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## A NEW WAY TO INTERPRET THE DIRAC EQUATION IN A NON-RIEMANNIAN MANIFOLD

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### Abstract

The idea of internal mass terms introduced in ref. [1], is shown not to be an appropriate hypothesis when it is placed in connection with the components of the generalized (matrix) vierbeins being proportional to the Riemannian (gravitational) vierbeins. It would result in an undesirable canceling of the Electromagnetic and the Yang-Mills components in the generalized metric. Another hypothesis is introduced where the wave function  $\psi$  is Taylor expanded in a small parameter  $p$ .

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# I. Introduction.

In the second part of the work entitled: “The Dirac Equation in a non-Riemannian manifold”, ref. [1], a generalized Dirac equation was obtained as an application of the geometrical properties of a tangent space local to a non-Riemannian manifold of the Einstein-Schroedinger (ES) nonsymmetric theory [2]. This generalized Dirac equation describes a spin 1/2 particle of mass  $m$ , with a wave function  $\psi(x)$ , placed in the above non-Riemannian manifold, and under the influence of an ( $n$ -dimensional) Yang-Mills field [4]. This is written as:

$$\begin{aligned}\gamma^\mu \bar{\nabla}_\mu \psi - \mu \psi &= 0 \quad , \\ \bar{\psi} \overleftarrow{\nabla}_\mu^\dagger \dot{\gamma}_\mu - \mu \bar{\psi} &= 0 \quad ,\end{aligned}\tag{1.1}$$

where,

$$\begin{aligned}\nabla_\mu &= \partial_\mu + \Delta_\mu + C_\mu + \Gamma_\mu \quad , \\ \nabla_\mu^\dagger &= \partial_\mu + \Delta_\mu - C_\mu - \Gamma_\mu \quad .\end{aligned}\tag{1.2}$$

These are the (covariant) differential operators acting on the wave function  $\psi$ , where  $\Delta_\mu$ ,  $C_\mu$ ,  $\Gamma_\mu$  are the connections for the Dirac, complex and Yang-Mills - internal spaces, respectively. In (1.1) the generalized Dirac  $\gamma$ -matrices are defined as:

$$\gamma_\mu = E_a^\mu \gamma^a \quad , \quad \dot{\gamma}^\mu = E_a^{\dagger \mu} \gamma^a \quad ,\tag{1.3}$$

where the generalized (matrix) vierbeins can be expanded in terms of the internal axes as:

$$\begin{aligned}E_a^\mu(x) &= k_{a0}^\mu \tau_0 + k_{ai}^\mu \tau_i \quad , \\ E_a^{\dagger \mu}(x) &= k_{a0}^{*\mu} \tau_0 + k_{ai}^{*\mu} \tau_i \quad , \\ i &= 1, \dots, n^2 - 1 \quad ,\end{aligned}\tag{1.4}$$

where  $n$  is the dimension of the symmetry group, here  $SU(n)$  (as in the Yang-Mills theory).

With the idea of making a possible interpretation of the theory, two suppositions were then made. One of them is that the mass term  $\mu$  in the Dirac equation also be a matrix-like term:

$$\mu = \mu_0 \tau_0 + \mu_i \tau_i \quad .\tag{1.5}$$

This has the effect splitting the Dirac equation into  $n^2$  parts, one corresponding to each of the Dirac-space axes.

The second supposition determines the amplitude of each of the (split) Dirac equations. This is done by the following definitions<sup>1</sup>:

$$\begin{aligned} k_{a0}^\mu &= k_{a0R}^\mu + ik_{a0I}^\mu \quad , \quad k_{a0I}^\mu = p\lambda n_{a0}^\mu \\ \mu_0 &= \mu_{0R} + i\mu_{0I} \quad , \quad \mu_{0I} = p\lambda m_0 \end{aligned} \quad (1.6)$$

and

$$\begin{aligned} k_{ai}^\mu &= i(p\lambda)^2 n_{ai}^\mu \quad \mu_i = i(p\lambda)^2 m_i \\ i &= 1, \dots, n^2 - 1 \end{aligned} \quad (1.7)$$

In this case  $p$  is considered a parameter, and  $\lambda$  is a constant with the (minimal) value of  $1/p$ ,

$$\lambda \sim \frac{e}{2\hbar} = 2.58 \times 10^{32} \text{ cm}^{-1} \quad , \quad (1.8)$$

where the maximal value for  $|p|$  is  $|p| = \left| \frac{-2\hbar}{e} \right| = 3.8 \times 10^{-32} \text{ cm}$ , in the normalization used in ref. [3].

