

## The Decay of Highly Excited Open Strings

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### Abstract

We calculate the decay rates of leading Regge trajectory states for very high level number in open bosonic string theories, ignoring tachyon final states. The optical theorem simplifies the analysis while enabling identification of the different mass level decay channels. Our main result is that (in four dimensions) the greatest single channel is the emission of a single photon and a state of the next mass level down. A simple asymptotic formula for arbitrarily high level number is given for this process. We also calculate the total decay rate exactly up to  $N=100$ . It shows little variation over this range but appears to decrease for larger  $N$ . We check our formalism in examples and calculate the decay rate of the first excited level for open superstring theories. The calculation may also have implications for high spin meson resonances.

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## Introduction

In all the years of string theory [1], remarkably little attention has been paid to the interactions of the massive string modes. Recent work [2,3,4] on the statistical mechanics of strings at very high densities has emphasised the fact that as the Hagedorn temperature is approached, a phase transition occurs with *most* of the energy going into very long strings. It is therefore of some importance to understand the interactions of very excited strings and in particular how fast they decay.

Quite independently of this it is interesting to try and understand whether and how a classical limit emerges in string theory - do highly excited strings behave in a 'classical' way?

In this paper we attempt a modest beginning to the effort at a better understanding of the massive string states by calculating the decay rate of a particular class of occupation number states of arbitrarily high energy - the states on the leading Regge trajectory of open string theories. A similar problem was considered in 1971 by Green and Veneziano [5] who argued that the high mass intermediate states in dual amplitudes had narrow decay widths and our result appears consistent with theirs. However their argument was rather incomplete.

Our main tool will be the optical theorem and a method for extracting the imaginary part of the string self-energy diagram which we explain in the first section. One advantage of our method is that it enables one to identify which piece of the imaginary part is responsible for the decay into a given mass level for each of the final states. Thus while we are saved considerable labour in calculating polarization sums and phase space integrals we can

recover quite a lot of detailed information on how massive strings decay.

This is actually essential to how we treat the tachyon. Of course tachyons, living on a continuous Lorentz hyperboloid<sup>1</sup>,  $p^2 = m^2$ , do not have a definite sign for their energy. Any decay process involving tachyons is infinite - one can obtain in the decay an arbitrarily negative energy tachyon plus an arbitrarily positive energy excited state in the 'decay' of a photon for example. This is quite a graphic way of seeing the 'vacuum' instability in theories with tachyons.

However we wish nevertheless to calculate physically meaningful quantities for bosonic strings since these are certainly the most tractable and allow us to proceed furthest. Since these theories have tachyons we will simply calculate the total decay rate of highly excited strings into everything *except* tachyons. A major advantage of our method is that it enables us to extract the 'tachyonic' part of the decay rate rather simply and calculate the remaining, finite part. This is perfectly well defined and in fact it is quite likely that the main features of our calculation will remain true for theories without tachyons - heterotic strings and superstrings for example.

We shall calculate in arbitrary  $d$ . The open bosonic string is of course inconsistent at the one loop level outside of  $d = 26$ , and in fact the need for  $d = 26$  was discovered in calculations by Lovelace[6] of one of the diagrams we consider. However we shall really only be using the loop diagrams as a convenient way of summing tree diagrams (we shall only use it's imaginary part and not the real part which is infinite), which do not require  $d = 26$ . Furthermore, because of the particular external states we consider the loop

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<sup>1</sup>Our metric convention throughout this paper is  $(-, +, +, + \dots)$ .

amplitude will not have the singularities of the type considered by Lovelace, corresponding to propagating closed strings. Our calculations would also apply to the case of open strings in  $d = 26$  with a subset of the 26 dimensions taken to be toroidally compactified, and with the compactification radius so small that degrees of freedom depending on the extra dimensions are never excited.

We shall discuss in detail the diagrammatics for the simplest case, the orientable  $U(1)$  open string. The extension of our result to other groups should be straightforward and not qualitatively different.

Our main result is that the largest single process is the decay into a photon and a state at the next level down - for this process the decay rate is inversely proportional to the mass of the initial string state. The total decay rate is much harder to say anything analytic about. We calculate it up to  $N=100$  and it appears to be slowly *decreasing* for large  $N$ . This means that the open string coupling constant *cannot* be thought of as a splitting probability per unit length - if this were the case long strings would fragment at a rate *increasing* as  $\sqrt{N}$ .

There may well be implications of our result for hadronic physics. It is well known that resonances of very high spin are observed rather more commonly than would be expected naively - their widths being generally no larger than resonances of lower mass.

One would expect that the bosonic string description of mesons would be good for the leading trajectory resonances since these are the most spatially extended states. Our procedure of throwing away the tachyon is certainly a fairly violent procedure at the low end of the mass spectrum so our calcula-

tions are probably a poor description of the interactions of low mass states. However the decay rates we find are not dominated by the emission of the lowest mass states as can be seen for example in Figure 7. It seems reasonable therefore that our calculations should provide a reasonable description of the decays of high spin mesons. It is gratifying that our results lead to an approximately constant decay rate as a function of level number, in qualitative agreement with experiment. It would be interesting to try to make the connection more precise - this could provide new information about hadronic strings.

As a final motivation for this work, it may eventually provide a consistent method for treating 'back-reaction' problems in radiation from cosmic strings - gravitational, electromagnetic or goldstone radiation. In the quantised string these processes are automatically finite, and the probability amplitudes for various final states should also tell us the nature of the configuration that the string ends up in. We leave this however for future work.

In a subsequent paper three of us [7] will extend the treatment presented below to closed strings where interesting issues connected with modular invariance arise.

The outline of this paper is as follows. In Section 1 we give a simple example of the optical theorem in  $d$ -dimensional scalar field theory and a method of extracting imaginary parts which we later use for strings. In Section 2 we calculate the one loop planar self-energy diagram for arbitrary highly excited external strings on the leading Regge trajectory, for which the vertex operators are particularly simple. In Section 3 we calculate its imaginary part, and in section 4 we discuss how the other topologically distinct diagram at

this order should be added. The total decay rate as a function of mass of the initial state is calculated exactly up to level 100, illustrated in Figures 6 to 8. In Section 5 we find an asymptotic formula for the single greatest term in the decay, due to the string emitting a photon and lowering its mass level by one. In Section 6 we calculate the decay of the first massive state in open superstring theories by the same method.

While this work was being completed we recieved a preprint by J. Polchinski [12] who discusses the interactions of macroscopic closed strings on a torus. He mentions that by a similar method one could calculate the decay of a closed string into open strings. His result seems quite different to ours - he claims it *is* interpretable as a splitting probability per unit length. It would be interesting to understand the reason for this apparent discrepancy.

## 1 The Optical Theorem: A Simple Example

In this section we present a simple example of the optical theorem for interacting scalar fields. This is of course very well known and in this example rather unnecessary - integrating over phase space is simpler. However the method we use will generalise directly to the case of strings, both open (in this paper) and closed (in a sequel [7]). It may also seem rather pedestrian for open strings, where a quicker approach would be to use a generalisation of the Cutkosky rules (see for example [11]). However both in the interests of giving a rigorous self contained account and because our approach generalises to closed strings, we prefer the method we shall explain below.

