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STUDIES AND CALCULATIONS OF TRANSVERSE EMITTANCE GROWTH IN PROTON STORAGE RINGS

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When high energy storage rings are used to collide beams of particles and antiparticles for high energy physics experiments, it is important to obtain as high an integrated luminosity as possible. Reduction of integrated luminosity can arise from several factors, in particular from growth of the transverse beam sizes (transverse emittances). We have studied the problem of transverse emittance growth in high energy storage rings caused by random dipole noise kicks to the beam. A theoretical formula for the emittance growth rate is derived, and agreement is obtained with experimental measurements where noise of known amplitude and power spectrum was deliberately injected into the Fermilab Tevatron, to kick the beam randomly. In the experiment, phase noise was introduced into the Tevatron RF system, and the measured dependence of horizontal emittance growth on phase noise amplitude is compared against the theoretically derived response.

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1. Introduction

When high energy storage rings are used to collide beams of particles and antiparticles for high energy physics experiments, it is important to obtain as high an integrated luminosity as possible. Reduction of integrated luminosity can arise from several factors, in particular from growth of the transverse beam sizes (transverse emittances). An example of this phenomenon was recently observed in the Tevatron [1,2] at Fermilab. During colliding beam operations in the Tevatron, it was noted that the luminosity decayed at an unexpectedly fast rate. Investigations showed that the horizontal and vertical beam emittances were growing linearly in size, and this growth was the dominant cause of the poor luminosity lifetime [3]. Since it was observed that the beam was undergoing externally driven betatron oscillations [4], a search for accelerator components which were capable of driving the beam transversely and causing emittance growth was initiated.

We have therefore investigated the problem of transverse emittance growth in high energy storage rings caused by random dipole noise kicks to the beam. A theoretical formula for the emittance growth rate is derived, and compared against experimental measurements. In order to obtain quantitative results, noise of known amplitude and power spectrum was deliberately injected into the Tevatron, to kick the beam randomly. The theoretical formula itself is applicable to arbitrary noise sources, provided they satisfy certain criteria, to be specified below. Measurements of transverse emittance growth as a function of noise amplitude were performed for a number of Tevatron systems. Unfortunately, most of these accelerator components have unknown transfer functions between a measurable device monitor and the effect of the device on the beam. For example, due to capacitance to ground and capacitance between the two conductors, the current ripple measured at a dipole magnet power supply does not all go into magnetic field ripple. In addition, the skin depth shielding effects of the beam pipe and the superconducting cables of the dipole also affect the magnetic field ripple sampled by the beam. Therefore, it was impossible to quantitatively compare magnet noise and emittance growth. It turns out that the only quantitatively understood system was the Tevatron RF system.

In the experiment, phase noise was introduced into the Tevatron RF system. Independent of previous work [5] which investigated the effect of RF noise on longitudinal beam dynamics, it was suspected that phase noise with the appropriate Fourier spectrum could induce horizontal emittance growth due to the existence of horizontal dispersion at the RF cavities. The measured dependence of horizontal emittance growth on phase noise amplitude is compared against the theoretically derived response. The theoretical derivation is presented below, followed by a description of the experiment.

2. Theory

2.1 Notation

First we shall describe the notation to be used below. The orbit of a particle is described by the functions [6]

$$\begin{pmatrix} x \\ p \end{pmatrix} \equiv \begin{pmatrix} x \\ \alpha x + \beta x' \end{pmatrix} = \sqrt{2I\beta} \begin{pmatrix} \sin(\Psi + \psi - \omega_\beta t) \\ \cos(\Psi + \psi - \omega_\beta t) \end{pmatrix} . \tag{1}$$

Here I and Ψ are the action and angle variables, respectively, ω_β is the linear dynamical oscillation frequency (linear tune times revolution frequency) and ψ is the Floquet phase. We make the approximation that the phase-space trajectories in $\{x, p\}$ space are circles with action-dependent tunes. The unnormalized emittance is given by $\epsilon = \langle I \rangle$, assuming $\langle x \rangle = \langle p \rangle = 0$. The angular brackets denote an average over the beam at fixed t. We always define the emittance to be averaged over the beam. The emittance growth rate is

$$r = \frac{d\epsilon}{dt} . \tag{2}$$

Note that to calculate r it is not necessary that $\langle x \rangle = 0$; it is sufficient if $\langle x \rangle$ and $\langle p \rangle$ are bounded, because then $d\langle x \rangle/dt$ and $d\langle p \rangle/dt$ average to zero.

2.2 Coherent betatron motion

Let us now study the excitation of coherent betatron oscillations, i.e. the motion of the beam centroid. Suppose there is a random horizontal dipole kick at location t, so that

$$\begin{aligned} x(t+\delta t) - x(t) &= 0 \\ p(t+\delta t) - p(t) &= N\delta t \, . \end{aligned} \tag{3}$$

It is sufficient to consider a kick which changes only p and not x. We shall add in the contribution of a kick which changes x below. We shall linearize the response of the beam with respect to the kicks, i.e. we calculate the changes to x and p to linear order in N only, and so the emittance growth rate will be of $O(N^2)$.

The changes to x and p at azimuth t, due to a kick $N\delta t$ at azimuth t', are

$$\begin{aligned} \delta\left(\frac{x}{\sqrt{\beta}}\right) &= \sin[\Phi(t) - \Phi(t')]\frac{N(t')}{\sqrt{\beta(t')}}\,\delta t \\ \delta\left(\frac{p}{\sqrt{\beta}}\right) &= \cos[\Phi(t) - \Phi(t')]\frac{N(t')}{\sqrt{\beta(t')}}\,\delta t \, . \end{aligned} \tag{4}$$

Here $\Phi = \Psi + \psi - \omega_\beta t$. The factors of $\sqrt{\beta}$ have been introduced because the quantity of real interest is $\langle x\rangle/\sqrt{\beta}$, because $\langle x\rangle$ itself is proportional to $\sqrt{\beta}$. To obtain the kick to the beam centroid $\langle x\rangle$, we must average over the beam. We shall assume that the kick N is the same for all the particles, i.e. $\langle Nx\rangle = N\langle x\rangle$, etc. This is a valid assumption for a dipole kick. The change to the beam centroid is

$$\delta\left(\frac{\langle x\rangle}{\sqrt{\beta}}\right) = D(t,t')\frac{N(t')}{\sqrt{\beta(t')}}\sin[\Phi_0(t) - \Phi_0(t')]\,\delta t \, , \tag{5}$$

where Φ_0 is the linear phase advance, i.e. $\Phi_0 = \Psi_0 + \psi - \omega_\beta t$, where Ψ_0 is the linear angle variable. The function $D(t,t')$ is a decoherence factor. It appears because the individual particles have different tunes, and so even though the kick N is the same for all particles, they get out of phase as time goes by.

The response to a sequence of kicks, using the linear response approximation, is

$$\frac{\langle x\rangle}{\sqrt{\beta}} = D(t,t_0)\left(\frac{\langle x\rangle}{\sqrt{\beta}}\right)_{t_0} + \int_{t_0}^{t} D(t,t')\frac{N(t')}{\sqrt{\beta(t')}}\sin[\Phi_0(t)-\Phi_0(t')]\,dt' \ . \tag{6}$$

We shall use only the asymptotic solution, i.e. we shall neglect the contribution from t_0 in eq. (6). Mathematically, we take $t_0 \to -\infty$. This is a formal limit, and simply means that we are assuming that after sufficient time the contribution of the initial state of the beam centroid decoheres and becomes negligible in comparison to the effect of the kicks experienced by the beam after injection.

