)\Phi(w,\bar{w})] \tag{8}$$

Let us now apply the operator product expansion:

$$A(z)\Phi(w,\bar{w}) = \sum_{k=l_{A\Phi}}^{\infty} (z-w)^k (A\Phi)_k(w,\bar{w}) \tag{9}$$

where we have introduced our notation for the product of two operators A and B as:

$$A(z)B(w) = \sum_{k=l_{AB}}^{\infty} (z-w)^k (AB)_k(w) \tag{10}$$

with $-l_{AB}$ being the sum of conformal dimensions of A and B.

By using the expression (9), equation (8) becomes

$$\partial_{\bar{z}} A = \lambda \partial_{\bar{z}} [\int d^2 w \sum_{k=1}^{-l_{A\Phi}} (z-w)^{-k} (A\Phi)_{-k}(w,\bar{w})] \tag{11}$$

Noting that

$$\partial_{\bar{z}} (z-w)^{-k} = \frac{1}{(k-1)!} \partial^{k-1} \delta(z-w)$$

we get

$$\partial_{\bar{z}} A = \lambda (A\Phi)_{-1} + \lambda \partial_z B \tag{12}$$

where

$$B = \sum_{k=2}^{-l_{A\Phi}} \frac{1}{(k-1)!} (-1)^{k-1} \partial_z{}^{k-2} (A\Phi)_{-k}(z,\bar{z}) \ .$$

Therefore, if $(A\Phi)_{-1}$ is a total derivative of the form $\partial X'$ then A is a conserved charge desity(current):

$$\partial_{\bar{z}} A = \lambda \partial_z X \ , \tag{13}$$

with $X = X' + B$, since its integral gives a conserved charge(integral of motion):

$$\frac{d}{d\bar{z}} \oint dz A(z,\bar{z}) = 0 \tag{14}$$

For a perturbation term

$$\Phi = e^{i\beta\phi}e^{i\beta\bar{\phi}} + e^{-i\beta\phi}e^{-i\beta\bar{\phi}} \tag{15}$$

the lowest-order perturbative calculation leads to the sine-Gordon equation[11]:

$$\bar{\partial}\partial\phi = \gamma \sin\beta\phi \tag{16}$$

with γ being some appropriate constant.

Now we choose the relevant perturbation for the $SU(2)$ WZW model as follows:

$$\Phi = e^{i\phi/\sqrt{2}}e^{i\bar{\phi}/\sqrt{2}} + e^{-i\phi/\sqrt{2}}e^{-i\bar{\phi}/\sqrt{2}} \tag{17}$$

This corresponds to the case $\beta = 1/\sqrt{2}$. Φ given by (17) has the conformal weight $(\frac{1}{4}, \frac{1}{4})$ and it is a conformal field which is invariant under the Z_2 transformation:ϕ , $\bar{\phi} \to -\phi$, $-\bar{\phi}$. This theory is a perturbed k=1 SU(2) WZW model and also a special case of a quantum sine-Gordon theory[10,11]. In the k=1 case, the primary fields are only 1 and $e^{\frac{i}{\sqrt{2}}\phi(z)}$, and hence the operator product can be written in terms of the descendants of those primary fields[20].

As discussed in ref.[10], this system has an infinite number of conserved quantities. But, in our particular case there exist much more conserved quantities than in the general case. For conformal weight 4 we have obtained 5 conserved currents, one of which coincides with the weight 4 conserved current found in the general case [10,11].

Before going to the study of the higher integrals of motion, let us note the following 'Leibniz' rule which turns out to be very useful:

$$((AB)_k C)_l = (A(BC)_l)_k + (-1)^k (B(AC)_{-1})_{k+l+1} \tag{18}$$

This can be easily shown by using the following relation:

$$\begin{aligned}
((AB)_k C)_l(z) &= \oint_z dy [\oint_y dx \frac{1}{(x-y)^{k+1}} A(x)B(y)] \frac{1}{(y-z)^{l+1}} C(z) \\
&= \oint dy_{|x|>|y|} \oint dx \frac{1}{(x-y)^{k+1}} \frac{1}{(y-z)^{l+1}} A(x) \sum_{k=l_{BC}}^{\infty} (y-z)^k (BC)_k(z) \\
&- \oint dy_{|x|<|y|} \oint dx \frac{1}{(x-y)^{k+1}} \frac{1}{(y-z)^{l+1}} B(y) \sum_{k=l_{AC}}^{\infty} (x-z)^k (AC)_k(z)
\end{aligned} \tag{19}$$

