Fermi National Accelerator Laboratory

FERMILAB–PUB–89/153-T
July 1989

Radial Quantization of the Ising model in the Scaling Regime, and Integrals of the Motion

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ABSTRACT

Radial Quantization of the massive Majorana fermion representation of the Ising model is developed to study the connection between Integrals of the Motion due to two-dimensional kinematics and non-critical Virasoro algebras. In the path integral approach to quantization, conserved charges arise as line integrals of fixed radius over the radial component of conserved currents. This formulation reduces to the analytic conformal field theory in the zero mass limit. Virasoro algebras constructed as bilinears of the fermion mode operators are spectrum generating with $c = \frac{1}{2}$; however they are charges with non-local associated currents densities. Virasoro charges with associated local currents densities are constructed ; they are similar to the scaling regime lattice Virasoro algebra current densities of Itoyama and Thacker constructed for the (Ising)$^2/XY$ model, however they are not spectrum generating, i.e. they have central charge $c = 0$. The Virasoro charges with local currents are imbedded in a larger algebraic structure which includes the integrals of the motion constructed by Zamolodchikov for this model. The physical origin of this algebraic structure is the conservation of the entire momentum distribution, including the 'angular momentum' associated with the Euclidean angular rotation operator. Further applications of this technology are discussed.

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

1. Introduction

The Virasoro algebra (VA) present in two-dimensional conformal field theories has powerful implications. As shown by Belavin, Polyakov, and Zamolodchikov[1] , Virasoro null vector constraints determine correlations of minimal conformal fields. Clearly the origin of the Virasoro structure is the infinite dimensional conformal symmetry of two dimensions. However, it is not necessary to have this conformal symmetry to have the Virasoro structure; this algebraic structure is not confined to the critical point. It was first noted by Itoyama and Thacker[2] that in the context of integrable lattice models[3] , the logarithm of the extended corner transfer matrix behaves as the central element L_0 of a VA . They were later able to explicitly construct a full Lattice VA for the (Ising)$^2/XY$ model, with a corresponding set of local current densities in the scaling regime.

There are two questions which immediately arise from the existence of non-critical VA structure. First, what is the connection between a non-critical VA and integrability in the usual sense, i.e. modes of the linear momentum distribution for a continuum field theory, or the transfer matrix of the lattice theory? For the Ising model it is shown in this paper that the VA's with local currents can be embedded in a larger kinematical algebra which expresses the conservation of not just modes of linear momentum , but modes of the Euclidean angular rotation operator. So in this context is becomes likely that other integrable models will have local Virasoro currents. Secondly, given the VA structure, can some of the techniques so successfully implemented at criticality to solve for the S-matrix, and for correlation functions be transferred to the non-critical cases? For the Ising model we will see that the VA's with local currents are not spectrum generating; i.e. $c = 0$. The representation theory of unitary Virasoro modules is therefore not applicable for these algebras; in particular the null vector equations do not apply. However in this paper, the $c = \frac{1}{2}$ VA's are also constructed from fermion bilinears. Their current densities are non-local and cannot be directly interpreted as being of kinematical origin. It is therefore less plausible that spectrum generating

Virasoro structure exists for other integrable models. However it is conceivable that it may still be the case, particularly in light of the Kyoto group expressions[4] of the local state probabilities of the restricted SOS models[5] in terms of unitary Virasoro characters.

Zamolodchikov has considered the deformation of conformal field theories via relevant operators. Infinite sets of conserved local densities are constructed via detailed analysis of null vector relations[6] . The existence of non-trivial integrals of the motion (IM) implies that the N-particle S-matrix factorizes into two-particle (elastic) scattering amplitudes. The two-particle S-matrix (the $\mathcal{R}$ matrix) must obey the Yang-Baxter relations for the factorization to be consistent.

Another intriguing connection between integrable systems and conformal field theory has been made recently in the context of the quantum group $\mathcal{U}(G)$ associated with the Lie algebra G. Originally introduced to describe the solution to the non-critical Toda field theories[7] , their Clebsh-Gordon coefficients (the q-$6j$ symbols) are also basis vectors of the Wess-Zumino-Witten conformal blocks[8]. Furthermore, in the context of the Sine-Gordon (SG) model, it has been argued[9] that the representations of $\mathcal{U}(su(2))$ which describe the minimal conformal series [10]also describe the restricted $\mathcal{R}$ marix of the SG model. A final example which is closest to what will be discussed below is the connection between the Temperley-Lieb algebra and $\mathcal{U}(su(2))$ of the XXZ Hamiltonian, and Virasoro representations and the Feigen-Fuchs construction of the minimal conformal series[11].

This paper considers another approach to understanding of the underlying structure behind conformal field theories and non-critical integrable systems, namely the existance and implications of a non-critical Virasoro algebra. In particular, we shall consider the scaling limit of the Ising model represented by the Majorana field $\Psi = \binom{\psi}{\bar\psi}$ with action*

* Throughout the complex coordinate $z = x_1 + ix_2 = re^{i\theta}$ with flat metric $g_{z\bar z} = \frac{1}{2}$ is used.

$$\mathcal{L} = \frac{-1}{2\pi} \int d^2z \ \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} + 2mi\bar{\psi}\psi \tag{1.1}$$

The mass term is a real energy perturbation $\epsilon = i\bar{\psi}\psi$. The mass is related to inverse temperature β and lattice spacing a as [12]

$$m = \frac{4(\beta_c - \beta)}{a} \tag{1.2}$$

where T_c is the critical temperature. Clearly as $\beta \to \beta_c$ and $a \to 0$ the mass remains finite.