2.3 Decoherence factor

The decoherence factor for a beam with an action-dependent tuneshift, i.e. $d\Psi/dt = \omega_\beta + \omega'_\beta I$, has been calculated in ref. [7]. The result is (with $\tau = t - t'$)

$$D(t,t') = D(t-t') = \frac{1}{1+(\tau/t_0)^2}\exp\left[-\frac{1}{2}\,\frac{(\tau/t_1)^2}{1+(\tau/t_0)^2}\right] \ , \tag{7}$$

where t_0 and t_1 are constants. Note that this expression is even under time-reversal, i.e. the beam decoheres even if $t < t'$, as expected for decoherence. From ref. [7], the exponential factor may be ignored ($t_1 \to \infty$) unless the beam centroid displacement is much greater than the beam size, and so

$$D(t,t') \simeq \frac{1}{1+(\tau/t_0)^2} \ . \tag{8}$$

Another model, which is motivated by radiation damping in synchrotron radiation theory, is to put [8]

$$\begin{aligned} D(t,t') &= e^{-\alpha_d(t-t')} \qquad && t > t' \\ &= 0 && t < t' \ . \end{aligned} \tag{9}$$

This has the disadvantage of not being even under time-reversal. A more reasonable approximation in this context would be

$$D(t,t') = \frac{1}{2\cosh[\alpha_d(t-t')]} \ . \tag{10}$$

Since we do not know the detailed decoherence mechanism in general, the choice of model is somewhat arbitrary. The expression eq. (9) is the simplest when evaluating eq. (6), because we shall decompose N into Fourier harmonics below, and an exponential decoherence factor yields the simplest analytical solution. In particular, we shall only be interested in $t > t'$, and so we shall use eq. (9). It must be understood that this is a phenomenological step.

There is an important caveat to the above statements. In all of the models of decoherence presented above, the decoherence factor approaches zero as $t - t_0 \to \infty$. Therefore we take the limit $t_0 \to -\infty$ in eq. 6 instead of keeping t_0 finite, because that end of the range of integration makes a negligible contribution to the total.

We note in passing that there are other phenomena which lead to decoherence, such as tune modulation due to synchrotron oscillations and chromaticity, where the beam recoheres after every synchrotron period. The expression for the decoherence factor of that mechanism is

$$D(t,t') = D(t-t') = \exp\Big(-2\sigma_s^2\xi^2\nu_s^{-2}\sin^2[\nu_s\omega_{rev}(t-t')/2]\Big) \tag{11}$$

where ν_s is the synchrotron tune, ω_{rev} is the revolution angular frequency, ξ is the chromaticity, and σ_s is the r.m.s. bunch length, assuming a Gaussian beam. We see that $D(t,t') = 1$ whenever $t - t'$ equals a multiple of $2\pi/(\nu_s\omega_{rev})$. In the absence of other mechanisms, if we were to examine the beam once every synchrotron period, the beam centroid betatron amplitude and emittance would be invariant. Therefore this mechanism does not contribute to long term emittance growth.

2.4 Solution of beam centroid motion

2.4.1 Fourier harmonics

It is useful at this point to decompose N into a Fourier spectrum, defined by

$$\begin{aligned} N(t) &= \int \frac{d\omega}{2\pi} \widetilde{N}(\omega) e^{i\omega t} \\ \frac{e^{i(\psi-\omega_\beta t)}}{\sqrt{\beta}} &= \sum_k L_k e^{ik\omega_{rev} t} \,. \end{aligned} \tag{12}$$

The latter function is periodic around the ring, and so one gets a sum, not integral, of harmonics. We assume that $\widetilde{N}(\omega)$ is a noise function, which has a random phase distributed uniformly between 0 and 2π. It means that $N(t)$ consists of wave trains that shift phase randomly from time to time. It will be more convenient below to deal with the combined transform

$$\begin{aligned} \frac{N e^{i(\psi-\omega_\beta t)}}{\sqrt{\beta}} &= \int \frac{d\omega}{2\pi} \widetilde{M}(\omega) e^{i\omega t} \\ \widetilde{M}(\omega) &= \sum_k \widetilde{N}(\omega - k\omega_{rev}) L_k \,. \end{aligned} \tag{13}$$

Note that $\widetilde{M}$ also has a random phase. In practice, one must interpret the Fourier transform as a sum over (angular) "frequency bins" of size $\Delta\omega$, i.e.

$$\int F(\omega)\, d\omega = \Delta\omega \sum_k F(k\Delta\omega) \,, \tag{14}$$

where $\Delta\omega$ is determined by the experimental apparatus used to measure, say, the power spectrum of the noise.

We substitute the above expression into eq. (6),

$$\begin{aligned} \frac{\langle x \rangle}{\sqrt{\beta}} &\simeq \int_{-\infty}^{t} dt'\, e^{-\alpha_d(t-t')} \mathrm{Im} \left\{ e^{i[\Psi(t)-\Psi(t')]} \frac{N(t') e^{i[\psi(t)-\psi(t')-\omega_\beta(t-t')]}}{\sqrt{\beta(t')}} \right\} \\ &= \int_{-\infty}^{t} dt'\, e^{-\alpha_d(t-t')} \mathrm{Im} \left\{ e^{i\omega_\beta(t-t')} e^{i[\psi(t)-\omega_\beta t]} \int \frac{d\omega}{2\pi} \widetilde{M}^*(\omega) e^{-i\omega t'} \right\} \\ &= \mathrm{Im} \left\{ \int \frac{d\omega}{2\pi} \widetilde{M}^*(\omega) e^{i[\psi(t)-\omega_\beta t-\omega t]} \int_{-\infty}^{t} e^{-\alpha_d(t-t')} e^{i\omega(t-t')} e^{i\omega_\beta(t-t')}\, dt' \right\} \end{aligned}$$

$$= \quad \mathrm{Im}\left\{ e^{i[\psi(t)-\omega_\beta t]} \int \frac{d\omega}{2\pi} \frac{\widetilde{M}^*(\omega)e^{-i\omega t}}{\alpha_d - i(\omega+\omega_\beta)} \right\} . \tag{15}$$

Let us try to visualize the beam centroid motion pictorially. Note that the integrand has a random phase. Hence if we average over the noise, $\langle x \rangle$ will average to zero. The physical picture is that of successive wave trains of coherent betatron motion, separated by random fluctuations in phase.

2.4.2 Time averages

Since $\langle x \rangle$ averages to zero, it is more useful to calculate $\langle x \rangle^2/\beta$, averaged over time. Note that this is *not* the beam emittance, but the r.m.s. beam centroid amplitude squared and divided by β. Formally, the time average of any function F is given by

$$F_{avg} = \lim_{t\to\infty} \left[\frac{1}{t} \int_0^t F(t')\, dt' \right] . \tag{16}$$

The limit $t \to \infty$ is again a formal limit. It simply means a long time, $t = 2\pi/\Delta\omega$, where $\Delta\omega$ is the "bin size" of (angular) frequencies in the Fourier transform. Beyond this time the Fourier harmonics are not well-defined. The approximations involved in the next few equations are also discussed in the section on approximations below.

