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Fusion rules and (sub)-modular invariant partition functions in non–unitary theories

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Abstract

Fusion rules in non–unitary Kac–Moody and minimal conformal theories are obtained from their modular properties. New modular invariant partition functions in the non–unitary Kac–Moody theories are classified. From the discrete symmetries of the system, we obtain new sub–modular invariant partition functions in the Kac–Moody and non–unitary conformal theories.

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I. Introduction

Near the critical temperature, the statistical systems in two dimensions exhibit the scaling behaviours. At the critical point, they are described by the conformal field theories[1]. Actually the conformal theories predict off-critical properties. The critical exponents[2] of the heat capacity ($C \propto t^{-\alpha}$), magnetization ($M \propto t^{\beta}$), susceptibility ($\chi \propto t^{-\gamma}$), correlators ($\langle\sigma(R)\sigma(0)\rangle \propto t^{-\eta}$), coherence length ($\xi \propto t^{-\nu}$) and magnetization ($M \propto h^{\sigma}$) ($t$ and h are the reduced temperature and magnetic field) are described by the spin and energy operator's conformal dimensions h_σ and h_ϵ as

$$\alpha = \frac{1-2h_\epsilon}{1-h_\epsilon}, \qquad \beta = \frac{h_\sigma}{1-h_\epsilon}, \qquad \gamma = \frac{1-2h_\sigma}{1-h_\epsilon},$$
$$\eta = 4h_\sigma, \qquad \nu = \frac{1}{2(1-h_\epsilon)}, \qquad \sigma = \frac{h_\sigma}{1-h_\sigma} \tag{1}$$

The numerical values of the critical exponents $(\alpha,\beta,\gamma,\eta,\nu,\sigma)$ in the Ising and three–state Potts models are known as $(0,\frac{1}{8},\frac{7}{4},\frac{1}{4},\frac{1}{15})$ and $(\frac{1}{3},\frac{1}{9},\frac{13}{9},\frac{4}{15},\frac{5}{6},\frac{1}{14})$ respectively, and agree with the conformal dimensions $(h_\sigma,h_\epsilon) = (\frac{1}{16},\frac{1}{2})$ and $(\frac{1}{15},\frac{2}{5})$ in Ising and three–state Potts models respectively. More generally correspondences between two-dimensional statistical systems and the unitary conformal theories are widely studied.

Relationships between the non–unitary conformal theories and statistical sytems have been less well studied. However there is at least one example in the statistical mechanics for which a minimal(i.e. with a finite number of primary fields) non–unitary theory shows up. It is the so called Lee–Yang singularity[3]. The properties of a system can be computed from the distribution of the zeroes of the grand partition *function where these zeroes are brought by the complex magnetic field(conjugate to* the spin operator). The density of zeroes exhibits the divergence near the critical complex magnetic field, and this led to identify[4] it with the $c = -\frac{22}{5}$ $h_{\sigma'} = -\frac{1}{5}$ non-unitary theory.

From the mathematical point of view, the minimal (both unitary and non-unitary) representations of Ref. [1] of the Virasoro algebra are the modular covariant representations; the $A_1^{(1)}$ unitary and non-unitary modular covariant representations are also described in Ref. [5].

Motivated by these aspects, we investigate properties of the non-unitary representations of Kac-Moody $A_1^{(1)}$ and minimal conformal theories in this letter. In a minimal representation, one can actually take subsets of operators whose choices are specified by the modular invariant properties of the partition functions. The dynamics of these theories are encoded into the operator product expansions. Using Verlinde's approach[6], we compute the fusion rules from the modular properties.

New non-unitary modular invariant partition functions of $A_1^{(1)}$ are found and are presented in a A-D-E classification similar to the one obtained in Ref. [7] for unitary $A_1^{(1)}$ representations.

Some of these theories enjoy additional symmetries which are also reflected in the fusion rules. We illustrate this with the well-known three-state Potts model. In presence of such symmetries, the fields at the opposite sides of the parallelogram can be different up to the action of these discrete symmetries(twisted boundary conditions). The partition function of the resulting theories are now invariant under a subgroup of the modular group, and we construct such submodular invariant partition functions.

