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Quantum Gravity at Large Distances and the Cosmological Constant

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Abstract

The large-distance behaviour of quantum corrections to the gravitational field equations with a cosmological constant term is studied within the formalism of the gauge-invariant Vilkovisky - DeWitt effective action. Implications for a recently proposed Coleman - Weinberg type suppression of the cosmological constant are discussed.

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1. Introduction

The General Theory of Relativity, formulated by Einstein in 1913-1916, provides a completely satisfactory description of classical gravitation. Its predictions agree with the laboratory and astronomical data within 1% of experimental errors. In addition to the increasing precision of the standard tests, like observations of precessions of planetary orbits, measurements of the radio echo delay *etc.*, the detection of gravitational radiation should soon provide further confirmation of Einstein's theory.

A simple dimensional argument explains very clearly why any attempt to incorporate the principles of Quantum Mechanics into General Relativity must inevitably lead to a puzzle. In the quantum-relativistic system of units with $\hbar = c = 1$, the Newton constant G defines the absolute length scale equal to the Planck distance of order 10^{-33}cm. The most general form of the macroscopic gravitational action, consistent with the principles of general covariance and locality, is:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{g}(2\Lambda - R) + \text{source terms}, \tag{1.1}$$

where we neglected possible terms of non-positive length dimensions, since for the manifolds with typical length scales much larger than the Planck distance such terms are negligibly small compared to the terms proportional to the cosmological constant Λ and to the Ricci scalar R. The natural value of Λ, expected on purely dimensional grounds from quantum physics, is of order $10^{66}/\text{cm}^2$. The problem is that such a cosmological constant would yield an unacceptably large contribution to the expansion rate of the Universe, which is experimentally known to be $10^{-10}/\text{yr} \approx 10^{-28}/\text{cm}$.

The failure of this naïve estimate creates a very challenging problem for Quantum Gravity (QG). One can hope however that some quantum effects may lead to a non-trivial rearrangement of the gravitational vacuum. Such a phenomenon does indeed exist

in quantum field theory, where it is known by the name of the Coleman - Weinberg mechanism [1]. In [2] we constructed a toy model for such a mechanism in QG, demonstrating that a mild logarithmic non-locality in the effective action might lead to a dramatic suppression of the effective cosmological constant. Clearly, a deeper understanding of quantum effects in the theory of gravity is necessary to decide whether such a solution to the cosmological constant is possible.

In this article we shall follow the path integral formulation of Euclidean Quantum Gravity. We shall consider quantum corrections to the macroscopic gravitational effective action. From our point of view, the choice of Euclidean signature of the metric should be regarded as a convenient method of regularizing the infra-red divergences. This is due to the fact that the spectra of kinetic energy (Laplace) operators on a compact manifold are discrete, with the mass gap determined by global properties of the manifold, like the volume *etc.* The solutions of Euclidean field equations should be continued to the Lorentzian signature; for instance, the four-sphere of radius r yields a DeSitter universe with a Hubble constant $1/r$.

In the case of QG, the construction of the effective action is complicated by the presence of non-renormalizable ultra-violet divergences. We shall regulate these divergences by introducing a physical short-distance cut-off. We will not make any attempt to remove the cut-off dependence. We shall assume the existence of a regularization that preserves the fundamental invariance of QG with respect to general coordinate transformations. String theory provides the proof of existence for such a regularization; indeed, the square root of the inverse string tension α' acts as a physical short-distance cut-off. This string-inspired regularization can be implemented into QG, at least at the one-loop level, by introducing a physical short-distance cut-off in the heat kernel formalism.

Another problem that complicates the construction of the gravitational effective action

is that its standard definition [3] suffers from the dependence on gauge fixing with respect to general coordinate transformations. Although the S-matrix is gauge-independent, the effective field equations, which contain the most important dynamical information, depend on the specific choice of gauge. This problem was discussed in detail by Vilkovisky [4] and later, by DeWitt [5], who proposed a new definition of the effective action that ensures the full off-shell gauge-independence. We shall apply the formalism of the gauge-independent Vilkovisky - DeWitt (VD) effective action in order to study the large-volume behaviour of quantum corrections to the gravitational field equations.

This paper is organised as follows. In Section 2, we discuss the problem of gauge-dependence of the effective action, and the VD definition of the gauge-independent effective action. In Section 3, we present the calculation of the one-loop off-shell VD effective action in the case of a spherical gravitational background of arbitrary radius. This is completely sufficient in order to determine the coefficients of the cosmological term and the scalar curvature term, which dominate the effective action at large volumes. In the on-shell limit, our result for the coefficient of the term that depends logarithmically on the ultra-violet cut-off differs from previous calculations. This discrepancy can be traced back to the formerly incorrect treatment of the zero ghost modes in the presence of $SO(5)$ spherical symmetry. In Section 4, we present a detailed discussion of the large volume limit of the one-loop VD effective action, and explain the rôle of G, Λ and the ultra-violet cut-off α' as the parameters of the loop expansion in QG. We discuss the effective cosmological constant and the effective Newton constant, which enter into the one-loop gravitational field equations. We find that the effective action contains some interesting non-local terms; however, these non-localities are not strong enough to cause a significant rearrangement of the gravitational vacuum. Hence the one-loop effects do not bring the desired solution to the cosmological constant problem. A possible way out of this impasse is discussed in Section 5, together with other conclusions. The Appendix, which is of a

purely technical character, contains the derivation of the small proper-time asymptotic expansion for the heat kernels of Laplace operators on the four-sphere.

2. The Gauge-Independent Effective Action

In this section, we give a brief review of the Vilkovisky – DeWitt (VD) version of the effective action formalism. We shall consider the quantum theory of a bosonic field φ^i. We adopt DeWitt's condensed notation [6], with the index i encompassing discrete and continuous (space-time) labels; for instance, $\varphi^i = g_{\mu\nu}(x)$ in the case of quantum gravity. The summation convention is extended to include integration over continuous labels as well .

Let $S(\varphi)$ be the classical action. In the path integral formulation, the quantum expectation values are defined as:

$$\langle \cdots \rangle \;=\; \frac{\int d\varphi \mu(\varphi)\,(\cdots)\,e^{-S(\varphi)}}{\int d\varphi \mu(\varphi)\,e^{-S(\varphi)}}, \tag{2.1}$$

where $\mu(\varphi)$ denotes the measure on the field configuration space $\mathcal{M}$. The standard effective action [3] $\Gamma(\overline{\varphi})$ is defined by the equation:

$$e^{-\Gamma(\overline{\varphi})} \;=\; \int d\varphi \mu(\varphi)\, e^{-S(\varphi)+(\varphi^i-\overline{\varphi}^i)\Gamma_{,i}(\overline{\varphi})}, \tag{2.2}$$

where $\Gamma_{,i}(\overline{\varphi}) = \delta\Gamma(\overline{\varphi})/\delta\overline{\varphi}^i$. The fundamental property of $\Gamma(\overline{\varphi})$, which justifies its name of the mean (background) field action, is that:

$$\overline{\varphi}^i_0 \;=\; \langle \varphi^i \rangle, \tag{2.3}$$

where $\overline{\varphi}_0$ is the solution of the equation:

$$\Gamma_{,i}(\overline{\varphi}_0) \;=\; 0. \tag{2.4}$$

Hence the effective action is a natural generalization of the classical action: the classical field equations derived by the variational principle from the classical action have now been replaced by the equations for quantum field expectation values, derived by varying $\Gamma(\overline{\varphi})$ with respect to the mean field $\overline{\varphi}$. This is especially useful when studying spontaneous symmetry breaking due to radiative effects [1]. The effective action formalism has, however, many other applications; for instance, it is a very efficient tool in the computations of quantum correlation functions.