In section 2 the model will be quantized via Ward identities which follow from the path integral approach, such that the analytic structure of the radial quantization at criticality is a smooth limit of the formalism. Section 3 begins by considering the non-critical VA's constructed of fermion bilinears with $c = \frac{1}{2}$. The element $L_0^+ + L_0^-$ is the generator of scale transformations, accompanied by a shift in scale factor m. The analysis leads to Virasoro IM's L^{sc}, L^{diff} and their non-commuting conjugate algebras, with conserved current densities and central charge $c = 0$. All four algebras share the same element L_0, the generator of Euclidean rotations. They are the Ising model version of the Lattice VA's found for the (Ising)^2/XY model in the scaling regime by Itoyama and Thacker [13]. In particular, the algebras L^{diff} are the generators of diffeomorphisms of the fermion rapidity cylinder, which is the continuum limit of the lattice rapidity torus. The larger symmetry algebra of IM's which contains both the VA's and the IM's found by Zamolodchikov for this model is constructed.

2. Radial Quantization

The fermion operator product expansion (OPE) $\psi_z\psi_w = 1/(z-w) + \cdots$ at criticality is equivalent to postulation of the cannonical anti-commutation relations for the massless Majorana fermions. As is well known, this OPE can be derived from the path integral formulation by demanding that the quantum theory is invariant with respect to the holomorphic or anti-holomorphic variations of the fermion. In this section the analysis is generalized to the massive case. Begin the path integral formulation by noting the action is invariant with respect to an infinite set of variations $\delta\psi = \epsilon$ and $\delta\bar{\psi} = \bar{\epsilon}$, which satisfy the linear constraints

$$\begin{aligned} &\bar{\partial}\epsilon - im\bar{\epsilon} = 0 \ , \\ &\partial\bar{\epsilon} + im\epsilon = 0 \ . \end{aligned} \tag{2.1}$$

At criticality these constraints reduce to holomorphic or anti-holomorphic variations. Let $E = \binom{\epsilon}{\bar{\epsilon}}$. A basis satisfying the constraints, which reduces to the conformal case, is given by the solutions to the equations of motion*

$$\begin{aligned}
E_n^+ &= \frac{\Gamma(n+1)}{m^n}\begin{pmatrix} e^{in\theta}I_n(2mr) \\ -ie^{i(n+1)\theta}I_{n+1}(2mr)\end{pmatrix} \underset{m=0}{\longrightarrow} \begin{pmatrix} z^n \\ 0 \end{pmatrix}, \\
E_n^- &= \frac{\Gamma(n+1)}{m^n}\begin{pmatrix} ie^{-i(n+1)\theta}I_{n+1}(2mr) \\ e^{-in\theta}I_n(2mr)\end{pmatrix} \underset{m=0}{\longrightarrow} \begin{pmatrix} 0 \\ \bar{z}^n \end{pmatrix}, \\
E_{-(n+1)}^+ &= \frac{2m^{n+1}}{\Gamma(n+1)}\begin{pmatrix} e^{-i(n+1)\theta}K_{n+1}(2mr) \\ ie^{-in\theta}K_n(2mr)\end{pmatrix} \underset{m=0}{\longrightarrow} \begin{pmatrix} z^{-(n+1)} \\ 0 \end{pmatrix}, \\
E_{-(n+1)}^- &= \frac{2m^{n+1}}{\Gamma(n+1)}\begin{pmatrix} -ie^{in\theta}K_n(2mr) \\ e^{i(n+1)\theta}K_{n+1}(2mr)\end{pmatrix} \underset{m=0}{\longrightarrow} \begin{pmatrix} 0 \\ \bar{z}^{-(n+1)} \end{pmatrix},
\end{aligned} \tag{2.2}$$

---

* As in the critical case we include solutions in the region $C - \{0\} - \{\infty\}$.

where $n = 0, 1, \cdots$; and for the antiperiodic sector,

$$E^{+}_{n-\frac{1}{2}} = \frac{\Gamma(n+\frac{1}{2})}{m^{n-\frac{1}{2}}} \begin{pmatrix} e^{i(n-\frac{1}{2})\theta} I_{n-\frac{1}{2}}(2mr) \\ -ie^{i(n+\frac{1}{2})\theta} I_{n+\frac{1}{2}}(2mr) \end{pmatrix} \underset{m=0}{\longrightarrow} \begin{pmatrix} z^{n-\frac{1}{2}} \\ 0 \end{pmatrix},$$
$$E^{-}_{n-\frac{1}{2}} = \frac{\Gamma(n+\frac{1}{2})}{m^{n-\frac{1}{2}}} \begin{pmatrix} ie^{-i(n+\frac{1}{2})\theta} I_{n+\frac{1}{2}}(2mr) \\ e^{-i(n-\frac{1}{2})\theta} I_{n-\frac{1}{2}}(2mr) \end{pmatrix} \underset{m=0}{\longrightarrow} \begin{pmatrix} 0 \\ \bar{z}^{n-\frac{1}{2}} \end{pmatrix}, \tag{2.3}$$

