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QUANTUM GROUPS, BRAIDING MATRICES and COSET MODELS [1,2]

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Abstract

We discuss a few results on quantum groups in the context of rational conformal field theory with underlying affine Lie algebras. A vertex-height correspondence - a well-known procedure in solvable lattice models - is introduced in the WZW theory. This leads to a new definition of chiral vertex operator in which the zero mode is given by the q-Clebsch Gordan coefficients. Braiding matrices of coset models are found to factorize into those of the WZW theories. We briefly discuss the construction of the generators of the universal enveloping algebra in Toda field theories.

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By now, there are two apparently distinct places in theoretical physics in which the quantum group *i.e.* the q- deformation of the universal enveloping algebra [1,2,3] provides a relevant structure. The one is physics of quantum spin chains (or equivalently the classical statistical mechanics) associated with the trigonometric solutions[4] of the Yang-Baxter equation[5,6]. This is the original place in which the quantum group structure was found. The underlying system can be viewed as a lattice-regulated quantum field theory. The other appears as monodromy properties[7,8] of a class of rational conformal field theories with underlying affine Lie algebras. The most concrete realization can be found in the WZW theory. It is well-known that conformal field theories predict finite size corrections to the macroscopic quantities of the system in the large volumes. On the other hand, there are in fact computations of finite size corrections based on Bethe ansatz in solvable lattice models, which agree with conformal field theory predictions. It is, therefore, not inconceivable to suspect that these two applications of the quantum groups are in fact related: the latter being obtained as the continuum limit of the former. I will not try to pursue this program in this talk. What I will discuss instead is a few results[9] which are inspired and transplanted from the structure of the solvable lattice models, but which have their own rationales solely in the context of rational conformal field theories. First, I will give a brief account on the construction of the guantum group generators in the generalized Toda system (GTS) as quantum field theory[10]. Then, without a logical connection, I turn to the discussion of the monodromy properties of WZW theory. I take the differential equation of Knizhinik and Zamolodchikov as a starting point and present a few attempts mentioned. The monodromy properties of the coset theories can be obtained from those of the WZW theory. Finally, I will briefly discuss a perspective for the more logical connection as well as the construction of universal enveloping algebra in Toda field theories[11]. Further references on the subjects discussed here can be found in [9,10].

1)° In a class of vertex models where fundamental variables are arrows taking the value $1 \sim N$, Yang-Baxter relation takes the form

$$\begin{aligned} X_j\, X'_{j+1}\, X''_j &= X''_{j+1}\, X'_j\, X_{j+1} \ , \\ X_i\, X'_j &= X'_j\, X_i \ . \end{aligned} \tag{1}$$

Here, the operators $X_j \equiv X_j(u)$, $X'_j \equiv X_j(u')$, and $X''_j \equiv X_j(u'' = u' - u)$ act nontrivially on the arrows in the j th column and the $j+1$ th column ($-M \leq j \leq M$) in a fixed row and take them to the ones in the next row. (See Fig.1). One-parameter family of the matrix elements labeled by u represents local Boltzman weights of the corresponding arrow configuration around a vertex. Here, we have drawn a row (fixed time slice) 45 degrees tilted to the x axis. The resultant time flow is to the southwest.

One way to characterize GTS is that the above infinite dimensional algebraic relation reduces to a finite dimensional algebra (or, to be more accurate, its completion) In A_n case ($N = n + 1$), eq. (1) reduces to $X_j = \rho(1 + y(u)U_j)$ with

$$U_j U_{j+1} U_j - U_j = U_{j+1} U_j U_{j+1} - U_{j+1} \ , \tag{2}$$

$$U_j^2 = 2\cosh\lambda U_j \ , \tag{3}$$

$$U_i U_j = U_j U_i \ , \ \text{for } i \neq j \pm 1 \ . \tag{4}$$

Here, λ is a coupling (or parameter) of the theory and ρ is an arbitrary constant. We also introduce $q \equiv -e^{\lambda}$, and $\Delta \equiv -\cosh\lambda$. The regions $-1 < \Delta < 1$ and $\Delta < -1$ represent respectively critical and antiferromagnetic regimes. In order not to introduce further relationship among U_j's, we have adopted here fixed boundary condition: $\mu'_{-M} = \mu_{M+1} =$ the highest value.

Up to a trivial rescaling, Equations (2), (3), (4) are defining relations of the Hecke algebra $\cup_m H_m$. The function $y(u)$ sets a momentum-rapidity relation of the system :

$$y(u) \equiv \frac{q(\zeta - 1)}{q^2\zeta - 1} \equiv e^{ip} \ . \tag{5}$$

Here, we introduced $\zeta \equiv e^{-2u}$. For $\Delta < -1$, u imaginary (real) corresponds to real (imaginary) p i.e. Minkowski (Euclidean) field theory. For $-1 < \Delta < 1$, set $\lambda = i\mu$ with μ real, and $u = i\mu/2 - \alpha/2$. Real α corresponds to $-(\pi - \mu) < p < (\pi - \mu)$. The solution to eq. (1) in A_n case is expressible as

$$U_j = \sum_{\alpha,\beta=1,\alpha\neq\beta}^{N} E^j_{\alpha,\beta} \otimes E^{j+1}_{\beta,\alpha} - q^{-1} \sum_{\alpha,\beta=1,\alpha>\beta}^{N} E^j_{\alpha,\alpha} \otimes E^{j+1}_{\beta,\beta} - q \sum_{\alpha,\beta=1,\alpha<\beta}^{N} E^j_{\alpha,\alpha} \otimes E^{j+1}_{\beta,\beta} \ . \tag{6}$$