To recap briefly, the optical theorem starts from the unitarity of the S

matrix:

$$S = 1 + iT \quad S^\dagger S = 1 \rightarrow i(T^\dagger - T) = T^\dagger T \quad (1)$$

Taking matrix elements and defining  $\langle f | T | i \rangle = T_{fi} \mathcal{V}^d(p_f - p_i)$  we obtain<sup>2</sup>

$$2ImT_{ii} = \sum_f |T_{if}|^2 \mathcal{V}^d(p_f - p_i) \quad (2)$$

where of course initial and final states must be normalised, producing the usual  $2E$  factors and the volume and time factors which turn the right hand side into a decay rate.

Now consider a theory with two scalar fields  $\Phi$  and  $\phi$  with masses  $M$  and  $m$  respectively, and a  $\frac{\lambda}{2}\Phi\phi^2$  interaction. Then equation (2) relates the imaginary part of the  $\Phi$  self-energy to the total decay rate of  $\Phi$  particles into  $\phi$  particles. To lowest order in the coupling, this is shown diagrammatically in Figure 1.

In its rest frame, the  $\Phi$  particle has momentum  $k^\mu = (M, 0)$  and we have for the one loop  $\Phi$  self-energy

$$iT_{ii} = \frac{\lambda^2}{4M} \int \frac{\mathcal{V}^d p}{(p^2 + m^2 - i\epsilon)((k-p)^2 + m^2 - i\epsilon)} \quad (3)$$

where  $\frac{1}{2M}$  is from normalising the  $\Phi$  particle initial state and  $\frac{1}{2}$  from the symmetry factor.

Using Schwinger's representation this is

$$-\frac{\lambda^2}{4M} \int \mathcal{V}^d p \int_0^\infty d\alpha \int_0^\infty d\beta e^{-i\alpha(p^2+m^2-i\epsilon)-i\beta((k-p)^2+m^2-i\epsilon)} \quad (4)$$

which is of course divergent for  $d \geq 4$ . However from (2) we know that its imaginary part, being a phase space integral over tree diagrams, must be

<sup>2</sup>our convention throughout is  $\mathcal{V}^d p = \frac{d^d p}{(2\pi)^d}$  and  $\mathcal{V}(p) = 2\pi\delta(p)$

finite. Performing a Wick rotation,  $p^0 = ip_E^0$  we regularise (4) with a Euclidean convergence factor  $e^{-\Lambda p^2}$ . We shall extract the imaginary part of the regularised expression and then set  $\Lambda$  to zero. The Schwinger representation arises naturally in string theory as we shall see, and this method of regularisation is close to what happens in the closed string theory, in which there are no ultraviolet divergences. After performing the Gaussian momentum integral we change variables to  $x = \alpha - i\Lambda + \beta, y = \frac{(\alpha - i\Lambda)}{(\alpha - i\Lambda + \beta)}$  to obtain

$$\begin{aligned}
T_{ii} &= \frac{\lambda^2}{4M(4\pi)^{\frac{d}{2}}} (i)^{2-\frac{d}{2}} \int_0^1 dy \int_{-i\Lambda}^{\infty} dx x^{1-\frac{d}{2}} e^{-iA(y)x - \epsilon x} & (5) \\
&\equiv \frac{\lambda^2}{4M(4\pi)^{\frac{d}{2}}} \int_0^1 dy I(y) \\
A(y) &= m^2 - k^2 y(1-y)
\end{aligned}$$

where we drop an overall real constant  $e^{\Lambda^2}$  which tends to 1 as  $\Lambda$  tends to zero. Taking the  $y$  integral along the real axis, the  $x$  integral converges for all  $A(y)$  on the contour (1) shown in Figure 2.

Now the function  $A(y)$  is a parabola (Figure 3); for  $M^2 < 4m^2$  it is always positive there is no imaginary part and consequently no decay - simply a reflection of energy conservation. However if  $M^2 > 4m^2$  then  $A(y)$  is *negative* over a range of  $y$  which results in an imaginary part for the self energy and the  $\Phi$  particle decays.

To see this is so, for  $A(y)$  positive we can rotate the  $x$  contour *downward* to (2), so  $x = -iz, \Lambda < z < \infty$ . Then  $T_{ii}$  is now clearly real (although of course divergent as  $\Lambda$  tends to zero) and there is no decay.

However if  $M^2 > 4m^2$  then  $A(y)$  is negative between  $y_{\pm} = \frac{1}{2}(1 \pm \sqrt{1 - \frac{4m^2}{M^2}})$ , and the contour may only be rotated *upward*, running around the origin on

(3). Integrating by parts, we can reduce  $I(y)$  in (4) to a real series in negative powers of  $\Lambda$ , and a remaining integral.

If  $d$  is even, setting  $z = |A| x$  we find

$$\begin{aligned}
I &= |A|^{\frac{d}{2}-2} (i)^{2-\frac{d}{2}} \int_{-i|A|\Lambda}^{i\infty} dz z^{1-\frac{d}{2}} e^{iz} \\
&= |A|^{\frac{d}{2}-2} \left( \left( \frac{r^{2-\frac{d}{2}}}{(2-\frac{d}{2})} + \frac{r^{3-\frac{d}{2}}}{(2-\frac{d}{2})(3-\frac{d}{2})} + \dots \right) e^r \right. \\
&\quad \left. + \frac{1}{\Gamma(\frac{d}{2}-1)} \left( \int_3 \frac{dz}{z} e^{iz} \right) \right) \quad (6)
\end{aligned}$$

where  $r = |A| \Lambda$ . As before, the series (in inverse powers of  $\Lambda$ ) is real. However for finite  $\Lambda$  the integral along (3) may be distorted to run up along the imaginary axis, along an infinitesimal semicircle around the origin and along the imaginary axis up to  $i\infty$ . The integral along the axis gives a *real* principal value (logarithmically divergent as  $\Lambda$  tends to zero), while the integral around the semicircle gives  $i\pi$ . Thus for  $\Lambda$  negative

$$\text{Im}(I) = \frac{|A|^{\frac{d}{2}-2}}{\Gamma(\frac{d}{2}-1)} i\pi \quad (7)$$

Similarly if  $d$  is odd we obtain a real series ending in

$$i \frac{|A|^{\frac{d}{2}-2}}{\Gamma(\frac{d}{2}-1)} \sqrt{\pi} \int_{-\Lambda}^{\infty} \frac{dx}{\sqrt{x}} e^{-x} \quad (8)$$

where we have distorted the integral to run along the imaginary axis. The integral is now finite as  $\Lambda$  tends to zero and the result is again given by (7). These results are also easily obtained in dimensional regularisation.