where $n \in Z$. The I_λ and K_n are standard modified Bessel functions. The antiperiodic sector is the mode expansion for a fermion in the presence of a background order operator located at the origin. The above functions satisfy $\partial\bar{\partial}E_\lambda = m^2 E_\lambda$, where λ is integer or half integer. Under the derivative operator ∂, $E^+_\lambda \to \lambda E^+_{\lambda-1}$ $(\lambda \neq 0)$, and $E^+_0 \to -imE^-_0$; $(\lambda+1)E^-_\lambda \to m^2 E^-_{\lambda+1}$ $(\lambda \neq -1)$, and $E^-_{-1} \to imE^+_{-1}$. Similarly for the operator $\bar{\partial}$, $E^-_\lambda \to \lambda E^-_{\lambda-1}$ $(\lambda \neq 0)$, and $E^-_0 \to imE^+_0$; $(\lambda+1)E^+_\lambda \to m^2 E^+_{\lambda+1}$ $(\lambda \neq -1)$, and $E^+_{-1} \to -imE^+_{-1}$. The functions take eigenvalues under the generator of Euclidean rotations

$$\mathcal{M} = \frac{1}{i}\frac{\partial}{\partial\theta} + \frac{1}{2}\sigma_3 \tag{2.4}$$

which are given by $\mathcal{M}E^+_\lambda = (\lambda + \frac{1}{2})E^+_\lambda$ and $\mathcal{M}E^-_\lambda = -(\lambda+\frac{1}{2})E^-_\lambda$.

At the classical level, Noether's theorem generates the currents $(J_z, J_{\bar{z}})$ satisfying $\bar{\partial}J_z + \partial J_{\bar{z}} = 0$. Let $\Psi = \binom{\psi}{\bar{\psi}}$ and it's conjugate $\Psi = \Psi^T\sigma_1$. The Noether currents are explicitly given by $J_z = \bar{\Psi}\sigma_z E$ and $J_{\bar{z}} = \bar{\Psi}\sigma_{\bar{z}} E$ for each function E, where $\sigma_z = g_{z\bar{z}}(\sigma_1 - i\sigma_2)$. The conserved charge in a given region A is the line integral $Q = \int dl \cdot j_\perp$ about A. The choice of contour is dictated by the eigenfunctions E which satisfy (2.1). Because (2.2) and (2.3) are eigenfunctions of the Euclidean angular rotation operator (2.4) (and not of the linear momentum operators) the conserved charges will have eigenvalues under (2.4) if we choose line integrals of fixed radius about the origin - $Q = \int_{-\pi}^{\pi} d\theta r J^r$, where J^r is the radial component of the current density. This expression for the charges clearly reduces to analytic contour integration about the origin in the zero mass limit.

For current densities bilinear in spinors A and B , charges are expressed as the inner product : $Q = (A, B)$, where

$$(A,B) = \frac{1}{2\pi}\int_{-\pi}^{\pi} r d\theta \bar{A} \begin{pmatrix} 0 & e^{-i\theta} \\ e^{-i\theta} & 0 \end{pmatrix} B \tag{2.5}$$

For the Noether currents given above, $Q_\lambda^\pm = \frac{1}{2}(E_{\lambda-\frac{1}{2}}^\pm, \Psi)$. The eigenfunctions (2.2) and (2.3) are orthonormal with respect to (2.5)

$$\begin{aligned}(E_\lambda^\pm, E_{\lambda'}^\pm) &= \delta_{\lambda+\lambda'},\\ (E_\lambda^+, E_{\lambda'}^-) &= 0\end{aligned} \tag{2.6}$$

That these relations are valid for arbitrary radius r is due to two Wronskian formulas for the modified Bessel functions

$$\begin{aligned}I_\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_\nu(z) &= 1/z,\\ I_\nu(z)I_{-(\nu+1)}(z) - I_{\nu+1}(z)K_{-\nu}(z) &= -2\sin(\nu)/\pi z.\end{aligned} \tag{2.7}$$

The first(second) of equations (2.7) applies to the periodic(anti-periodic) sector. These properties continue in the critical limit, where they follow for analytic contour integration from Cauchy's theorem.

The mode expansion in each sector is given as

$$\Psi = \sum_\lambda Q_\lambda^+ E_{-(\lambda+\frac{1}{2})}^+ + Q_\lambda^- E_{-(\lambda+\frac{1}{2})}^- \tag{2.8}$$

The functions $E_{-(\lambda+\frac{1}{2})}$ diverge at $r = 0$ for $\lambda \geq 0$ and at $r = \infty$ for $\lambda < 0$. However, the fermion field should be defined at every point on the complex plane, so that bilinears representing physical observables are well defined. In the quantum theory, the coefficients Q_λ are operators acting in a Hilbert space. The fermion field can be made well defined in the quantum theory by the introduction of highest weight vacuum states $|0\rangle$ and $\langle 0|$ at the points zero and infinity such that

$$\begin{aligned} Q_\lambda^{\pm}\,|0\rangle &= 0, \quad \lambda \geq 0, \\ \langle 0|\,Q_\lambda^{\pm} &= 0, \quad \lambda < 0 \end{aligned} \tag{2.9}$$

Asymptotic states are defined as fields evaluated at $r = 0$(incoming) and $r = \infty$ (outgoing) as in the critical case. The generators of translations and rotations are particular bilinear operators which have zero expectation values in the presence of the vacuum states. These operators shall be given explicitly in the next section. Radial ordering of the fermion modes for composite operators is sufficient to enforce the vanishing of these expectation values; this is clear by analogy with the critical theory. Radial ordering is also consistent with the angular symmetry of the analysis. Hence the vacuum structure is essentially equivalent to the conformal case; the addition of mass is an integrable perturbation away from the critical point.