It is worthwhile to note that $\lim_{\rho=iq^{-1}, u\to+\infty} X_j(u) = i(U_j + 1/q) \equiv \sigma^j$, safely provides an operator obeying Braid relation :

$$\sigma^j\sigma^{j+1}\sigma^j = \sigma^{j+1}\sigma^j\sigma^{j+1} , \tag{7}$$

$$\sigma^i\sigma^j = \sigma^j\sigma^i , \quad i \neq j \pm 1 . \tag{8}$$

It is well known that a sequence of mutually commuting conserved charges is generated by one-parameter family of commuting transfer matrices. In the current formalism, the transfer matrix is simply

$$T(\zeta) = \lim_{M\to+\infty} X_{-M} \cdots X_M \quad . \tag{9}$$

The Hamiltonian can be defined to be the lowest nontrivial term in the expansion of $T(\zeta)$ around $\zeta = 1$. It consists of the nearest neighbour interations only. In A_n case,

$$H = \sum_j \mathcal{H}_j \equiv \frac{1}{q - q^{-1}} \sum_j U_j \ . \tag{10}$$

This identifies the generators of the Hecke algebra with the Hamitonian density. The higher order terms tell us

$$\sum_j U_j \ , \sum_j U_j U_{j+1} \ , \cdots, \sum_j U_j U_{j+1} \cdots U_{j+\ell}, \ \cdots \qquad , \tag{11}$$

are a set of bases for the conserved charges.

Another useful bases for the sequence of conserved charges are

$$Q_n = \oint \frac{d\zeta}{2\pi i} \zeta^{n-1} S T(\zeta) \ , n \in Z \ . \tag{12}$$

Here, S is the shift operator which shifts the arrow labeling by one unit. The boost operator $\mathcal{L}_0 = \sum_j j\mathcal{H}_j$ acts on Q_n as

$$[\mathcal{L}_0, Q_n] = nQ_n \quad . \tag{13}$$

Let us now examine the symmetries of the system. For simplicity, we restrict ourselves to A_n case. An inspection on expression (6) immediately tells that there are rather obvious n abelian generators which commute with Hamiltonian and the transfer matrix :

$$h_\ell = \sum_j h_\ell^j , \quad 1 \le \ell \le n \ , \tag{14}$$

$$\exp h_\ell^j \equiv \mathrm{diag}\left(1, \cdots, \frac{1}{q}, q, 1, \cdots\right)^j \ . \tag{15}$$

What is much less obvious is that, for an arbitrary value of q, we can construct conserved operators which are analogs of the step operators of $SU(n+1)$. Let

$$\begin{aligned} e^j_{\ell,\ell+1} &\equiv \exp\left(\sum_{i=-\infty}^{j-1} h^i_\ell\right) E^j_{\ell,\ell+1} = f^j_\ell \ , \\ e^j_{\ell+1,\ell} &\equiv \exp\left(-\sum_{i=j+1}^{\infty} h^i_\ell\right) E^j_{\ell+1,\ell} = e^j_\ell \ . \end{aligned} \tag{16}$$

The generators corresponding to simple roots are $e_\ell = \sum_j e^j_\ell$ and $f_\ell = \sum_j f^j_\ell$. The choice of the exponential factors is crucial in proving vanishing commutator with the Hamiltonian and the transfer matrix:

$$\begin{aligned} [\,H, e_\ell\,] &= [\,H, f_\ell\,] = 0 \\ [\,T(\zeta), e_\ell\,] &= [\,T(\zeta), f_\ell\,] = 0 \ . \end{aligned} \tag{17}$$

At $\Delta = 1$, e_ℓ's and f_ℓ's become the ordinary group generators of $SU(n+1)$. At $\Delta = 0, n = 1$, eq. (16) reduces to the familiar Jordan -Wigner transformation for the Pauli spin operators. The generators h_ℓ, e_ℓ, and f_ℓ constructed above are the Chevally generators of $\mathcal{U}(A_n)$ and form an algebra

$$\begin{aligned} \exp(h_\ell/2)\, e_{\ell'} \exp(-h_\ell/2) &= q^{a_{\ell,\ell'}/2} e_{\ell'} \\ \exp(h_\ell/2)\, f_{\ell'} \exp(-h_\ell/2) &= q^{-a_{\ell,\ell'}/2} f_{\ell'} \\ [e_\ell, f_{\ell'}] &= \delta_{\ell,\ell'} \frac{\exp h_\ell - \exp -h_\ell}{q - q^{-1}} \ . \end{aligned} \tag{18}$$

Here, the matrix $a_{\ell,\ell'}$ is a Cartan matrix for A_n. The remaining generators of $\mathcal{U}(g)$ are given recursively by $t_{\ell,\ell'} = t_{\ell,\ell''} t_{\ell'',\ell'} - q t_{\ell'',\ell'} t_{\ell,\ell''}$ $\quad (\ell \le \ell'' \le \ell')$ with $t_{\ell,\ell+1} \equiv e_\ell$ and $t_{\ell,\ell-1} \equiv f_\ell$. This choice is dictated by the q-analog of the Chevally relation.