The remaining  $y$  integral in (5) may be performed as explained in Appendix A to obtain

$$\Gamma = \frac{\lambda^2 \pi M^{d-5}}{(16\pi)^{\frac{d-1}{2}} \Gamma(\frac{d-1}{2})} \left( 1 - \frac{4m^2}{M^2} \right)^{\frac{d-3}{2}} \quad (9)$$

It is simple enough to check that this is identical with the answer obtained from the right hand side of (2)

$$\Gamma = \frac{\lambda^2}{2} \frac{1}{2M} \int \frac{d^{d-1}p_1}{2E_1} \int \frac{d^{d-1}p_2}{2E_2} \delta^d(p - p_1 - p_2) \quad (10)$$

The  $\frac{1}{2}$  comes from the identical  $\phi$  final particles, with momentum  $p_1$  and  $p_2$ .

One slightly surprising consequence of (9) is that decay rates *decrease* rapidly in high dimensions, if the masses and the dimensionless coupling  $\lambda m^{\frac{4-d}{2}}$  are kept fixed. This is a result of the fact that the surface area of a unit sphere decreases rapidly in high dimensions. For just this trivial reason massive string modes are rather long lived in 26 dimensions!

## 2 One Loop String Amplitudes and Decay Rates

We now turn to the problem of evaluating one loop string amplitudes with two identical excited external particles. We shall focus on open bosonic string theories with external particles of definite occupation number, and furthermore on the states with highest angular momentum for given mass, the “leading Regge trajectory” states. Classically they correspond to a straight rotating rod where of course the ends move at the speed of light, although they are of course occupation number rather than position eigenstates.

We construct the one loop amplitude by inserting two vertex operators to produce the external states and then tracing over all states circulating in the loop. This is a well known procedure for external tachyons or vector particles. In this section we present the calculation for arbitrarily excited

external states and although for the most part this is mostly familiar, there are some new aspects and we shall highlight these.

We work in d-dimensional spacetime - we shall not see the requirement  $d=26$  in this process as we discuss later. The states we calculate the decay of are, in standard notation[1] at level  $N$

$$| \psi \rangle_N = \eta_{\mu\nu\rho\dots\lambda} \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\rho \dots \alpha_{-1}^\lambda | 0 \rangle \quad (11)$$

with momentum  $k$  obeying the mass shell condition  $k^2 = -2\pi T(N - 1)$  and the (symmetric) polarisation tensor  $\eta$  has  $N$  indices.  $T$  is the string tension. In what follows we will set  $T\pi = \frac{1}{2\alpha'} = 1$  and restore  $T$  by dimensional analysis later. Requiring (11) to be physical (i.e.  $L_n | \psi \rangle_N = 0, n > 0$ ) results in the transversality and tracelessness conditions

$$\begin{aligned} k^\mu \eta_{\mu\nu\dots\lambda} &= 0 \\ \eta_{\mu}^{\mu\nu\rho\dots} &= 0 \end{aligned} \quad (12)$$

The vertex operator for these states is simply

$$V_k = \eta_{\mu\nu\rho\dots\tau} : P^\mu P^\nu P^\rho \dots P^\tau e^{ik \cdot X} : \quad (13)$$

where dots denote normal ordering and the notation is standard[1].

The conditions (12) are sufficient, with the mass-shell condition, to guarantee (13) has the correct conformal dimension. There are no normal ordering singularities between any of the different terms in (13), because of (12), so the naive conformal dimension is the correct one. It is also easily checked that applying the vertex (13) to the vacuum at  $\tau = i\infty$  in the usual way [1] produces the state (11). Again following standard procedure we construct

the one loop amplitude by sewing together the ends of the tree diagram (Figure 4) with loop momentum  $p$ . Including ghosts in the propagators[9,1], we obtain

$$iT = g^2 \int_0^1 dx_1 \int_0^1 dx_2 \int \mathcal{D}^d p \text{Tr}(x_1^{L_0} V_k x_2^{L_0} V_{-k}) x_1^{-2-i\epsilon} x_2^{-2-i\epsilon} \quad (14)$$

The evaluation of (14) is well known for external tachyons or photons[10,1] and we shall not repeat it here. Generalising this method to the case at hand we write

$$V_k = \eta^{\mu\nu\dots\lambda} \left( \frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial \xi^\nu} \dots \frac{\partial}{\partial \xi^\lambda} \right) : e^{ik \cdot X + \xi \cdot P} : |_{\xi=0} \quad (15)$$

This allows considerable simplification of the trace calculation. Indeed we now arrive at

$$\begin{aligned} iT &= \frac{g^2}{4} \int_0^1 dx_1 \int_0^1 dx_2 \int \mathcal{D}^d p x_1^{-2-i\epsilon} x_2^{-2-i\epsilon} x_1^{p^2/2} x_2^{(p-k)^2/2} f^{2-d}(w) G \\ G &= \eta^{\mu\nu\dots\lambda} \bar{\eta}^{\sigma\tau\dots\kappa} \left( \frac{\partial}{\partial \xi^\mu} \dots \frac{\partial}{\partial \xi^\lambda} \right) \left( \frac{\partial}{\partial \bar{\xi}^\sigma} \dots \frac{\partial}{\partial \bar{\xi}^\kappa} \right) e^{\xi \cdot p} e^{\bar{\xi} \cdot (p-k)} H \\ H &= e^{\sum_{n=1}^{\infty} A_n(\xi, \bar{\xi})} |_{\xi=\bar{\xi}=0} \end{aligned} \quad (16)$$

where  $\bar{\eta}$  denotes the complex conjugate of  $\eta$  and

$$\begin{aligned} w &= x_1 x_2 \\ f(w) &= \prod_{n=1}^{\infty} (1 - w^n) \\ A_n(\xi, \bar{\xi}) &= \frac{1}{n(1-w^n)} [k^2(x_1^n + x_2^n - 2x_1^n x_2^n) + n(\xi + \bar{\xi}) \cdot k(x_1^n - x_2^n) \\ &\quad + n^2 \xi \cdot \bar{\xi} (x_1^n + x_2^n) + (\xi^2 + \bar{\xi}^2) n^2 x_1^n x_2^n] \end{aligned} \quad (17)$$

We note that due to the conditions of transversality and tracelessness of the polarisation tensor, (16) can be rewritten as

$$G = \eta^{\mu\nu\dots\lambda} \bar{\eta}^{\sigma\tau\dots\kappa} \left( \frac{\partial}{\partial \xi^\mu} \dots \frac{\partial}{\partial \xi^\lambda} \right) \left( \frac{\partial}{\partial \bar{\xi}^\sigma} \dots \frac{\partial}{\partial \bar{\xi}^\kappa} \right) e^{\xi \cdot p} e^{\bar{\xi} \cdot p} Q |_{\xi=\bar{\xi}=0}$$

$$Q = e^{\sum_{n=1}^{\infty} n(x_1^n + x_2^n) \xi \cdot \xi / (1-w^n)}$$

Let us define  $B = \sum_{n=0}^{\infty} \frac{n(x_1^n + x_2^n)}{(1-w^n)}$ . Performing the differentiations and paying careful attention to the combinatorial factors we arrive at

$$G = \sum_{r=0}^N \frac{(N!)^2}{(r!)^2 (N-r)!} \overbrace{\eta_{\mu\nu\dots}}^{N \text{ indices}} \overbrace{\bar{\eta}_{\sigma\tau\dots}}^{r \text{ terms}} (\overbrace{p^\mu p^\nu \dots}^r) (p^\sigma p^\tau \dots) B^{N-r}$$

with all "spare" indices being contracted. Hence (16) can be written

$$\begin{aligned} iT &= \frac{g^2}{4} \int_0^1 dx_1 \int_0^1 dx_2 \int d^d p x_1^{-2-i\epsilon} x_2^{-2-i\epsilon} x_1^{p^2/2} x_2^{(p-k)^2/2} f^{2-d}(w) U \\ U &= \sum_{r=0}^N \frac{(N!)^2}{(r!)^2 (N-r)!} \eta^{\mu\nu\dots} \bar{\eta}^{\sigma\tau\dots} (p^\mu p^\nu \dots) (p^\sigma p^\tau \dots) B^{N-r} C^{2(N-1)} \\ C &= \prod_{m=1}^{\infty} \frac{(1-x_1 w^{m-1})(1-x_2 w^{m-1})}{(1-w^m)^2} \end{aligned} \quad (18)$$

