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Light propagation and the distance-redshift relation in a realistic inhomogeneous universe

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ABSTRACT

We investigate the propagation of light rays in a clumpy universe constructed by cosmological version of the post-Newtonian approximation. We show that the linear approximation to the propagation equations is valid in the region $z \lesssim 1$ even if the density contrast is much larger than unity. Based on a general order-of-magnitude statistical consideration, we argue that the linear approximation is still valid for $z \gtrsim 1$. Then we give a general formula for the distance-redshift relation in a clumpy universe and derive an explicit expression for a simplified situation in which the effect of the gravitational potential of inhomogeneities dominates. In the light of the derived relation we discuss the validity of the Dyer-Roeder distance. Furthermore, we consider a simple model of an inhomogeneous universe and investigate statistical properties of light rays. We find that the result of this specific example also supports the validity of the linear approximation.

1. Introduction

In the last several years large-scale structures of the universe has become an active area of research in cosmology. In spite of extensive theoretical as well as observational efforts, we have not yet understood the formation of the structures. Further observational information such as more complete survey of galaxies at high redshifts is definitely necessary in order to improve the situation. Fortunately it is expected that rapid progress in the observational techniques and the appearance of new telescopes of the next generation will bring about vital information on the structure of the universe in near future and open a new era for the observational cosmology.

On the other hand it seems that careful attention has not been paid for theoretical aspects of the observational cosmology. One of the difficulties is that light rays from distant galaxies might have propagated through intergalactic space in which the density is much lower than the average density of the universe. Thus it is not clear at all if the averaged homogeneous, isotropic Friedmann-Robertson-Walker (FRW) metric may be used as an appropriate metric on which light propagates or not. In fact the averaged FRW metric coincides nowhere with the real inhomogeneous metric. Since light feels local metric not the averaged metric, there is no justification for using the FRW metric to calculate the propagation. Nonetheless it has been customary to compare observations of distant galaxies with the predictions of a FRW universe. Otherwise rather crude descriptions of inhomogeneities such as the Dyer-Roeder model[1] or Swiss cheese model are used to interpret the observational data. It is not known in what sense such descriptions approximate inhomogeneities of the real universe.

There is also another problem. Even if one has a realistic description of inhomogeneities, it is not straightforward to relate theoretical quantities calculated on the inhomogeneous metric with actual observables. Without such relations, we will not have the correct interpretation of observational data. In view of rapid progress in the observational side it seems urgent to develop a consistent theoretical framework of observational cosmology.

Recently a consistent theoretical derivation of important relations in the observational cosmology such as the magnitude-redshift relation was given in a linearly perturbed FRW universe.[2,3] Also an approximation method for an inhomogeneous universe beyond linearized theory which is applicable to the present clumpy universe as well as to a linearly perturbed FRW universe is developed.[4] The purpose of the present paper is to study the light propagation in a realistic inhomogeneous universe constructed by the approximation method mentioned above and to extend the derivation of the magnitude-redshift (or distance-redshift) relation in the linearized case to the nonlinear case in which the density contrast is much larger than unity.

The paper is organized as follows. In §2, we review the method for constructing an approximate metric for an inhomogeneous universe and clarify the condition for the cosmological version of the Newtonian approximation. In §3, assuming that the condition for the Newtonian approximation is satisfied, we consider the propagation of light rays in an inhomogeneous universe and argue that the linear perturbation can be applied to the light propagation equations even in a highly inhomogeneous universe. In §4, we give the basic formula for the distance-redshift relation in an inhomogeneous universe, derive an explicit expression for a simplified case and compare it with the Dyer-Roeder distance. Our distance coincides with Dyer-Roeder's in the region $z \lesssim 1$. Then we consider a simple model of an inhomogeneous universe and compute the probability distribution of fluctuations in the distance-redshift relation. We find the result is in agreement with the general discussion of §3 and supports the validity of the linear approximation, even for very high redshifts. Finally, §5 is devoted to conclusions.

2. Approximation of inhomogeneous metric

2.1. General scheme

In order to make this paper self-contained, we shall briefly explain the approximation method for constructing the metric of an inhomogeneous universe in general relativity developed by one of us.[4] Since the calculational detail has been presented in Ref.[4], we discuss mainly the physical idea behind the method and present the results. The method applies for the nonlinear stage as well as linear stage as far as the metric deviation from the FRW background metric. As shown below, this does not of course impose the smallness of the density contrast.

We assume that the spacetime considered here may be parametrized by two independent small parameters ϵ and κ. The ϵ is associated with the amplitude of the gravitational potential (ϕ) generated by inhomogeneous distribution of matter, $\phi \sim \epsilon^2$. The κ is the ratio between the typical scale of the inhomogeneities (ℓ) and the scale of the background spacetime (L), $\kappa = \ell/L$. The relative size of ϵ and κ depends on the system we have in mind. Since the metric fluctuation is generated by the density fluctuation $\delta\rho$ via Poisson equation, the density contrast $\delta\rho/\rho_b$ may be evaluated from $\Delta\phi/(G\rho_b) \sim \epsilon^2/\kappa^2$, where ρ_b is the averaged density (see below). Thus the linear and nonlinear stage may be characterized by the condition $\kappa \gg \epsilon$ and $\epsilon \gg \kappa$, respectively. For example, if we take a supercluster whose size is about $30\,\mathrm{Mpc}h^{-1}$, then κ will be $\sim 30/3000 \sim 10^{-2}$, where $L \sim 3000\,\mathrm{Mpc}h^{-1}$ is the present horizon size. We do not know the order of the gravitational potential for such a system, but the density contrast seems to be of the order of unity. Thus the gravitational potential would be $\epsilon^2 \sim \kappa^2 \sim 10^{-4}$ for such a system. The size of supercluster seems to be the boundary between the linear and nonlinear regions. We also note that typical values of ϵ and κ for galaxies are $\epsilon \sim 10^{-3}$ and $\kappa \sim 10^{-4.5}$.

We make the following ansatz for the metric,

$$g_{\mu\nu} = a^2(\eta)(\gamma_{\mu\nu} + h_{\mu\nu}) , \tag{2.1}$$

where a is the scale factor which describes the averaged global expansion and is assumed to be a function of the conformal time η. It is also assumed that $a'/a = O(1/L)$ where the prime means the derivative with respect to η. The h's are supposed to be generated by inhomogeneous distribution of matter and by possibly gravitational waves. We do not consider the latter possibility in this paper and assume that $h_{\mu\nu} = O(\epsilon^2)$ and $h_{\mu\nu,\rho} = O(\epsilon^2/\ell)$. We assume that the spacetime considered here reduces to the closed, flat or open FRW spacetime depending on the curvature of the spatial section $K = +1, 0$ or -1, respectively, when the matter distributes homogeneously and $h_{\mu\nu}$ vanish identically. Thus the $\gamma_{\mu\nu}$ is the standard metric for one of the FRW universes.