We have presented here the quantum group structure of GTS in the vertex representation. The transition to the path representation is made through the q-Clebsch Gordan coefficients as an intertwiner. This is a special case of the well-known vertex-height correspondence. We will see the same structure in the WZW theory.

2)° Monodromy properties of Wess-Zumino-Witten model: Let $\hat{g}$ be a simple affine Lie algebra and g_α^Λ be a primary field of the WZW theory transforming in the representation labelled by the highest weight Λ. Here, a weight index α, which is generally denoted by a set of integers, labels the components of the primary fields. The object we study is the n-point correlation function

$$\mathcal{G}_n \begin{pmatrix} \Lambda_1\Lambda_2\cdots\Lambda_n \\ \alpha_1\alpha_2\cdots\alpha_n \end{pmatrix} (z_1, z_2, \cdots z_n; \bar{z}_1, \bar{z}_2, \cdots \bar{z}_n) \equiv \langle g_{\alpha_1}^{\Lambda_1}(z_1, \bar{z}_1)\, g_{\alpha_2}^{\Lambda_2}(z_2, \bar{z}_2) \cdots\, g_{\alpha_n}^{\Lambda_n}(z_n, \bar{z}_n)\rangle \quad , \tag{19}$$

obeying the differential equation of Knizhnik and Zamolodchikov[12]:

$$\left(\frac{\partial}{\partial z_i} + \sum_{j(\neq i)} \frac{2}{(k+\tilde{h})\theta^2} \frac{1}{z_i - z_j} t_i^a \otimes t_j^a\right)_{\alpha_1,\alpha_2,\cdots,\alpha_n}^{\alpha_1',\alpha_2',\cdots,\alpha_n'} \mathcal{G}_n \begin{pmatrix} \Lambda_1\Lambda_2\cdots\Lambda_n \\ \alpha_1'\alpha_2'\cdots\alpha_n' \end{pmatrix} = 0 \tag{20}$$

Here, k and $\tilde{h}$ are the level of the affine Lie algebra and the dual Coxeter number respectively, and θ^2 is defined through $f_{ac}^d f_{bd}^c = -\tilde{h}\theta^2 g_{ab}$. The correlator is originally defined in the region $|z_1| < |z_2| < \cdots < |z_n|$ and analytically continued to the other regions. The differential equation is, therefore, defined over the domain $X_n \equiv \{(z_1, \cdots z_n) \in \mathcal{C}_n \,;\, z_i \neq z_j \text{ if } i \neq j\}$. The fundamental group of X_n is the pure braid group with n strands. From now on, we suppress the dependence on the $\bar{z}_i$'s. Eq. (20) can be written as

$$(d+\omega)_{\alpha_1,\alpha_2,\cdots,\alpha_n}^{\alpha_1',\alpha_2',\cdots,\alpha_n'} \mathcal{G}_n \begin{pmatrix} \Lambda_1\Lambda_2\cdots\Lambda_n \\ \alpha_1'\alpha_2'\cdots\alpha_n' \end{pmatrix} (z_1, z_2, \cdots, z_n) = 0 \quad . \tag{21}$$

by introducing a one-form $\omega = \sum_{1\leq i<j\leq n} \frac{2}{(k+\tilde{h})\theta^2} t_i^a \otimes t_j^a d\log(z_i - z_j)$, taking the value in

$End\left(V^{\Lambda_1} \otimes \cdots V^{\Lambda_n}\right)$. The solution can formally be written as

$$\mathcal{G}_n \begin{pmatrix} \Lambda_1 \Lambda_2 \cdots \Lambda_n \\ \alpha_1 \alpha_2 \cdots \alpha_n \end{pmatrix} (z_1, z_2, \cdots z_n) = \left(\mathcal{P} \exp\left(-\int_{\mathbf{z}_0}^{\mathbf{z}} \omega\right) \mathcal{G}_n\right) \begin{pmatrix} \Lambda_1 \Lambda_2 \cdots \Lambda_n \\ \alpha_1 \alpha_2 \cdots \alpha_n \end{pmatrix} (z_{10}, z_{20}, \cdots z_{n0}) \tag{22}$$

with respect to the one at base point $\mathbf{z}_0 = (z_{10}, z_{20}, \cdots z_{n0})$ and the symbol $\mathcal{P}$ implies the path ordering of the exponential. We introduce $q \equiv \exp(-\frac{\pi i}{k+h})$.