Therefore, when the above quantities are placed in the Dirac equation (1.1), we obtain:

$$[k_{a0R}^\mu \gamma^a \nabla_\mu \psi - \mu_{0R} \psi] \tau_0 + ip\lambda [n_{a0}^\mu \gamma^a \nabla_\mu \psi - m_0 \psi] \tau_0 + i(p\lambda)^2 [n_{ai}^\mu \gamma^a \nabla_\mu \psi - m_i \psi] \tau_i = 0 \quad . \quad (1.9)$$

In the limit of the parameter  $p \rightarrow 0$ , we would get the standard Dirac equation in the presence of gravitation, electromagnetism, and Yang-Mills fields.

Suppose now that:

$$k_{a0R}^\mu = n_{a0}^\mu = n_{ai}^\mu \sim h_a^\mu \quad ,$$

and

$$m_0 = m_i \equiv \mu_{0R} \quad ,$$

for each  $i$ , and where  $h_a^\mu$  and  $\mu_{0R}$  are taken as the vierbeins and mass term of General Relativity. This situation will produce projections of the Dirac equation on the internal space axes proportional to the one obtained in GR. The intensity, or amplitude, of this projection will be determined by the parameter  $p$ . However, it is easy to show that this special choice cancels the antisymmetrical parts of the generalized metric of the theory, which means the

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<sup>1</sup>With these definitions we split the generalized Dirac equation in  $n^2 + 1$  parts.

electromagnetic and Yang-Mills terms of the matrix metric would be eliminated. This is not desirable, as it would imply the canceling of the related terms in the Lagrangian density used for the Action principle.

## II. An interpretation of the Dirac equation through the expansion of $\psi$ .

Let us propose now, that the mass term be maintained on the unitary axis ( $\tau_0$ ), and that the wave function be expanded in powers of the parameter  $p$ , which is supposed to be small. The expansion of  $\psi$  can be written as:

$$\psi(x) = \psi_0(x) + p\psi_1(x) + \frac{1}{2}p^2\psi_2(x) + \dots \quad (2.1)$$

Also, the Dirac equation (1.1) can be expanded in terms of its components on the internal axis as:

$$\tau_0 k_{a0R}^\mu \gamma^a \nabla_\mu \psi_1 + \tau_0 i k_{a0I}^\mu \gamma^a \nabla_\mu \psi - \tau_0 \mu \psi + \tau_i k_{ai}^\mu \gamma^a \nabla_\mu \psi = 0 \quad . \quad (2.2)$$

Now placing (2.1) in (2.2), and separating terms in orders of  $p$ , we obtain the following system of equations:

$$1 : \quad \tau_0 (k_{a0R}^\mu \gamma^a \nabla_\mu \psi - \mu \psi) = 0 \quad (2.3)$$

$$p : \quad \tau_0 (i n_{a0}^\mu \gamma^a \nabla_\mu \psi_0 + k_{a0R}^\mu \gamma^a \nabla_\mu \psi_1 - \mu \psi_1) = 0 \quad (2.4)$$

$$p^2 : \quad \tau_i n_{ai}^\mu \gamma^a \nabla_\mu \psi_0 + \tau_0 (i n_{a0}^\mu \gamma^a \nabla_\mu \psi_1 + \frac{1}{2} k_{a0R}^\mu \gamma^a \nabla_\mu \psi_2 - \frac{1}{2} \mu \psi_2) = 0 \quad (2.5)$$

$$p^3 : \quad \dots \text{and so on} \dots$$

Therefore, in the limit  $p \rightarrow 0$  we would obtain the Dirac equation as in General Relativity.

The advantage of this proposition in comparison to the former splitting of the mass term, is that the zero-order wave function  $\psi_0$  corresponds exactly to the GR-Dirac equation in presence of the Electromagnetism and Yang-Mills fields, when we take  $k_{a0R}^\mu$  as being equivalent to the vierbeins of GR. The other equations should be interpreted as being the result of a more exact Dirac equation, that is corrections to the Dirac equation obtained in a Gauge theory local to a curved Riemannian space-time.

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