Now we change variables to the 'Schwinger' variables  $\alpha$  and  $\beta$

$$\begin{aligned} x_1 &= e^{-i\alpha} \\ x_2 &= e^{-i\beta} \end{aligned}$$

Next we Wick rotate and regularise with a Euclidean convergence factor  $e^{-\Lambda p^2}$  as we did in Section 1. Performing the Gaussian integrals and defining  $x = \alpha - i\Lambda + \beta$ ,  $y = \frac{(\alpha - i\Lambda)}{(\alpha - i\Lambda + \beta)}$  again as in Section 1 we obtain

$$\begin{aligned} T &= \frac{1}{2E_{\text{ext}}} \frac{g^2}{4(2\pi)^{d/2}} \sum_{r=0}^N \int_0^1 dy \int_{-i\Lambda}^{\infty} dx (i)^{2+r-d/2} x^{-r-d/2+1} e^{-i\alpha x} P \\ \alpha &= [-y(1-y)(1-N) + 1 - i\epsilon] \\ P &= f(w)^{2-d} C(x_1, x_2)^{2(N-1)} B(x_1, x_2)^{N-r} \frac{(N!)}{r!(N-r)!} \\ C &= f(w)^{-3} \left( \sum_{n=-\infty}^{\infty} ((-1)^n x_1^{\frac{n(n+1)}{2}} x_2^{\frac{n(n-1)}{2}}) \right) \end{aligned} \quad (19)$$

Here we have set  $\eta^2 = \frac{1}{2E_{\text{ext}}N!}$  in order to normalise the external states (11) and have used Jacobi's identity [8] to reexpress the infinite product  $C$  in terms of the partition function and an infinite series ( a Jacobi  $\theta_4$  function).

This is our final expression for the one loop amplitude with identical excited external particles. We shall now determine how to extract the imaginary part of it just as in the field theory example.

### 3 The Imaginary Part of the String One Loop amplitude

Let us concentrate on the expression for the amplitude (19). In particular let us consider the factor  $P$ . This is simply a polynomial in  $x_1$  and  $x_2$ , the term  $x_1^r x_2^m$  having a coefficient that also depends on the summation variable  $r$  i.e.,

$$P = \sum_{p,q=0} C_{pq}^r x_1^p x_2^q \quad p, q = 0, 1, 2, \dots \quad (20)$$

where the  $C_{pq}^r$  are numbers.

In terms of  $x$  and  $y$  we have, recalling from 19 that  $\alpha = xy + i\Lambda, \beta = x(1-y)$ ,

$$\begin{aligned} x_1 &= e^{-ixy} \\ x_2 &= e^{-ix(1-y)} \end{aligned}$$

where we drop an overall real constant  $e^\Lambda$  in  $x_1$ , which tends to unity as  $\Lambda$  tends to zero. This allows us to include the factor of  $P$  in (19) as

$$\begin{aligned} T &= \frac{1}{2E_{\text{ext}}} \frac{g^2}{4(2\pi)^{d/2}} \sum_{r=0}^N \sum_{p,q=0} C_{pq}^r \int_0^1 dy \int_{-i\Lambda}^{\infty} dx (i)^{2+r-d/2} x^{-r-d/2+1} K \\ K &= e^{-iA(y)x - \epsilon x} \end{aligned} \quad (21)$$

following the field theory example we set

$$A(y) = -(N - 1)y(1 - y) - 1 + py + q(1 - y) \quad (22)$$

The positive and negative regions of this function will determine whether or not our string can decay. However the -1 in (22) is a reflection of the tachyon: if this is not removed then the decay rate will be infinite. To see this, return to (14) where the integrand contains

$$Tr(x_1^{L_0} V_k x_2^{L_0} V_{-k})$$

and the trace is over states propagating around the loop. Now  $x_1^{L_0}$  is just  $x_1^{p^2/2+N}$  where  $N$  is the level number operator. Inserting a complete set of occupation number states one sees that the power of  $x_1$  occurring in the trace corresponds to the level number of the particle circulating on that side of the loop, and similarly for  $x_2$ . In particular terms independent of  $x_1$  or  $x_2$  correspond to tachyonic states propagating on either side of the loop. Thus if we wish to extract those parts of  $P$  which do *not* correspond to tachyonic decays we simply discard the terms in  $P$  which do not have at least one power of  $x_1$  and  $x_2$ . We emphasise that this results in a perfectly well defined (and totally finite) calculation, and corresponds to summing up all tree diagrams not involving tachyons and integrating over phase space. We perform an explicit check of this for the simplest case, the decay of the first massive state into photons, in Appendix B.

The only terms in  $P$  we keep are those with at least one power of  $x_1$  and  $x_2$ . This results in the -1 in (22) being cancelled. One can now perform the

$x$  integral and take the imaginary part just as in section 1 to get

$$Im(T) = \frac{1}{2E_{ext}} \frac{g^2 \pi}{4(2\pi)^{d/2}} \sum_{r=0}^N \sum_{p,q} C_{pq}^r \pi \int_{y_-}^{y_+} dy \frac{|A(y)|^{d/2+r-2}}{\Gamma(\frac{d}{2} + r - 1)} \quad (23)$$

For what values of  $p$  and  $q$  does  $A(y)$  go negative?. We find, for  $N > 1$  that this happens for

$$\frac{(p-q)^2}{(N-1)^2} - \frac{2(p+q-2)}{(N-1)} + 1 > 0 \quad (24)$$

This is the string generalisation of the constraint  $k^2 > 4m^2$  we had before. The values of  $p$  and  $q$  correspond to different mass levels in the decay products and (24) just tells us if certain decays are energetically allowed. This is not the full story - for a given  $p, q$  that satisfies (24) we calculate the range of  $y$  for which  $A$  is negative (see Figure 3)

$$y_+ - y_- = \sqrt{\left[ \frac{(p-q)^2}{(N-1)^2} - \frac{2(p+q-2)}{(N-1)} + 1 \right]} \equiv 2a \quad (25)$$

Returning to (23) we have to evaluate the  $y$  integral, just as in the field theory example. From Appendix A we find

$$\int_{y_-}^{y_+} dy |A|^{d/2+r-2} = a^{d+2r-3} \sqrt{\pi} \frac{\Gamma(\frac{d}{2} + r - 1)}{\Gamma(\frac{d}{2} + r - \frac{1}{2})} (N-1)^{\frac{d}{2}+r-2}$$

where  $a$  is defined in (25).