The monodromy of the conformal block around the singularity $z_i = z_{i+1}$ is defined to be $\mathcal{M}^{(i)} = \mathcal{P} \exp\left(-\oint_{\gamma_0^{(i)}} \omega\right)$. Here $\gamma_0^{(i)}$ denotes a closed path which starts and ends at base point $\mathbf{z}_0$ and goes around the line $z_i = z_{i+1}$. The braiding matrix $\sigma^{(i)}$ is defined to be the square root of $\mathcal{M}^{(i)}$ times the permutation matrix. It is straightforward to evaluate the eigenvalues of a *particular* $\sigma^{(i)}$ from the above expression. But such evaluation does not provide a conceptual explanation of the coincidence noted by many people. What we would like to explain is that *all* elements of $\sigma^{(i)}$'s are, up to a similarity transformation, equal to the vertex Boltzman weight of the corresponding GTS realizing the quantum group. For definiteness, we consider the case A_{N-1} in which all Λ's are in the fundamental representation. Here, we only give essential logical points. For a full explanation, see in ref.[8,9].

The first step is to show that $\sigma^{(i)}$'s form a representation of Hecke algebra H_n: $(\sigma_i - q)(\sigma_i + q^{-1}) = 0$. For that purpose, unambiguous bases for the conformal blocks must first be determined. The ordinary definition of chiral vertex operator[7] provides the path bases which qualify this. In this bases, all elements σ_r's are related to σ_1 simply by a similarity transformation given by the fusion matrix[13]. These arguments are essentially due to Kohno[8]. The second step is the point we already discussed before: the $\check{R}$ matrix or vertex Boltzman weight in the integrable lattice models at infinite spectral parameter in general provides a representation of the braid group B_n through $End\left(V^{\Lambda_1} \otimes \cdots V^{\Lambda_n}\right)$. In the A_{N-1} case, it also provides a representation of Hecke algebra. The third step, which is due to Wenzl[14], is that any finite dimensional representation of Hecke algebra can be obtained through this procedure.

The above argument is sufficient to tell us that there exists a similarity transformation S which brings all elements of the braid group into the $\check{R}$ matrix of the quantum group:

$$\sigma^{(i)\alpha_i}_{\alpha'_i}|_{\alpha'_{i+1}}{}^{\alpha_{i+1}}(\mathcal{S}\mathcal{G}_n)\begin{pmatrix}\Lambda_1\Lambda_2\cdots\Lambda_n\\ \alpha_1\alpha_2\cdots\alpha_n\end{pmatrix}=\mathcal{S}\check{R}(u\to\infty)_{\alpha'_i}{}^{(i)\ \alpha_i}|_{\alpha'_{i+1}}{}^{\alpha_{i+1}}(\mathcal{G}_n)\begin{pmatrix}\Lambda_1\Lambda_2\cdots\Lambda_n\\ \alpha_1\alpha_2\cdots\alpha_n\end{pmatrix}\quad . \tag{23}$$

On the other hand, the existence of the operator formalism of WZW theory guarantees the factorization of n-point correlator into four point blocks, and we may write

$$g^{\Lambda_i}_{\alpha_i}(z_i)g^{\Lambda_{i+1}}_{\alpha_{i+1}}(z_{i+1})=\sigma^{(i)\alpha'_i}_{\alpha_i}|_{\alpha_{i+1}}{}^{\alpha'_{i+1}}g^{\Lambda_{i+1}}_{\alpha'_i}(z_{i+1})g^{\Lambda_i}_{\alpha'_{i+1}}(z_i)\quad . \tag{24}$$

We have seen that the product of the primary fields of WZW theory carrying weight indices of the classical Lie algebra has turned out to be the representation space (of the vertex type) for the Hecke algebra and therefore for quantum group. The intertwiner to the path representation is given by q- Clebsch-Gordan coefficients.

Let the composed operator in eq. (24) be acting on a direct sum of the irreducible $\hat{g}$ modules $\mathcal{H}_d$'s with the highest weight Λ_d's. The above discussion tells us that , in accordance with the decomposition of the tensor module ,

$$\mathcal{H}_a\otimes\mathcal{H}_i=\sum_b\mathcal{H}_b,\quad \mathcal{H}_b\otimes\mathcal{H}_{i+1}=\sum_d\mathcal{H}_d\quad , \tag{25}$$

it is legitimate to expand eq. (24) as

$$g^{(\Lambda_i)}_{\alpha_i}(z_i)g^{(\Lambda_{i+1})}_{\alpha_{i+1}}(z_{i+1})=\sum_{a,b,d,}\mathcal{V}^{(\Lambda_i)}_{a,b}(z_i)\mathcal{V}^{(\Lambda_{i+1})}_{b,d}(z_{i+1})\mathcal{S}_{\alpha_i,\alpha_{i+1}}{}^{\alpha'_i,\alpha'_{i+1}}\Phi^{(\Lambda_i)}_{a,b}{}_{\alpha'_i}\Phi^{(\Lambda_{i+1})}_{b,d}{}_{\alpha'_{i+1}}c_b^{(a,d,\Lambda_i,\Lambda_{i+1})}\quad . \tag{26}$$