Commutation relations follow from the assumption that $\delta\psi = E$ is an exact symmetry of the quantum theory. Consider variation of the one point function in the path integral

$$0 = \delta\langle \Psi(w, \bar{w})\rangle \ . \tag{2.10}$$

Two terms, the explicit variation of the field Ψ and the variation of the effective action, contribute. The measure is assumed invariant with respect to these symetries. The variation of the action contributes a divergence of the Noether current integrated over the punctured plane $C - \{w\}$. This has support at the boundary which is defined as the contours $C_> = (|w| + |\epsilon|)e^{i\theta}$ and $C_< = (|w| - |\epsilon|)e^{i\theta}$, for small ϵ, which excises the point w while preserving rotational symmetry (see fig. 1). Then with this choice of boundary (2.10) implies the Ward identity

$$E_\lambda^{\pm}(w, \bar{w}) = \underset{C_> - C_<}{\int_{-\pi}^{\pi} r\, d\theta}\, \langle \Psi(w, \bar{w}) J_\lambda^{\pm}(r, \theta)\rangle \ . \tag{2.11}$$

This is the integral form of Gauss's law for the current J_λ. Multiplication of (2.11) by $E_p^{\pm\,T}(w, \bar{w})$ and integration over θ_w about the origin, with measure factor to form the inner product (2.5) , results in the anti-commutation relations

$$
\begin{aligned}
\{Q^{\pm}_{\lambda}, Q^{\pm}_{\lambda'}\} &= \delta_{\lambda+\lambda'}, \\
\{Q^{\pm}_{\lambda}, Q^{\mp}_{\lambda'}\} &= 0.
\end{aligned}
\tag{2.12}
$$

Radial ordering and the Grassman property of fermions has been assumed.

Propagators follow from the anticommutation relations (2.12) and the highest weight conditions (2.9) in the periodic sector:

$$
\begin{aligned}
\langle \psi(z,\bar{z})\psi(0,0)\rangle &= m(\bar{z}/z)^{\frac{1}{2}}K_1(mr), \\
\langle \bar{\psi}(z,\bar{z})\psi(0,0)\rangle &= imK_0(mr), \\
\langle \bar{\psi}(z,\bar{z})\bar{\psi}(0,0)\rangle &= m(z/\bar{z})^{\frac{1}{2}}K_1(mr).
\end{aligned}
\tag{2.13}
$$

They are normalized with respect to the vacuum amplitude $\langle 0|\,1\,|0\rangle = 1$.

3. Virasoro Algebras and Local Current Densities

In this section the Virasoro Algebras with local current densities are derived and their properties are discussed. The most important results are that they have central charge $c = 0$ and a kinematical interpretation. Consider the normal (radial) ordered bilinear operators

$$
L^{\pm}_{n} = \frac{1}{2}\sum_{\lambda}(n-\lambda+\tfrac{1}{2}) : Q^{\pm}_{\lambda}Q^{\pm}_{(n-\lambda)} :
\tag{3.1}
$$

These are the spectrum generating Virasoro algebras with central charge $c = \frac{1}{2}$ in both the periodic ($\lambda \in Z + \frac{1}{2}$) and anti-periodic ($\lambda \in Z$) sectors. At criticality, they are contour integrals over local currents $z^{n+1}T_{zz}$ and $\bar{z}^{n+1}T_{\bar{z}\bar{z}}$. In the non-critical case, can these operators be written as $L = (\Psi, \mathcal{O}\Psi)$, where (,) is the inner product (2.5) and $\mathcal{O}$ is a local operator? If this is the case, then associated local current densities are $J_z = \bar{\Psi}\sigma_z \mathcal{O}\Psi$ and $\bar{J}_{\bar{z}} = \bar{\Psi}\sigma_{\bar{z}}\mathcal{O}\Psi$. Consider for example L_0^+ in the periodic sector. It can be rewritten as

$$L_0^+ = \frac{1}{2}(\Psi, \mathcal{M} \sum_{q=-\infty}^{\infty} Q^+_{q+\frac{1}{2}} E^+_{-(q+1)}) \tag{3.2}$$

where the definition of $Q^+_{n+\frac{1}{2}}$, and the properties of (2.4) have been applied. When the corresponding result for L_0^- is subtracted from this result, then by completeness (2.8) the sums in the bilocal expression add up to yield the fermion Ψ:

$$L_0^+ - L_0^- = \tfrac{1}{2}(\Psi, \mathcal{M}\Psi) \tag{3.3}$$

As expected, $L_0^+ - L_0^-$ is the generator of Euclidean angular rotations. It is also possible to evaluate the sum of the two operators in this way, because $m\frac{\partial}{\partial m}$ has simple properties when acting on the functions (2.2) and (2.3).

$$L_0^+ + L_0^- = \frac{1}{2}(\Psi, [r\frac{\partial}{\partial r} - m\frac{\partial}{\partial m}]\Psi) \tag{3.4}$$

This is the broken scale invariance of the model, which is still a useful symmetry for the calculation of order operator correlators[14]. The Hilbert space of fermion operators described in the previous section is graded under this operator in the sense that fermion modes with positive eigenvalue under $L_0^+ + L_0^-$ are lowering operators and annihilate the vacuum $|0\rangle$, etc. Both (3.3) and (3.4) are also valid in the anti-periodic sector. Other remaining Poincare currents $T_{\mu\nu} = \frac{1}{2}\bar{\Psi}\sigma_{(\mu}\partial_{\nu)}\Psi$ have integrals of the motion which are combinations of a Virasoro operator and a fermion bilinear which vanishes for zero mass. For example in the periodic sector

$$\frac{1}{2}\int_{-\infty}^{\infty} r d\theta (e^{i\theta}T^{\bar{z}\bar{z}} + e^{-i\theta}T^{\bar{z}z}) = -2L^+_{-1} + m^2 \sum_{q\neq 0} \frac{1}{q} Q^-_{q+\frac{1}{2}} Q^-_{1-(q+\frac{1}{2})} \tag{3.5}$$