The above ansatz for the metric is used to expand the Einstein equations in terms of ϵ and κ as follows:

$$\begin{aligned} 3\{2(\frac{a'}{a})^2 + K\}(\bar{h}^{\eta\eta} + \frac{1}{2}\bar{h}) + 3\{(\frac{a'}{a})^2 + K\} \\ - \frac{a'}{a}(\bar{h}^{\eta\eta} + \frac{1}{2}\bar{h})_{|\eta} - \frac{1}{2}\bar{h}^{\eta\eta|\rho}_{\ \ |\rho} = 8\pi G\tau^{\eta\eta}\,, \end{aligned} \tag{2.2}$$

$$\{3(\frac{a'}{a})^2 + 2K\}\bar{h}^{\eta i} + (\frac{a'}{a})(\bar{h}^{\eta\eta} + \frac{1}{2}\bar{h})^{|i} + \frac{1}{2}\bar{h}^{\eta i|\rho}_{\ \ |\rho} = 8\pi G\tau^{\eta i}\,, \tag{2.3}$$

$$\begin{aligned} \{2\frac{a''}{a} - (\frac{a'}{a})^2\}(\bar{h}^{ij} + \gamma^{ij}\bar{h}^{\eta\eta} - \gamma^{ij}\bar{h}) + K(\bar{h}^{ij} - \frac{1}{2}\gamma^{ij}\bar{h}) \\ + \{(\frac{a'}{a})^2 + 2\frac{a''}{a} - K\}\gamma^{ij} + \frac{a'}{a}(2\bar{h}^{\eta(i|j)} - \bar{h}^{ij|\eta} + \frac{1}{2}\gamma^{ij}\bar{h}^{|\eta}) \\ - \frac{1}{2}\bar{h}^{ij|\rho}_{\ \ |\rho} = 8\pi G\tau^{ij}\,, \end{aligned} \tag{2.4}$$

where we have used the trace reversed metric perturbation defined by $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\gamma^{ij}h$ and worked in the harmonic gauge $\bar{h}^{\mu\nu}{}_{|\nu} = 0$. The indices on h and $\bar{h}$ are shifted by the background metric $\gamma_{\mu\nu}$ and the bar indicates a covariant derivative with respect to $\gamma_{\mu\nu}$. $\tau^{\mu\nu} = a^4T^{\mu\nu} + t^{\mu\nu}$ is the total effective stress energy pseudotensor. The $t^{\mu\nu}$ consists of terms quadratic in $\bar{h}$ and may be interpreted as a gravitational stress energy pseudotensor. In deriving the above equations, we have neglected terms of order higher than $O(\epsilon^4/\kappa^2)$ with the assumption that $\epsilon^2 \ll \kappa$ which we assume throughout the present paper. Provided that one focuses on cosmological problems in which structures of interest are above galactic scales, this assumption gives practically no restriction for applicability of the present scheme to the real universe.

The equations for the averaged global expansion is obtained by taking the spatial average of the above equations. The results may be written as follows:

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\left\langle \tau^{\eta\eta}\right\rangle - K\,, \tag{2.5}$$

$$\frac{a''}{a} = \frac{4\pi G}{3}\left\langle \tau^{\eta\eta} - \tau^k_k\right\rangle - K\,, \tag{2.6}$$

$$\frac{1}{a^2}\{a^2\left\langle \bar{h}^{ij}\right\rangle_{|\eta}\}_{|\eta} + (4\frac{a''}{a} - 2\left(\frac{a'}{a}\right)^2 - 2K)\left\langle \bar{h}^{ij}\right\rangle = 16\pi G\left\langle \hat{\tau}^{ij}\right\rangle\,. \tag{2.7}$$

In deriving the above equations we have assumed $\left\langle \tau^{\eta i}\right\rangle = 0$ expressing no coherent motion over the volume to be averaged and we have required that $\left\langle \bar{h}^{\eta\eta}\right\rangle = \left\langle \bar{h}^k_k\right\rangle = 0$ by choosing appropriate time variable and the scale factor. The spatial average of the line element takes then the following form,

$$\left\langle ds^2\right\rangle = a^2\left[-d\eta^2 + (\gamma_{ij} + \langle h_{ij}\rangle)dx^i dx^j\right]\,, \tag{2.8}$$

Thus $\left\langle \bar{h}_{ij}\right\rangle$ express the deviation from the isotropic expansion due to the inhomogeneities $\left\langle \hat{\tau}^{ij}\right\rangle$ and the averaged spacetime expands anisotropically except if $\left\langle \bar{h}_{ij}\right\rangle$ vanishes identically. Equations (2.5) and (2.6) are the same with the equations of the FRW model except that the source terms are replaced by the total effective stress energy pseudotensor including gravitational contribution. Thus the effect of local inhomogeneity on the global expansion may be expressed by the effective density $\rho_{eff} = a^2\left\langle \tau^{\eta\eta}\right\rangle$ and the effective pressure $p_{eff} = \frac{1}{3}a^2\left\langle \tau^k_k\right\rangle$. The equations which determine the local metric may be derived by subtracting the above averaged equation from the original equations (2.2) $\sim$ (2.4). These are as follows:

$$\Box\bar{h}^{\eta\eta} = -16\pi G(\tau^{\eta\eta} - \langle\tau^{\eta\eta}\rangle) + \frac{a'}{a}(\bar{h}^{\eta i}_{|i} - \bar{h}^{k}_{k|\eta}) + 3(2(\frac{a'}{a})^2 - K)(\bar{h}^{\eta\eta} + \bar{h}^k_k)\,, \tag{2.9}$$

$$\Box\bar{h}^{\eta i} = -16\pi G\tau^{\eta i} + \frac{a'}{a}(\bar{h}^{\eta\eta|i} + \bar{h}^{k|i}_k) + 6(\frac{a'}{a})^2\bar{h}^{\eta i}\,, \tag{2.10}$$

$$\Box\hat{\bar{h}}^{ij} = -16\pi G\hat{\tau}^{ij} + \frac{a'}{a}2\hat{\bar{h}}^{ij}_{|\eta} + 4(\bar{h}^{\eta(i|j)} + \frac{1}{3}\gamma^{ij}\bar{h}^{\eta k}_{|k}) - 2\{(\frac{a'}{a})^2 - 2\frac{a''}{a} - K\}\hat{\bar{h}}^{ij}\,, \tag{2.11}$$