Thus we actually evaluate

$$\begin{aligned} \left[\frac{\langle x \rangle^2}{\beta} \right]_{avg} &\simeq \frac{\Delta\omega}{2\pi} \int_0^{2\pi/\Delta\omega} \frac{\langle x \rangle^2}{\beta}\, dt' \\ &= \frac{\Delta\omega}{4\pi} \int_0^{2\pi/\Delta\omega} dt' \int\int \frac{d\omega d\omega'}{(2\pi)^2} \frac{\widetilde{M}^*(\omega)\widetilde{M}(\omega')e^{i(\omega'-\omega)t'}}{[\alpha_d - i(\omega+\omega_\beta)][\alpha_d + i(\omega'+\omega_\beta)]} . \end{aligned} \tag{17}$$

We have neglected terms in $\widetilde{M}(\omega)\widetilde{M}(\omega')$ etc. which average to zero. We now need to interchange the orders of integration, which involves further assumptions about the uniformity of the convergence of the integrals involved. Doing so, we obtain

$$\begin{aligned} \left[\frac{\langle x \rangle^2}{\beta} \right]_{avg} &\simeq \frac{\Delta\omega}{4\pi} \int\int \frac{d\omega d\omega'}{(2\pi)^2} \frac{\widetilde{M}^*(\omega)\widetilde{M}(\omega')}{[\alpha_d - i(\omega+\omega_\beta)][\alpha_d + i(\omega'+\omega_\beta)]} \\ &\qquad \times \int_0^{2\pi/\Delta\omega} e^{i(\omega'-\omega)t'}\, dt' . \end{aligned} \tag{18}$$

The integral over t' averages to zero unless $\omega = \omega'$. More precisely, if we recall that the integration over frequencies consists of sums over bins of size $\Delta\omega$, then the integrals over ω' and t' yield

$$\begin{aligned}\int \frac{d\omega'}{2\pi} F(\omega') \int_0^{2\pi/\Delta\omega} e^{i(\omega'-\omega)t'}\, dt' &\simeq \frac{\Delta\omega}{2\pi} \sum_{\omega'} F(\omega') \frac{\pi}{\Delta\omega} \delta_{\omega,\omega'} \\ &\simeq \frac{1}{2} F(\omega' = \omega) \ . \end{aligned} \tag{19}$$

If we were to take $\Delta\omega \to 0$ in the above integrals, *then* we would get true δ-functions

$$\begin{aligned}\int \frac{d\omega'}{2\pi} F(\omega') \int_0^{\infty} e^{i(\omega'-\omega)t'}\, dt' &= \int \frac{d\omega'}{2\pi} F(\omega')\, \pi\delta(\omega' - \omega) \\ &= \frac{1}{2} F(\omega' = \omega) \ , \end{aligned} \tag{20}$$

but we have a global factor of $\Delta\omega$ in eqs. (17) and (18), and so we must keep $\Delta\omega \neq 0$.

Using the above results in eq. (18),

$$\left[\frac{\langle x\rangle^2}{\beta}\right]_{avg} = \frac{\Delta\omega}{8\pi} \int \frac{d\omega}{2\pi}\, \frac{|\widetilde{M}(\omega)|^2}{\alpha_d^2 + (\omega + \omega_\beta)^2} \ . \tag{21}$$

The integrand is a Lorentzian with a maximum at $\omega = -\omega_\beta$ and width α_d. It no longer has a random phase. Thus the conclusion is that the harmonics of the noise which contribute significantly to the coherent betatron motion are those in a range $\pm\alpha_d$ around the betatron tune ω_β. Since $\alpha_d \ll 1$ in practice, and $\widetilde{M}(\omega)$ is slowly varying in the interval $|\omega + \omega_\beta| < \alpha_d$, we can set $M(\omega) \simeq M(-\omega_\beta)$ and pull it out of the Lorentzian integral, whose value is π/α_d. Then

$$\left[\frac{\langle x\rangle^2}{\beta}\right]_{avg} \simeq \frac{1}{16\pi\alpha_d} |\widetilde{M}(-\omega_\beta)|^2 \Delta\omega \ . \tag{22}$$

This result agrees with Siemann's finding [4] that $x_{r.m.s.} \propto \sqrt{\tau_d}$, where $\tau_d = \alpha_d^{-1}$ is the decoherence time. The negative frequency in $\widetilde{M}(-\omega_\beta)$ is required because we have used complex numbers to describe the betatron motion. The quantity $|\widetilde{M}(-\omega_\beta)|^2\Delta\omega$ is the power (times various lattice functions) in the noise source in the angular frequency bin at $\omega = -\omega_\beta$. Using eq. (13), this means that the original noise source N

must have a nonzero harmonic at some $k\omega_{rev} - \omega_\beta$, i.e., the lower betatron sideband from some multiple of the revolution frequency.

2.4.3 Zero binwidth

Note that if we were able to measure frequencies to infinite precision ($\Delta\omega = 0$), we would be led to conclude that $[\langle x\rangle^2/\beta]_{avg} = 0$. This is a correct conclusion for this model, and has the following interpretation. To measure a Fourier harmonic with a bin size $\Delta\omega \to 0$, it would truly take an infinite number of turns, and on *this* time scale even the most slowly varying harmonics in $\langle x\rangle^2/\beta$ would average to zero. In practice, we cannot observe the beam over such a long time, and so any harmonic that does not average to zero rapidly over the interval of experimental measurement $2\pi/\Delta\omega$ survives the average over t in eq. (17).

2.4.4 Kicks to x

It is straightforward to modify the above result to include the effect of a noise source which kicks both x and p. Suppose that $\delta x = N_1\delta t$ and $\delta p = N_2\delta t$, and one defines

$$\begin{aligned} \frac{N_1 e^{i(\psi-\omega_\beta t)}}{\sqrt{\beta}} &= \int \frac{d\omega}{2\pi}\,\widetilde{M}_1 e^{i\omega t} \\ \frac{N_2 e^{i(\psi-\omega_\beta t)}}{\sqrt{\beta}} &= \int \frac{d\omega}{2\pi}\,\widetilde{M}_2 e^{i\omega t}\ , \end{aligned} \tag{23}$$

then one can show that the final result is

$$\left[\frac{\langle x\rangle^2}{\beta}\right]_{avg} \simeq \frac{1}{16\pi\alpha_d}\Big(|\widetilde{M}_1(-\omega_\beta)|^2 + |\widetilde{M}_2(-\omega_\beta)|^2\Big)\Delta\omega\ . \tag{24}$$

Effectively, we just add the two sources of noise power.

3. Emittance growth

Next, let us calculate the rate of emittance growth. For this we need to study $x^2 + p^2$. Now

$$[x^2 + p^2]_{t+\delta t} \;=\; [x^2 + (p + N\delta t)^2]_t \;\simeq\; [x^2 + p^2]_t + 2pN\delta t \tag{25}$$

and

$$\frac{p}{\sqrt{\beta}} = \int_{-\infty}^{t} \frac{N(t')}{\sqrt{\beta(t')}} \cos[\Phi_0(t) - \Phi_0(t')]\, dt' \,. \tag{26}$$

Combining these results with the fact that $\delta(x^2 + p^2) = 2\beta\, \delta I$,

$$\delta I = \frac{pN\delta t}{\beta} = \frac{N\delta t}{\sqrt{\beta}} \int_{-\infty}^{t} \frac{N(t')}{\sqrt{\beta(t')}} \cos[\Phi(t) - \Phi(t')]\, dt' \,. \tag{27}$$

Averaging over the beam,

$$\frac{d\epsilon}{dt} = \frac{d\langle I\rangle}{dt} = \frac{N}{\sqrt{\beta}} \int_{-\infty}^{t} D(t,t') \frac{N(t')}{\sqrt{\beta(t')}} \cos[\Phi_0(t) - \Phi_0(t')]\, dt' \,. \tag{28}$$

We want the time average of the growth rate. The function ϵ itself grows indefinitely, and does not have a finite time average.