The only systematic method available so far for solving the integro-differential eq.(3.2) for $\Gamma(\overline{\varphi})$, is the semiclassical approximation based on the loop expansion in the powers of small Planck constant $\hbar$. One assumes that $S(\varphi)$ is of order $\mathcal{O}(\hbar^{-1})$, when $\varphi-\overline{\varphi}\sim\mathcal{O}(1)$ and $\overline{\varphi}\sim\mathcal{O}(\hbar^{-1/2})$. Then the classical action may be expanded as $S(\varphi)=S(\overline{\varphi})+(\varphi^i-\overline{\varphi}^i)S_{,i}+\ldots$, with each term suppressed by an additional factor of $\hbar^{1/2}$. This implies that $\Gamma(\overline{\varphi})=S(\overline{\varphi})+\mathcal{O}(1)$, which allows solving eq.(3.2) iteratively, by changing the integration variables to $\varphi-\overline{\varphi}$. For instance, in the one-loop approximation the solution is:

$$\Gamma(\overline{\varphi}) \;=\; S(\overline{\varphi}) \;+\; \log\mu(\overline{\varphi}) \;-\; \tfrac{1}{2}\mathrm{Tr}\log[S_{,ij}(\overline{\varphi})] \;+\; \mathcal{O}(\hbar)\,. \tag{2.5}$$

Consider the case of gauge theory, with the classical action $S(\varphi)$ invariant under the group $\mathcal{G}$ of local gauge transformation such that $\delta\varphi^i=R^i(\varphi)$. Notice, however, that the argument of the exponent on the *r.h.s.* of eq.(3.2) is *not* gauge invariant, due to the presence of the term $(\varphi^i-\overline{\varphi}^i)\Gamma_{,i}(\overline{\varphi})$. Therefore, as emphasized by Fradkin and Tseytlin [7], the functional integration over φ does not *a priori* necessitate gauge fixing. Without gauge fixing, however, the resulting loop expansion would be divergent in the on-shell limit $\Gamma_{,i}(\overline{\varphi})=0$. This is the reason why the usual procedure of performing the integration of eq.(3.2) involves first imposing the gauge fixing condition $\chi^i(\varphi)=0$, which ensures the finiteness of the integral in the on-shell limit. The major deficiency of such a procedure is that the resulting effective action $\Gamma(\overline{\varphi})$ is *gauge-dependent*: the definition of

eq.(3.2) involves averaging a gauge-non-invariant functional. In particular, the effective field equations are gauge-dependent [4]. The popular choices of background covariant gauges $\chi(\varphi,\overline{\varphi})=0$ [8] cure this problem only partially; in this case, $\Gamma(\overline{\varphi})$ is invariant under the background gauge transformations $\delta\overline{\varphi}^i=R^i(\overline{\varphi})$, nevertheless it depends on the choice of the gauge fixing function χ.

From the geometric point of view, the gauge-dependence of the standard definition of the effective action is due to the fact that the integral on the *r.h.s.* of eq.(3.2) depends on the parametrization of the space of group orbits $\mathcal{M}/\mathcal{G}$. This observation prompted Vilkovisky [4] to propose a new definition of the effective action. Suppose that $\mathcal{M}$ admits some $\mathcal{G}$-invariant metric. The Vilkovisky effective action $\tilde{\Gamma}(\overline{\varphi})$ is defined by:

$$\mathrm{e}^{-\tilde{\Gamma}(\overline{\varphi})} \;=\; \int \mathrm{d}\varphi\mu(\varphi)\,\mathrm{e}^{-S(\varphi)+\sigma^i(\varphi,\overline{\varphi})\tilde{\Gamma}_{,i}(\overline{\varphi})}, \tag{2.6}$$

where $\sigma^i(\varphi,\overline{\varphi})$ is the vector at $\overline{\varphi}$, pointing towards φ tangentially to the geodesic connecting $\overline{\varphi}$ and φ, whose length is equal to the the geodesic distance between $\overline{\varphi}$ and φ [6]. The mean field condition, analogous to eq.(3.3), reads now:

$$\langle\sigma^i(\varphi,\overline{\varphi}_0)\rangle \;=\; 0\,, \tag{2.7}$$

where $\overline{\varphi}_0$ is the solution of the effective field equation:

$$\tilde{\Gamma}_{,i}(\overline{\varphi}_0) \;=\; 0\,. \tag{2.8}$$

Note that $\sigma^i(\varphi,\overline{\varphi})$ transforms as a vector at $\overline{\varphi}$ and as a scalar at φ, therefore the integrand on the *r.h.s.* of eq.(3.6) is invariant under the quantum gauge transformations. This makes gauge fixing necessary even off-shell, as contrasted with the previous definition of eq.(3.2).

A covariant version of the loop expansion follows from the covariant Taylor expansion of the classical action: $S(\varphi)=S(\overline{\varphi})+\sigma^i(\varphi,\overline{\varphi})S_{,i}+\ldots$, where the derivatives are covariant with respect to the connection on $\mathcal{M}/\mathcal{G}$. The resulting perturbative series for $\tilde{\Gamma}(\overline{\varphi})$ is

invariant under the background gauge transformations and does *not* depend on the choice of the gauge fixing condition [1,7].

The Vilkovisky effective action was further analysed by DeWitt [5], who proposed some modifications of the original definition. DeWitt's definition overlaps with eq.(3.6) at the one-loop level. In this article, we do not intend to go beyond the one-loop approximation, therefore we refer to [5] for the details of DeWitt's construction.[1] We would like to point out, however, that there is a price to pay for the geometric constructions of gauge-independent effective actions à la Vilkovisky and DeWitt. As is clear from the previous discussion, $\tilde{\Gamma}(\overline{\varphi})$ depends on the choice of metric on the "supermanifold" $\mathcal{M}$ of field configurations. In the case of quantum gravity, there exists a family of gauge-invariant metrics:

$$\gamma_{ij}d\varphi^i d\varphi^j = \int d^4x\sqrt{g}\,(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} + a\,g^{\mu\nu}g^{\alpha\beta})\,h_{\mu\nu}(x)h_{\alpha\beta}(x)\,, \tag{2.9}$$

labelled by one real parameter a, whose "naturalness" has been advocated in many places [4,6], particularly for $a = -1$. One should bear in mind, however, that other metrics cannot be a priori excluded. In the VD effective action formalism, the dynamics of gravitational fields depend on the supermanifold metric.