II. Fusion rules

A. Kac-Moody system

The integrable representations of SU(2) affine Kac-Moody algebra in level $m \in N$ are denoted as ϕ_n with $n = 0, 1, \cdots, m$. Here n is twice of SU(2) spin. Their fusion rules are known[8,6]:

$$\phi_n \otimes \phi_{n'} = \sum_{n''=|n-n'|,\ n+n'-n''=0\, mod 2}^{min(n+n', 2m-n-n')} \phi_{n''} \tag{2}$$

More generally one can consider rational level $m = \frac{t}{u}$ with relatively prime $t \in Z$ and $u \in N$ satisfying $2u + t - 2 \geq 0$. The modular invariant or admissible representations[5] are labeled by two positive integers k and n such that

$$\begin{aligned} \phi_{k:n}\ ; \quad & 0 \leq k \leq u-1 \\ & 0 \leq n \leq 2u+t-2 \end{aligned} \tag{3}$$

The unitary integrable representation case is recovered by putting $u = 1, \quad t \in N$.

The chracters for the admissible representation (3) are given by

$$\chi_{\ k:n}\ (\tau, z) = \frac{\theta_{b_+,a}(\tau, \frac{z}{u}) - \theta_{b_-,a}(\tau, \frac{z}{u})}{\theta_{1,2}(\tau, z) - \theta_{-1,2}(\tau, z)} \tag{4}$$

with

$$\theta_{b,a}(\tau, z) = \sum_{j \in Z + \frac{b}{2a}} e^{2\pi i \tau a (j^2 + jz)} \tag{5}$$

and

$$a = u^2(m+2), \qquad b_\pm = u\{\pm(n+1) - k(m+2)\} \tag{6}$$

Due to the following properties of the theta functions

$$\theta_{b,a}(\tau, z) = \theta_{-b,a}(\tau, -z) = \theta_{b+2a,a}(\tau, z) = \theta_{-b+2a,a}(\tau, -z) \tag{7}$$

the characters for $1 \leq k \leq u-1$ satisfy the relation

$$\chi_{k:n}(\tau, z) = -\chi_{u-k\ :\ 2u+t-2-n}(\tau, -z) \tag{8}$$

The character is dominated in the large $Im\tau$ limit by

$$e^{2\pi i\tau(-\frac{c}{24}+h)} \tag{9}$$

where

$$c = \frac{3m}{m+2}, \qquad h = \frac{\{(n+1)-k(m+2)\}^2-1}{4(m+2)} \tag{10}$$

The h value becomes negative for $k(m+2)-2 < n < k(m+2)$. The character (4) may contain negative coefficients in the power exponsion of $e^{2\pi i\tau}$. For $u=1, h$ in eq.(10) is reduced to the Casimir eigenvalue of the spin $\frac{n}{2}$ representation normalized with respect to $1/(m+2)$ as

$$h = \frac{\frac{n}{2}(\frac{n}{2}+1)}{(m+2)} \tag{11}$$

The general modular transformation is

$$S_{(k,n)(k',n')} = \sqrt{\frac{2}{a}}\ \frac{1}{2i}\{e^{\frac{-i\pi b_+ b'_-}{a}} - e^{\frac{i\pi b_+ b'_+}{a}}\} \tag{12}$$

and satisfies $SS^\dagger = 1$. The fusion rules

$$\phi_i \otimes \phi_j = \sum_k A_{ij}^{\ k}\ \ \phi_k$$

are computed from eq.(12) with

$$A_{ij}{}^k = \sum_r \frac{S_{ir}S_{jr}S^*_{kr}}{S_{0r}}, \tag{13}$$

where index i denotes a representation $(k:n)$ and $i=0$ is the index of the identity representation. Note that A is symmetric in i and j, and one obtains:

$$\phi_{k:n} \otimes \phi_{k':n'} = \sum_{n''=|n-n'|,\ n+n'-n''=\ 0\,mod 2}^{min\{n+n',\,2(2u+t-2)-n-n'\}} (-)^{[(k+k')/u]} \phi_{k+k'\ mod\ u:n''} \tag{14}$$

where $[(k+k')/u]$ denotes integer division without remainder(e.g. $[4/3] = 1$). When $[(k+k')/u] = 1$, the n' in the right hand side has to be replaced by $n' \to (2u+t-2)-n'$. Eq. (14) is of course reduced to eq. (2) for the integrable representation $u = 1$ and $t \in N$. Let us illustrate eq. (14) in the case $m = \frac{1}{2}$ with $(t,u) = (1,2)$. The fusion rules then read:

$$\begin{aligned} \phi_{1:3}\ \phi_{1:2} &= -\phi_{0:2} \\ \phi_{1:0}\ \phi_{0:3} &= +\phi_{1:3} \end{aligned} \tag{15}$$