The approximations used here are the same as those used above in analyzing the behavior of $[\langle x\rangle^2/\beta]_{avg}$. Further, as noted in the previous section, since the beam centroid motion is bounded, we can get the emittance growth rate directly from the above equation, without worrying about $d\langle x\rangle/dt$ etc. The time-averaged growth rate is

$$\begin{aligned}
r = \left[\frac{d\epsilon}{dt}\right]_{avg} &\simeq \frac{\Delta\omega}{2\pi} \int_0^{2\pi/\Delta\omega} dt\, \frac{N(t)}{\sqrt{\beta(t)}} \int_{-\infty}^{t} D(t,t') \frac{N(t')}{\sqrt{\beta(t')}} \cos[\Phi_0(t) - \Phi_0(t')]\, dt' \\
&\simeq \frac{\Delta\omega}{2\pi} \mathrm{Re}\left\{ \int\!\!\int \frac{d\omega d\omega'}{(2\pi)^2}\, \widetilde{M}(\omega)\widetilde{M}^*(\omega') \int dt\, \frac{e^{i(\omega-\omega')t}}{\alpha_d - i(\omega' + \omega_\beta)} \right\} \\
&\simeq \frac{\Delta\omega}{4\pi} \int \frac{d\omega}{2\pi}\, |\widetilde{M}(\omega)|^2 \mathrm{Re}\left\{ \frac{1}{\alpha_d - i(\omega + \omega_\beta)} \right\} \\
&\simeq \frac{\Delta\omega}{4\pi} \int \frac{d\omega}{2\pi}\, |\widetilde{M}(\omega)|^2 \frac{\alpha_d}{\alpha_d^2 + (\omega + \omega_\beta)^2}
\end{aligned}$$

$$\simeq \; |\widetilde{M}(-\omega_\beta)|^2 \frac{\Delta\omega}{8\pi} \; . \tag{29}$$

Measuring frequencies in Hz, and adding the effect of a noise source which kicks x also, in the same way as for $[\langle x \rangle^2/\beta]_{avg}$, the above result becomes

$$r = \frac{1}{4}\Big(|\widetilde{M}_1(f_b)|^2 + |\widetilde{M}_2(f_b)|^2\Big)\Delta f \; . \tag{30}$$

Here f_b is the betatron frequency in Hz, which separate both the upper and lower betatron sidebands from the revolution harmonics. One can visualize this using the following argument: eq. (30) is a manifestation of energy balance. The emittance growth is caused by an increase of the amplitudes (energy) of the betatron oscillations of the particles, and this increase comes from the power in the noise, which is proportional to $|\widetilde{M}(f_b)|^2\Delta f$. Hence, the above result is independent of the mechanism and magnitude of the decoherence affecting the trajectory of the beam centroid.

We see that the integrand above is also a Lorentzian with a maximum at $\omega = -\omega_\beta$ and a width α_d, and so the emittance growth is driven by harmonics of the noise in a range $\pm\alpha_d$ centered on the betatron frequency plus multiples of the revolution frequency.

3.1 Approximations

In this section we recapitulate and discuss the approximations made in the above calculations. To begin with, we are only interested in the behavior of the beam long after injection, so that the initial state of the beam does not matter. We are assuming that a time interval exists where the contribution of the "transient" (initial value of $\langle x \rangle$ in eq. (6)) is negligible. Now the "damping" of the beam centroid is really a decoherence of individual particle orbits, not true damping of $\langle x \rangle$. We need to take various averages over time to obtain the average values for the beam centroid amplitude (eqs. (17) – (22)) and emittance growth rate (eq. (30)). These averages are supposed to extend over a sufficiently long time interval so that the oscillatory components in the observed values of $\langle x \rangle^2/\beta$ and r cancel out. The timescale is $2\pi/\Delta\omega$. The value of $\Delta\omega$ must therefore be much less than α_d in order for the Lorentzian integrals in eqs. (21) and (30) to be meaningful, i.e. in order to justify the approximations made to the integrands in the integrations over ω' and ω. The interchange of the orders of integration between eqs. (17) and (18) relies on the uniqueness of Fourier transforms, i.e. on the relations

$$\int_{-\infty}^{\infty} e^{-i\omega t}\,dt = 2\pi\delta(\omega)\ , \qquad \int_{-\infty}^{\infty} e^{i\omega t}\,\frac{d\omega}{2\pi} = \delta(t)\ . \tag{31}$$

Since we only integrate from $t = 0$ to $t = 1/\Delta f = 2\pi/\Delta\omega$, we actually have

$$\begin{aligned} \int_0^{1/\Delta f} e^{-i\omega t}\,dt = \int_0^{2\pi/\Delta\omega} e^{-i\omega t}\,dt &\simeq \frac{\pi}{\Delta\omega}\delta_{\omega,0} \qquad && (|\omega| < \Delta\omega/2) \\ &\simeq 0 && \text{otherwise.} \end{aligned} \tag{32}$$

Thus the interchange of the orders of integration involves errors of $O(\Delta\omega/\pi)$ because the δ-functions are not "pointlike."

Hence the global picture is: we first wait a time much longer than $2\pi/\alpha_d$, so that the initial beam conditions "damp out," then define Fourier transforms over a time interval $2\pi/\Delta\omega$ which must exceed $2\pi/\alpha_d$. The use of ∞ or $-\infty$ as a limit of integration simply means that the contribution from that end of the range of integration is being neglected. It does not imply tracking of a particle, or the beam, for an infinite number of turns.

4. Measurements

4.1 Phase modulation

Each accelerator revolution the beam sampled a random phase shift introduced into the RF voltage waveform. Mathematically, this voltage is equal to the real part of

$$V(t) = V_0 e^{i[\omega_0 t + \phi_0 + \phi_n(t)]}\ . \tag{33}$$

The phase waveform was a random noise signal generated by a Hewlett-Packard 3561A

Dynamic Signal Analyzer in the frequency band surrounding the lowest betatron sideband frequency (near 20 kHz). The RF voltage was measured with a RF cavity gap monitor. Figure 1 is a photograph of the frequency domain effect of this phase noise modulation measured by such a gap monitor and a Hewlett-Packard 8568B Spectrum Analyzer. At each modulation angular frequency (ω_n) the effect on the RF voltage can be written as

$$V(t) = V_0 e^{i[\omega_0 t + \phi_0]} e^{i\phi_n \cos(\omega_n t)} \ . \tag{34}$$

The Jacobi-Anger expansion states that

$$e^{iz\cos(\omega_n t)} = \sum_{m=-\infty}^{\infty} i^m J_m(z) e^{im\omega_n t} \ , \tag{35}$$

so therefore the amplitude of the RF waveform and the first two sidebands are given by

$$V(t) = V_0 e^{i\phi_0} \left[J_0(\phi_n) e^{i\omega_0 t} + iJ_1(\phi_n) e^{i(\omega_0+\omega_n)t} + iJ_1(\phi_n) e^{i(\omega_0-\omega_n)t} \right] \ . \tag{36}$$

The ratio of the sideband amplitudes to the RF carrier amplitude is the ratio of the Bessel functions $J_1(\phi_n)$ and $J_0(\phi_n)$. Since the ϕ_n injected into the Tevatron RF system was on the order of 10^{-3}, the small angle approximations of the Bessel functions

$$J_0(x) = 1 \ , \qquad J_1(x) = \frac{x}{2} \tag{37}$$

can be used. Therefore, the sum of the first sideband amplitudes divided by the central RF amplitude is equal to the phase modulation amplitude ϕ_n. Using superposition arguments, the above single frequency derivation can be generalized to a band of frequencies. Figure 1 is a photograph of the Tevatron RF gap spectrum when phase noise is added.