In the next section we shall study the gravitational VD effective action, in the one-loop approximation. In order to see how quantum corrections affect the gravitational field equations, we shall consider the simplest case of a spherical background of arbitrary radius r. This turns out to be completely sufficient to determine the coefficients of the cosmological term and the scalar curvature term, which dominate the effective action in the large volume limit. Fradkin and Tseytlin [7] have shown that in the presence of spherical symmetry, the calculation of the one-loop VD effective action simplifies enormously. The one-loop $\tilde{\Gamma}(g)$ can be evaluated from the standard effective action with the

[1] See also [9].

gauge fixing term:

$$S_f = \frac{1}{32\pi G\alpha}\int d^4x\sqrt{g}\,\nabla^\rho\tilde{h}_{\rho\mu}\nabla^\sigma\tilde{h}_\sigma{}^\mu\,, \tag{2.10}$$

where $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h^\sigma{}_\sigma$, by taking the limit of parameter $\alpha \to 0$. The resulting $\tilde{\Gamma}(g)$ corresponds to the "canonical" supermanifold metric of eq.(3.9), with $a = -1$. In the next section, we present details of this calculation.

3. The One-Loop Effective Action

The Euclidean, classical field equations with the cosmological constant term admit a spherical solution, with the radius of the sphere determined by the cosmological constant: $r^2 = 3\Lambda^{-1}$. In this section, we will evaluate the Vilkovisky - DeWitt effective action for a spherical background of arbitrary radius. As it becomes clear later on, this "minimal" off-mass-shell extension is completely sufficient to determine the leading large-distance behaviour of quantum corrections in the case of an arbitrary background. We will also clarify some subtleties which appear when the enhanced $SO(5)$ spherical symmetry is present in the background geometry.

Let $h_{\mu\nu}$ denote the fluctuations (the graviton field) around the background metric $g_{\mu\nu}$ of the four-dimensional sphere with the radius conveniently normalized to $r = 1$. The functional Taylor expansion of the classical action begins with the linear term:

$$S_1 = \frac{1}{16\pi G}\int d^4x\sqrt{g}\,(\Lambda - 3)h\,, \tag{3.1}$$

where $h = h^\sigma{}_\sigma$. The quadratic term is given by:

$$S_2 = \frac{1}{32\pi G}\int d^4x\sqrt{g}\left[\tfrac{1}{2}\tilde{h}^{\mu\nu}(-\Box - 2\Lambda + 8)h_{\mu\nu} + h^2 - \nabla^\rho\tilde{h}_{\rho\mu}\nabla^\sigma\tilde{h}_\sigma{}^\mu\right]\,, \tag{3.2}$$

where $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h$. It has been shown in [7] that for the spherical background under consideration, the one-loop Vilkovisky - DeWitt effective action can be obtained

from the conventional effective action involving the gauge-fixing term:

$$S_f = \frac{1}{32\pi G\alpha}\int d^4x\sqrt{g}\,\nabla^\rho \tilde{h}_{\rho\mu}\nabla^\sigma \tilde{h}_\sigma{}^\mu\,, \tag{3.3}$$

by taking the limit $\alpha \to 0$ at the end of the calculation. The corresponding ghost term is given by:

$$S_g = \frac{1}{32\pi G}\int d^4x\sqrt{g}\,v^*_\mu(-\Box - 3)v^\mu\,. \tag{3.4}$$

Let $h_n^{\mu\nu(i)}$ (transverse-traceless), $\xi_n^{\mu(i)}$ (transverse), and $\phi_n^{(i)}$ denote the pure spin-2, spin-1, and spin-0 eigenfunctions of the Laplacian on the background sphere, corresponding to the eigenvalues $\lambda_n^{(2)}$, $\lambda_n^{(1)}$, and $\lambda_n^{(0)}$, respectively. These functions are normalized in the standard way:

$$\delta_{ij}\delta_{nm} = \int d^4x\sqrt{g}\,h_n^{\mu\nu(i)}h_{\mu\nu}^{m(j)} = \int d^4x\sqrt{g}\,\xi_n^{\mu(i)}\xi_\mu^{m(j)} = \int d^4x\sqrt{g}\,\phi_n^{(i)}\phi_m^{(j)} \tag{3.5}$$

Since the spectrum of the Laplacian on the sphere, as well as the properties of the eigenfunctions, have already been extensively discussed in the literature [10,11], we recall here some basic facts only, related to the spherical symmetry of the background; see the Appendix for more details. The lowest eigenvalue, $\lambda_0^{(1)} = -3$, spin-1 eigenfunctions $\xi_0^{\mu(i)}$, $i = 1-10$, correspond to the ten Killing vectors which satisfy $\nabla^{(\mu}\xi_0^{\nu)} = 0$, and generate the $SO(5)$ group of rotations of the sphere. The lowest non-zero eigenvalue, $\lambda_1^{(0)} = -4$, spin-0 eigenfunctions $\phi_1^{(i)}$, $i = 1-5$, satisfy the condition $\nabla^\mu\nabla^\nu\phi_1 = -g^{\mu\nu}\phi_1$, and their gradients correspond to the five conformal Killing vectors; the zero mode ϕ_0 is constant, as usual.

Following [11] we define the symmetric tensor modes:

$$\begin{aligned}
v_n^{\mu\nu} &= [-\tfrac{1}{2}(\lambda_n^{(1)}+3)]^{-1/2}\,\nabla^{(\mu}\xi_n^{\nu)} \quad \text{for } n = 1,2,\ldots,\\
w_n^{\mu\nu} &= [\lambda_n^{(0)}(\tfrac{3}{4}\lambda_n^{(0)}+3)]^{-1/2}\,(\nabla^\mu\nabla^\nu - \tfrac{1}{4}g^{\mu\nu}\Box)\phi_n \quad \text{for } n = 2,3,\ldots,\\
z_n^{\mu\nu} &= \tfrac{1}{2}g^{\mu\nu}\phi_n \quad \text{for } n = 0,1,\ldots.
\end{aligned} \tag{3.6}$$

Together with the transverse-traceless modes $h_n^{\mu\nu}$, these form the orthonormal basis of symmetric tensors. In eqs.(3.6) and following, we suppress the indices that label degenerate sets of eigenfunctions. We define also the longitudinal vector modes:

$$l_n^\mu = [-\lambda_n^{(0)}]^{-1/2}\,\nabla^\mu\phi_n \quad \text{for } n = 1,2,\ldots, \tag{3.7}$$

which, together with the transverse modes ξ_n^μ, form the orthonormal vector basis. The graviton field can be represented now as:

$$h^{\mu\nu} = \sum_{n=0}^{\infty} a_n h_n^{\mu\nu} + \sum_{n=1}^{\infty} b_n v_n^{\mu\nu} + \sum_{n=2}^{\infty} c_n w_n^{\mu\nu} + \sum_{n=0}^{\infty} e_n z_n^{\mu\nu}. \tag{3.8}$$