Consider the expression (3.1) for L_n^+ where n is positive and the fermions are in the periodic ($\lambda = p + \frac{1}{2}$) sector. Each $Q^+_{n-(p+\frac{1}{2})}$ can be rewritten as an inner product (2.5) between fermion and function $E_{n-(p+1)}$. The operator $\mathcal{M}+\frac{1}{2}$ acting on $E^{+}_{n-(p+1)}$ generates the $n-p$ coefficient in the expression (3.2). Is there a local operator $\mathcal{O}$ which maps $E^{+}_{n-(q+1)}$ to $E^{+}_{-(q+1)}$? If such an operator exists then we have half of the completeness relation (2.8) in the expression for L^{+}_{n}. Only a corresponding albeit possibly non-trivial expression of the $Q^{-}_{p+\frac{1}{2}}$ modes is needed , as in the Poincare case (3.5), to apply completeness and subsequently form a local expression.

In the critical case, mapping $E^{+}_{n-(q+1)}$ to $E^{+}_{-(q+1)}$ is achieved by multiplication by z^{-n}. In the non-critical case for the periodic sector this cannot be accomplished in general because this requires the existence of a local operator which converts modified Bessel functions of type $K_{\lambda}(mr)$ to type $I_{\lambda'}(mr)$. There do however, exist operators which map functions E_{λ} to other $E_{\lambda'}$ of the same type: $\{\partial, \mathcal{Z}, \bar{\partial}, \bar{\mathcal{Z}}\}$, where

$$
\begin{aligned}
\mathcal{Z} &= \frac{1}{m^2}(\mathcal{M}-\tfrac{1}{2})\bar{\partial} \\
\bar{\mathcal{Z}} &= \frac{1}{m^2}(-\mathcal{M}-\tfrac{1}{2})\partial
\end{aligned}
\tag{3.6}
$$

Under the action of the operator $\mathcal{Z}$, $E^{+}_{\lambda} \rightarrow \lambda E^{+}_{\lambda+1}$ $(\lambda \neq -1)$, and $E^{+}_{-1} \rightarrow 0$; $m^2 E^{-}_{\lambda} \rightarrow -\lambda^2 E^{-}_{\lambda-1}$ $(\lambda \neq 0)$, and $-imE^{-}_{0} \rightarrow E^{+}_{0}$. Similarly for the operator $\bar{\mathcal{Z}}$,$E^{-}_{\lambda} \rightarrow \lambda E^{-}_{\lambda+1}$ $(\lambda \neq -1)$, and $E^{-}_{-1} \rightarrow 0$; $m^2 E^{+}_{\lambda} \rightarrow -\lambda^2 E^{+}_{\lambda-1}$ $(\lambda \neq 0)$, and $imE^{+}_{0} \rightarrow E^{-}_{0}$. These relations are valid in both periodic and anti-periodic sectors. Acting on the fermion, $[\partial, \mathcal{Z}]\Psi = \Psi$, similarly for $\bar{\partial}$ and $\bar{\mathcal{Z}}$. Note that ∂ and $\bar{\mathcal{Z}}$ do not commute unless $m=0$.

Let $\Psi = \Psi^{+} + \Psi^{-}$ where $\Psi^{+}(\Psi^{-})$ contains only the $E^{+}(E^{-})$ eigenfunctions. This is a non-local decomposition unless $m=0$. In this case the local operators σ_z and $\sigma_{\bar{z}}$ project onto chirality. The variation $\delta_{L^{+}_{n}}\Psi^{+} = [L^{+}_{n}, \Psi^{+}]$ can be expressed in the periodic sector as

$$
\begin{aligned}
\delta_{L^{+}_{n}}\Psi^{+} = & \mathcal{Z}^{n+1}\partial\Psi^{+} + \tfrac{1}{2}[\partial, \mathcal{Z}^{n+1}]\Psi^{+} \\
& + \sum_{q=0}^{n}[\tfrac{1}{2}(n+1)-(q+1)]Q^{+}_{q+\frac{1}{2}}E^{+}_{n-(q+1)} \ ,
\end{aligned}
\tag{3.7}
$$

and in the anti-periodic sector

$$\delta_{L_n^+}\Psi^+ = \mathcal{Z}^{n+1}\partial\Psi^+ + \tfrac{1}{2}[\partial, \mathcal{Z}^{n+1}]\Psi^+ \ . \tag{3.8}$$

valid for $n \geq -1$. Hence the operator $\mathcal{Z}$ behaves very much like the coordinate z. (The extra terms in (3.7) are due to the annihilation of E_{-1}^+ when acted upon by $\mathcal{Z}$.) Below, the operators (3.6) are used to solve the local Virasoro currents problem.