Thus (23) becomes

$$Im(T) = \frac{1}{2E_{ext}} \frac{g^2 \pi^{\frac{3}{2}}}{4(2\pi)^{d/2}} \sum_{r=0}^N \sum_{p,q} C_{pq}^r \frac{a^{2r+d-3} (N-1)^{\frac{d}{2}+r-2}}{\Gamma(\frac{d}{2} + r - \frac{1}{2})} \quad (26)$$

where the  $p, q$  sums only run over values for which the energy constraint (24) holds.

## 4 Twisted Diagrams

Equation (26) is not the final answer because it comes from only one diagram, the *planar* diagram shown in Figure 4. In general for open strings there are two other classes of diagrams, the orientable and nonorientable nonplanar diagrams. These are illustrated in Figure 5. To construct these diagrams one has to use the *twist* operator. Let us assume for simplicity that we are dealing with (orientable)  $U(1)$  strings - which amounts to imagining the strings have two oppositely charged particles ('quarks') on either end. Charge conjugation symmetry ( $C$ ) reverses the string - it is the same as the operation  $(-1)^N$  where  $N$  is the level number operator. For example tachyons may be thought of as neutral point particles, even under  $C$ . Photons are odd under  $C$  and so on[1].

To see what this means in tree amplitudes, consider the general 3 point vertex. The general prescription in string theory is to add all cyclically inequivalent diagrams contributing to a given process. In this case there are two diagrams,  $\langle 1 | V(2) | 3 \rangle$  and  $\langle 3 | V(2) | 1 \rangle$  in bra and ket notation. These correspond to the string  $| 3 \rangle$  breaking in 2 ways - emitting state 2 from one end and state 1 from the other and the reverse. The second diagram is however related to the first:

$$\langle 3 | V(2) | 1 \rangle = (-1)^{N_1+N_2+N_3} \langle 1 | V(2) | 3 \rangle \quad (27)$$

by a generalisation of the cyclic symmetry proof. Here  $N_i$  is the level number of the  $i$ th state.

This means that the total amplitude of a given process is zero unless  $(-1)^{N_3} = (-1)^{N_2+N_1}$ , an expression of  $C$  conservation. For example the

amplitude for a tachyon (and more generally any string state) to scatter off a photon is zero, indicating that the string states carry no net charge, consistent with the picture of their coupling to photons as “dipoles”, carrying oppositely charged particles at either end.

The second diagram may be written in terms of the first diagram by using the *twist* operator  $\Omega = (-1)^N$

$$\langle 3 | V(2) | 1 \rangle = (-1)^{N_2} \langle 1 | \Omega V(2) \Omega | 3 \rangle \quad (28)$$

How does this relate to loop diagrams? For oriented open strings, the nonorientable diagram (Figure 5a) does not contribute - cutting the loop reveals a string with two *like* charges at the ends. So there are just the planar and nonplanar orientable (Figure 5b) diagrams to consider. Now it is seen that squaring the two tree diagrams and summing over intermediate states will result in two *direct* terms, equal to the planar diagram, and two *interference* terms, equal to the nonplanar orientable diagram. This shows that for our loop diagrams to give a unitary result, we must add in the loop diagram with two twists with the same weight as the planar diagram but with a level-dependent phase factor  $(-1)^N$  as well. Thus for the nonplanar orientable diagram the trace in (14) is replaced with

$$\text{Tr}(x_1^{L_0} \Omega V_k x_2^{L_0} \Omega V_{-k}) (-1)^N \quad (29)$$

where  $N$  is the level number of the external particle.

The result of all this is very simple. The effect of the twist operator in the trace part of the loop amplitude is to change  $x_1$  and  $x_2$  to  $-x_1$  and  $-x_2$ . When we add the two loop diagrams, then terms involving odd total powers of  $N + p + q$  cancel, just as they should.

For other groups the diagrammatics is more complicated (see [1]) and in general the nonorientable diagram is essential for unitarity. We will not pursue this here - the group theoretic factors should not qualitatively alter our result.

As is well known, the nonplanar orientable diagram we have included has singularities corresponding to propagation of *closed string* intermediate states (as may be seen heuristically by lifting the inner circle of Figure 5b). In fact it was this observation, and the requirement that the singularities be poles that led to the realisation of the importance of  $d = 26$  in string theory. At tree level, this corresponds to a *two point* coupling between open and closed string states, necessary for unitarity, which means that open string states can in principle mix with closed string states to form the true mass eigenstates of the theory. However in our case it is clear that mixing cannot occur. This is because for the open string the states on the leading Regge trajectory have  $J = \alpha' M^2 + 1$  and  $M^2 = \frac{1}{\alpha'}(N - 1)$  whereas for the closed string the relation is  $J = \frac{1}{2}\alpha' M^2 + 2$ , and  $M^2 = \frac{4}{\alpha'}(N - 1)$ . It follows that for  $M^2 > 0$  no closed string state has the correct mass and spin to mix with the leading Regge trajectory open string states. Thus we are not faced with singularities corresponding to intermediate closed string states and the consequent requirement that  $d = 26$ .

## 5 Decay Rates

Equation (26) with the added nonplanar diagram contains all the information pertaining to the decay rate of a leading trajectory state. However, it is not

in an immediately useable form for we have not given an analytic expression for the coefficients  $C_{pq}^r$ . To try to understand which terms dominate the sum in (26) we have written a computer program to explicitly compute the various coefficients for different external states, namely those with level number  $N$  up to 100 (Figure 6). We chose  $d=4$  in order that the variation with  $N$  could clearly be seen. The results for  $N = 6$  and  $N = 20$  are shown in Figures 7 and 8. The decay rate for a string at level  $N$  is dominated by decay into a photon ( $p, q = 1$ ) and a massive string with  $q, p = N - 1$ . The rate for this process alone is also shown in Figure 6 as well as an analytic approximation to it we shall derive below.

If we define, for given  $p, q$

$$\Theta = \sum_{r=0}^N \sum_{\text{twists}} f^{2-d-6(N-1)} B^{N-r} \Psi(r) \left( \sum_{n=-\infty}^{\infty} (-1)^n x_1^{\frac{n(n+1)}{2}} x_2^{\frac{n(n-1)}{2}} \right)^{2(N-1)} \quad (30)$$

where

$$\Psi(r) = \frac{\Gamma(N+1) a^{2r+d-3} (N-1)^{\frac{d}{2}-2+r}}{\Gamma(r+1) \Gamma(N-r+1) \Gamma(\frac{d}{2} + r - \frac{1}{2})} \quad (31)$$

then the coefficient of  $x_1^p x_2^q$  gives the rate of decay into a string at level  $p$  and one at level  $q$ . The crucial point is very simple.  $\Psi(r)$  is a rapidly decreasing function of  $r$ , and this effect dominates in  $d=4$ .