The ghost field can be represented as:

$$v^\mu = \sum_{n=0}^{\infty} g_n \xi_n^\mu + \sum_{n=1}^{\infty} f_n l_n^\mu\,, \tag{3.9}$$

where the coefficients g_n and f_n are complex numbers. After substituting eq.(3.8) into eqs.(3.2) and (3.3), and integrating over the background manifold, we obtain the following expression for the quadratic part of the graviton action:

$$\begin{aligned}
64\pi G\,(S_2 + S_f) = &\sum_{n=0}^{\infty} a_n^2\,[-\lambda_n^{(2)} - 2\Lambda + 8]\\
&+ \sum_{n=1}^{\infty} b_n^2\,[\alpha^{-1}(-\lambda_n^{(1)} - 3) - 2\Lambda + 6]\\
&+ \sum_{n=2}^{\infty} c_n^2\,[\alpha^{-1}(-\tfrac{3}{2}\lambda_n^{(0)} - 6) + \tfrac{1}{2}\lambda_n^{(0)} - 2\Lambda + 6]\\
&+ \sum_{n=0}^{\infty} e_n^2\,[\tfrac{1}{2}(3 - \alpha^{-1})\lambda_n^{(0)} + 2\Lambda]\\
&+ \sum_{n=2}^{\infty} 2c_n e_n\,(\alpha^{-1} - 1)[\lambda_n^{(0)}(\tfrac{3}{4}\lambda_n^{(0)} + 3)]^{1/2}\,.
\end{aligned} \tag{3.10}$$

Similarly, the ghost action is:

$$32\pi G\,S_g = \sum_{n=0}^{\infty} g_n^* g_n\,[-\lambda_n^{(1)} - 3] + \sum_{n=1}^{\infty} f_n^* f_n\,[-\lambda_n^{(0)} - 6]\,. \tag{3.11}$$

The one-loop Vilkovisky - DeWitt effective action $\tilde{\Gamma}_1$ is given by:

$$\exp[-\tilde{\Gamma}_1] \;=\; \lim_{\alpha\to 0}\int \mathcal{D}h_{\mu\nu}\mathcal{D}v^*_\mu\mathcal{D}v_\mu\,\exp[-(S_2+S_f+S_g)]\,, \tag{3.12}$$

where the functional integration measure is [11]:

$$\begin{aligned}\mathcal{D}h_{\mu\nu}\mathcal{D}v^*_\mu\mathcal{D}v_\mu \;=\;& \prod_{n=0}^{\infty}\mathrm{d}a_n\prod_{n=1}^{\infty}\mathrm{d}b_n\prod_{n=2}^{\infty}\mathrm{d}c_n\prod_{n=0}^{\infty}\mathrm{d}e_n\\ &\times\prod_{n=1}^{\infty}\mathrm{d}g^*_n\prod_{n=1}^{\infty}\mathrm{d}g_n\prod_{n=1}^{\infty}\mathrm{d}f^*_n\prod_{n=1}^{\infty}\mathrm{d}f_n\,.\end{aligned} \tag{3.13}$$

Notice that there is *no* integration over the zero modes g^*_0 and g_0 of S_g [11]. The corresponding ghost fields are proportional to the ten Killing vectors ξ^μ_0. The infinitesimal metric fluctuations $v^{\mu\nu}_0$, which would generate rotations of the background sphere, are already absent by definition, see eqs.(3.6) and (3.8). The integration over the rotational degrees of freedom of the spherical background should be performed ***globally***, giving rise to the overall factor $[V_{SO(5)}]^{-1}$ in front of the *r.h.s.* of eq.(3.12) [12].

The quadratic forms of eqs.(3.10) and (3.11) are unbounded from below, therefore the integration of eq.(3.12) necessitates some field redefinitions and contour rotations [13,12,14]. In practice, such a procedure amounts to the replacement of every unstable real mode by a purely imaginary one; we have nothing new to say for its justification. In the limit of $\alpha = 0$, the result of integration is:

$$\begin{aligned}\exp[-\tilde{\Gamma}_1] \;=\;& [2\Lambda]^{-1/2}\,[-\tfrac{1}{2}\lambda^{(0)}_1]^{-5/2}\,[\lambda^{(0)}_1+6]^5\;\times\;[V_{SO(5)}]^{-1}\\ &\times\left\{\frac{\det_1[-\Box^{(1)}-3]\,\det_2[-\Box^{(0)}-6]}{\det_0[-\Box^{(2)}-2\Lambda+8]\,\det_2[-\Box^{(0)}-2\Lambda]}\right\}^{1/2}\,,\end{aligned} \tag{3.14}$$

Here, the subscript k in $\det_k$ means that the determinant is restricted to the subspace of functions orthogonal to the first $k-1$ modes, for instance:

$$\det{}_2[-\Box^{(0)}-6] \;=\; \prod_{n=2}^{\infty}\left[-\lambda^{(0)}_n-6\right]^{D^{(0)}_n}\,, \tag{3.15}$$

where $D^{(0)}_n$ denotes the n-th level degeneracy.

The determinants involved in eq.(3.14) contain ultra-violet divergences. We adopt here the proper-time regularization [15], with the physical short-distance cut-off α'. The regularized determinant of a non-negative self-adjoint elliptic operator Δ is defined as:

$$\det\Delta \;=\; \exp\left[-\int_{\alpha'}^{\infty}\frac{\mathrm{d}t}{t}\,K(t)\right]\,, \tag{3.16}$$

with the heat kernel $K(t)$ given by:

$$K(t) \;=\; \sum_{n=0}^{\infty}{}' D_n\,\exp(-\lambda_n t)\,, \tag{3.17}$$

where the prime means that summation runs over the ***positive*** (D_n-fold degenerate) eigenvalues (λ_n) only. The heat kernels involved in the calculation of $\tilde{\Gamma}$ correspond to the operators of the form $\Delta = -\Box + X$, where X is a c-number. In this formalism, the contribution to the determinant from a single mode of eigenvalue $\lambda \ll 1/\alpha'$ is:

$$\exp\left[-\int_{\alpha'}^{\infty}\frac{\mathrm{d}t}{t}\,\mathrm{e}^{-\lambda t}\right] \;=\; \exp[\mathrm{Ei}(-\alpha'\lambda)] \;=\; \alpha'\lambda\;\times\;\mathcal{O}(1)\,, \tag{3.18}$$

therefore the prefactor of eq.(3.14) should be rewritten as $[2\alpha'\Lambda]^{-1/2}\ldots\times\mathcal{O}(1)$, *i.e.* with each eigenvalue multiplied by α' to form a dimensionless quantity.

This is the right place to discuss the asymptotic behaviour of heat kernels associated with the Laplace operator. Consider the heat kernel $K_0(t)$ of the operator $-\Box^{(0)}$, acting on the space of scalar functions on a four-dimensional background manifold M: $K_0(t) = \sum_{n=1}^{\infty} D^{(0)}_n\exp(\lambda^{(0)}_n t)$. The large proper-time (IR) asymptotics of this heat kernel is determined by the lowest eigenvalue $-\lambda^{(0)}_1$ of $-\Box^{(0)}$:

$$K_0(t) \;\sim\; D^{(0)}_1\exp(\lambda^{(0)}_1 t) \;+\;\ldots \qquad (t\to\infty)\,. \tag{3.19}$$

Although we are primarily interested in the case of a spherical background, it may be useful to point out that there exist some very good estimates of $-\lambda^{(0)}_1$ in more general

cases. For example, it can be shown [16] that for a manifold M with $\partial M = \emptyset$ and non-negative Ricci curvature:

$$\frac{\pi^2}{4d^2} \leq -\lambda_1^{(0)} \leq \frac{4\pi^2}{d^2}, \tag{3.20}$$

where d is the diameter of M, *i.e.* the diameter of the largest geodesic ball inscribed in M. In the case of a sphere of radius r, $d = \pi r$, and the lowest eigenvalue $-\lambda_1^{(0)} = 4/r^2$ saturates the upper bound of eq.(3.20).