$$\Box\bar{h}^k_k = -16\pi G(\tau^k_k - \left\langle\tau^k_k\right\rangle) + \frac{a'}{a}(\bar{h}^k_{k|\eta} - \bar{h}^{\eta k}_{|k}) + 4\{\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\}\bar{h}^k_k + K(3\bar{h}^{\eta\eta} - \bar{h}^k_k)\,, \tag{2.12}$$

where $\hat{\bar{h}}^{ij} = \bar{h}^{ij} - \frac{1}{3}\gamma^{ij}\bar{h}^k_k$ is the spacial trace free part of the perturbation and may be regarded as the gravitational wave degrees of freedom. Above equations are of course supplemented by the equations of motion $T^{\mu\nu}{}_{;\nu} = 0$ or

$$T^{\mu\nu}{}_{|\nu} + \frac{a'}{a}(6T^{\eta\mu} + \gamma^{\eta\mu}T) + (\bar{h}^{\mu}{}_{\rho|\sigma} + \frac{1}{2}\bar{h}_{\rho\sigma}{}^{|\mu})T^{\rho\sigma} - (\bar{h}_{|\rho}T^{\rho\mu} + \frac{1}{4}\bar{h}^{|\mu}T) = 0\,. \tag{2.13}$$

Then the calculated perturbations $\bar{h}^{\mu\nu}$ are used to construct the local metric

$$ds^2 = a^2(\eta)(\gamma_{\mu\nu} + h_{\mu\nu})\,, \tag{2.14}$$

where a and $\langle h_{\mu\nu}\rangle$ are calculated from the global equations (2.5) $\sim$ (2.7).

2.2. Linear and post-Newtonian approximation

In this subsection, we construct the linear and post-Newtonian approximation from the equations derived in the above. First we give a characterization for the linear and nonlinear regimes. Since $\bar{h}^{mn} \sim \epsilon^2$ and $G\rho_b \sim 1/L^2$, ρ_b the averaged density, by definition, we can easily obtain the order of the density contrast from the local equation (2.9),

$$\frac{\delta\rho}{\rho_b} \sim \frac{\epsilon^2}{\kappa^2}, \tag{2.15}$$

Thus the linear and nonlinear regimes are characterized by the condition $\kappa \gg \epsilon$ and $\epsilon \gg \kappa$ ($\gg \epsilon^2$ by assumption), respectively.

We first consider the linear approximation which applies for the linear regime $\kappa \gg \epsilon$. In this regime the dynamical timescale of the density fluctuations will be of the order of the timescale of cosmic expansion. Then the equations of motion is used to evaluate the order of the velocity as ϵ^2/κ. Thus we have the following ordering in the linear regime,

$$\bar{h}^{\mu\nu} \sim \epsilon^2 \ll 1, \quad \frac{\delta\rho}{\rho_b} \sim \frac{\epsilon^2}{\kappa^2} \ll 1, \quad v^i \sim \frac{\epsilon^2}{\kappa} \ll 1. \tag{2.16}$$

Every perturbed quantities are much less than unity and their second order terms are safely ignored. For example, $\tau^{\mu\nu}$ may be approximated by $a^4 T^{\mu\nu}$. This is of corse what we call the linear approximation. If we take the perfect fluid form for the stress energy tensor, we may neglect the spacial trace free part because these are second order in velocity and thus the averaged spacial trace free part of the metric perturbation $\left\langle \hat{\bar{h}}^{ij} \right\rangle$ may be neglected. Thus the spacetime expands isotropically in this case. The expansion equations (2.5) and (2.6) reduce to the usual equations for a FRW model.

Next we consider a cosmological version of the post-Newtonian approximation which applies for the nonlinear regime $\epsilon \gg \kappa$. In this case the dynamics of the density fluctuation is totally governed by its self gravity and thus the dynamical time scale will be the Newtonian time defined by $\tau = \epsilon\eta$.[5] The equations of motion then give the proper relation $\phi \sim v^2$ where ϕ is the Newtonian potential generated by the density fluctuation. Thus we have the following ordering in the nonlinear regime,

$$\phi \sim \epsilon^2 \ll 1, \quad \bar{h}^{ij} \sim \epsilon^4 \ll 1, \quad v^i \sim \epsilon \ll 1, \quad \frac{\delta\rho}{\rho_b} \sim \frac{\epsilon^2}{\kappa^2} \gg 1\,. \tag{2.17}$$

This ordering allows us to neglect terms like $(a'/a)^2\bar{h}^{\mu\nu}$ in the equations (2.9) $\sim$ (2.11) and to solve the equations perturbatively. The lowest order equations are just the Newtonian equations and the next order is order ϵ^2 smaller than the Newtonian order. Thus we may safely use the Newtonian approximation as far as ϵ is sufficiently small, but much larger than κ. In such a region the approximated line element is given by

$$ds^2 = a^2(\eta)\left[-(1+2\Psi)d\eta^2 + (1-2\Psi)\gamma_{ij}dx^i dx^j\right] , \tag{2.18}$$

where a and Ψ are determined by the following equations in the lowest order:

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho_b - K \tag{2.19}$$

$$\overset{(3)}{\Delta}\,\Psi = 4\pi G a^2 \delta\rho\,, \tag{2.20}$$

where $\overset{(3)}{\Delta}$ is the Laplacian operator in the spacial section. Thus Ψ is the Newtonian potential generated by the density inhomogeneity. The equations of motion in the lowest order are the usual conservation law and the Newtonian equation:

$$\rho' + 3\frac{a'}{a}\rho + (\rho v^i)_{,i} = 0\,, \tag{2.21}$$

$$v^i_{,\eta} + \frac{a'}{a}v^i + v^j v^i_{,j} + \frac{1}{\rho}p^{,i} = -\Psi^{,i}\,, \tag{2.22}$$

where $v^i = dx^i/d\eta$. In particular, Eq.(2.21) implies $\rho_b \propto a^{-3}$ at this order. Thus the conventional FRW model is justified as a model for the averaged background universe. The higher order corrections are calculated elsewhere.

It should be pointed out that we may write the line element in the above form (2.18) even in the linear approximation by choosing an appropriate gauge, namely the so-called Newtonian gauge.[2,8] It is known in the linearized theory that the potential satisfies the same Poisson equation (2.20) and is interpreted as the Newtonian potential. The difference between the linear approximation and Newtonian approximation is then the equations of motion. We may thus use the above expression for the line element in the linear as well as the nonlinear regions.