Now we will show there exist higher integrals of motion in our perturbed theory. We have obtained following 5 conserved charge densities with the conformal weight 4:

$$\begin{aligned}
J_4^{(++)} &= (J^+J^+)_2 = ((J^+J^+))_2 \\
J_4^{(+)} &= -2[(J^+J^3)_2 + (J^3J^+)_2] = -4((J^+J^3))_2 \\
J_4^{(0)} &= -2[(J^+J^-)_2 + (J^-J^+)_2 - 4(J^3J^3)_2] = -4[((J^+J^-))_2 - 2((J^3J^3))_2] \\
J_4^{(-)} &= -2[(J^-J^3)_2 + (J^3J^-)_2] = -4((J^-J^3))_2 \\
J_4^{(--)} &= (J^-J^-)_2 = ((J^-J^-))_2
\end{aligned} \tag{20}$$

where we have introduced the symmetrized product

$$((AB))_k \equiv \frac{1}{2}[(AB)_k + (BA)_k] \ .$$

In terms of the field ϕ, these conserved currents turn out to be

$$\begin{aligned}
J_4^{(++)}(z) &=: e^{i2\sqrt{2}\phi(z)} : \\
J_4^{(+)}(z) &=: [-\frac{i}{3\sqrt{2}}\partial^3\phi - (\partial\phi)(\partial^2\phi) + \frac{i\sqrt{2}}{3}(\partial\phi)^3](z) e^{i\sqrt{2}\phi(z)} : \\
J_4^{(0)}(z) &=: [(\partial^2\phi)^2 - \frac{2}{3}(\partial\phi)^4 - \frac{2}{3}(\partial\phi)(\partial^3\phi)](z) : \\
J_4^{(-)}(z) &=: [\frac{i}{3\sqrt{2}}\partial^3\phi - (\partial\phi)(\partial^2\phi) - \frac{i\sqrt{2}}{3}(\partial\phi)^3](z) : e^{-i\sqrt{2}\phi(z)} : \\
J_4^{(--)}(z) &=: e^{-i2\sqrt{2}\phi(z)} :
\end{aligned} \tag{21}$$

For example, the expression for $J_4^{(++)}(z)$ is obtained from

$$: e^{i\sqrt{2}\phi(z)} :: e^{i\sqrt{2}\phi(w)} :\sim (z-w)^2 : e^{i\sqrt{2}\phi(z) + i\sqrt{2}\phi(w)} : \tag{22}$$

First of all, let us show that $J_4^{(++)} =: e^{i2\sqrt{2}\phi} :$ is a conserved current under perturbation.

Since

$$: e^{i2\sqrt{2}\phi(z)} : \Phi(w,\bar{w})$$

$$= \frac{1}{(z-w)^2} : e^{i\frac{3}{\sqrt{2}}\phi(w)} :: e^{-\frac{i}{\sqrt{2}}\bar{\phi}(\bar{w})} : + \frac{1}{(z-w)} i2\sqrt{2}\partial_w\phi(w) : e^{i\frac{3}{\sqrt{2}}\phi(w)} e^{-\frac{i}{\sqrt{2}}\bar{\phi}(\bar{w})} : \qquad (23)$$

we have

$$(J_4^{(++)}\Phi)_{-1} = \frac{4}{3}\partial(e^{i\frac{3}{\sqrt{2}}\phi} e^{-\frac{i}{\sqrt{2}}\bar{\phi}}) \qquad (24)$$

and therefore $J_4^{(++)}$ is a conserved current. Possible another proof for conservation of $J_4^{(++)}$ is to show that $\bar{\partial}((J^+J^+))_2$ is proportional to a total derivative by taking into account the conformal weight of $((J^+J^+))_2$ and the dimension of the coupling constant λ following the Zamolodchikov's argument [7,8,9]. Conservation of the remaining currents can now be easily seen by noting the following expressions

$$J_4^{(++)} = (J^+J^+)_2\ ,\ J_4^{(+)} = (J^-(J^+J^+)_2)_{-1}$$

$$J_4^{(0)} = (J^-(J^-(J^+J^+)_2)_{-1})_{-1} = (J^+(J^+(J^-J^-)_2)_{-1})_{-1} \qquad (25)$$

$$J_4^{(-)} = (J^+(J^-J^-)_2)_{-1}\ ,\ J_4^{(++)} = (J^-J^-)_2$$

and using the Leibniz rule. The conservation can be demonstrated by explicit calculation using the boson representation. But more general arguement goes as follows.

If X is a conserved current or equivalently if $(X\Phi)_{-1}$ is a total derivative, then $(J^-X)_{-1}$ is also another conserved current. This can be seen in the following way. By using the Leibniz rule (18) for $k = l = -1$ we have

$$((J^-X)_{-1}\Phi)_{-1} = (J^-(X\Phi)_{-1})_{-1} - (X(J^-\Phi)_{-1})_{-1} \qquad (26)$$

Noting that in general $(A\partial B)_{-1}$ is a total derivative, the first term of (26) becomes a total derivative. And also we note

$$(J^-\Phi)_{-1} =: e^{-\frac{i}{\sqrt{2}}\phi} e^{\frac{i}{\sqrt{2}}\bar{\phi}} : \qquad (27)$$

Since $(X\Phi)_{-1}$ is a total derivative, $(Xe^{\pm\frac{i}{\sqrt{2}}\phi})_{-1}$ are also total derivatives. Therefore the second term turns out to be a total derivative. Hence eq.(26) is also a total derivative, which implies that $(J^-X)_{-1}$ is a conserved current. From this argument, starting from $J_4^{(++)}$, we can show the conservation of $J_4^{(+)}$ and $J_4^{(0)}$. In a similar manner,$J_4^{(-)}$ and $J_4^{(--)}$ are seen to be conserved currents.

