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The Running of the Cosmological Constant

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Abstract

In a quantum theory, the cosmological constant is scale-dependent. The leading behavior of the scaling in Coleman's mechanism for the vanishing of the asymptotic cosmological constant is calculable.

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Of the many unexplained small numbers that appear in modern physics, the cosmological constant is undoubtedly one of the smallest. The most recent measurements by Loh and Spillar of the density of galaxies [1] put a bound of $10^{-9}\mathrm{eV}^4$ on the cosmological constant*. Coleman [2], building on earlier work of Baum [3], Hawking [4], and Linde [5], has presented an argument that the asymptotic low-energy value of the cosmological constant is zero. The question I wish to address here is, how zero is zero? What are the corrections to Coleman's result when the theory is being considered not at zero energy but at some finite energy? In a quantum theory, the cosmological constant is not a constant, rather it depends on the scale at which one probes the theory. The presence of quantum corrections is after all the reason the 'cosmological constant problem' exists in the first place; and quantum corrections are scale-dependent.

Coleman's argument is based on the Euclidean functional formulation of quantum gravity, As is evident in much of the contemporary literature on the subject, Euclidean gravity has many pitfalls and poorly understood embarrassments; adding wormholes only exposes more weaknesses. I will have nothing to say about these issues, the unboundedness of the gravitational action, the appropriateness of the Euclidean continuation, the supposed existence of a phase in the sum over spheres [6], or the possible dangers of large wormholes [7, 8, 9, 10]. Rather, I will blithely make all the standard assumptions of wormhole physics, namely that all of these technical problems can be solved. In addition, I will assume a mechanism for keeping the gravitation coupling constant G_N bounded away from zero [11, 8, 12]. With all these assumptions, let us ask: how does the cosmological constant run?

For the purposes of the present work, it is most convenient to follow the approach of Klebanov, Susskind, and Banks [13]. The expectation value of some observable, $\mathcal{O}$, in a Euclidean gravitational theory is given by a path integral,

$$\langle\mathcal{O}\rangle_\lambda = \frac{\int[dg][d\phi]\, e^{-S_E(g,\phi;\lambda)}\,\mathcal{O}}{\int[dg][d\phi]\, e^{-S_E(g,\phi;\lambda)}} \tag{1}$$

where the subscript denotes the coupling constants of the theory (including the cosmological constant), and where ϕ denotes all the matter fields. I am implicitly assuming a physical, gauge-fixed form of the gravitational path integral, along the lines of Arnowitt, Deser, and Misner [14] and Schleich [15], with a cut-off of order the Planck mass.

What is the effect of wormholes in the dilute gas approximation? They lead to a sum over large manifolds connected by small wormholes; the sum exponentiates, giving us the form

$$\langle\mathcal{O}\rangle = \mathcal{N}\int\prod_i d\alpha_i\ \exp(-\sum_i \alpha_i^2/C_i)\ \int[dg][d\phi]\, e^{-S_E(g,\phi;\lambda+\alpha)}\ \exp\left(\int[dg'][d\phi']e^{-S_E(g',\phi';\lambda+\alpha)}\right)\ \mathcal{O} \tag{2}$$

* Particle physics conventions are used throughout.

where the gravitational path integrals are now taken over connected large manifolds (and fluctuations about them) with no wormhole insertions. The α_i (in spite of ref. [13]) label wormhole types rather than operators in an operator expansion. This equation tells us that an observable is given by a weighted sum over theories with coupling constants shifted by the wormhole parameters α_i. The weighting is given by the probability distribution

$$P(\alpha) = \mathcal{N} \exp(-\sum_i \alpha_i^2/C_i) \int [dg][d\phi]\, e^{-S_E(g,\phi;\lambda+\alpha)} \exp\left(\int [dg'][d\phi']e^{-S_E(g',\phi';\lambda+\alpha)}\right) \tag{3}$$

If we now approximate the gravitational path integral by summing over Euclidean de Sitter spaces — four-spheres — then we find Coleman's double exponential,

$$P(\alpha) = \mathcal{N} \exp(-\sum_i \alpha_i^2/C_i) \exp\left(\frac{3}{8G_N^2\Lambda(0)}\right) \exp\left[\exp\left(\frac{3}{8G_N^2\Lambda(0)}\right)\right] \tag{4}$$

where I have indicated explicitly the dependence of the cosmological constant on the renormalization scale but have left the dependence on the α's implicit. If G_N is bounded away from zero, and if there are values of the αs for which $\Lambda(0)$ can vanish, then the distribution will be infinitely peaked about those values, and $\Lambda(0)$ will indeed vanish. This is Coleman's solution to the cosmological constant problem.

However, experiments do not measure $\Lambda(\mu)$ at a scale $\mu = 0$, but rather at some finite scale. What does Coleman's argument tell us about $\Lambda(\mu)$? Were we to calculate the cosmological constant ignoring the effect of wormholes, we would find that the effective cosmological constant is a sum of 'bare' and 'fluctuation' contributions,

$$\Lambda(\mu) = \Lambda_0 + \Lambda_{\text{quantum}} \tag{5}$$

To be a bit more precise, in changing the renormalization scale from M to μ, we would find new contributions to Λ arising from integrating out quantum fluctuations of energies M through μ:

$$\Lambda(\mu) = \Lambda(M) + \int_\mu^M dk\, \delta\Lambda(k) \tag{6}$$

In the absence of a symmetry, such as supersymmetry, or an Atkin-Lehner symmetry [16], $\Lambda(M)$ will typically be of order M^4.