Here, we have introduced an operator $\mathcal{V}^{(\Lambda_i)}_{a,b}(z_i)$ which might be called a q-version of the chiral vertex operator discussed before. This operator carries over the normalization of the ordinary chiral vertex operator, but has no reference to the weight indices. A set of coefficients $\Phi^{(\Lambda_i)}_{a,b}{}_{\alpha_i}=<a,M_i\mid b,\ M_i-\alpha_i\ ;\ \Lambda_i,\ \alpha_i>$ is a q-version of the Clebsch-Gordan coefficients. They serve as basis vectors of flat sections (zero mode part) of the trivial vector bundle over X_n. The coefficient $c_b^{(a,d,\Lambda_i,\Lambda_{i+1})}$ reflects representation dependent normalization of the chiral vertex operators. [2]

Eq. (26) permits us to translate eqs. (23),(24), into the exchange algebra of the

[2] To determine this, one usually has to solve the connection problem of the attendant differential equation. See [7].

q-vertex operators :

$$\mathcal{V}_{a,b}^{(\Lambda_i)}(z_i)\mathcal{V}_{b,d}^{(\Lambda_{i+1})}(z_{i+1}) = \sum_{b'} B^{(\Lambda_i,\Lambda_{i+1})}\begin{bmatrix} a,d \\ b,b' \end{bmatrix} \mathcal{V}_{a,b'}^{(\Lambda_{i+1})}(z_{i+1})\mathcal{V}_{b',d}^{(\Lambda_i)}(z_i) \ ,$$

$$B^{(\Lambda_i,\Lambda_{i+1})}\begin{bmatrix} a,d \\ b,b' \end{bmatrix} = q^{-3/2}(c_b^{(a,d,\Lambda_i,\Lambda_{i+1})})^{-1} W^{(\Lambda_i,\Lambda_{i+1})}\begin{bmatrix} a,b' \\ b,d \end{bmatrix}(u\to\infty)\, c_{b'}^{(a,d,\Lambda_i,\Lambda_{i+1})} \tag{27}$$

The braiding matrix $B^{(\Lambda_i,\Lambda_{i+1})}\begin{bmatrix} a,d \\ b,b' \end{bmatrix}$ is, up to the diagonal similarity transformation due to the normalization of the vertex operator, given by the face Boltzmann weight $W^{(\Lambda_i,\Lambda_{i+1})}\begin{bmatrix} a,b' \\ b,d \end{bmatrix}(u\to\infty)$ of the corresponding GTS at infinite spectral parameter. In the case where all representations Λ_i's are in the fundamental representation, the explicit answer for the Boltzman weight can be extracted from ref. [15]. In the simplest case $A_1^{(1)}$ in which Λ_i's are in the spin 1/2 representation, the answer $B^{(1/2,1/2)}\begin{bmatrix} j_0, & J \\ j, & j' \end{bmatrix}$ agrees with the result by Tsuchiya and Kanie [7] :

$$\begin{array}{ll} q^{-3/2}\begin{pmatrix} q^{-2j_0}/[2j_0+1], & c_{j_0+1/2}^{-1}c_{j_0-1/2}q\sqrt{[2j_0][2j_0+2]}/[2j_0+1] \\ c_{j_0-1/2}^{-1}c_{j_0+1/2}q\sqrt{[2j_0][2j_0+2]}/[2j_0+1], & -q^{2j_0+2}/[2j_0+1] \end{pmatrix} & \text{for} \quad 1/2\le j_0=J\le k/2 \\ \qquad -q^{1/2}\delta_{j,j'} & \text{for} \quad J=j_0\pm 1/2 \\ \qquad q^{-3/2}\delta_{j,j'} & \text{for} \quad j_0=J=0,\ k/2\ . \end{array} \tag{28}$$

Here, $c_{j_0+1/2}=\Gamma(-\frac{2j_0+1}{k+2})/\sqrt{\Gamma(-\frac{2j_0+2}{k+2})\Gamma(-\frac{2j_0}{k+2})}$, and $c_{j_0-1/2}=\Gamma(\frac{2j_0+1}{k+2})/\sqrt{\Gamma(\frac{2j_0+2}{k+2})\Gamma(\frac{2j_0}{k+2})}$, and the q-number j is defined by $[j]=\frac{q^j-q^{-j}}{q-q^{-1}}$.

Among the primary fields of of a given RCFT, the most relevant operator i.e. the one carrying the lowest conformal dimension has been given a special meaning: we can regard it as an elementary field out of which the rest of the primary fields is expressed as its composites. The formula (28) given above is for those cases in which the external primary fields are the most relevant ones. The braiding matrices for the arbitrary primary fields can be obtained from those for the most relevant ones. This is certainly true for WZW and the coset $\frac{G(k)\otimes G(1)}{G(k+1)}$. All primary fields can then be obtained as repeated products of the most relevant ones. The problem now is to find braiding matrices for these "composite" (higher spin) fields. The relevant matrices satisfying the braid relation can be obtained from the (trigonometric) solution of the Yang-Baxter equation for the higher representations by sending the spectral parameter to infinity. There exists a well-known procedure called fusion procedure in the integrable lattice models which generates these solutions from the fundamental ones[16]. We will not review it here. The solutions essentially consist of product of R matrices with prescribed shifts of the spectral parameter. The product begins and ends with projectors. The identical procedure, modulo the problem of the phase, can be implemented solely in the context of RCFT. In Fig. 2, we indicate how the procedure goes through. Take, for instance, a five point block and regard that $\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{6}$, are most relevant fields. We obtain