It is clear from the above analysis that there exist a large set of charges with local conserved current densities which are bilinear in fermion fields: $T_n = -\frac{1}{2}(\Psi, t_n\Psi)$ where the operator $t_n = f\{\partial, \bar{\partial}, \mathcal{M}, I\}$ is any smooth function of the arguments. In particular four Virasoro Algebras can be constructed which are of the form $L_n = -\frac{1}{2}(\Psi, l_n\Psi)$:

$$\begin{aligned} l_n^{sc} &= \mathcal{Z}^{n+1}\partial, \\ \bar{l}_n^{sc} &= \bar{\mathcal{Z}}^{n+1}\bar{\partial}, \\ l_n^{diff} &= (-\partial)^{n+1}\mathcal{Z}, \\ \bar{l}_n^{diff} &= (-\bar{\partial})^{n+1}\bar{\mathcal{Z}} \end{aligned} \tag{3.9}$$

The L_n^{sc} Algebras are clearly the scaling regime IM's with local charges which correspond to the critical algebras that generate spacetime diffeomorphisms. They are formally defined for $n \geq -1$ in the periodic sector because $\mathcal{Z}$ has no inverse (since $\mathcal{Z}E_{-1}^+ = 0$) and for all n in the antiperiodic sector, although the inverse of $\mathcal{Z}$ is non-local in this case. The l_n^{diff} algebras are defined for all n in both sectors, where the local inverse of $-\partial$ is $-\frac{1}{m^2}\bar{\partial}$ (similarly for $\bar{\partial}$). They are the scaling regime IM's with local currents which correspond to the critical algebras that generate momentum space diffeomorphisms. These algebras are the generators of diffeomorphisms of a complex rapidity cylinder, as will be shown below. The commutation relations are determined via the identity

$$[L_n, L_m] = -\tfrac{1}{2}(\Psi, [l_n, l_m]\Psi) \tag{3.10}$$

This implies that the central charge for these algebras is zero. The proof is by construction. Required are integration by parts properties stated as follows : For conserved charge (A, B) with corresponding conserved current densities, and operator $\mathcal{O} = \{\partial, \bar{\partial}, \mathcal{M}\}$, then $(A, \mathcal{O}B) = -(\mathcal{O}A, B)$.

First consider the periodic sector. In the periodic sector the central charge vanishes since the L_n^{sc} contain no pairs of raising or pairs of lowering operators ; i.e. neither ∂_z or $\mathcal{Z}$ can convert raising to lowering. To make this statement concrete, define operators $B, \tilde{B}$:

$$\begin{aligned} B_{p+\frac{1}{2}} &= Q^{+}_{p+\frac{1}{2}}/\Gamma(p+1), \;\; B_{-(p+\frac{1}{2})} = Q^{-}_{p+\frac{1}{2}} m^{2p+1}/i\Gamma(p+1), \\ \tilde{B}_{-(p+\frac{1}{2})} &= Q^{+}_{-(p+\frac{1}{2})}\Gamma(p+1), \;\; \tilde{B}_{p+\frac{1}{2}} = Q^{-}_{p+\frac{1}{2}} i\Gamma(p+1)/m^{2p+1} \end{aligned} \tag{3.11}$$

where $p \geq 0$. The B modes all annihilate the conformal 'in' vacuum while the $\tilde{B}$ modes all annihilate the 'out' vacuum. They have non-vanishing anti-commutation relations $\{B_{p+\frac{1}{2}}, \tilde{B}_{q+\frac{1}{2}}\} = \delta_{p+q+1}$. Define functions $F, \tilde{F}$:

$$\begin{aligned} F_p &= E^{+}_{p}/\Gamma(p+1), \;\; F_{-(p+1)} = E^{-}_{p} m^{2p+1}/i\Gamma(p+1), \\ \tilde{F}_{-(p+1)} &= E^{+}_{-(p+1)}\Gamma(p+1), \;\; \tilde{F}_{p} = E^{-}_{p+\frac{1}{2}} i\Gamma(p+1)/m^{2p+1} \end{aligned} \tag{3.12}$$

where $p \geq 0$. They have the nice properties $\partial F_n = F_{n-1}$ and $\partial \tilde{F}_n = \tilde{F}_{n-1}$ $\forall n \in Z$, and non-vanishing inner products $(F_n, \tilde{F}_p) = \delta_{n+p+1}$. The fermion can be written as

$$\Psi = \sum_p B_{(p+\frac{1}{2})}\tilde{F}_{-(p+1)} + \tilde{B}_{(p+\frac{1}{2})}F_{-(p+1)} \,. \tag{3.13}$$

The utility of this basis is observed when the expression L_n^{sc} is evaluated:

$$L_n^{sc} = \sum_q \tilde{B}_{n-(q+\frac{1}{2})} B_{(q+\frac{1}{2})} (F_{(q-n)}, l_n^{sc}\tilde{F}_{-(q+1)}) \tag{3.14}$$

Note the crucial fact that no normal ordering symbol is required since the B's($\tilde{B}$'s) are all annihilation (raising) operators. The Virasoro generators (3.14) are formally like the Virasoro generators of a holomorphic Dirac fermion *without* normal ordering. (For such a Dirac fermion , the central charge $c = 1$ is due entirely to this normal ordering.) Hence by explicit computation one finds

$$[L_n^{sc}, L_m^{sc}] = \sum_q \tilde{B}_{n+m-(q+\frac{1}{2})} B_{(q+\frac{1}{2})} (F_{(q-n-m)}, [l_n, l_m] \tilde{F}_{-(q+1)}) \tag{3.15}$$

This analysis can be reproduced for the L_n^{diff} algebras in the periodic sector. In the antiperiodic sector the central charge cancels between the Q^+ and Q^- contributions. This is most easily seen by defining

$$\bar{Q}_p^+ = m^{2p} Q_{-p}^- \pi / \Gamma^2(p + \tfrac{1}{2}) \tag{3.16}$$

The algebra L_n^{sc} is then given by

$$L_n^{sc} = L_n^+ - \tfrac{1}{2} \sum_p (-n - p + \tfrac{1}{2}) : \bar{Q}_p^+ \bar{Q}_{-n-p}^+ : \tag{3.17}$$

Hence L_n^{sc} is of the form $L_n = L_n^+ - L_{-n}^-$, where L_n^+ and L_n^- are commuting algebras with $c = \frac{1}{2}$, and the central charge cancels between the two algebras. Similar analysis shows that the central charge also vanishes for L_n^{diff} .