The associativity

$$\sum_k A_{ij}^{\ \ k}\ \ A_{kl}^{\ \ m} = \sum_k A_{il}^{\ \ k}\ \ A_{kj}^{\ \ m} \tag{16}$$

of the fusion rules in eq.(14) has been confirmed even in presence of negative fusion rules.

A computation of modular invariant partition functions for rational level $m = t/u$ leads to a A–D–E classification generalizing the one obtained in Ref. [7] for unitary $A_1^{(1)}$ representations. Note that the unitary cases are re-obtained from eq. (17) by restricting t to be positive integer and u to be one, the sum over $k = 0, ..., u-1$ then disappearing.

$$A_{2u+t-1}: \qquad \sum_{k=0}^{u-1}\sum_{n=0}^{2u+t-2}|\chi_{k:n}|^2 \tag{17}$$

$$D_{2\rho+2}: \quad 4\rho = 2u+t-2 \geq 4: \quad \sum_{k=0}^{u-1}\sum_{n\,even=0}^{2\rho-2}\{|\chi_{k:n}+\chi_{k:4\rho-n}|^2 + 2|\chi_{k:2\rho}|^2\}$$

$$D_{2\rho+1}: \quad 4\rho-2 = 2u+t-2 \geq 6: \quad \sum_{k=0}^{u-1}\{\sum_{neven=0}^{4\rho-2}|\chi_{k:n}|^2 + |\chi_{k:2\rho-1}|^2$$

$$+\sum_{nodd=1}^{2\rho-3}(\chi_{k:n}\chi^*_{k:4\rho-n}+c.c)\}$$

$$E_6: \quad 2u+t-2=10 \quad \sum_{k=0}^{u-1}\{|\chi_{k:0}+\chi_{k:6}|^2+|\chi_{k:3}+\chi_{k:7}|^2+|\chi_{k:4}+\chi_{k:10}|^2\}$$

$$E_7: \quad 2u+t-2=16 \quad \sum_{k=0}^{u-1}\{|\chi_{k:0}+\chi_{k:16}|^2+|\chi_{k:4}+\chi_{k:12}|^2$$

$$+|\chi_{k:6}+\chi_{k:10}|^2+|\chi_{k:8}|^2+[(\chi_{k:2}+\chi_{k:14})\chi^*_{k:8}+c.c.]\}$$

$$E_8: \quad 2u+t-2=28 \quad \sum_{k=0}^{u-1}\{|\chi_{k:0}+\chi_{k:10}+\chi_{k:18}+\chi_{k:28}|^2$$

$$+|\chi_{k:6}+\chi_{k:12}+\chi_{k:16}+\chi_{k:22}|^2\ \}$$

It is worthwhile to remark that to a given algebra of A–D–E type are now asociated a infinite(discrete) set of models. For example, to the D_4 algebra, correspond theories such that

$$(t,u) = (-4,5),(-8,7),(-16,11),(-20,13),....$$

We note that two different allowed models do not involve the same number of primary fields, k index in the partition function taking the values $k=0,1,...,u-1$. We have not yet checked whether eq. (17) gives the complete list of modular invariant partition functions relative to the admissible representations in $A_1^{(1)}$.

B. Minimal conformal system

Let us remind that a minimal conformal theory[1] is characterized by a pair (p, p') of two relatively prime positive integers. The central charge then reads:

$$c = 1 - \frac{6(p-p')^2}{pp'} \tag{18}$$

and the primary fields are labelled as:

$$(r,s) = (p'-r, p-s) \quad \text{with} \quad \begin{array}{l} 1 \leq r \leq p'-1 \\ 1 \leq s \leq p-1. \end{array} \tag{19}$$

The case $|p-p'| = 1$ corresponds to the unitary series[9].