3. Light propagation in an inhomogeneous universe

We now consider the propagation of light rays in an inhomogeneous universe constructed by the previous method. Thus the line element takes the form (2.18) in the linear as well as the nonlinear regimes. We may safely adopt the geometric optics approximation in the cosmological context. Since the constructed metric takes the form $g_{\mu\nu} = a^2(\gamma_{\mu\nu}+h_{\mu\nu}) = a^2\tilde{g}_{\mu\nu}$ and the light propagation is unaffected by conformal transformations, it is rather convenient to work in the conformally related spacetime $\tilde{g}_{\mu\nu}$. In the following, quantities with tilde are quantities in the conformally related world.

The basic equations for the propagation of light rays are

$$\frac{d}{d\lambda}\tilde{k}^\mu + \tilde{\Gamma}^\mu_{\alpha\beta}\tilde{k}^\alpha\tilde{k}^\beta = 0\,, \tag{3.1}$$

$$\frac{d}{d\lambda}\tilde{\theta} = -\tilde{R}_{\alpha\beta}\tilde{k}^\alpha\tilde{k}^\beta - \frac{1}{2}\tilde{\theta}^2 - 2\tilde{\sigma}^2\,, \tag{3.2}$$

$$\frac{d}{d\lambda}\tilde{\sigma}_{\alpha\beta} = -\tilde{C}_{\alpha\rho\beta\sigma}\tilde{k}^\rho\tilde{k}^\sigma - \tilde{\theta}\tilde{\sigma}_{\alpha\beta}\,, \tag{3.3}$$

where the first equation is the null geodesic equation and the second and third are the equations for the expansion $\tilde{\theta}$ and shear tensor $\tilde{\sigma}_{\alpha\beta}$ of the bundle of light rays. Our strategy here is to show first that fluctuations in Eqs.(3.1) and (3.3) due to the presence of inhomogeneities are small. Then we consider Eq.(3.2) and look for conditions for the validity of the linear approximation for the propagation of light rays.

We write the 4-momentum of the light rays as follows:

$$\tilde{k}^\mu = k^\mu + \delta k^\mu \tag{3.4}$$

where k^μ is the 4-momentum in the unperturbed metric $\Psi = 0$ and satisfies the geodesic equation,

$$\frac{d}{d\lambda} k^\mu + \Gamma^\mu_{\rho\sigma} k^\rho k^\sigma = 0 \tag{3.5}$$

where $\Gamma^\mu_{\rho\sigma}$ is the Christoffel symbol of the metric $\gamma_{\mu\nu}$. This equation then tells us that $k^\mu = O(1)$. The δk^μ satisfies the following equation,

$$\frac{d}{d\lambda}\delta k^\mu + \delta\Gamma^\mu_{\rho\sigma} k^\rho k^\sigma + 2\Gamma^\mu_{\rho\sigma} k^\rho \delta k^\sigma = 0\,, \tag{3.6}$$

where we have neglected higher order terms and $\delta\Gamma^\mu_{\rho\sigma}$ is given by

$$\delta\Gamma^\mu_{\rho\sigma} = \frac{1}{2}\gamma^{\mu\nu}(h_{\nu\rho|\sigma} + h_{\nu\sigma|\rho} - h_{\rho\sigma|\nu})\,. \tag{3.7}$$

From this expression we find that $\delta\Gamma^\mu_{\rho\sigma} = O(\epsilon^2/\ell)$.

As for the shear, it vanishes on the unperturbed background since the Weyl tensor is identically zero for $\gamma_{\mu\nu}$, provided the light rays are originally shear-free. Hence the propagation equation at the lowest non-vanishing order is

$$\frac{d}{d\lambda}\tilde{\sigma}_{\alpha\beta} = -\tilde{C}_{\alpha\rho\beta\sigma} k^\rho k^\sigma - \theta\tilde{\sigma}_{\alpha\beta}\,, \tag{3.8}$$

where we have $\tilde{C}_{\alpha\rho\beta\sigma} = O(\epsilon^2/\ell^2)$.

Now consider the propagation equation for the expansion. The Ricci tensor is approximately written as

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + \delta R_{\mu\nu} \,, \tag{3.9}$$

where $R_{\mu\nu} = 2K\gamma_{ij}\delta^i_\mu\delta^j_\nu$ is the Ricci tensor of the unperturbed spacetime and $\delta R_{\mu\nu}$ is given by

$$\delta R_{\mu\nu} = \frac{1}{2}\gamma^{\rho\sigma}(2h_{\sigma(\mu|\nu)\rho} + h_{\mu\nu|\rho\sigma} - h_{\rho\sigma|\mu\nu}) \,. \tag{3.10}$$

Note that $\delta R_{\mu\nu} = O(\epsilon^2/\ell^2)$. We decompose the expansion as

$$\tilde{\theta} = \theta + \delta\theta \,, \tag{3.11}$$

where θ is the expansion on the unperturbed background. Then θ and $\delta\theta$ satisfy the following equations,

$$\frac{d}{d\lambda}\theta = -2K - \frac{1}{2}\theta^2 \,, \tag{3.12}$$

$$\frac{d}{d\lambda}\delta\theta = -\delta(R_{\mu\nu}k^\mu k^\nu)_\lambda - \theta\delta\theta - \frac{1}{2}\delta\theta^2 - 2\tilde{\sigma}^2 \,, \tag{3.13}$$

where

$$\delta(R_{\mu\nu}k^\mu k^\nu)_\lambda = \delta R_{\mu\nu}k^\mu k^\nu + 4K\frac{d}{d\lambda}(k_i\delta x^i) \,, \tag{3.14}$$

with δx^i being the perturbation of the geodesic path and we have used the fact that the shear vanishes in the unperturbed FRW universe and assumed that the perturbation δk^μ is small, which we shall verify shortly. Note that the expansion on the unperturbed background is $\theta = O(1/\lambda)$.

First we consider δk^μ. Imagine a universe filled with objects (galaxies) of size ℓ whose density is $\delta\rho = (\epsilon^2/\kappa^2)\rho_b$. Hence the mean separation distance is $r_0 = (\epsilon/\kappa)^{2/3}\ell$. Then for a geodesic affine distance of λ, the light gravitationally encounters such objects $N_g = \lambda/r_0 = \epsilon^{-2/3}\kappa^{-1/3}(\lambda/L)$ times in average. In each encounter, the integral of the geodesic equation gives the contribution $(\epsilon^2/b) \times b = \epsilon^2$ to δk^μ where b is the impact parameter ($b \lesssim r_0$). However, since the sign of each contribution will be random, the total contribution to δk^μ will be $\sqrt{N_g}\epsilon^2 = \epsilon^{4/3}(\epsilon^2/\kappa)^{1/6}(\lambda/L)^{1/2}$, which is always much smaller than unity, since $\epsilon^2 \ll \kappa$ by assumption and $\lambda \lesssim L$ (note that the affine distance to the source object λ is bounded from above due to the fact that the age of our universe is finite). It should be mentioned that since the contribution of each encounter is independent of b, the direct encounters which are much rarer events are totally unimportant in this case. Thus we conclude that the linear approximation is valid for the evaluation of δk^μ.