A closer view of the noise band is shown in figure 2. Centered at 20 kHz with a width of 3 kHz, the noise spectrum is uniform in the region of the betatron frequency. As shown in figure 3, the other noise sidebands in figure 1 are due to the output characteristics of the random noise generator.

Based upon photographs such as the one in figure 1, the measured phase noise band amplitudes, as a ratio of the fundamental RF voltage amplitude, were measured three times. The results are listed in table 1.

4.2 Emittance Growth

The horizontal emittance of the beam is measured using devices called flying wires [9,10]. A 1 mil diameter carbon wire, oriented vertically, passes through the beam horizontally at a velocity of approximately 3 m/sec. As the wire traverses the proton beam, protons collide with wire atoms producing particle showers detected by scintillator/phototube monitors. The flying wire system is set up such that the phototube output voltage is proportional to the local proton density at the wire. By digitizing the wire position and phototube voltage on a turn by turn basis, one can map out the beam's horizontal density distribution. Figure 4 is an example of the data from such a beam. Typically, only the r.m.s. width of the beam is recorded. The horizontal emittance is calculated using r.m.s. beam widths at two flying wire monitors. These wires are placed at points in the Tevatron where the dispersion is very different, so that the contributions of horizontal emittance and momentum spread to the total beam size can be separated. By flying the wires periodically over the span of an hour or more, and doing a linear least square fit of emittance vs. time, the emittance growth rate of the beam is determined. Figure 5 is an example of horizontal emittance as a function of time, with the result of such a fit superimposed.

Using the results of such fits, the horizontal emittance growth rates were measured three times, corresponding to the three phase noise amplitudes tabulated above (figure 6). The growth data are listed in table 2.

5. Analysis

In order to compare the above experimental data to theoretical predictions, the relationship between betatron position changes and RF phase modulation must be specified. Since the energy change u of a synchronous proton traversing an RF voltage V_0 with a phase error ϕ_n is

$$u = eV_0 \sin\phi_n \ , \tag{38}$$

and since ϕ_n is much less than unity, $u = eV_0\phi_n$. If the horizontal position of this synchronous particle is described by

$$x = \sqrt{2I\beta}\cos(\omega_\beta t) \ , \tag{39}$$

then the change in amplitude due to the energy change u is calculated in ref. [11] to be

$$\delta I = \frac{1}{2}\frac{u^2}{E_0^2}H(s) \ , \tag{40}$$

where E_0 is the beam energy and $H(s)$ depends on the beta and dispersion (η) functions and their slopes

$$H = \frac{1}{\beta}\left\{\eta^2 + \left[\beta\eta' - \frac{1}{2}\beta'\eta\right]^2\right\} \ . \tag{41}$$

The above expression depends, however, on the assumption that the effects of distinct kicks are uncorrelated, whereas in our case we assume that the particles undergo undisturbed betatron oscillations except for the RF kicks, hence there are correlations between the effects of successive kicks — this is why only the noise harmonic at the betatron frequency contributes to emittance growth. We therefore find that, following the notation used in the previous sections,

$$\begin{aligned} N_1\delta t &= \eta\frac{eV_0}{E_0}\phi_n \\ N_2\delta t &= (\alpha\eta + \beta\eta')\frac{eV_0}{E_0}\phi_n \end{aligned} \tag{42}$$

where $\delta t = 1/f$ in this context. Thus

$$\int\frac{d\omega}{2\pi}\widetilde{M}_1 e^{i\omega t} = f\frac{eV_0}{E_0}\left[\frac{\eta}{\sqrt{\beta}}e^{i(\psi-\omega_\beta t)}\right]_{RF}\phi_n$$

$$
\begin{aligned}
\widetilde{M}_1(\omega) &= f\frac{eV_0}{E_0}\left[\frac{\eta}{\sqrt{\beta}}e^{i(\psi-\omega_\beta t)}\right]_{RF}\tilde{\phi}(\omega) \\
|\widetilde{M}_1(\omega)|^2 &= f^2\left(\frac{eV_0}{E_0}\right)^2\left[\frac{\eta^2}{\beta}\right]_{RF}|\tilde{\phi}(\omega)|^2 \\
|\widetilde{M}_2(\omega)|^2 &= f^2\left(\frac{eV_0}{E_0}\right)^2\left[\frac{(\alpha\eta+\beta\eta')^2}{\beta}\right]_{RF}|\tilde{\phi}(\omega)|^2 ,
\end{aligned} \tag{43}
$$

where $\tilde{\phi}(\omega)$ is Fourier transform of $\phi_n(t)$, and so

$$
\begin{aligned}
\frac{d\epsilon}{dt} &= \frac{1}{4}\Big(|\widetilde{M}_1(f_b)|^2+|\widetilde{M}_2(f_b)|^2\Big)\Delta f , \\
&= \frac{f^2H}{4}\left(\frac{eV_0}{E_0}\right)^2|\tilde{\phi}(f_b)|^2\Delta f ,
\end{aligned} \tag{44}
$$

Now the experimental noise harmonic $\tilde{\phi}_{sa}(\omega)$, as recorded by the spectrum analyzer, is actually the integral of $\tilde{\phi}(\omega)$ over the frequency bins Δf at the frequencies ω and $-\omega$, i.e.

$$
|\tilde{\phi}_{sa}(\omega)|^2 = \Big[|\tilde{\phi}(\omega)|^2+|\tilde{\phi}(-\omega)|^2\Big](\Delta f)^2 = 2|\tilde{\phi}(\omega)|^2(\Delta f)^2 , \tag{45}
$$

because $\tilde{\phi}(-\omega) = \tilde{\phi}^*(\omega)$ for a real function of time $\phi_n(t)$. Hence $\tilde{\phi}_{sa}$ has the same dimension as $\phi_n(t)$. Thus the time-averaged emittance growth rate is

$$
r = \frac{f^2H}{8\Delta f}\left(\frac{eV_0}{E_0}\right)^2|\tilde{\phi}_{sa}(f_b)|^2 . \tag{46}
$$

The Tevatron conditions at the time of this experiment are lised in table 3. Substituting these values into eq. (46) yields the theoretical prediction

$$
r = (4.3\pm 0.9)\times 10^{-7}\,|\tilde{\phi}_{sa}(f_b)|^2 \quad \text{(m-rad/sec)} . \tag{47}
$$

Experimentally, one obtains the result (figure 6 and table 2)

$$
r = (3.3\pm 0.7)\times 10^{-7}\,|\tilde{\phi}_{sa}(f_b)|^2 \quad \text{(m-rad/sec)} . \tag{48}
$$

The agreement is within the errors quoted.