The small proper-time (UV) behaviour of the heat kernel is governed by the asymptotic expansion [17]:

$$K_0(t) \sim \sum_{k=0}^{\infty} a_k^{(0)} t^{k-2} \qquad (t \to 0), \tag{3.21}$$

where the coefficients $a_k^{(0)}$ of length dimension $4-2k$ are determined by the geometry of M, for instance:

$$a_0^{(0)} = \frac{1}{16\pi^2}\int d^4x\sqrt{g}, \qquad a_1^{(0)} = \frac{1}{96\pi^2}\int d^4x\sqrt{g}\,R, \qquad etc. \tag{3.22}$$

The asymptotic expansions of the form analogous to eqs.(3.21) and (3.19) apply to the heat kernels associated with the Laplace operators acting on higher spin functions as well. Only the first few UV coefficients a_k, $k = 0-2$, are known explicitly. This is sufficient to extract the ultra-violet divergent terms in the effective action, proportional to $(\alpha')^{-2}$, $(\alpha')^{-1}$, and $\log(\alpha')$, respectively. Our scope, however, goes beyond the usual procedure of extracting the coefficients of UV divergent terms - we intend to study the dependence of $\tilde{\Gamma}$ on background geometry to as great a degree of accuracy as possible.

The properties of the heat kernels $K_\sigma(t)$, corresponding to the operators $-\Box^{(\sigma)}$ acting on pure spin-σ ($\sigma = 0, 1, 2$) functions on the four-sphere, are discussed in the Appendix.

Define the function:

$$P(t) = \sum_{n=0}^{\infty}(2n+1)\,e^{-n(n+1)t}, \tag{3.23}$$

In the Appendix, we prove that:

$$K_0(t) = -\tfrac{1}{6}e^{2t}P_t - 1, \tag{3.24}$$

$$K_1(t) = -\tfrac{1}{2}e^{3t}(P_t + 2P - 2), \tag{3.25}$$

$$K_2(t) = -\tfrac{5}{6}e^{4t}(P_t + 6P - 6 - 12e^{-2t}). \tag{3.26}$$

Here, the subscript t denotes the derivative with respect to t. In eq.(3.24), the zero mode has been subtracted. The IR asymptotics follow directly from eq.(3.23):

$$P(t) \sim 1 + 3e^{-2t} + 5e^{-6t} + \ldots \qquad (t \to \infty). \tag{3.27}$$

The UV behaviour of heat kernels can be determined by using the following asymptotic expansion:

$$P(t) \sim \frac{1}{t} + \sum_{k=0}^{\infty} c_k t^k \qquad (t \to 0), \tag{3.28}$$

where:

$$c_k = \frac{1}{2^{k+1}(2k+1)!!} + (-1)^{k+1}\sum_{m=0}^{[k/2]}\frac{B_{2k-2m+2}}{(k-2m)!\,(2m)!\,(k-m+1)}, \tag{3.29}$$

with B_n denoting the n-th Bernoulli number ($B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$ *etc.*), so that $c_0 = \frac{1}{3}$, $c_1 = \frac{1}{15}$ *etc.* The derivation of this expansion is sketched in the Appendix. The UV asymptotics obtained in this way agree with the existing calculations [6,18] of leading short-distance coefficients. For example, eqs.(3.24) and (3.28) yield $a_0^{(0)} = \frac{1}{6}$ and $a_1^{(0)} = \frac{1}{3}$, in agreement with eq.(3.22) in the case of a sphere with unit radius, for which $R = 12$ and $\int d^4x\sqrt{g} = 8\pi^2/3$. Particular care should be taken when comparing the constant parts of asymptotic series, since they are sensitive to the zero-mode subtraction.

The calculations of determinants involve integrals of the corresponding heat kernels over the proper-time variable t, with the integration measure dt/t, see eq.(3.16). It is convenient to consider $t > 1$ and $\alpha' < t < 1$ separately, and use IR and UV asymptotic expansions in the respective regions of integration. To illustrate this procedure, we consider $\det_2[-\Box^{(0)} - 2\Lambda]$, one of the determinants involved in the evaluation of $\tilde{\Gamma}_1$, eq.(3.11). The starting point is:

$$\det_1[-\Box^{(0)} - 2\Lambda] \;=\; \exp\left[-\int_{\alpha'}^{\infty} \frac{dt}{t}\, e^{2\Lambda t} K_0(t)\right]. \tag{3.30}$$

By using eqs.(3.23-29), one obtains the following asymptotic expansions:

$$e^{2\Lambda t} K_0(t) \;\sim\; e^{2\Lambda t}\left(5e^{-4t} + 14e^{-10t} + \ldots\right) \qquad (t \to \infty), \tag{3.31}$$

$$e^{2\Lambda t} K_0(t) \;\sim\; \tfrac{1}{6} e^{2(\Lambda+1)t}\left(t^{-2} - \tfrac{1}{15} + \ldots\right) - e^{2\Lambda t} \qquad (t \to 0). \tag{3.32}$$

Notice that for $\Lambda > 2$ the integral on the *r.h.s.* of eq.(3.30) diverges at $t \to \infty$, reflecting the presence of additional negative graviton modes. We will handle this problem by first considering $\Lambda < 0$ and then, whenever necessary, analytically continuing to $\Lambda > 0$. By using the UV asymptotics of eq.(3.32), we obtain:

$$\begin{aligned}\int_{\alpha'}^{1} \frac{dt}{t}\, e^{2\Lambda t} K_0(t) \;&=\; \tfrac{1}{12}(\alpha')^{-2} \;+\; \tfrac{1}{3}(\Lambda+1)(\alpha')^{-1} \\ &\quad - \tfrac{1}{90}(30\Lambda^2 + 60\Lambda - 61)\log(\alpha' M^2) \;+\; \mathcal{O}(1),\end{aligned} \tag{3.33}$$

where the mass scale $M^2 \approx \max\{|\Lambda|, 1\}$. Notice that for $-\Lambda \gg 1$, $M^2 \approx -\Lambda$, reflecting the fact that the integral of eq.(3.33) is effectively cut off at $t \sim -1/\Lambda$. The remaining part of the integral, over $t > 1$, can be estimated by using the IR asymptotics of eq.(3.31). It turns out to be of order $\mathcal{O}(1)$, irrespectively of the value of M^2. After subtracting the contribution of the five $n = 1$ modes with $\lambda_1^{(0)} = -4$, we obtain:

$$\log\{\det_2[-\Box^{(0)} - 2\Lambda]\} \;=\; -\int_{\alpha'}^{1} \frac{dt}{t}\, e^{2\Lambda t} K_0(t) \;-\; 5\log(\alpha' M^2) \;+\; \mathcal{O}(1), \tag{3.34}$$

with the integral given by eq.(3.33). The techniques employed in the derivations of eqs.(3.33) and (3.34) allow straightforward evaluation of all remaining determinants.