The rapidity cylinder appears via the integral representations of Bessel functions :

$$\begin{aligned}
e^{in\theta} I_n(2mr) &= \frac{1}{2\pi i} \int_{C_1} \frac{d\gamma}{\gamma} e^{mz\cdot\gamma} \gamma^n, \\
e^{in\theta} K_n(2mr) &= \frac{(-1)^n}{2} \int_{C_2} \frac{d\gamma}{\gamma} e^{mz\cdot\gamma} \gamma^n, \\
e^{i(n-\frac{1}{2})\theta} I_{n-\frac{1}{2}}(2mr) &= \frac{1}{2\pi i} \int_{C_3} \frac{d\gamma}{\gamma} e^{mz\cdot\gamma} \gamma^{n-\frac{1}{2}}
\end{aligned} \tag{3.18}$$

The contours of integrations over the complex variable γ are shown in fig. $2a-2c$. The contour C_1 is about the unit circle, $C_2(\theta)$ is from the origin to infinity in the direction $arg\gamma = \pi + \theta$, and $C_3(\theta)$ is about the unit circle and around the square root branch cut terminating at the origin. The contours depend upon θ as to make $z \cdot \gamma \equiv z\gamma^{-1} + \bar{z}\gamma$ damp the exponent at the origin and infinity, which are essential singularities. The rapidity cylinder is the argument of γ . In the periodic sector define two ' chiral ' fermions in terms of the $B, \tilde{B}$ modes (3.12)

$$
\begin{aligned}
\chi_P &= \sum_q B_{q+\frac{1}{2}}/(-m)^{q+1}\gamma^{q+1}, \\
\tilde{\chi}_P &= \sum_q \tilde{B}_{q+\frac{1}{2}}/(+m)^{q+1}\gamma^{q+1}.
\end{aligned}
\tag{3.19}
$$

The fermion in the periodic sector can be written as

$$
\Psi = \int\limits_{C_1} \frac{d\gamma}{\gamma} e^{mz\cdot\gamma} \begin{pmatrix} 1 \\ -i\gamma \end{pmatrix} \tilde{\chi}_P(\gamma) + \int\limits_{C_2} \frac{d\gamma}{\gamma} e^{mz\cdot\gamma} \begin{pmatrix} 1 \\ i\gamma \end{pmatrix} \chi_P(\gamma) \tag{3.20}
$$

In this representation it is clear that the L^{diff} algebras are diffeomorphisms of γ since ∂^{n+1} maps to $\gamma^{-(n+1)}$ in rapidity space. The non-critical periodic fermion is represented by a periodic, holomorphic Dirac fermion with non-standard normal ordering. In the anti-periodic sector define the ' chiral ' fermions

$$
\begin{aligned}
\chi_A^+ &= \sum_q Q_q^+ \Gamma(-n + \tfrac{1}{2}) m^{q+\frac{1}{2}}/\gamma^{q+1}, \\
\chi_A^- &= \sum_q Q_q^- \Gamma(-n + \tfrac{1}{2}) m^{q+\frac{1}{2}}/\gamma^{q+1}
\end{aligned}
\tag{3.21}
$$

In terms of these two anti-periodic massless majorana fermions, the Ising fermion can be written as

$$
\Psi = \frac{1}{2\pi i} \int\limits_{C_3} \frac{d\gamma}{\gamma} e^{mz\cdot\gamma} \begin{pmatrix} 1 \\ -i\gamma \end{pmatrix} \chi_A^+(\gamma) + \begin{pmatrix} i\gamma \\ 1 \end{pmatrix} \chi_A^-(\gamma) \tag{3.22}
$$

It is reasonable to conjecture that the the algebras (3.9) are the lattice Virasoro algebras of Itoyama and Thacker in the scaling limit, for the Ising model.

These Virasoro Algebras are clearly a subset of an Extended Algebra $E_{npq} = (\Psi, e_{npq}\Psi)$ where $e_{npq} = \partial^n \bar{\partial}^p \mathcal{M}^q$. Some interesting subsets are E_{n00} and E_{0p0}, which commute with the Hamiltonian. Their physical interpretation is that for this model, all modes of left and right light-cone momentum are conserved in a scattering process. They are the continuum analog of lattice row-row transfer matrices. Similarly, E_{00q} implies that all modes of Euclidean angular rotation operator are conserved. They commute with the logarithm of the corner transfer matrix $L_0^+ - L_0^-$. Since the integrals of motion constructed by Zamolodchikov[6] have similar kinematical interpretations they are expected to lie in the set $\{E_{npq}\}$. For this model the operators T_{2n} are given as

$$T_{2n} \cong : \partial_z^{n-1} \psi \partial_z^n \psi : \qquad (3.23)$$

and satisfy $\partial_{\bar{z}} T_{2n} = \partial_z Q_{2n-1}$. The conserved IM's of these currents are $E_{(2n-1)00}$. All of the mentioned integrals of the motion can be thought of as arising from 2-dimensional kinematics, in a general sense. Of course the differentiation between kinematical and dynamical symmetry depends upon the choice of initial hypersurface; kinematical symmetries preserve the form of the initial hypersurface. For each of the classes of IMs E_{n00}, E_{0p0} and E_{00q} hypersurfaces can clearly be selected so that they are kinematical. For general E_{npq} , construction of such a surface is not obvious, however it is probably intuitively correct to think of these as of kinematical origin likewise. The underlying structure is the simplicity of the relativistic Poincare algebra in two dimensions.