Our strategy is to extract, for  $r = 0, 1, 2, \dots$ , the coefficient of  $x_1^{N-1} x_2$  in (30). We denote this coefficient  $Q_r$ . In this term,  $p = N - 1$  and  $q = 1$  or vica versa so from (25)  $a = \frac{1}{2(N-1)}$ . We find

$$Q_0 = \frac{N}{(N-1)\sqrt{\pi}}$$

$$Q_1 = \frac{-2N}{3(N-1)\sqrt{\pi}}$$

$$\begin{aligned}
Q_2 &= \frac{N(3N-4)}{24(N-1)^2\sqrt{\pi}} \\
Q_3 &= \frac{-N(N-2)(19N-40)}{1260(N-1)^3\sqrt{\pi}}
\end{aligned}
\tag{32}$$

The sum of these terms appears to converge rapidly with increasing  $N$ . The terms above add to  $0.443/\sqrt{\pi}$  for large  $N$ , so that from (26) we have

$$\Gamma_{N \rightarrow N-1,1} \approx \frac{1}{E_{ext}} \frac{0.443g^2}{16\pi}
\tag{33}$$

including an extra factor of two because of decay into  $x_1x_2^{N-1}$  and  $x_2x_1^{N-1}$  a factor of two from the optical theorem and a 'symmetry' factor of  $\frac{1}{2}$  (see Appendix B). This result is compared with the exact result as evaluated on a computer in Figure 6.

This result certainly provides a lower bound on the total decay rate of this string state. The total decay rate is not dominated by this process for  $N < 100$ . It does however appear to decrease more steeply as one goes to larger  $N$  - the magnitude of the slope of  $\Gamma$  versus  $N$  *increases* in the ratios 1 : 1.24 : 1.53 for the ranges 60 - 80, 80 - 90, 90 - 100, faster than a simple power law (the magnitude of the slope any power law *decreases* with increasing  $N$ ). If we extrapolate the slope at  $N = 100$  to higher  $N$  we would find the total decay rate reached (33) at  $N \approx 270$ . The total decay rate certainly must flatten out for  $N$  of this order. Unfortunately the exact computation becomes prohibitive at such large  $N$ . It does seem possible therefore that the total decay rate approaches (33) (possibly times a constant) for very large  $N$ . It seems very difficult to check this conjecture in our approach. In fact Green and Veneziano [5] argued that the total decay rate should go like  $1/E_{ext}$  - our result seems consistent with this. However

they applied this to meson resonances with  $N < 5$  - we have shown a far better way to calculate the result for small  $N$ .

In any case our results are rather surprising from the point of view of string interactions as splitting and joining - our result certainly conflicts with the idea of a splitting probability per unit length. This would lead to  $\Gamma \propto \sqrt{N}$ . Unless something very strange happens at large  $N$  we have the bound  $const/\sqrt{N} < \Gamma < const$ .

If the lower bound is saturated for very large  $N$  this means that the decay is completely dominated by the ends of the string - if one determines from (33) a rate of energy loss

$$P = \Gamma E_{ext} \approx \frac{0.443g^2}{16\pi} \quad (34)$$

then this is completely independent of the mass (and thus the 'size'[2] of the rotating string). Thus the lifetime of a string would be proportional to its length.

It would also be interesting to calculate decay rates for strings in more general states, to see whether and how the picture changes.

In Figures 7 and 8 we show plots of the amplitudes of the various terms in (30) for  $N = 6$  and  $N = 20$ .

As we discussed in Section 1, we expect the decay rate to be very much suppressed in higher dimensions from phase space. We calculated the total decay rate in  $d = 8$  and  $d = 26$  for  $N = 20$  and 40 to check this. The result, in the same units as Figure 6 but with appropriate powers of  $\sqrt{2\alpha'}$  to make up dimensions, were for  $d = 8$ :  $\Gamma_{20} = 1.37 \cdot 10^{-2}$ ,  $\Gamma_{40} = 1.31 \cdot 10^{-2}$  and for  $d = 26$ :  $\Gamma_{20} = 3.78 \cdot 10^{-10}$ ,  $\Gamma_{40} = 7.79 \cdot 10^{-10}$ , much smaller as expected. These

cases are also different in that the  $N - 1, 1$  term that we saw was largest in  $d = 4$  is no longer largest here. These results give some indication that in high dimensions string cosmology may be rather complicated - large strings can have quite long lifetimes. Of course all this would also be affected by powers of the compactification radius in any 'realistic' model of open strings.

## 6 Superstrings

In this section we sketch the calculation for the decay rate of the analagous first excited level in the open superstring model. This is harder because the general vertex is not known, and the fermionic traces difficult. Thus we shall only deal with the simplest case. There are no surprises here - the first massive state decays in a Planck time. As in the bosonic case we use the optical theorem, and thus we compute the imaginary part of the two point one-loop amplitude. We perform the calculation of the amplitude using the vertex operator formalism in the light-cone gauge. As a first step we have to determine the vertex, in the light-cone gauge, for the emission of the first excited state of mass  $m = \sqrt{2}$  (we set  $2\alpha' = 1$  throughout this section). To do this we write the most general combination of  $P^i$  and  $R^{ij} k^j$  operators [1] with conformal spin 1 compatible with the symmetric polarisation tensor  $\zeta^{ij}$

$$V(\zeta^{ij}, k^l) = g\zeta^{ij}(\lambda_1 P^i P^j + \lambda_2(P^i R^{jl} k^l + P^j R^{il} k^l) + \lambda_3 R^{il} k^l R^{jn} k^n)e^{ikX} \quad (35)$$

with

$$\begin{aligned} P^i &= \partial X^i / \partial \tau \\ R^{ij} &= \frac{1}{4} \gamma_{ab}^{ij} S^a S^b \end{aligned} \quad (36)$$

Our notation as before is that of [1].  $\zeta^{ij}$  is the traceless symmetric polarisation tensor with  $\zeta^{ij}k^j = \zeta^{ij}k^i = 0$ . The coefficients  $\lambda_1, \lambda_2$  and  $\lambda_3$  can be determined by evaluating the matrix element of the coupling between two massless particles and one massive particle in two different ways. First we take the above vertex between two massless states

$$\begin{aligned} & \zeta_3^n \langle k_3, n | V(\zeta^{ij}, k_1^i) | k_2, m \rangle \zeta_2^m \\ &= g\lambda_1(\zeta_2\zeta_3)k_2^i\zeta_1^{ij}k_2^j + g\lambda_2[(k_1\zeta_3)k_3^i\zeta_1^{ij}\zeta_2^j + (k_1\zeta_2)\zeta_3^i\zeta_1^{ij}k_2^j - \\ & \quad (k_1\zeta_3)\zeta_2^i\zeta_1^{ij}k_2^j] - g\lambda_3k_1^2\zeta_3^i\zeta_1^{ij}\zeta_2^i \end{aligned} \quad (37)$$