The final result for the one-loop effective action, in the case of a four-sphere of arbitrary radius r, is:

$$\begin{aligned}\tilde{\Gamma}_1 \;&=\; r^4\left[-\tfrac{1}{12}(\alpha')^{-2} - \Lambda(\alpha')^{-1} + \Lambda^2\log(\alpha' M^2)\right] \\ &\quad + r^2\left[\tfrac{17}{3}(\alpha')^{-1} - 8\Lambda\log(\alpha' M^2)\right] \;+\; \tfrac{1679}{90}\log(\alpha'/r^2) \\ &\quad + \tfrac{257}{15}\log(M^2 r^2) \;+\; \tfrac{1}{2}\log(r^2\Lambda) \;+\; \mathcal{O}(1),\end{aligned} \tag{3.35}$$

where:

$$M^2 \;\approx\; \max\{|\Lambda|, 1/r^2\}. \tag{3.36}$$

This is the main result of this section. Its implications will be discussed in the next section.

We will now establish a connection with [7] and [19], which contain calculations of the ultra-violet ($\alpha' \to 0$) divergent loop corrections in Einstein's theory of general relativity with the cosmological constant term. Fradkin and Tseytlin (FT) [7] have actually calculated the UV divergent part of $\tilde{\Gamma}_1$ in the case of an arbitrary background. The terms proportional to $(\alpha')^{-2}$, $(\alpha')^{-1}$, and $\log(\alpha')$ in eq.(3.35) agree with [7], with the only exception of the constant coefficient of $\log(\alpha')$, which in our case is $\frac{1679}{90}$, while FT obtain $\frac{779}{90}$.[2] The difference between these two results is 10, which is exactly the number of zero Killing (complex) modes of the ghost action S_g. We have already pointed out before that the ghost modes proportional to the ten Killing vectors should be omitted in the functional integral, and that the integration over the rotational degrees of freedom of the spherical background yields an additional contribution to the effective action, equal to

[2] α' corresponds to the square of the short-distance cut-off in [7] and [19].

$\log[V_{SO(5)}]$. This is a finite correction of order $\mathcal{O}(1)$. In another work [19], Christensen and Duff (CD) have integrated out the quantum fluctuations around the backgrounds satisfying classical field equations, which in the case of a four-sphere fix the radius to $r^2 = 3\Lambda^{-1}$. We obtain:

$$\tilde{\Gamma}[r^2 = 3\Lambda^{-1}] \;=\; -\tfrac{3}{4}(\alpha'\Lambda)^{-2} \;+\; 8(\alpha'\Lambda)^{-1} \;+\; \tfrac{329}{90}\log(\alpha'\Lambda) \;+\; \mathcal{O}(1)\,, \tag{3.37}$$

with the coefficient of the logarithm equal $\frac{329}{90}$, as compared to $-\frac{571}{90}$ reported by CD. Once again, the difference is equal to 10, due to the fact that CD explicitly include the zero ghost modes. To summarize, the only discrepancy between our result and the existing calculations can be traced back to the formerly incorrect (with the exception of [11]) treatment of enhanced symmetry in the case of a spherical background.

4. The Large Distance Limit of Quantum Gravity

The definition of a large distance limit in Quantum Gravity seems, at first sight, a rather subtle issue owing to the fact that, in QG, the metric itself (which defines what "distance" is) is a dynamical variable. A little thought, as well as the string-theory example, teaches us of a general definition of the large distance limit as an expansion of the effective action in the number of derivatives acting on the various space-time fields. In string theory these are the background fields in which the string propagates, each derivative being multiplied by the ultra-violet string cut-off $\sqrt{\alpha'}$.

The fact that the metric is a dynamical variable means that, even for smooth "external" metrics, the functional integral runs over all, arbitrarily rough, "virtual" metrics. In a theory without UV cut-off this integration diverges due to the presence of unsuppressed contributions of rapidly fluctuating metrics, while in a string-like theory such a cut-off

makes a typical QG n-loop correction $\mathcal{O}(\alpha_G^n)$, where

$$\alpha_G = G/\alpha'. \tag{4.1}$$

We shall assume that the parameter α_G is sufficiently small, $\alpha_G \ll 1$, so that the loop expansion makes sense. In [20] we argued that the only mechanism that can possibly a QG correction is the presence of IR singularities. In such cases we expect large logarithms to develop, whose arguments depend on the ratio of the UV and IR cut-offs.

In the two previous sections we have been able to compute the leading terms of the one-loop effective action $\tilde{\Gamma}_1$ expanded in inverse powers of the radius r of a large spherical background. Since, for such backgrounds, $\partial g \propto r^{-1}g$, this is indeed a large distance expansion in the sense specified above.

The first problem that one is facing is that of converting the spherical background result (3.35) into one valid for arbitrary, yet smooth (small curvature) metrics. A priori one is facing serious ambiguities since, for example, a term proportional to r^2 can be interpreted either as $\int d^4x\sqrt{g}R$ or as $(\int d^4x\sqrt{g})^{1/2}$. Fortunately, these ambiguities can be resolved for the factors sitting in front of $\log(\alpha')$ owing to the fact that the UV divergences are exactly known from the small t-expansion of the heat kernel [eqs.(3.21,22)].

The only important remaining ambiguity concerns interpreting the r-dependence of the IR cut-off appearing under the logarithms, but the following physical argument can be used to solve it. As discussed in Section 3, the IR behaviour of $\tilde{\Gamma}_1$ depends on the low-lying eigenvalues of the Laplacian and these, in turn, depend on the global properties of the manifold {see [16] and eq.(3.20)}. It would be absurd therefore to interpret $r^4\log r$ as $\int d^4x\sqrt{g}\log R(x)$, which would imply a divergence if R is singular (or vanishing) at a single point. The only possible interpretation of such logarithms is

$$\int d^4x\sqrt{g}\cdot\log\left(\int d^4x\sqrt{g}\right) \;=\; V\log V\,, \tag{4.2}$$

i.e. a *non-local* term. With the above remarks in mind it is straightforward to convert our result (3.35) in the following form that should hold for general, smooth manifolds:

$$
\begin{aligned}
16\pi^2\,\tilde{\Gamma}_1 \;=\; & \int d^4x\sqrt{g}\,\left[-\tfrac{1}{2}(\alpha')^{-2} - 6\Lambda(\alpha')^{-1} + 6\Lambda^2\log(\alpha' M^2)\right] \\
& + \int d^4x\sqrt{g}\,R\left[\tfrac{17}{6}(\alpha')^{-1} - 4\Lambda\log(\alpha' M^2)\right] \qquad (4.3) \\
& + \; R^2\log \;\; \text{terms},
\end{aligned}
$$

where

$$
M^2 \;\equiv\; \max\{\,|\Lambda|, V^{-1/2}\,\}\,. \qquad (4.4)
$$

Let us now discuss eq.(4.3), starting from the non-logarithmic terms. The quartically divergent contribution to the cosmological constant {of order $O[(\alpha')^{-2}]$} has to be disregarded in a theory where there is a protection (*e.g.* from broken supersymmetry) of the cosmological constant making $\Lambda \ll (\alpha')^{-1}$. It is an artifact of our simplified non-supersymmetric model. By contrast, the $\Lambda(\alpha')^{-1}$ term represents a typical radiative correction to Λ/G which is, however, small under our assumption $\alpha_G \ll 1$. The same is true for the $(\alpha')^{-1}$ correction to $1/G$.