4. Discussion

The non-critical Virasoro structure with local currents is clearly imbedded in a larger algebra which is of *kinematical* origin. The factorization of the S-matrix into a product of two-particle amplitudes is probably necessary and sufficient for this algebra to appear since the larger algebra generically appears in two dimensional elastic scattering. In this context, the connection between the Sine-Gordon/ MKdV quantum IMs and the Virasoro generators[15] is of great interest.

Whether the non-critical radial quantization of section 2 can be achieved for more non-trivial models is clearly an open question. A particularly promising class are the Z_n parafermion (PF) models of Zamolodchikov and Fateev[16] which describe the antiferromagnetic critical behavior of the ABF [5] restricted SOS models. The point of such quantization is to extract as much non-trivial information from the critical point where a model can be solved via powerful conformal techniques and then to move away from criticality in a relatively simple way.

This paper has focused on the $c = 0$ Virasoro IMs with local currents. However the $c = \frac{1}{2}$ spectrum generating VAs must place constraints on correlation functions; as in the critical case, null vector constraints are non-trivial. It has been shown by McCoy and Perk [17] that the Painlevé V equation, a second order nonlinear differential equation satisfied by the Ising model two spin correlators in the scaling regime, reduce to the level two null vector equation $[L^2_{-1} - \frac{3}{4}L_{-2}]\,|\sigma\rangle = 0$, where $|\sigma\rangle$ is the scaling dimension $1/8$ conformal field. It would clearly be of interest to determine if the same null vector equations for the non-critical spectrum generating VAs yield this nonlinear equation. This is not inconsistent with the linearity of the null vector equations because $L^{\pm}_{-2}$ is non-local. Such analysis begins with the definition of the local operator product structure of the Ising system in the scaling regime, determined purely from its critical properties[18]. It should be mentioned that there already exists a beautiful formulation of spin correlation functions for the Ising model in the scaling regime due to the Kyoto group[19] based on isomonodromic deformation theory. However, even at the critical point this analysis does not generalize to the Z_n PF series because of degeneracy in monodromy between PF highest weights and parafermions.

I would like to thank H. Itoyama, H. Riggs, G. Savvidy, C. Wendt and particularly H. Thacker for enlightening conversations.

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Figure Captions

Fig. 1 : Boundary of integration $C_> - C_<$ for Noether currents.

Fig. 2 : Contours for integral representations of modified Bessel functions.

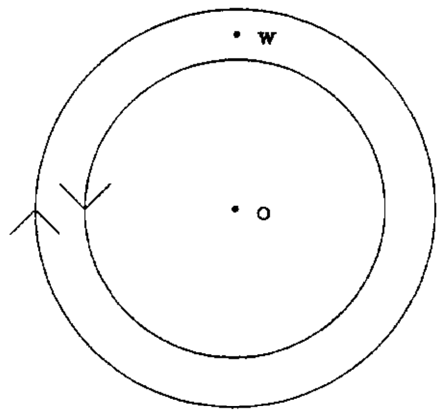

Fig. 1

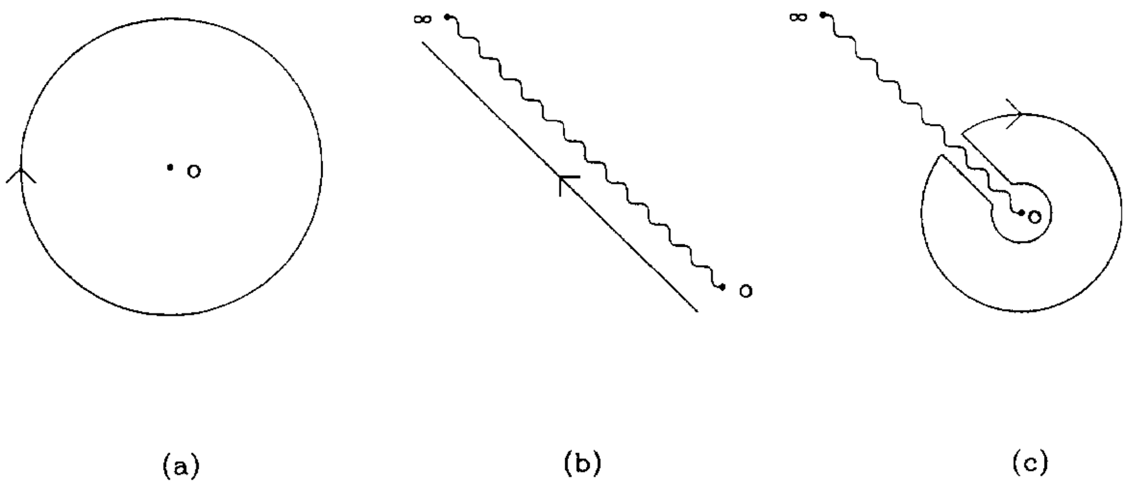

Fig. 2