$$B^{(\mathbf{2},\mathbf{3})}\begin{bmatrix} \mathbf{1},\mathbf{4} \\ a,b \end{bmatrix} = \sum_{c,d} F^{-1\ (\mathbf{5},\mathbf{6})}\begin{bmatrix} a,\mathbf{4} \\ c,\mathbf{3} \end{bmatrix} B^{(\mathbf{2},\mathbf{5})}\begin{bmatrix} \mathbf{1},c \\ a,d \end{bmatrix} B^{(\mathbf{2},\mathbf{6})}\begin{bmatrix} d,\mathbf{4} \\ c,b \end{bmatrix} F^{(\mathbf{5},\mathbf{6})}\begin{bmatrix} \mathbf{1},b \\ d,\mathbf{3} \end{bmatrix} . \tag{29}$$

We represent this equation by Fig. 3. The fusion matrix F in eq. (29) play the role of projection and inclusion operators. They are the 6-j symbols of the underlying quantum group.

3)° Monodromy properties of coset models: Coset models form an interesting class of RCFT. Here we would like to show how the braiding properties of the G/H coset models can be obtained from the ones of the G-WZW theory and the ones of H-WZW theory. For definiteness, we consider a coset $G/H = \frac{A^{(1)}_{N-1}(k)\otimes A^{(1)}_{N-1}(1)}{A^{(1)}_{N-1}(k+1)}$ with diagonal embedding. The arguments $k, 1$, and $k+1$ refer respectively to the levels of the Kac-Moody algebras.

The factorization formula of the characters by GKO implies that , for a given primary state in the coset theory under consideration and a primary state in H theory, one can find a unique state in G theory. This state, however, is not necessarily primary: the diagonal embedding of the H theory into G theory is in the sense of Kac-Moody modules. A point worth making here is that the monodromy properties of the conformal block is insensitive to this ambiguity as conformal dimensions for the primary fields and the ones for their descendents differ only by integers. The formula we give below should be understood in this sense. One can also show the factorization of fusion coefficients, starting from the Verlinde's formula[17].

Therefore, the q-vertex operator of the G theory introduced in eq. (26) naturally factorizes into the one of the H theory and the one of the coset theory. We may, therefore, write

$$\mathcal{V}_{\mathbf{a}_1,\mathbf{a}_2;\ \mathbf{b}_1,\mathbf{b}_2}^{(\Lambda,1)}(z) = \sum_{a,b,\lambda} V_{a,b}^{(\lambda)}(z)\, P_{\alpha,\beta}^{(\sigma)}(z) \quad . \tag{30}$$

Here, α, β and σ respectively denote triplets of integers $\alpha = \begin{pmatrix} \mathbf{a}_1,\mathbf{a}_2 \\ a \end{pmatrix}$, $\beta = \begin{pmatrix} \mathbf{b}_1,\mathbf{b}_2 \\ b \end{pmatrix}$, and $\sigma = \begin{pmatrix} \Lambda,1 \\ \lambda \end{pmatrix}$.

Let the exchange algebra of the coset theory be

$$P_{\alpha,\ \beta}^{(\sigma_1)}(z_1)P_{\beta,\ \delta}^{(\sigma_2)}(z_2) = \sum_{\beta'} B_{G/H}^{(\sigma_1,\sigma_2)} \begin{bmatrix} \alpha, \delta \\ \beta, \beta' \end{bmatrix} P_{\alpha,\ \beta'}^{(\sigma_2)}(z_2)P_{\beta',\ \delta}^{(\sigma_1)}(z_1) \quad . \tag{31}$$

Here, $B_{G/H}^{(\sigma_1,\sigma_2)} \begin{bmatrix} \alpha, \delta \\ \beta, \beta' \end{bmatrix}$ is a braiding matrix for the coset theory which we would like to express in terms of the one $B_G^{(\Lambda_1,1),(\Lambda_2,1)} \begin{bmatrix} \mathbf{a}_1,\mathbf{a}_2;\mathbf{d}_1,\mathbf{d}_2 \\ \mathbf{b}_1,\mathbf{b}_2;\mathbf{b}_1',\mathbf{b}_2' \end{bmatrix}$ for the G theory and the one $B_H^{(\lambda)} \begin{bmatrix} a, d \\ b, b' \end{bmatrix}$ for the H theory.