The conclusion of the above discussion is that we can limit ourselves to the logarithmic terms appearing in $\tilde{\Gamma}_1$. The coefficients of the logarithms depend on the particle content of the theory; however, in order for a loop to produce the desired large logarithms, it is necessary for the particle circulating in the loop to be lighter than the low-lying eigenstates of Laplacian, i.e. than $1/r$. Finally, the particle in question has to couple to gravitons with a coupling controlled by Λ. It is thus hard to think of particles other than the graviton itself, but maybe one should not exclude other massless particles coupled to our observable matter only via Λ. In order to allow for the latter possibility, we thus

write the relevant part of $\tilde{\Gamma}_1$ together with the tree action in the form:

$$
\begin{aligned}
\tilde{\Gamma} \;=\; S + \tilde{\Gamma}_1 \;=\; & \int d^4x\sqrt{g}\,[\,\Lambda(8\pi G)^{-1} + \beta_\Lambda\Lambda^2\log(\alpha' M^2)\,] \\
& - \int d^4x\sqrt{g}R\,[\,(16\pi G)^{-1} + \beta_G\Lambda\log(\alpha' M^2)\,]\,, \qquad (4.5)
\end{aligned}
$$

where M is defined in (4.4) and, for pure gravity,

$$
16\pi^2\beta_\Lambda \;=\; 6\,, \qquad\qquad 16\pi^2\beta_G \;=\; 4\,. \qquad (4.6)
$$

One way to re-express the above results is in terms of effective (*i.e.* distance-dependent) cosmological and Newton constants, to read:

$$
\Lambda_{eff}(8\pi G_{eff})^{-1}|_V \;=\; \Lambda(8\pi G)^{-1} + \beta_\Lambda\Lambda^2\log(\alpha' M^2)\,, \qquad (4.7)
$$

$$
(16\pi G_{eff})^{-1}|_V \;=\; (16\pi G)^{-1} + \beta_G\Lambda\log(\alpha' M^2)\,. \qquad (4.8)
$$

Equivalently,

$$
\Lambda_{eff}|_V \;=\; \Lambda\,\frac{1+\beta_\Lambda 8\pi G\Lambda\log(\alpha' M^2)}{1+\beta_G 16\pi G\Lambda\log(\alpha' M^2)}\,, \qquad (4.9)
$$

$$
(G\Lambda)_{eff}|_V \;=\; (G\Lambda)\,\frac{1+\beta_\Lambda 8\pi G\Lambda\log(\alpha' M^2)}{[\,1+\beta_G 16\pi G\Lambda\log(\alpha' M^2)\,]^2}\,. \qquad (4.10)
$$

According to these equations, the bare G and Λ are to be identified with the physical couplings at the scale α'. We can distinguish two cases (always keeping $G > 0$ and assuming $0 < \beta_\Lambda < 2\beta_G$, as in the pure gravity case):

a) $\Lambda > 0$. Both Λ_{eff} and $(G\Lambda)_{eff}$ *increase* in the IR. This result, which is in contrast with some claims in the literature [19] - and in agreement with others [21] - would imply that the inflation-driving positive cosmological constant of the early universe is *not* washed out by quantum effects.

b) $\Lambda < 0$. In this case, both Λ_{eff} and $(G\Lambda)_{eff}$ *decrease* in magnitude in the IR. Taking seriously the one-loop formulae for very large logarithms, Λ_{eff} approaches a finite fraction of Λ, while $(G\Lambda)_{eff}$ goes to zero as an inverse logarithm. It is thus G, rather than Λ, that goes to zero at large distances!

Furthermore, at least in the one-loop approximation, the running *stops* at $V \approx \Lambda^{-2}$. Beyond that, the couplings remain frozen at

$$\Lambda_{eff}|_{V\to\infty} = \Lambda \frac{1+\beta_\Lambda 8\pi G\Lambda \log(\alpha'\Lambda)}{1+\beta_G 16\pi G\Lambda \log(\alpha'\Lambda)}, \tag{4.11}$$

$$(G\Lambda)_{eff}|_{V\to\infty} = (G\Lambda) \frac{1+\beta_\Lambda 8\pi G\Lambda \log(\alpha'\Lambda)}{[1+\beta_G 16\pi G\Lambda \log(\alpha'\Lambda)]^2}. \tag{4.12}$$

Under the usual assumption, $G \ll \alpha'$, all radiative corrections remain bounded by $G\Lambda \log(G\Lambda) \ll 1$. Thus the one-loop effects seem to be inadequate for inducing the desired vacuum rearrangement.

One may be sceptical about the language of effective coupling constants in Quantum Gravity. A more rigorous procedure consists in looking at eq.(4.5) as at an effective action whose stationary points represent candidate vacua. One looks for spherical solutions of the type

$$R_{\mu\nu} = \Lambda_{ph}\, g_{\mu\nu}, \tag{4.13}$$

where Λ_{ph} represents the observed cosmological constant. Inserting this ansatz into the variation of (4.5) one finds the equation:

$$\Lambda_{ph} = \Lambda \frac{1+\beta_\Lambda 8\pi G\Lambda \log(\alpha' M_{ph}^2)}{1+\beta_G 16\pi G\Lambda \log(\alpha' M_{ph}^2)}, \tag{4.14}$$

where $M_{ph}^2 = \max\{|\Lambda|, \Lambda_{ph}\}$. One would like to have a solution with a very small $\Lambda_{ph} \ll |\Lambda|$. Unfortunately, such a solution does not exist, since in this case $M_{ph}^2 = |\Lambda|$ and the logarithms are too small. The physical cosmological constant Λ_{ph} is equal to the

asymptotic value of the effective cosmological constant, eq.(4.11), of the order of the bare cosmological constant Λ.

5. Conclusions

In this paper we studied the large distance limit of the gravitational effective action in the presence of the cosmological constant term. We have carried out a one-loop calculation within the framework of gauge-independent formalism developed by Vilkovisky and DeWitt [4,5].