What is remarkable here is that the conserved quantities (20) form a SU(2) multiplet. This is supposed to be due to a remnant of SU(2) symmetry existing in the unperturbed system.

In refs.[10,11], the energy-momentum tensor contains a Feigin-Fuchs linear term $\partial^2\phi$ in order to connect the sine-Gordon theory with the minimal theories for which $c < 1$. Their energy- momentum tensor which we denote by $\tilde{T}$ is related to our $T = -\frac{1}{2} : (\partial\phi)^2 :$ as

$$\tilde{T} = T + \frac{3}{2}\partial J^3 = -\frac{1}{2} : (\partial\phi)^2 : +i\frac{3\sqrt{2}}{4}\partial^2\phi \tag{28}$$

The weight 4 conserved quantity in the sine-Gordon theory is calculated to be

$$\tilde{T}_4 = (\tilde{T}\tilde{T})_0 =: -\frac{5}{8}(\partial^2\phi)^2 + \frac{1}{4}(\partial\phi)^4 : \tag{29}$$

up to total derivatives. This coincides with the $J_4^{(0)}$: $J^3 = 0$ component of our weight 4 multiplet up to an overall factor and total derivatives. The conserved charges obtained from $\tilde{T}_4$ and $J_4^{(0)}$ are the same up to an overall constant factor. The conserved quantities obtained in refs.[10,11] are all expressed by T and J^3 in our notation. Although we only investigated the case of weight 4, we expect that for higher conformal weights these quantities form SU(2) multiplets as well. Thus it is conjectured that there exist an infinite number of conserved quantities which contain the conserved charges of ref.[10] as a subset.

Now some comments are in order. It would be interesting to extend the present analysis to general simply-laced algebras as Eguchi and Yang studied in the case of sine-Gordon theory. Secondly, although we only studied the $k = 1$ case in this paper, it should be explored for higher levels. In particular, the $k = 2$ case is described by three fermions[20] , and the explicit computation might be carried out. Although our analysis is still at preliminary stage, it would be intriguing to study the possible connection of the extra conserved charges with spectrum structure of sine-Gordon theory. Finally, the present analysis is totally based on the lowest-order perturbation theory. Therefore it is extremely important to see how the present result will be affected by possible higher-order corrections.

Acknowledgement

One of us (T.U.) would like to thank W. A. Bardeen and the Fermilab Theory Group for their kind hospitarity. He is also grateful to P. Griffin for useful conversation on the present subject. One of us(K.K.) thanks S. K. Yang for useful discussion. This work was partially supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (#63540216 and #591160202066).

References

[1] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. **241** (1984)333.

[2] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. **52**(1984)1575.

[3] A. B. Zamolodchikov, JETP Lett.**43**(1986)730 ;Sov. J. Nucl. Phys.**46** (1987)1090.

[4] A. W. W. Ludwig and J. L. Cardy, Nucl. Phys. **B285**[FS19](1987)687.

[5] D. A. Kastor, E. J. Martinec and S. H. Shenker, Nucl. Phys.**B316**(1989)590;
E. J. Martinec, Phys. Lett. **217B**(1989)431;
C. Vafa and N. Warner, Phys. Lett. **218B**(1989)51.

[6] H. Itoyama and H. Thacker, Phys. Rev. Lett. **58**(1987)1395;Nucl. Phys. **B320**[FS](1989)541.

[7] A. B. Zamolodchikov, JETP Lett.**46**(1987)160.

[8] A. B. Zamolodchikov, Kiev preprint ITF-87-65P(1987).

[9] A. B. Zamolodchikov, 'Integrable Field Theory from Conformal Field Theory', in the Taniguchi Conference on Integrable Models(1988).

[10] R. Sasaki and I. Yamanaka, in Advanced Studies in Pure Mathematics **16** (1988)271.

[11] T. Eguchi and S. K. Yang, Phys. Lett. **224B**(1989)373.

[12] E. Witten, Commun. Math. Phys. **92**(1984)455.

[13] A. M. Polyakov and P. B. Wiegmann, Phys. Lett. **131B**(1983)223 ; **141B**(1984)223.

[14] V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. **B247**(1984)83.

[15] J. L. Cardy, Phys. Rev. Lett. **60**(1988)2709.

[16] J. Kodaira, Y. Sasaki and H. Sato, Hiroshima University preprint(HUPD-8902, 1989)

[17] P. Griffin, Fermilab preprint, FERMILAB-PUB-89/153-T(1989).

[18] P. Ginsparg, Nucl. Phys. **B295**[FS21](1988)153.

[19] R. Dijgkraaf, E. Verlinde and H. Verlinde,Commun. Math. Phys. **115**(1988)649.

[20] A. B. Zamolodchikov and V. A. Fateev, Sov. J. Nucl. Phys. **43**(1986)657.