The fusion rule for a general pair (p, p') can be deduced from the modular transformation properties of characters, and reads

$$\phi_{r,s} \otimes \phi_{r',s'} = \sum_{r''=|r-r'|+1}^{min\{r+r'-1,\ 2p'-1-r-r'\}} \sum_{s''=|s-s'|+1}^{min\{s+s'-1,\ 2p-1-s-s'\}} \phi_{r'',s''} \tag{20}$$

with the constraint $r + r' - r'' = s + s' - s'' = 1$ mod2. The relation between eq. (20) and the eq. (6.7) in Ref. [1] is the following: one applies eq. (6.7) of Ref. [1] for $(r,s)\otimes(r',s')$, $(r,s)\otimes(p'-r',p-s')$, $(p'-r,p-s)\otimes(r',s')$, $(p'-r,p-s)\otimes(p'-r',p-s')$ and take common terms in the product after making use of the identity (19).

III. Symmetries

Minimal theories cannot have a continuous symmetry since a $h = 1, \bar{h} = 1$ field, which would be associated to a conserved current, never shows up, but they do admit discrete symmetries.

The well known examples of such symmetries are the Z_2 symmetry in the Ising model and the Z_3 or more precisely S_3 symmetry in the three-state Potts models[10]. Actually these models belong to the class of the parafermionic theories[11].

Note that such a discrete symmetry can help to break the degeneracy in a theory where more than one primary field $\phi_{i,\bar{i}}(z, \bar{z})$ with the same conformal weights h_i, $h_{\bar{i}}$ are present[12].

As an example consider the (D_4, A_4) model (i.e. the three-state Potts model) with partition function

$$Z = |\chi_0 + \chi_3|^2 + |\chi_{2/5} + \chi_{7/5}|^2 + 2|\chi_{1/15}|^2 + 2|\chi_{2/3}|^2 \tag{21}$$

where the subscripts denote the conformal weights of the corresponding primary field. We will denote the primary fields associated to the linear combination of characters $(\chi_0 + \chi_3)$, $(\chi_{2/5} + \chi_{7/5})$ by ϕ_1 and ϕ_2, and by $\phi_3{}^{\pm}$ and $\phi_4{}^{\pm}$ those associated to $\chi_{1/15}$ and $\chi_{2/3}$.

The modular transformation matrix relative to the six vectors $(\phi_1, \phi_3^+, \phi_3^-, \phi_2, \phi_4^+$ and

ϕ_4^-) is

$$S = \frac{2}{\sqrt{15}} \begin{bmatrix} \sin\frac{\pi}{5} & \sin\frac{2\pi}{5} & \sin\frac{2\pi}{5} & \sin\frac{2\pi}{5} & \sin\frac{\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{2\pi}{5} & -\omega\sin\frac{\pi}{5} & -\omega^2\sin\frac{\pi}{5} & -\sin\frac{\pi}{5} & \omega\sin\frac{2\pi}{5} & \omega^2\sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & -\omega^2\sin\frac{\pi}{5} & -\omega\sin\frac{\pi}{5} & -\sin\frac{\pi}{5} & \omega^2\sin\frac{2\pi}{5} & \omega\sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & -\sin\frac{\pi}{5} & -\sin\frac{\pi}{5} & -\sin\frac{\pi}{5} & \sin\frac{2\pi}{5} & \sin\frac{2\pi}{5} \\ \sin\frac{\pi}{5} & \omega\sin\frac{2\pi}{5} & \omega^2\sin\frac{2\pi}{5} & \sin\frac{2\pi}{5} & \omega\sin\frac{\pi}{5} & \omega^2\sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & \omega^2\sin\frac{2\pi}{5} & \omega\sin\frac{2\pi}{5} & \sin\frac{2\pi}{5} & \omega^2\sin\frac{\pi}{5} & \omega\sin\frac{\pi}{5} \end{bmatrix} \tag{22}$$

The S matrix satisfies

$$S\,S^{\dagger} = 1, \qquad SS = C, \tag{23}$$

where C is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The Z_3 group leaves invariant ϕ_1 and ϕ_2, and provides a phase-(eigenvalue) $\omega = \exp(i2\pi/3)$ to ϕ_3^+ and ϕ_4^+, and ω^2 to ϕ_3^- and ϕ_4^-. Such an action can be seen as an automorphism on the fusion rules[13]. of the model.