One of the main motivations for the present work was to find some solid justification for a recently proposed mechanism [2] for suppressing the cosmological constant. In [2] we presented a toy model capable of quenching a negative cosmological constant (without affecting a positive one). This model was defined, in our present notations, by the effective action:

$$\tilde{\Gamma} = S + \tilde{\Gamma}_1 = S + 2\beta\Lambda^2 V \log(\alpha' V^{-1/2}) \tag{5.1}$$

Comparing the toy model with the one-loop result (4.5,6) we notice two important differences:

a) The sign of β making the toy model work ($\beta < 0$) is opposite to the sign of β_Λ which we have found to hold for pure gravity.

b) In the actual loop calculations the scale $V^{-1/2}$ is replaced by $M^2 \equiv \max\{|\Lambda|, V^{-1/2}\}$. This prevents the logarithms from growing, and leads to negligible radiative corrections.

Thus, at face value, our conclusions appear to be rather negative: the large distance quantum corrections to Einstein's action with a cosmological constant Λ are kept under control by Λ itself which plays the role of an infra-red cut-off.

Before conceding defeat, however, we should ask ourselves whether higher loop effects can change the lowest order result. We certainly expect that, as higher loops are taken into account, the physical, rather than the bare, cosmological constant will play the role of IR cut-off. This would be enough (modulo the sign question) to yield the physics of our toy model provided the coefficient of the logarithm remains $\beta\Lambda^2$. Unfortunately, when we tried to derive a self-consistent equation of the Dyson or Nambu - Jona-Lasinio type, the coefficient of the logarithms became $\beta\Lambda_{eff}^2$ (or, at best $\beta\Lambda\Lambda_{eff}$) again rendering the logarithmically enhanced terms hopelessly small.

Indeed, something radically different seems to be needed if quantum effects have to save the idea that a sort of phase transition is taking place at $\Lambda = 0$, in analogy with what happens in scalar field theory at $m^2 = 0$. We would like to devote the end of this paper to some ideas on such a phase transition.

The basic point is that the tree-level action is bounded (unbounded) from below for positive (negative) Λ for large smooth manifolds (flat, infinite space is the unbounded action configuration). Unlike the unboundedness of Einstein's action for very rapidly fluctuating conformal factors, the above one cannot be cured by short-distance modification of General Relativity or by Wick rotations: it is a bona-fide, large distance, infra-red divergence.

Classically, such infinite action configurations are irrelevant, since one is restricted to considering stationary points. However, at the quantum level, an unbounded action should lead to instabilities. Paradoxically, no sign of instability appears in our one-loop calculation: actually, IR convergence turns out to be better for $\Lambda < 0$ than for $\Lambda > 0$. At some perturbative or maybe non-perturbative level, however, the basic fact that there are *physical* runaway directions must show up.

An analogy can be drawn here with the case discussed a few years ago by Dimopoulos

and Georgi [22] of a supersymmetric theory to which one adds soft supersymmetry breaking terms of the "wrong" sign, so that the classical action is unbounded from below. As shown in [22], quantum corrections manage, in this case, to rescue the sick classical theory by making the effective action bounded from below. This phenomenon was termed by the authors as "quantum resuscitation". The outcome was a new stable vacuum sitting at a scale exponentially different from the original one.

It could very well be that something of this sort might happen to a theory with a negative cosmological constant: important quantum corrections could emerge at very large volumes endowing the theory with a new ground state far away from the classical (anti-DeSitter) vacuum. Such a vacuum could correspond to an exponentially small cosmological constant.

Unravelling the full structure and significance of a mechanism of this sort represents a big challenge, which goes far beyond the perturbative analysis of quantum gravitational corrections presented in this paper.

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Appendix

In this Appendix, we derive eqs.(3.24-26) and outline the derivation of the UV asymptotic expansion, eqs.(3.28) and (3.29), for the function $P(t)$ defined in eq.(3.23).

The heat kernels for the Laplace operators $-\Box^{(\sigma)}$, $\sigma = 0, 1, 2$, are defined as:

$$K_\sigma(t) \;=\; \sum_{n=0}^{\infty}{}' D_n^{(\sigma)} \exp[\lambda_n^{(\sigma)} t]\,, \tag{A.1}$$

where the prime means that summation runs over the positive eigenvalues only. For the four-sphere of unit radius, the eigenvalues and corresponding degeneracies are [10]:

$$-\lambda_n^{(0)} = n^2 + 3n \quad ; \quad D_n^{(0)} = \tfrac{1}{6}(n+1)(n+2)(2n+3), \tag{A.2}$$

$$-\lambda_n^{(1)} = n^2 + 5n + 3 \quad ; \quad D_n^{(1)} = \tfrac{1}{2}(n+1)(n+4)(2n+5), \tag{A.3}$$

$$-\lambda_n^{(2)} = n^2 + 7n + 8 \quad ; \quad D_n^{(2)} = \tfrac{5}{6}(n+1)(n+6)(2n+7). \tag{A.4}$$

Eqs.(3.24-26) follow from eqs.(A.1-4) after a little algebra. It is worth mentioning that the prefactor e^{2t} in the scalar heat kernel $K_0(t)$, eq.(3.24), is in agreement with [18], where it has been argued that the presence of prefactors $e^{Rt/6}$ is universal for scalar heat kernels on arbitrary backgrounds.

The derivation of the asymptotic expansion for the function $P(t)$ of eq.(3.23) is based on the Poisson summation technique. The function $P(t)$ can be written as:

$$P(t) = \sum_{n=-\infty}^{+\infty} f(n,t)\,, \quad \text{where} \quad f(y,t) = |y|\, e^{-ty(y+1)}. \tag{A.5}$$

The function $f(y,t)$ is not infinitely differentiable with respect to y; nevertheless it can be shown that the Poisson summation formula remains valid, so that:

$$P(t) = \sum_{n=-\infty}^{+\infty} \hat{f}(n,t)\,, \quad \text{where} \quad \hat{f}(x,t) = \int_{-\infty}^{+\infty} dy\, e^{2\pi i x y} f(y,t) \tag{A.6}$$

The Fourier transform $\hat{f}(x,t)$ is given by a rather lengthy formula [23] which involves some elementary functions and the error function $\mathrm{Erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty dy\, e^{-y^2}$. The arguments of the error function are of the form $z = \frac{1}{2}(\sqrt{t} \pm 2\pi i x/\sqrt{t})$. For $x = 0$, we use the Taylor series [24] representation of the error function in the neighbourhood of $t = 0$. The corresponding $\hat{f}(0,t)$ gives rise to the leading $1/t$ term in eq.(3.28), and contributes the first term to the *r.h.s.* of eq.(3.29). For $x \neq 0$, we use the asymptotic series expansion [24]:

$$\sqrt{\pi}\, z\, e^{z^2} \mathrm{Erfc}(z) \;\sim\; 1 + \sum_{m=1}^{\infty} (2m-1)!!\,(-2z^2)^{-m} \qquad (z \to \infty), \tag{A.7}$$

and subsequently Taylor-expand around $t = 0$. The resulting sum $\sum_{n\neq 0} \hat{f}(n,t)$ contributes the second term to the *r.h.s.* of eq.(3.29). It is important to remember that the UV expansion in the neighbourhood of $t = 0$ is only *asymptotic*, therefore it makes no sense in "resumming" it, as is sometimes mistakingly suggested in the literature. For t much larger than the inverse of lowest non-zero eigenvalue, the IR asymptotics of eq.(3.27) should be used.

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