As for the magnitude of the shear, the same argument as above leads to the estimate $\tilde{\sigma}^2 \sim (\epsilon^2/\kappa)(\lambda/L^3)$ as the contribution from gravitational scattering, where we have assumed that the mean impact parameter $\langle b \rangle$ is of the order r_0. The contribution from the direct encounters can be similarly estimated by noting that the average number of encounters is $N_d = (\ell^2/r_0^3)\lambda = (\kappa/\epsilon^2)(\lambda/L)$ with each encounter contributing $(\epsilon^2/\ell^2) \times \ell = \epsilon^2/\ell$ with random sign. The result turns out to be the same as that of gravitational distant encounters. Hence we conclude that $\tilde{\sigma}^2 \sim (\epsilon^2/\kappa)(\lambda/L^3)$.

Finally let us consider the expansion. From the above estimate for the shear and from the fact $\theta = O(1/\lambda)$, we find $\delta\theta/\theta = (\epsilon^2/\kappa)(\lambda/L)^3 \ll 1$ as the contribution of the shear. Thus the linear approximation for the evaluations of the shear and its contribution to $\delta\theta$ is justified. To evaluate the contribution from the perturbed Ricci tensor, it is important to know the explicit form of $\delta R_{\mu\nu}k^\mu k^\nu$ (see Eq.(4.3) below);

$$\delta R_{\mu\nu}k^\mu k^\nu = 2 \overset{(3)}{\Delta} \Psi + O\left(\kappa \frac{\epsilon^2}{\ell^2}\right). \tag{3.15}$$

Hence from the Poisson equation (2.20), the dominant contribution from $\delta R_{\mu\nu}$ to $\delta\theta$ comes only from regions where $\delta\rho/\rho$ is non-vanishing, *i.e.*, there is no contribution from distant gravitational encounters at the leading order.

The contribution from the direct encounters with galaxies can be estimated similarly as above. We obtain $\delta\theta \sim (\epsilon^2/\ell^2) \times \ell \times N_d$, where note that each contribution to $\delta\theta$ is negative definite (since $\delta\rho/\rho_b$ is positive definite) in this case, hence adds up secularly as $\propto N_d$. This gives $\delta\theta = O(\lambda/L^2)$ and we obtain $\delta\theta/\theta = O(\lambda^2/L^2)$. Hence the linear approximation is valid if $(\lambda/L)^2 \ll 1$, *i.e.*, $z^2 \ll 1$. On the other hand, we have $\delta R_{\mu\nu}k^\mu k^\nu = O(1/L^2)$ in intergalactic space. Hence it gives $\delta\theta \sim (1/L^2) \times \lambda$ which is the same as the contribution from galaxies in magnitude but has the opposite sign (since $\delta\rho/\rho_b$ is negative definite). Therefore, provided the number of encounters of a light ray with galaxies is sufficiently large ($N_d \sim 30(\lambda/L)$ for galactic scale objects), we can expect these two contributions to cancel each other on average. The resudial fluctuation in $\delta\theta$ will be proportional to $\sqrt{N_d}$; $\delta\theta \sim (\epsilon^2/\ell^2) \times \ell \times \sqrt{N_d}$, hence $(\delta\theta/\theta)^2 \sim (\epsilon^2/\kappa)(\lambda/L)^3 \ll 1$. Note that the assumption $\epsilon^2 \ll \kappa$ guarantees $N_d \gg 1$ for $z \gtrsim 1$. As a result the linear approximation will be valid even for $\lambda = O(L)$, *i.e.*, for $z \gg 1$.

The above argument may sound too naive. However, as for the shear contribution, there exists a more detailed theoretical argument[7] which supports our result. As for the perturbed Ricci tensor contribution, in the next section, we shall investigate statistical properties of light rays in a simple but reasonable model universe and show that the results are indeed consistent with the above argument. Further, recent numerical calculations by Watanabe[8] also seem to support the validity of the linear approximation. To summerize, we conclude that except for a statistically very rare kind of light rays, the linear approximation can be safely used to study the propagation of light rays in an inhomogeneous universe in which the density contrast is much larger than unity as long as one focuses on a region sufficiently smaller than the horizon scale, *i.e.*, $z \ll 1$, and it is a very good (if not the best) approximation even for $z \gg 1$, provided that our approximation based on the expansion in terms of ϵ is applicable to the universe, *i.e.*, $\epsilon^2 \ll \kappa$.

4. Distance-redshift relation

4.1. Basic formulas

As discussed in §2, the approximate metric (2.18) gives a sufficiently accurate description of the real universe. Further, we have seen in §3 that the linear approximation can be used to investigate the propagation of light rays in the region $z \lesssim 1$ (and probably even for $z \gtrsim 1$, provided one is interested in average light rays). These imply that most of the arguments given up to §4 of Ref.[2] hold also for a highly inhomogeneous universe, under the assumption that $\epsilon^2 \ll \kappa$, since only geometrical considerations but non of the Einstein equations were used there. The only modifications we have to make are to include the contribution of the shear in the propagation equation for the expansion and to retain terms of $O(v^2)$ in the expressions for perturbed four-velocities of the source and observer. Note that $O(\Psi) \sim O(v^2)$ in the Newtonian situation.

It follows that the distance-redshift relation in an inhomogeneous universe, written in the Newtonian gauge, is expressed as[2,9]

$$\frac{\delta d_L(z,\gamma^i)}{d_L(z)} = \frac{\delta d_A(z,\gamma^i)}{d_A(z)} = \sqrt{-K}\delta\lambda_s \coth\sqrt{-K}\lambda_s + I\,, \tag{4.1}$$

with $\delta\lambda_s$ and I given by

$$\begin{aligned}\delta\lambda_s = &\left(\left(\frac{a'}{a}\right)_s^{-1} - \lambda_s\right)\left\{\left[\Psi - v_i\gamma^i - \frac{1}{2}v^2\right]_0^{\lambda_s} + 2\int_0^{\lambda_s} d\lambda\, \Psi'(\lambda)\right\}\\ &+ 2\int_0^{\lambda_s}\lambda\left(\Psi - \Psi_0 + \int_0^{\lambda} d\lambda_1\, \Psi'(\lambda_1)\right) + \lambda_s(\Psi + v_i\gamma^i + \frac{1}{2}v^2)_0\,,\end{aligned} \tag{4.2}$$