Start from the exchange algebra of G theory (*cf* eq. (27)). We apply the factorization property (eq. (30)) to the right hand side. As for the left hand side, we first use the factorization property and subsequently the exchange algebra of the H theory and the one of the coset theory *i.e.* eq. (31). Taking matrix elements with respect to an arbitrary state in the coset theory, and subsequently in H theory, we conclude

$$\begin{aligned} &\sum_{\mathbf{b}_1'',\mathbf{b}_2''} B_G^{(\Lambda_1 1,\Lambda_2 1)} \begin{bmatrix} \mathbf{a}_1,\mathbf{a}_2;\mathbf{d}_1,\mathbf{d}_2 \\ \mathbf{b}_1,\mathbf{b}_2;\mathbf{b}_1'',\mathbf{b}_2'' \end{bmatrix} \delta_{b',b''}\theta\left(b' \subseteq (\mathbf{b}_1'',\mathbf{b}_2'')\right) \\ &= \sum_{b\subseteq \mathbf{b}_1'',\mathbf{b}_2''} B_H^{(\lambda_1,\lambda_2)} \begin{bmatrix} a, d \\ b, b' \end{bmatrix} B_{G/H}^{(\sigma_1,\sigma_2)} \begin{bmatrix} \alpha, \delta \\ \beta, \beta'' \end{bmatrix} \quad . \end{aligned} \tag{32}$$

The braiding matrices for the coset models are given by solving the above factorized formula. The symbol $\theta(b' \subseteq (b_1'', b_2''))$ refers to an embedding restriction.

It is straightforward to apply the formula eq. (32) to the minimal discrete series described by the coset $\frac{A_1^{(1)}(k)\otimes A_1^{(1)}(1)}{A_1^{(1)}(k+1)}$. The most relevant primary field is labeled by

$\sigma = \begin{pmatrix} \frac{1}{2},0 \\ \frac{1}{2} \end{pmatrix}$. Let the incoming and outgoing primary states of the G theory be $(\mathbf{a}_1, \mathbf{a}_2) = (\text{spin} j_i, 0)$ and $(\mathbf{d}_1, \mathbf{d}_2) = (\text{spin} j_f, 0)$ respectively. Likewise, $(\mathbf{b}_1, \mathbf{b}_2) = (j = j_i \pm 1/2,\ 0)$, $(\mathbf{b}''_1, \mathbf{b}''_2) = (j'' = j_i \pm 1/2,\ 0)$. The H-primary fields a, d, b, b'' are diaginal embeddings of the above ones. Since both B_G and B_H are known from the WZW model (eq. (28)), the above formula (eq. (32)) determines the braiding matrices of the most relevant field for the arbitrary incoming and outgoing primary states in the minimal series. It is instructive to compare this result with the one from the Coulomb gas approach in the case of the Ising model. We denote the braiding matrices $B^{(\sigma,\sigma)}_{G/H}\begin{bmatrix} \alpha, \delta \\ \beta, \beta'' \end{bmatrix}$ by $\begin{smallmatrix} & (\sigma) & \\ \beta & \Diamond & \beta'' \\ & (\sigma) & \end{smallmatrix}$ for $1 = \begin{pmatrix} 0,0 \\ 0 \end{pmatrix}$, $\sigma = \begin{pmatrix} \frac{1}{2},0 \\ \frac{1}{2} \end{pmatrix}$, and $\epsilon = \begin{pmatrix} 0,0 \\ 1 \end{pmatrix}$. We obtain

$$\begin{smallmatrix} & \sigma & \\ \beta & \Diamond & \beta'' \\ & \sigma & \end{smallmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-\pi i/8} , & \frac{\sqrt{6}}{4} e^{-5\pi i/8} \\ \sqrt{6}\, e^{-5\pi i/8} , & \frac{1}{\sqrt{2}} e^{-\pi i/8} \end{pmatrix}$$

$$\begin{smallmatrix} & 1 & \\ \sigma & \Diamond & \sigma \\ & 1 & \end{smallmatrix} = \begin{smallmatrix} & \epsilon & \\ \sigma & \Diamond & \sigma \\ & \epsilon & \end{smallmatrix} = e^{\pi i/8} \qquad \begin{smallmatrix} & 1 & \\ \sigma & \Diamond & \sigma \\ & \epsilon & \end{smallmatrix} = \begin{smallmatrix} & \epsilon & \\ \sigma & \Diamond & \sigma \\ & 1 & \end{smallmatrix} = e^{-3\pi i/8} \tag{33}$$

This agrees with the the result from the Coulomb gas approach[18] except for the $-\sqrt{3}$ factors in $\begin{smallmatrix} & \sigma & \\ \beta & \Diamond & \beta'' \\ & \sigma & \end{smallmatrix}$ for $(\beta, \beta'') = (1, \epsilon), (\epsilon, 1)$. These factors can be attributed to the normalization of the bases different from the one employed here. The braiding matrices for general external primary fields of coset models can be obtained from the one given in eq. (32) by the fusion procedure described in eq. (29) for the WZW theory.

4)° So far, our discussion on quantum groups consists of the two diconnected parts which one can relate only by an analogy. Let me suggest here a more direct connection between the braiding matrices of WZW theory and $\check{R}$ matrix of the GTS.

Our main proposal eq. (26) is partly suggested by drawing an analogy to the well-known vertex-height correspondence in a class of integrable lattice models. It would be desirable if the exchange algebra eq.(27) follows directly from the structure of the lattice correlation functions. The braid relation appearing in WZW conformal blocks is nothing but the Yang-Baxter relation in the infinite momentum frame. Moreover, the notion of monodromy is already in the lattice correlation functions despite the absence of the complex z-plane[19]. One can presumably deduce monodromy structure of the conformal block, by studying lattice correlation function in the analytic rapidity plane and taking a continuum limit in the end. Some of the techniques developed in [20] appear to be relevant.