6. Conclusions

The theory surrounding stimulated transverse emittance growth of proton beams has been presented. Given a random dipole kick each turn, quantitative predictions for the r.m.s. beam centroid betatron oscillation amplitude and average emittance growth rate are made. Because the particles undergo deterministic betatron oscillations between kicks, the effects of successive kicks are correlated, hence only the noise harmonics at the betatron frequency plus multiples of the revolution frequency contribute to the emittance growth. The final result can be understood at a heuristic level in terms of an energy balance argument.

An experiment was performed at the Fermilab Tevatron, where the effect of RF phase noise on transverse emittance growth was measured. The RF system was chosen because it was quantitatively understood, thus enabling an absolute calibration between the emittance growth rate and the injected phase noise amplitude. It also had the advantage that the kicks to the beam were localized at one point in the ring, and were of sufficient magnitude to dominate over other sources of dipole kicks, thus simplifying the analysis. Applying the above theory to this experiment, it was found that the predicted and measured emittance growth rates were in agreement.

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Table 1: Ratio of phase noise band amplitudes to fundamental RF voltage amplitude.

Measurement	Relative Amplitude (db)	R.M.S. Phase Noise in 100 Hz (mrad)
1	-62 ± 1	1.1 ± 0.1
2	-68 ± 1	0.56 ± 0.06
3	< -90	< 0.04

Table 2: Emittance growth rate measurements corresponding to phase noise measurements in table 1.

Measurement	Growth Rate (10^{-12} m-rad/sec)
1	0.54 ± 0.04
2	0.16 ± 0.03
3	0.12 ± 0.01

Table 3: Tevatron parameters used in experiment.

Parameter	Value	Unit
f	47.713	kHz
Δf	100	Hz
E_0	900	GeV
eV_0	1.16	MeV/turn
H_{RF}	0.090 ± 0.020	m

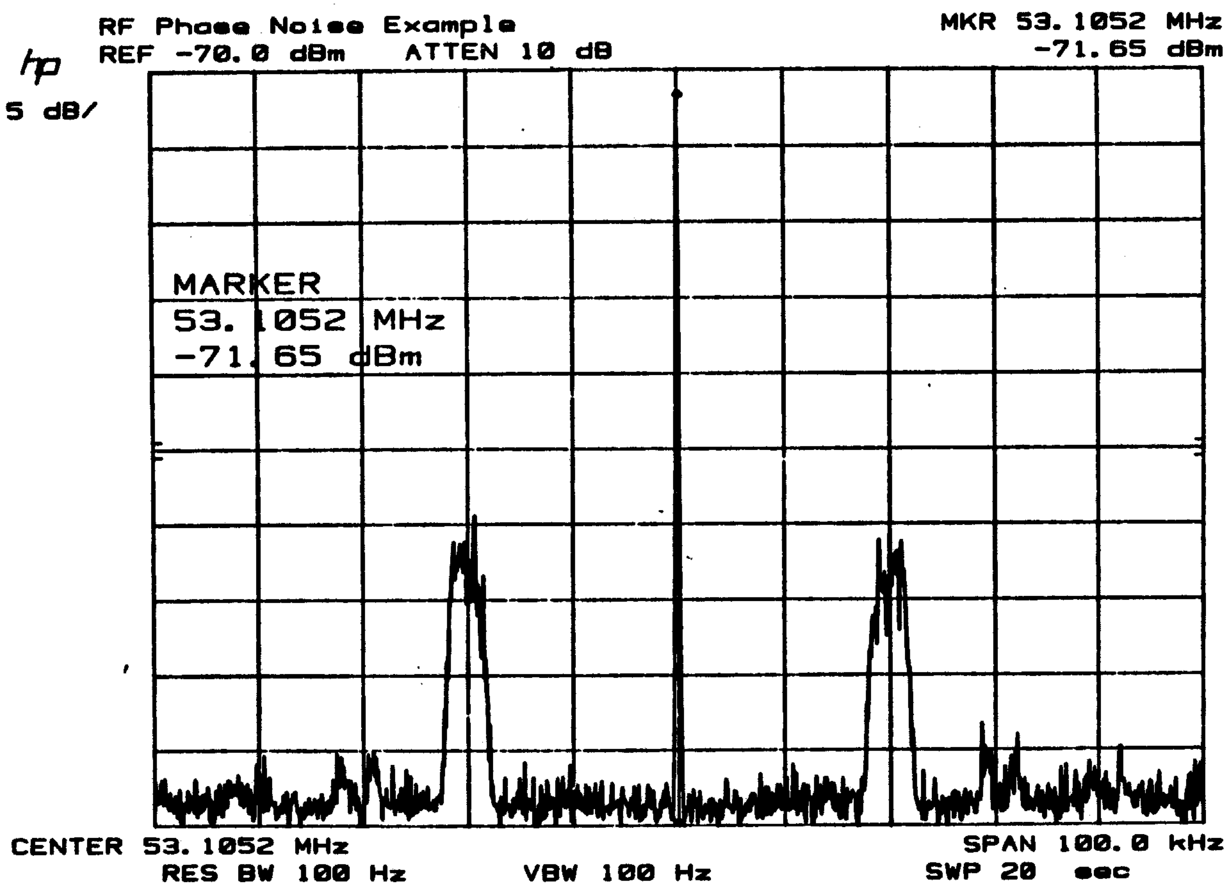

Figure 1: Photograph of the Tevatron RF gap voltage spectrum when phase noise was being added. The center frequency is 53.104 MHz (the RF frequency), the horizontal scale is 10 kHz/div, the vertical scale is 10 db/div, and the binwidth is 100 Hz.