$$
\begin{aligned}
I =& \frac{-1}{\sqrt{-K}} \int_0^{\lambda_s} d\lambda \, \sinh^2 \sqrt{-K}\lambda \, (\coth \sqrt{-K}\lambda - \coth \sqrt{-K}\lambda_s) \\
& \times \left\{ \frac{1}{2} \delta(R_{\mu\nu}k^\mu k^\nu)_\lambda + \tilde{\sigma}^2 \right\} ; \\
\frac{1}{2}\delta(R_{\mu\nu}k^\mu k^\nu)_\lambda =& \overset{(3)}{\Delta} \Psi - (\Psi'' + 2\frac{d}{d\lambda}\Psi') \\
& + 2K \left\{ (\Psi - v_i\gamma^i - \frac{1}{2}v^2)_s + 2\int_\lambda^{\lambda_s} d\lambda_1 \, \Psi'(\lambda_1) \right\} ,
\end{aligned}
\tag{4.3}
$$

where $\delta d_L/d_L$ $(\delta d_A/d_A)$ is the anisotropy in the luminosity (angular diameter) distance to a given redshift z measured in the direction of γ^i, $v^2 = v_i v^i$, $\lambda_s = \eta(z) - \eta_0$ is the conformal distance to the redshift z in the averaged background universe, the suffix s denotes a quantity at the source of redshift z, the suffix 0 for $z = 0$, $\Psi'(\lambda) = \partial\Psi(\eta(\lambda), x^i(\lambda))/\partial\eta$, *etc.*. Note that $d_L = (1+z)^2 d_A$ holds for arbitrary spacetimes,[10] which guarantees $\delta d_L/d_L = \delta d_A/d_A$ $(\equiv \delta d/d)$. The difference between the case of linear density perturbations and the non-linear case is that Ψ and v^i are determined from the Newtonian equations (2.21) and (2.22) in the latter, while they are determined from the linearized Einstein equations in the former.

For distances in the range $\ell \ll \lambda_s \lesssim L$, the examination of the order of magnitude of each term in Eq.(4.1) $\sim$ (4.3), in the non-linear case, shows that the leading terms, up to the order of ϵ^2/κ, are those involving $\overset{(3)}{\Delta} \Psi$, $\tilde{\sigma}^2$ and $v_i\gamma^i$. Neglecting the other terms we find

$$
\begin{aligned}
\frac{\delta d}{d} =& -\left(\frac{a'}{a}\right)_s^{-1} \sqrt{-K} \coth \sqrt{-K}\lambda_s \left\{ (v_i\gamma^i)_s - (v_i\gamma^i)_0 \right\} + (v_i\gamma^i)_s \\
& - \frac{1}{\sqrt{-K}} \int_0^{\lambda_s} d\lambda \, \sinh^2 \sqrt{-K}\lambda \, (\coth \sqrt{-K}\lambda - \coth \sqrt{-K}\lambda_s) \\
& \times \left\{ 4\pi G \delta\rho a^2 + \tilde{\sigma}^2 \right\} ,
\end{aligned}
\tag{4.4}
$$

where the Poisson equation (2.20) was used to replace $\overset{(3)}{\Delta}\Psi$ by $\delta\rho$. Further comparison of the remaining terms, with the same strategy we took in the previous section, shows the first term ($\propto \{(v_i\gamma^i)_s - (v_i\gamma^i)_0\}$) is generally most important for

$$z \sim \lambda_s/L \lesssim \epsilon^{1/3} \equiv z_{cr}\,, \tag{4.5}$$

and the density term ($\propto \delta\rho$), otherwise. However, the other terms can be important at high redshifts if the integral of the density term gives zero in average as we discussed in §3. We note that the first term describes the usual Doppler effect due to the peculiar velocities of the source object and the observer, which is used in the determination of the peculiar velocity field on large scales ($\lesssim 50\,\mathrm{Mpc}h^{-1}$). Note also that with typical values of ϵ for galaxies, one has $z_{cr} \sim 10^{-1}$, or a distance of $\sim 300\,\mathrm{Mpc}h^{-1}$.

Now let us discuss the relation of Eq.(4.4) to the Dyer-Roeder distance.[1] To do so, we must assume the same physical situation as the one assumed in deriving the Dyer-Roeder distance. That is, we assume that light propagates only through the intergalactic space where the density is uniform and given by $\rho_{IG} = \alpha\rho_b$ ($0 \leq \alpha < 1$). Hence $\delta\rho = -(1-\alpha)\rho_b$. In addition we assume that the potential gradients can be neglected in the intergalactic space so that there is no shear contribution. Further we consider a spatially flat universe ($K = 0$) for simplicity. Then Eq.(4.4) is easily evaluated to yield

$$\begin{aligned}\frac{\delta d}{d} = &-\frac{1}{2(\sqrt{1+z}-1)}\left[(v_i)_s - (v_i)_0\right]\gamma^i + (v_i)_s\gamma^i \\ &+ 3(1-\alpha)\left[\frac{\sqrt{1+z}+1}{\sqrt{1+z}-1}\ln(1+z) - 4\right]\,.\end{aligned} \tag{4.6}$$

In accordance with the general argument used to derive Eq.(4.5), the above equation clearly shows that the Doppler term dominates for $z \lesssim v^{1/3} \sim \epsilon^{1/3}$ but becomes unimportant for $z \gtrsim v^{1/3}$.

Let us concentrate on the case when the peculiar velocities are negligible and compare our result with the Dyer-Roeder distance. The Dyer-Roeder (luminosity) distance for the spatially flat universe is given by[1]

$$d^{DR}(z;\alpha) = \frac{2}{H_0\beta}(1+z)^{3/4}\left[(1+z)^{\beta/4} - (1+z)^{-\beta/4}\right], \qquad (4.7)$$

where $\beta = \sqrt{25-24\alpha}$. It is then easily shown that our result (4.6) coincides with Eq.(4.7) in either of the limit $1-\alpha \ll 1$ or $z \ll 1$. In fact, the coincidence is much more impressive than one would formally expect; the relative error is less than 0.5% at $z=1$ and about 10% at $z=5$ even for the extreme case of $\alpha = 0$. Of course, this coincidence is not accidental. If we examine the essential assumption which lead to Eq.(4.6)(with the velocity terms neglected), we find it is the fact that the perturbation in the affine parameter distance $\delta\lambda_s$ is negligible, *i.e.*, one can use the geodesic equation on the background FRW spacetime to relate the redshift of the source object with the affine distance. This is just the assumption used to derive the Dyer-Roeder distance. Hence apart from the fact that we employed the linear approximation in evaluating the expansion of light rays, there is no essential difference between the two.