Let us finally present a result[11] in Toda field theories which is motivated from the form of the quantum group generators (eq. (16)) in GTS. This expression bears a striking resemblance to Mandelstam's soliton operator in sine-Gordon theory. In fact, the following expression turns out to be a density for the universal enveloping algebra which is a symmetry of $su(2)$ Toda field theories:

$$\Psi(x) =: \exp[\gamma \int_{-\infty}^{x} d\xi p(\xi) + C \int_{-\infty}^{x} \partial_\xi \phi(\xi)] :$$
$$\tilde{\Psi}(y) =: \exp[\gamma' \int_{y}^{+\infty} d\xi p(\xi) + C' \int_{y}^{+\infty} \partial_\xi \phi(\xi)] : \quad . \tag{34}$$

Here, γ, γ', C, and C' are constants fixed by the requirements of symmetry generators and statistics. For details, see ref.[11].

In this talk, I discussed the deformation of symmetry generators due to the change of the coupling constant which is the only parameter (except the spectral parameter) in the trigonometric solutions of the Yang-Baxter relation. In the general elliptic case, there is another deformation or distortion due to the modulus. Physically, this is a mass parameter and one realization of such deformation is the deformation of conformal field theory which has received much attention now. Integer eigenvalue structure of the corner transfer matrix(CTM) in the eight-vertex model and its relatives is the statement that the CTM spectrum stays invariant under the both of the deformations. The relevant operator algebra is the noncritical Virasoro algebra proposed a while ago[21]. All these developments together with the quantum group seem to point towards a single appealing and coherent framework of operator algebras based on noncommutative geometry.

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Figure Captions

Fig. 1: Procedure for obtaining braiding matrices for primary fields belonging to higher representations

Fig. 2: Graphical representation of eq. (29).

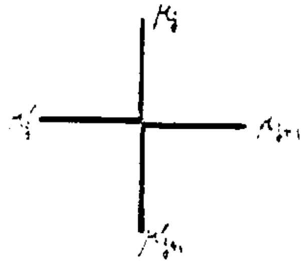

Fig. 1

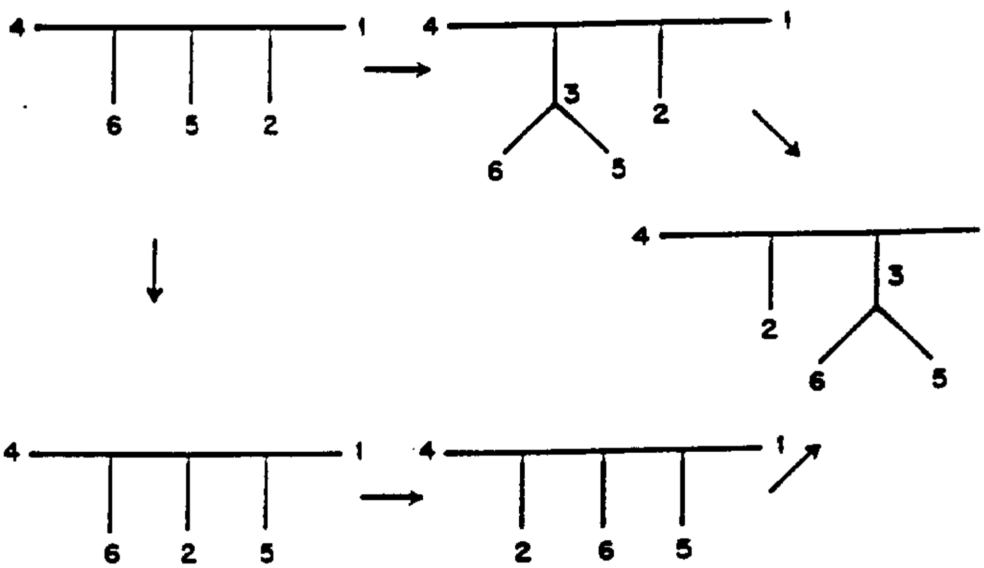

Fig. 2

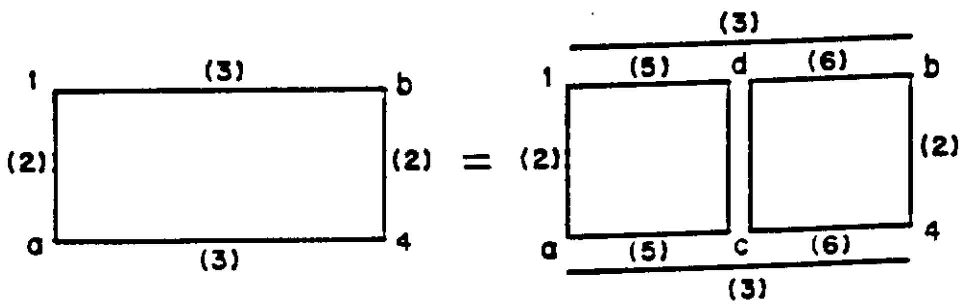

Fig. 3