(we used momentum conservation  $\sum_i k_i = 0$ ) or by taking the matrix element of the vertex for the emission of a massless state between an excited and a massless state in a cyclic permutation of (37)

$$\begin{aligned} & \zeta_2^i\zeta_3^i \langle k_2, n | g(P^i - R^{ij}k_3^j)e^{ik_3X}\alpha_{-1}^i | k_1, m \rangle \zeta_1^m \\ &= g\zeta_2^i\zeta_1^{ij}\zeta_3^j + g(k_2\zeta_3)\zeta_2^i\zeta_1^{ij}k_3^j - g(k_3\zeta_2)\zeta_3^i\zeta_1^{ij}k_3^j + g(\zeta_2\zeta_3)k_3^i\zeta_1^{ij}k_3^j \end{aligned} \quad (38)$$

Comparing the results of (37) and (38) we find  $\lambda_1 = 1, \lambda_2 = \frac{1}{2}$  and  $\lambda_3 = \frac{1}{2}$ . With the above vertex we can compute the two point one-loop amplitude defined as

$$A_k = Tr(\Delta_1 V_k \Delta_2 V_{-k}) \quad (39)$$

where  $V_k$  is the vertex 35 for the emission of a particle corresponding to the first excited state and  $\Delta_a$  is the string propagator between neighboring vertices

$$\Delta_a = \frac{1}{2} \int_0^1 dx_a x_a^{L-1} \quad (40)$$

and

$$L = \frac{1}{2}p^2 + \sum_{n=1}^{\infty} (\alpha_{-n}^i \alpha_n^i + n S_{-n}^a S_n^a) \quad (41)$$

Due to the fact that  $TrR^n = 0$  for  $n \leq 3$  only the last term, proportional to  $\lambda_3$  contributes to the amplitude. Due to supersymmetry the partition function,  $f(w) = \prod_{n=1}^{\infty}(1 - w^n)$ , cancels out. The trace over the oscillators can be easily carried out.

For the amplitude we get

$$A_k = \frac{g^2 K}{16} \int d^{10}p \int_0^1 dx_1 \int_0^1 dx_2 x_1^{\frac{z^2}{2}-1} x_2^{\frac{(z-k)^2}{2}-1} C \quad (42)$$

$$C = \prod_{n=1}^{\infty} \left[ \frac{(1 - w^{n-1} x_2)(1 - w^n x_2^{-1})}{(1 - w^n)^2} \right]^2 \quad (43)$$

using  $k^2 = -2$  with  $w = x_1 x_2$  and the kinematical factor

$$K = \zeta^{ij} \zeta^{lm} k^n k^q k^t k^z \epsilon^{ijnlqms} \quad (44)$$

where

$$\epsilon^{ijnlqms} = Tr[R_o^{in} R_o^{jt} R_o^{lq} R_o^{ms}] \quad (45)$$

as defined in [1]. Using the mass shell and gauge conditions we can evaluate K,

$$K = 8\zeta^2 = \sqrt{8} \quad (46)$$

using  $\zeta^2 = \frac{1}{2E_{\text{ext}}} = \frac{1}{\sqrt{8}}$  The momentum integration is carried out as in section 2 and the polynomial  $C$  expanded as in section 3. The only term contributing an imaginary part is the first; it gives 1. We thus get

$$\frac{1}{(2\pi)^5} \frac{g^2 K}{16} i^{-3} \int_0^1 dy \int_{-i\Lambda}^{\infty} dx x^{-4} e^{-ixA - \epsilon x} \quad (47)$$

with  $A = y^2 - y$ . The imaginary part is then, following the method presented in section 1

$$i \frac{1}{(2\pi)^5} \frac{\pi g^2 K}{16.6} \int_0^1 dy (y - y^2)^3 = i \frac{g^2 K \pi}{(2\pi)^5 16.6.140} \quad (48)$$

The same result is obtained from the diagram with one twist (which has an additional minus sign) , and the diagram with two twists.

The first excited level decays into 2 massless ground state particles. The decay rate, in units where  $2\alpha' = 1$ , is given by

$$\Gamma = \frac{g^2 \pi \sqrt{2}}{(2\pi)^5 6720} \quad (49)$$

The extension of the above calculation to higher excited states appears to be difficult in this formalism - being complicated in particular by the fermionic traces.

## 7 Appendix A

In general the function  $A(y)$  is a parabola with zeros at  $y_{\pm}$ . Shifting  $y$  to  $\bar{y} = y - \frac{y_+ + y_-}{2}$  and defining  $a = \frac{y_+ - y_-}{2}$  we obtain

$$I_n = \int_{-a}^a dy (a^2 - y^2)^n$$

Changing variables to  $z = (\frac{y}{a})^2$  this is

$$I_n = a^{2n+1} \int_0^1 dz z^{-\frac{1}{2}} (1-z)^n$$

which is just

$$I_n = a^{2n+1} B\left(\frac{1}{2}, n+1\right) = a^{2n+1} \frac{\pi^{\frac{1}{2}} \Gamma(n+1)}{\Gamma(n + \frac{3}{2})}$$

the required result.

## 8 Appendix B

As we argued in Section (4) the two one loop diagrams must be added with equal weight. The overall weight of these diagrams can be found by explicitly calculating the tree amplitudes for the lowest excited states and then comparing this answer to that derived via the optical theorem. We shall choose the simplest well-defined case - the decay of an  $N = 2$  state into photons. The result we find is that if the tree level coupling constant is  $g$  ( $\frac{g}{2}$  for each diagram, see section 4) then each of the loop diagrams must be added in with a coupling  $(\frac{g}{2})^2$  i.e. one could take one loop diagram with a symmetry factor of  $\frac{1}{2}$  and the rule that  $\mathbf{C}$  be conserved. If nothing else this appendix should convince the reader that calculating total decay rates by summing over polarisations and integrating over phase space is considerably more complicated than using the optical theorem! We shall also calculate in arbitrary  $d$  as a useful check of the formalism above. By calculating  $C_{11}^0$ ,  $C_{11}^1$  and  $C_{11}^2$  explicitly we find that

$$\Gamma_{2 \rightarrow 1,1} = \frac{g^2 \xi^2 \pi^{\frac{3-d}{2}}}{2^{\frac{3d+1}{2}}} [8(d+1)(d-1) - 16(d+1) + (d+6)] \frac{1}{\Gamma(\frac{d+3}{2})} \quad (50)$$

where an extra factor of  $\frac{1}{2E_{\text{ext}}}$  has been added to correctly normalise the external state. Here  $\xi_{\alpha\beta}$  is the polarisation tensor of the initial massive state.