Indeed one obtains the following fusion rules from eq. (22):

$$\begin{array}{lll}
\phi_1 \cdot \phi_1 = \phi_1 & \phi_1 \cdot \phi_2 = \phi_2 & \phi_2 \cdot \phi_2 = \phi_1 + \phi_2 \\
\phi_1 \cdot \phi_3^\pm = \phi_3^\pm & \phi_2 \cdot \phi_3^\pm = \phi_3^\pm + \phi_4^\pm & \\
\phi_1 \cdot \phi_4^\pm = \phi_4^\pm & \phi_2 \cdot \phi_4^\pm = \phi_3^\pm & \\
\phi_3^+ \cdot \phi_3^- = \phi_1 + \phi_2 & \phi_3^+ \cdot \phi_4^- = \phi_3^- \phi_4^+ = \phi_2 & \phi_4^+ \cdot \phi_4^- = \phi_1 \\
\phi_3^+ \cdot \phi_3^+ = \phi_3^- + \phi_4^- & \phi_3^+ \cdot \phi_4^+ = \phi_3^- & \phi_4^+ \cdot \phi_4^+ = \phi_4^- \\
\phi_3^- \cdot \phi_3^- = \phi_3^+ + \phi_4^+ & \phi_3^- \cdot \phi_4^- = \phi_3^+ & \phi_4^- \cdot \phi_4^- = \phi_4^+
\end{array} \tag{24}$$

Fusion rule A_{23}^1 in $\phi_3^+ \cdot \phi_3^- = \phi_1$ is computed from eq. (13) as $A_{23}^1 = (SS)_{23} = C_{23} = 1$, and this partially 'explains' why $SS = C$ in eq. (23). The fusion rules in eq.(20) exhibit the following S_3 symmetries ;

$$Z_3 \; : \; \begin{pmatrix} \phi_1 \\ \phi_3^+ \\ \phi_3^- \end{pmatrix} \rightarrow \begin{pmatrix} \phi_1 \\ \omega\phi_3^+ \\ \omega^2\phi_3^- \end{pmatrix} \text{and} \begin{pmatrix} \phi_2 \\ \phi_4^+ \\ \phi_4^- \end{pmatrix} \rightarrow \begin{pmatrix} \phi_2 \\ \omega\phi_4^+ \\ \omega^2\phi_4^- \end{pmatrix} \tag{25}$$

$$Z_2 \; : \; \begin{pmatrix} \phi_1 \\ \phi_3^+ \\ \phi_3^- \end{pmatrix} \rightarrow \begin{pmatrix} \phi_1 \\ \phi_3^- \\ \phi_3^+ \end{pmatrix} \text{and} \begin{pmatrix} \phi_2 \\ \phi_4^+ \\ \phi_4^- \end{pmatrix} \rightarrow \begin{pmatrix} \phi_2 \\ \phi_4^- \\ \phi_4^+ \end{pmatrix} \tag{26}$$

Note that eqs.(25) and (26) corresponds to cyclic group of order 3 and 2 respectively.

Products of transformations (25) and (26) also leave the fusion rule (24) invariant.

IV. Submodular invariant partition functions

If a system is invariant under a group G, one may impose G-twisted boundary conditions on its conformal fields [10,14,8]. In the case where $G = Z_2$ (resp. Z_3) the associated partition functions are then invariant under a subgroup of the modular group[14], namely $\Gamma_0(2)$ (resp. $\Gamma_0(3)$) generated by

$$\begin{aligned} T' : \tau \rightarrow \quad & \tau + 2 \quad \text{in} \quad \Gamma_0(2) \\ & \tau + 3 \quad \text{in} \quad \Gamma_0(3) \end{aligned} \tag{27}$$

and S' which is constructed from the S and T of full modular group as

$$\tau \rightarrow \tau + 1 \rightarrow -\frac{1}{\tau+1} \rightarrow 1 - \frac{1}{\tau+1} \tag{28}$$