(The reader may worry that this is all a fake; after all, in dimensional regularization, one would throw away such terms. Dimensional regularization is not appropriate here, because it amounts to throwing away an infinite constant, whereas the appropriate subtraction is completely determined by Coleman's result. Another way of saying this is that the difference between $\Lambda(0)$ and $\Lambda(\mu)$ is finite and well-determined. Unlike ordinary flat-space field-theory, we are *not* free to make different subtractions for different μ.)

Returning to equation (2), we may put in two independent (infrared) scales, μ, and μ':

$$\langle \mathcal{O} \rangle = \mathcal{N} \int \prod_i d\alpha_i \, \exp(-\sum_i \alpha_i^2 / C_i) \int_{k \geq \mu} [dg][d\phi] \, e^{-S_E(g,\phi;\lambda+\alpha)} \times \exp\left(\int_{k' \geq \mu'} [dg'][d\phi'] e^{-S_E(g',\phi';\lambda+\alpha)} \right) \mathcal{O} \tag{7}$$

(Recall that we are assuming a physical gauge-fixed form for the gravitational integral, so that the notion of a momentum cut-off makes sense.) The first scale, μ, sets the scale of the effective Lagrangian which determines the results of observations made in our universe (the large manifold in which we live). The other scale, μ', determines an infrared cut-off on the sum over large four-spheres. It is important to note that μ and μ' are unrelated, once we are evolving below the wormhole scale, since the different large manifolds are not connected by any physics below that scale. Indeed, while μ is fixed by the 'momentum transfer' in any observation we perform, we cannot fix μ', but must take the limit $\mu' \to 0$, otherwise we are truncating the heat bath of disconnected universes which give rise to Coleman's result.

Equation (7) implicitly yields a formula for $\Lambda(\mu, \mu')$. This comes about because, as shown by Preskill [8], the sum over spheres smaller than μ'^{-1} gives rise to a probability distribution for the α's that is peaked (although not infinitely so) for values which give a cosmological constant of order $G_N^2 \mu'^{-2}$. (Because the double exponential dominates over the single exponential, μ plays essentially no role in determining the most probable value of $\Lambda(\mu, \mu')$.) We thus have

$$\Lambda(\mu, \mu') = \Lambda(0, \mu') - \int_0^\mu dk \, \delta\Lambda(k) \tag{8}$$

Passing to the limit $\mu' \to 0$, we find

$$\Lambda(\mu) = \lim_{\mu' \to 0} \Lambda(\mu, \mu') = - \int_0^\mu dk \, \delta\Lambda(k) \tag{9}$$

The leading order contributions to $\delta\Lambda(k)$ come from closed loops of particles interacting only with the background gravitational field. The coupling here comes solely from the kinetic terms of the particles, and involves no coupling constant, dimensionful or otherwise. For dimensional reasons, we thus have $\int_0^\mu \delta\Lambda(k) = \mathcal{O}(\mu^4)$. In order to determine it precisely, we must separate the contributions to Λ from those which renormalize the background Ricci scalar and higher-order terms. This has been done by Fradkin and Tseytlin [17]; from their work, we may extract

$$\Lambda(\mu) = -\frac{1}{64\pi^2} \left(2N_2 + 2N_1 - 2N_{1/2} + N_0 \right) \mu^4 \tag{10}$$

where N_s is the number of massless fields of spin s in the theory; $N_{1/2}$ counts the number of Weyl degrees of freedom, and N_0 the number of real scalars. This equation holds in the region between mass (and symmetry-breaking) thresholds; near such thresholds, it will be modified. The relative sign of Bose and Fermi terms is as expected, but the over-all sign may seem a bit surprising; it is a reflection of the fact that Coleman's mechanism has over-compensated for the short-distance quantum fluctuations by subtracting contributions to Λ coming from energy scales lower than μ.

What is the correct choice of μ for the observations of Loh and Spillar? The answer is rather obscure, but Strominger's analysis [18] suggests the choice $\mu \sim H^{-1}$, where H is the Hubble time. This is *not* the present-day scale factor R, which is still related to the photon temperature T_γ by $R \sim T_\gamma^{-1}$ (because the ratio of the number of baryons to the number of photons is so small), but it is probably still appropriate, as the photons have long since decoupled from matter, and the observations track the matter density.

It is worthwhile contrasting the result obtained above for $\Lambda(\mu)$ with that which would emerge from a Baum-Hawking analysis. In that case, the exponential of the path integral is absent, and so there is no μ'; the probability distribution is determined entirely by μ. In this case, we will find a cosmological constant of order $G_N^2 \mu^2$, although it is not precisely computable since the probability distribution is not infinitely peaked for finite μ. This differs from the prediction following from Coleman's mechanism, but is it significant? With $\mu \sim H^{-1}$, both predictions are in accord with present-day observational bounds, and so it might appear that we cannot distinguish between the two possibilities. However, in the radiation-dominated era, the temperature, rather than the Hubble time, presumably set the scale for the cosmological constant. If so, then the Baum-Hawking analysis would lead to a cosmological constant of order $G_N^2 T_\gamma^2$, which is much too large, while Coleman's mechanism leads to an effective cosmological constant which is acceptable even in the early universe. Strominger's analysis leads to a present-day cosmological constant similar to that obtained in a Baum-Hawking analysis, presumably for much the same reason: it excludes configurations which give rise to Coleman's double exponential.

Because the effective cosmological constant and the thermal energy density act differently as sources for Einstein's equation, a (small) effective cosmological constant will modify the equation of state in the radiation-dominated era from $\rho \sim R^{-4}$ to $\rho \sim R^{-4+\epsilon}$. Equation (9) shows that ϵ would be rather small in practice, so it is not clearly observable. Nonetheless, it would be interesting to examine the bounds on ϵ arising from nucleosynthesis constraints.

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