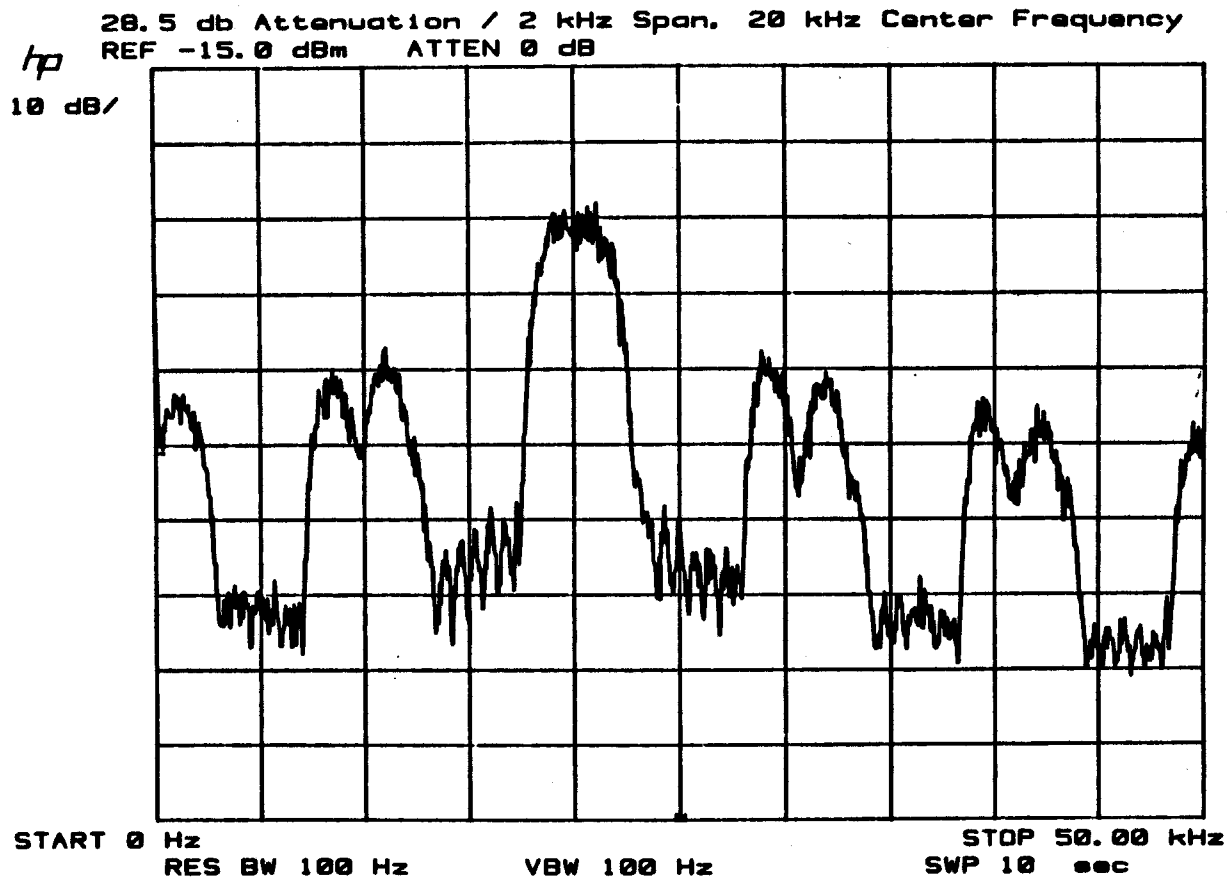

Figure 2: Fundamental noise band generated by the Hewlett-Packard 3561A Dynamic Signal Analyzer random noise source and injected into the Tevatron phase shifter.

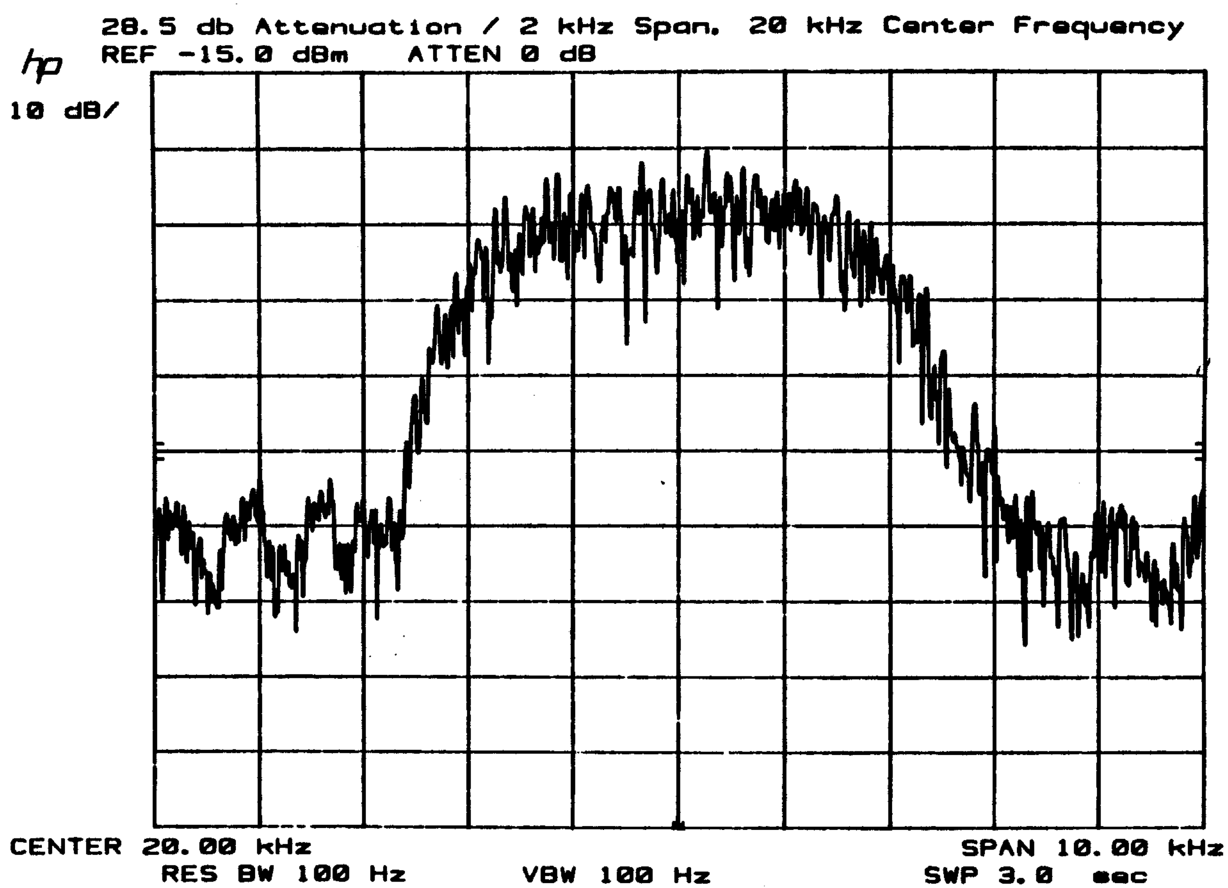

Figure 3: Broader view of the random noise generated by the HP 3561A. Note that there are noise sidebands roughly 20 db below the fundamental noise band at 20 kHz.

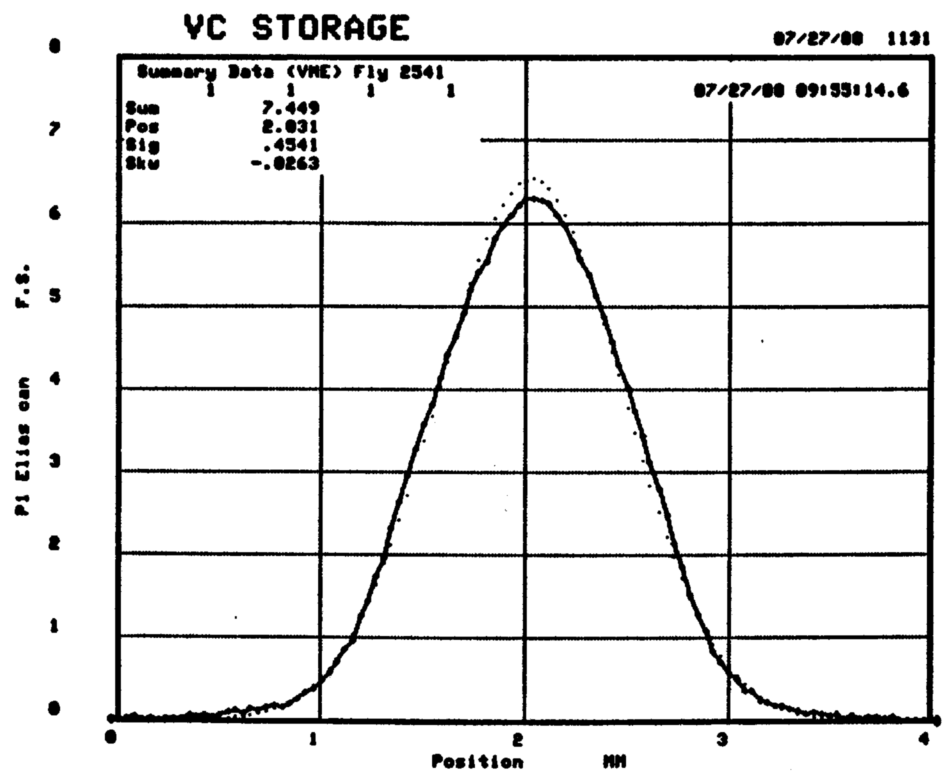

Figure 4: Example of the data generated by a flying wire scan through the beam. The small detached dots are an attempt by a computer program at fitting the data to a Gaussian.