The above discussion, together with the considerations given in §3, shows the validity and the limitation of the use of the Dyer-Roeder distance. First of all, it is valid only if $\epsilon^2 \ll \kappa$ so that the contribution of the shear can be neglected. Then it is valid for $1 \gtrsim z \gtrsim z^{cr} \sim \epsilon^{1/3}$ if light rays which reach us came only through intergalactic space. Since the fact that δk^μ is small is always true even for $z \gtrsim 1$, the Dyer-Roeder distance is, in the mathematical sense, probably valid also for $z \gtrsim 1$. However, because the condition $\epsilon^2 \ll \kappa$ implies that the expected number of encounters of a light ray with galactic objects are large; $N_d \sim \kappa/\epsilon^2 \gg 1$ for $z \gtrsim 1$ (see §3), it becomes increasingly rare for a light ray to pass only through intergalactic space. Consequently, the Dyer-Roeder distance becomes physically almost useless for $z \gtrsim 1$.

4.2. Statistical consideration

We now focus on the contribution of the density term, which is seemingly not only dominant but also non-linear, to the propagation of light rays and investigate their statistical properties in terms of the distance-redshift relation, under the linear approximation, in a simple model of an inhomogeneous universe (see Ref.[11] for a similar analysis). In particular, our main intention is to show that the results are consistent with the linear approximation. Hence our basic equation is

$$I \equiv \left(\frac{\delta d}{d}\right)_{\overset{(3)}{\Delta}\Psi} = \frac{1}{\lambda_s}\int_0^{\lambda_s} d\lambda(\lambda - \lambda_s)\lambda \overset{(3)}{\Delta} \Psi \,. \tag{4.8}$$

We consider a spatially flat universe whose average density is ρ_b, in which stellar objects (galaxies) of equal mass are placed on a cubic lattice of constant comoving size, and the rest of space (intergalactic space) has a uniform density $\alpha\rho_b$. The galaxies are assumed to be spherical and uniform in density with a constant proper radius R_*. Then, we employ the so-called Wigner-Seitz approximation, frequently used in nuclear physics, and replace each cube of the lattice by a sphere of comoving radius r_0. With this approximation, the Einstein equations in each Wigner-Seitz cell become

$$\begin{aligned} \left(\frac{a'}{a}\right)^2 &= \frac{8\pi G}{3}\rho_b a^2 \,, \\ \overset{(3)}{\Delta} \Psi &= 4\pi G(1-\alpha)\rho_b a^2 \left[\frac{r_0^3 a^3}{R_*^3}\theta(R_* - ar) - 1\right] \,; \quad r \leq r_0 \,, \end{aligned} \tag{4.9}$$

where $\theta(x)$ is the step function and the center of the cell is chosen to be the origin.

We divide the integral in Eq.(4.8) into intervals $[\lambda_i,\ \lambda_i + \Delta\lambda]$ $(\Delta\lambda \sim r_0 \ll \lambda_s)$;

$$I = \sum_i I_i \,; \quad I_i = \frac{\lambda_i(\lambda_i - \lambda_s)}{\lambda_s} \int_{\lambda_i}^{\lambda_i + \Delta\lambda} d\lambda \overset{(3)}{\Delta} \Psi \,. \tag{4.10}$$

In order to avoid inessential complications, we further replace the sphere of radius

r_0 by a cylinder of radius r_0 and height $\Delta\lambda$ such that the volumes of the two are equal and assume that a light ray comes into the cylinder parallel to its axis. Thus we take $\Delta\lambda = (4/3)r_0$. Then in each interval, if the light ray passes through the galaxy, on average it will give the integral the contribution,

$$\begin{aligned}\int_{\lambda_i}^{\lambda_i+\Delta\lambda} d\lambda \overset{(3)}{\Delta} \Psi =& 4\pi G(1-\alpha)\rho_b a^2 \frac{(ar_0)^3}{R_*^3} \times \frac{2}{3} \times 2\frac{R_*}{a} \\ =& \frac{16}{3}\pi G(1-\alpha)\rho_b a^2 r_0 \frac{(ar_0)^2}{R_*^2}\,.\end{aligned} \tag{4.11}$$

This occurs with the probability $p = R_*^2/(ar_0)^2$. On the other hand, regardless of whether the ray hits the galaxy or not, there is always a contribution to the integral given by

$$\begin{aligned}\int_{\lambda_i}^{\lambda_i+\Delta\lambda} d\lambda \overset{(3)}{\Delta} \Psi =& -4\pi G(1-\alpha)\rho_b a^2 \times \frac{4}{3} r_0 \\ =& -\frac{16}{3}\pi G(1-\alpha)\rho_b a^2 r_0\,.\end{aligned}$$

Thus we may rewrite I_i as $I_i = c_i x_i$ where

$$\begin{aligned} c_i =& 8(1-\alpha)\frac{r_0}{\lambda_s}\frac{\lambda_i(\lambda_i-\lambda_s)}{(\eta_0-\lambda_i)^2}\,, \\ x_i =& \left[\frac{1}{p_i}\theta(R_* - a_i r) - 1\right]\;; \quad p_i = \frac{R_*^2}{(a_i r_0)^2}\,,\end{aligned} \tag{4.12}$$

and the back ground equations ($\rho_b \propto a^{-3}$, $a \propto \eta^2$) have been used. Then the probability distribution of x_i is given by

$$P_i(x_i) = p_i \delta\left(x_i - \frac{1-p_i}{p_i}\right) + (1-p_i)\delta(x_i+1)\,, \tag{4.13}$$

and the probability distribution of $I = \sum_i c_i x_i$ is expressed as

$$P(I)dI = \int \prod_i dx_i P_i(x_i)\, \delta\left(I - \sum_i c_i x_i\right) dI\,. \tag{4.14}$$

We note that Eq.(4.13) implies $\langle x_i \rangle = 0$, which in turn implies $\langle I \rangle = 0$, the result in accordance with the argument given in §3.