The submodular $\Gamma_0(2)$ invariant partition functions for the minimal (unitary or non-unitary) conformal theories associated to the $(A_{p'-1}, A_{p-1})$ series are:

$$\sum_{r=1}^{p'-1} \sum_{s=1}^{p-1} \chi_{r,s} \chi^*{}_{r,\, p-s} = \sum_{r=1}^{p'-1} \sum_{s=1}^{p-1} \chi_{r,s} \chi^*{}_{p'-r,\, s} \,. \tag{29}$$

When the p and p' are restricted to $|p - p'| = 1$(unitary series), eq.(29) include eq.(9) in the first paper in Ref.[14]. The spin of the operators(=difference in conformal dimensions of the chiral and anti-chiral sectors) is a multiple of $\frac{1}{2}$.

The submodular $\Gamma_0(3)$ invariant partition functions have been obtained for $(p, p') = (p, 6)$ conformal theories in the series (D_4, A_{p-1}), and read:

$$\sum_{i=1}^{[p/2]} \{ (\chi_{1\, i} + \chi_{1\, p-i}) \chi^*_{3i} + c.c + |\chi_{3i}|^2 \} \tag{30}$$

where $[p/2]$ denotes the integer division without remainder. As mentioned before, the p should be co-prime with respect to 6. Operators in such theories have spin which is a multiple of $\frac{1}{3}$. The $p = 5$ and 7 cases agree with the unitary $\Gamma_0(3)$ invariant partition function given in Ref. [14,11] and $p = 5$ case is related to the three-state Potts model with Z_3 symmetry considered in eq.(25).

The simplest non-unitary $\Gamma_0(3)$ invariant partition function corresponds to the theory $(p, p') = (11, 6)$ with central charge $c = \frac{6}{11}$ and reads:

$$(\chi_{1,1} + \chi_{1,10})\chi_{31}{}^* + (\chi_{1,2} + \chi_{1,9})\chi_{3,2}^* + \cdots$$

$$+ (\chi_{1,5} + \chi_{1,6})\chi_{3,5}^* + c.c + \{|\chi_{3,1}|^2 + \cdots + |\chi_{3,5}|^2\} \tag{31}$$

We end up this section with the $\Gamma_0(3)$ invariant partition function associated to the level $m = 4$ representations of $A_1^{(1)}$ affine Kac–Moody algebras:

$$(\chi_0 + \chi_4)\chi_2^* + \chi_2(\chi_0 + \chi_4)^{\ *} + \chi_3\chi_3{}^* \tag{32}$$

V. Conclusion

Some properties of the Kac–Moody $A_1^{(1)}$ as well as minimal Virasoro theories have been generalized to the non-unitary case. Fusion rules have been constructed for the non-unitary representations of affine Kac–Moody theories and their modular invariant partition functions are constructed. Submodular invariant partition functions with twisted boundary conditions have been worked out for the non-unitary conformal and Kac-Moody theories. The above results can be considered as a first step in a general study of the minimal non-unitary theories. In particular, one may wonder whether our classification of non-unitary $A_1^{(1)}$ modular invariant partition functions is complete. Furthermore one should compute the fusion coefficients in the operator product expansions which are typically ratios of gamma functions[1,15], while the fusion coefficients in eq. (13) is normalized to the integer.

From the given list of the sub-modular invariant partition functions, which deserves to be completed, it looks likely that the symmetry group of a minimal conformal theory is isomorphic to the invariance group of the A-D-E Dynkin diagram associated to the model. Pasquier's approach[16] of the A-D-E classification[7] suggests this conjecture. Note that, if Z_2 and S_3 are the symmetry group of the Ising model and three-state Potts model respectively, the four-state Potts model involves the $D_4^{(1)}$ Kac-Moody algebra[17], the symmetry of the Dynkin diagram of which is S_4. One may wonder whether the five-state Potts model could be related to the hyperbolic algebra[18] D_4^H whose Dynkin diagram, obtained from the diagram of $D_4^{(1)}$ by additions of a sixth dot connected to the central dot by one line, admits a S_5 symmetry.

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