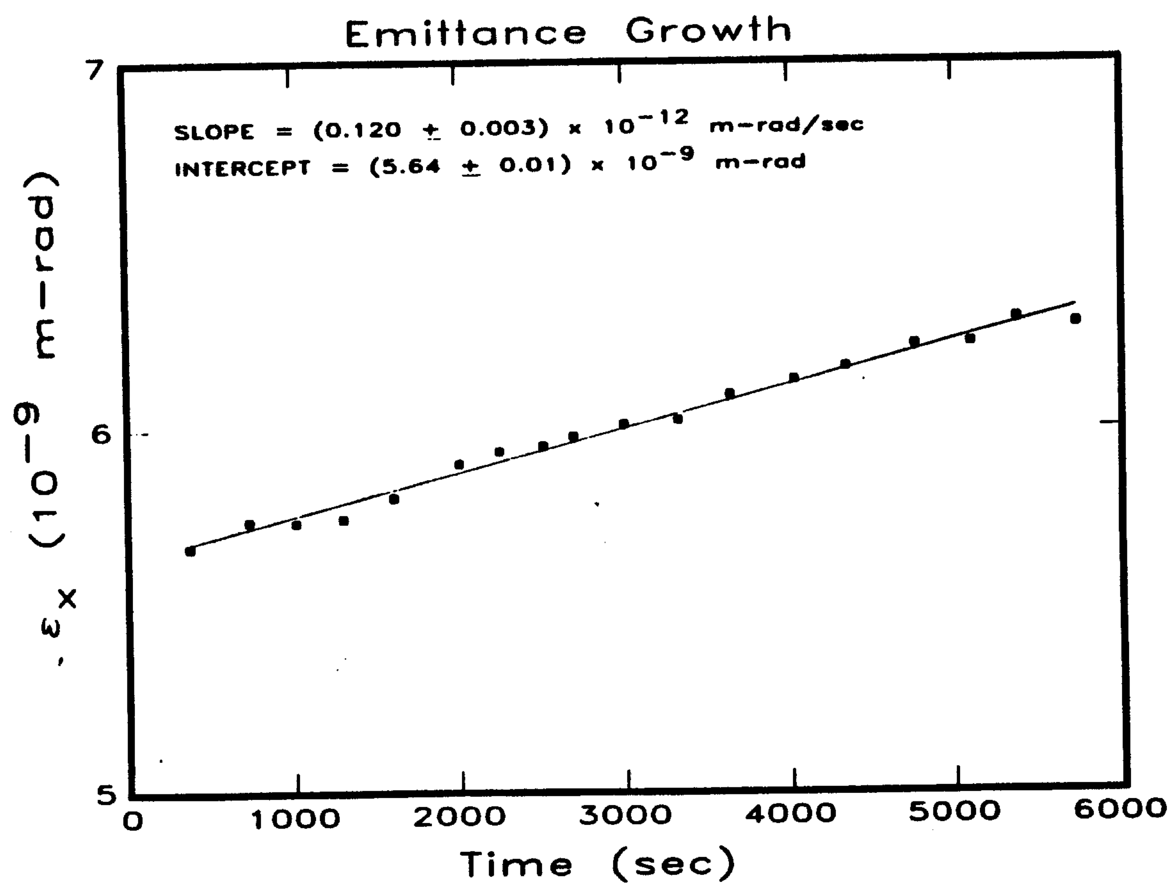

Figure 5: Horizontal emittance as a function of time in the Tevatron.

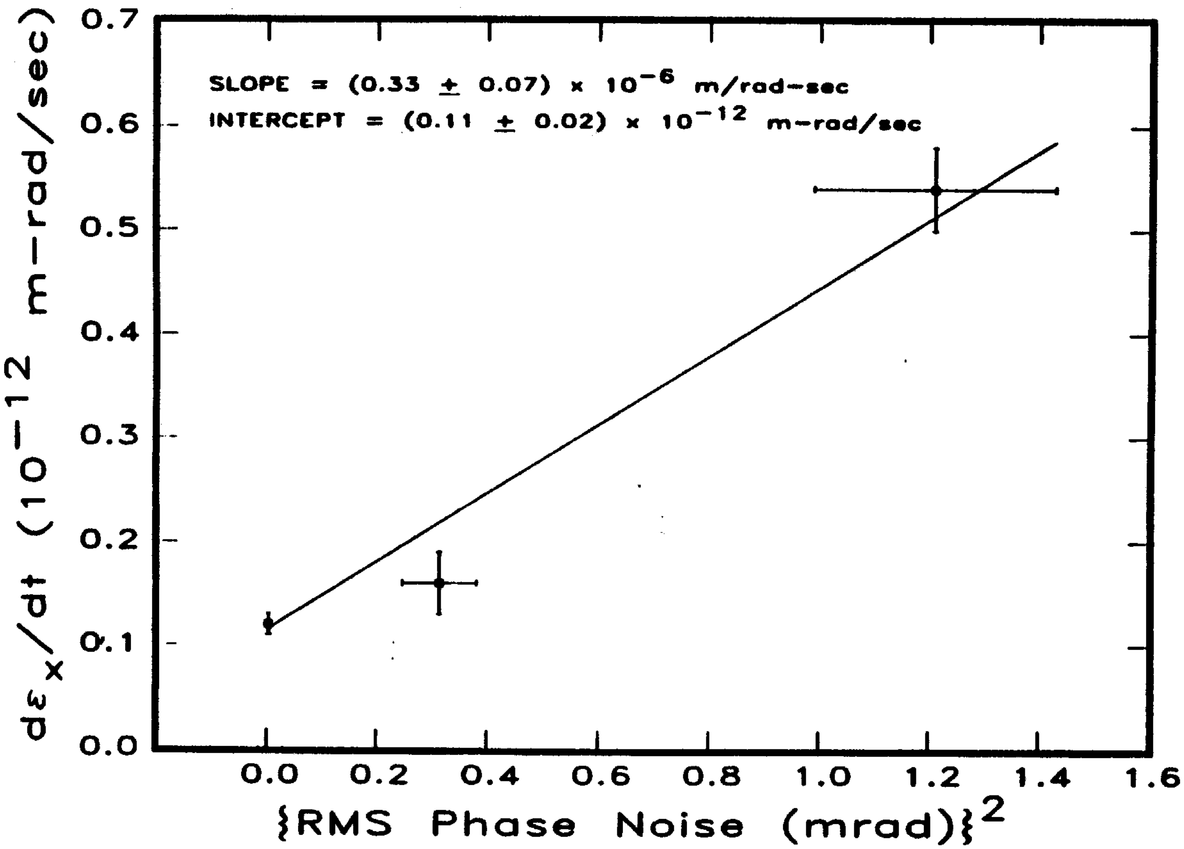

Figure 6: Fit of horizontal emittance as a function of phase noise amplitude in 100 Hz binwidth.