To evaluate the probability distribution function $P(I)$, we consider the characteristic function of $P(I)$. We find

$$
\begin{aligned}
F(q) \equiv & \int_{-\infty}^{\infty} P(I) e^{iqI} dI \\
= & \prod_i \left[\int_{-\infty}^{\infty} dx_i \, P(x_i) e^{iqc_i x_i} \right] \\
= & e^{-iq\sum_i c_i} \exp\left(\sum_i \ln\left[1 + p_i(e^{iq(c_i/p_i)} - 1)\right] \right) .
\end{aligned}
\tag{4.15}
$$

Using the fact $p_i \ll 1$, the above can be approximated as

$$
F(q) = e^{-iq\sum_i c_i} \exp\left[\sum_i p_i(e^{iq(c_i/p_i)} - 1) \right] . \tag{4.16}
$$

Hence, at $q \ll 1$, we obtain

$$
\begin{aligned}
F(q) \approx & e^{-iq\sum_i c_i} \exp\left[\sum_i \left(ic_i q - \frac{c_i^2}{2p_i} q^2 \right) \right] \\
= & \exp\left[-\sum_i \frac{c_i^2}{2p_i} q^2 \right] \\
\equiv & \exp\left[-\frac{1}{2} f(z) q^2 \right] ,
\end{aligned}
\tag{4.17}
$$

where from Eqs.(4.12), the function $f(z)$ is given by

$$
\begin{aligned}
f(z) = & (1-\alpha)^2 \frac{34}{5} \frac{H_0 R_0^3}{R_*^2} \left(\frac{\sqrt{1+z} - 1}{\sqrt{1+z}} \right)^3 \\
\approx & 0.1(1-\alpha) h^{-2} \left(\frac{50\,\mathrm{kpc}}{R_*} \right)^2 \left(\frac{R_0}{0.5\,\mathrm{Mpc}h^{-1}} \right)^3 \left(\frac{\sqrt{1+z} - 1}{\sqrt{1+z}} \right)^3 ,
\end{aligned}
\tag{4.18}
$$

where $R_0 = a_0 r_0$ is the present proper radius of the comoving cell, *i.e.*, half the mean separation distance of galaxies and $H_0 = 100\,h\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$ is the Hubble constant. On the other hand, for $q \to \infty$, the exponential factor in the exponent of $F(q)$ oscillates rapidly and vanishes on average. Hence we find

$$
\begin{aligned}
F(q) \approx & e^{-\sum_i p_i} e^{-iq\sum_i c_i} \\
\equiv & e^{-g(z)} e^{+iqI_0(z)} ,
\end{aligned}
\tag{4.19}
$$

where $g(z)$ and $I_0(z)$ are given by

$$
\begin{aligned}
g(z) =& \frac{R_*^2}{2R_0^3 H_0}\left[(1+z)^{3/2}-1\right] \\
\approx & 30h^2\left(\frac{R_*}{50\,\mathrm{kpc}}\right)^2\left(\frac{0.5\,\mathrm{Mpc}h^{-1}}{R_0}\right)^3\left[(1+z)^{3/2}-1\right] , \\
I_0(z) =& 3(1-\alpha)\left[\frac{\sqrt{1+z}+1}{\sqrt{1+z}-1}\ln(1+z)-4\right] ,
\end{aligned}
\tag{4.20}
$$

where I_0 is just the Dyer-Roeder part of the distance-redshift relation given in Eq.(4.6). It is also clear that $g(z)$ represents the "optical depth" associated with encounters of light rays with galaxies.[7]

Now combining Eqs.(4.17) and (4.19), and taking into account the facts that $\langle 1\rangle = \int P(I)dI = 1$ and $\langle I\rangle = \int I\, P(I)dI = 0$, we deduce that

$$
\begin{aligned}
P(I) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} F(q)e^{-iqI}dq \\
\approx & (1-e^{-g(z)})\frac{1}{\sqrt{2\pi f(z)}}\exp\left[-\frac{(I-I_1(z))^2}{2f(z)}\right] \\
& + e^{-g(z)}\delta\Big(I-I_0(z)\Big) ,
\end{aligned}
\tag{4.21}
$$

where I_1 is given by

$$
I_1(z) = -\frac{1}{e^{g(z)}-1}I_0(z) . \tag{4.22}
$$

The interpretation of the above distribution function is very easy. The term proportional to the delta function represents light rays which never encounter a galaxy, hence giving the Dyer-Roeder distance for I. Of course, the appearance of the delta function is due to the simplification we adopted for our model and not real. For example, any contribution from the shear or the density fluctuations in intergalactic space will develop a finite dispersion. The other term of a Gaussian form in Eq.(4.21) represents those light rays which encounter galaxies at least once. Thus $I_1(z)$ and $f(z)$ are the mean value and the dispersion, respectively, of I for such rays.

What we have to show is that the probability distribution function (4.21) is consistent with the assumption used to derive it, *i.e.*, the linear approximation. The consistency demands $\langle I^2 \rangle \ll 1$. The expectation value $\langle I^2 \rangle$ is easily calculated to be

$$\langle I^2 \rangle = \frac{1}{e^{g(z)} - 1} I_0(z)^2 + \left(1 - e^{-g(z)}\right) f(z) . \tag{4.23}$$

It is easy to see that the first term proportional to I_0^2 is always very small at any redshift. Incidentally, this implies I_1 is very small at any redshift (an order-of-magnitude evaluation shows $|I_1|_{max} = O(\epsilon^4/\kappa^2)$ at $z \sim \epsilon^2/\kappa$). The smallness of the second term will be guaranteed if $f(z) \ll 1$ for any z. As given in Eq.(4.18), for characteristic values of R_* and R_0, we find $f(z)$ is always smaller than unity (note that Eq.(4.18) is indeed in the form $f(z) = O(\epsilon^2/\kappa)$, in agreement with the order estimate given in §3). Thus not only the general order-of-magnitude argument of §3 but also the specific model consideration of this section support strongly the validity of the linear approximation for study of the propagation of light rays in a highly inhomogeneous universe, and hence the validity of the formula (4.1).

5. Conclusions

We have carefully investigated the propagation of light rays in a realistic inhomogeneous universe. We have found that the linear approximation to the propagation equations is valid even in a universe with very high density contrast. Then we have derived a general expression for the distance-redshift relation (4.1). It should be stressed that the relation has been derived totally within the framework of general relativity without any *ad hoc* assumption.

Based on the drived relation, we have been able to clarify the validity and the limitation of the use of the Dyer-Roeder distance; it is numerically valid only for $1 \gtrsim z \gtrsim \epsilon^{1/3}$ if the light rays which reach us have traveled only through intergalactic space. In more general situations, one has to use Eq.(4.1) to construct the distance-redshift relation. The equation can be evaluated explicitly once the density and velocity distributions are given throughout the region of interest.

We have given some plausible arguments that thus constructed relation is valid for any redshift provided $\epsilon^2 \ll \kappa$ is satisfied and the condition gives practically no restriction for its applicability to the real universe as far as regions of interest are above galactic scales. Although the validity of Eq.(4.1) should be checked more rigorously (for example, by a careful numerical analysis which takes full account of non-linearity in the propagation equations), if it is indeed justified, it will play a fundamental role in the correct interpretation of cosmological observations.

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