Now we check this using the tree amplitude method. The polarisation vectors of the photons will be called  $\epsilon_\alpha$ ,  $\epsilon'_\gamma$ . The matrix element for the transition is given by

$$T^{\kappa\lambda} = g \langle k_3 | \alpha_1^\alpha : P^\kappa P^\lambda e^{ik_2 X} : \alpha_2^\lambda | -k_1 \rangle \xi_{\kappa\lambda} \epsilon_\alpha^3 \epsilon_\gamma^1 \delta^d(k_2 - k_1 - k_3) \quad (51)$$

It is important to remember all the constraints on the polarisation tensor and

vectors, namely that they are transverse and that the tensor has no trace. Calculating all the various terms in (51) we arrive at

$$T^{\kappa\lambda} = g(\eta^{\alpha\gamma}k_1^\alpha k_1^\lambda + 2\eta^{\alpha\kappa}\eta^{\lambda\gamma} - 2k_2^\alpha\eta^{\kappa\gamma}k_1^\lambda + 2k_2^\gamma\eta^{\alpha\kappa}k_1^\lambda - k_1^\kappa k_1^\lambda k_2^\alpha k_2^\gamma) \xi_{\kappa\lambda}\epsilon_\alpha^3\epsilon_\gamma^1\delta^d(k_3 + k_1 - k_2) \quad (52)$$

and so the decay rate is simply

$$\Gamma = \frac{1}{4E_2} \sum_{\epsilon^1, \epsilon^3} \int d^d k_1 d^d k_3 \delta(k_1^2)\theta(k_1^0)\delta(k_3^2)\theta(k_3^0)\delta^d(k_2 - k_1 - k_3) |T^{\kappa\lambda}|^2 \quad (53)$$

where we sum over the final polarisation vectors and include a factor of  $\frac{1}{2}$  for the two identical final particles. The various identities we will need are:

$$\begin{aligned} k_2^\mu \xi_{\mu\nu} &= 0 \\ \sum_\epsilon \epsilon_\alpha \epsilon_{\bar{\alpha}}^* &= \bar{\delta}_{\alpha\bar{\alpha}} - \frac{\bar{k}_\alpha \bar{k}_{\bar{\alpha}}}{\bar{k}^2} \end{aligned} \quad (54)$$

where the arrow over any quantity indicates that only the spatial components are nonzero. We now choose the frame  $k_2 = (M, 0, \dots)$  which implies that  $\bar{k}_2 = 0$  so

$$k_2^\alpha \sum_\epsilon \epsilon_\alpha \epsilon_{\bar{\alpha}}^* = 0 \quad (55)$$

Hence we can rewrite (53)

$$\begin{aligned} \Gamma &= \frac{1}{4E_2} \int \frac{d^{d-1}\bar{k}_1}{2E_1} \frac{d^{d-1}\bar{k}_3}{2E_3} \delta(E_2 - E_1 - E_3) \delta^{D-1}(\vec{k}_1 + \vec{k}_3) \\ &\quad (2\eta^{\alpha\kappa}\eta^{\lambda\gamma} + \eta^{\alpha\gamma}\bar{k}_1^\kappa\bar{k}_1^\lambda)(2\eta^{\bar{\alpha}\bar{\kappa}}\eta^{\bar{\lambda}\bar{\gamma}} + \eta^{\bar{\alpha}\bar{\gamma}}\bar{k}_1^{\bar{\kappa}}\bar{k}_1^{\bar{\lambda}})(\bar{\delta}_{\gamma\bar{\gamma}} - \frac{\bar{k}_1^\gamma\bar{k}_1^{\bar{\gamma}}}{\bar{k}_1^2}) \\ &\quad (\bar{\delta}_{\alpha\bar{\alpha}} - \frac{\bar{k}_1^\alpha\bar{k}_1^{\bar{\alpha}}}{\bar{k}_1^2})\xi_{\kappa\lambda}\xi_{\bar{\kappa}\bar{\lambda}} \end{aligned} \quad (56)$$

but  $E_2 = \sqrt{2}$ , thus after a little algebra we find

$$\begin{aligned}
\Gamma = & \frac{1}{16\sqrt{2}} \int \frac{d^{d-1}\vec{k}_1}{E_1^2} \delta(E_2 - 2E_1) \left[ 4\eta^{\kappa\bar{\kappa}}\eta^{\lambda\bar{\lambda}} - \frac{4k_1^{\bar{\kappa}}k_1^{\bar{\lambda}}\eta^{\lambda\bar{\lambda}}}{k_1^2} \right. \\
& \frac{-4k_1^{\bar{\lambda}}k_1^{\bar{\kappa}}\eta^{\kappa\bar{\kappa}}}{k_1^2} + \frac{4k_1^{\bar{\kappa}}k_1^{\bar{\lambda}}k_1^{\bar{\lambda}}k_1^{\bar{\kappa}}}{k_1^4} \\
& \left. \frac{-4k_1^{\bar{\kappa}}k_1^{\bar{\lambda}}k_1^{\bar{\lambda}}k_1^{\bar{\kappa}}}{k_1^2} + (d-2)k_1^{\bar{\kappa}}k_1^{\bar{\lambda}}k_1^{\bar{\lambda}}k_1^{\bar{\kappa}}\xi_{\kappa\lambda}\xi_{\bar{\kappa}\bar{\lambda}} \right] \quad (57)
\end{aligned}$$

we now use the identity

$$\int d^{d-1}\vec{k} \overbrace{\vec{k}\vec{k}\dots}^{2n \text{ indices}} f(k^2) = \frac{2^{1-n} \pi^{\frac{d-1}{2}}}{(2\pi)^{d-2} \Gamma(\frac{d-1+2n}{2})} \int k^{2n+d-2} f(k^2) d\vec{k} \quad (58)$$

where ‘‘spare’’ indices from the LHS are used up in all distinct permutations of metric tensors e.g

$$\begin{aligned}
\int d^{d-1}\vec{k}_1 k_1^{\bar{\kappa}}k_1^{\bar{\lambda}}k_1^{\bar{\lambda}}k_1^{\bar{\kappa}}f(k^2) &= \frac{2^{-1} \pi^{\frac{d-1}{2}}}{(2\pi)^{d-2} \Gamma(\frac{d+3}{2})} \\
& \int k_1^{d+2} f(k_1^2) d\vec{k}_1 (\eta^{\kappa\bar{\kappa}}\eta^{\lambda\bar{\lambda}} + \eta^{\kappa\bar{\lambda}}\eta^{\lambda\bar{\kappa}}) \quad (59)
\end{aligned}$$

all other  $\eta\eta$  giving zero due to the tracelessness of  $\xi_{\kappa\lambda}$ . Doing this for all terms we recover exactly (50).

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## Figure Captions

Figure 1

The optical theorem in lowest order for a  $\frac{\lambda}{2}\Phi\phi^2$  coupling.

Figure 2

Contours for integration in determining the imaginary part of the one loop amplitude.

Figure 3

The function  $A(y)$

Figure 4

Constructing the planar self energy loop by sewing the ends of a tree diagram.

Figure 5

(a) The nonorientable nonplanar self energy diagram (b) The orientable nonplanar self energy diagram Diagram (a) is excluded for U(1) open strings since cutting it reveals a string with two *like* charges on the ends.

Figure 6

The total decay rate in  $d = 4$  as a function of level number  $N$  up to  $N = 100$  (crosses on solid line). It is given in units of  $(g^2/16\pi)\sqrt{2\alpha'}$ . Also shown is the decay rate into a single photon and a state at level  $N - 1$ , the largest single process, in the same units (crosses on dashed line), and our analytic approximation to it, equation (33) (plain dashed line), which becomes very good for large  $N$ .

Figure 7

The decay of an  $N = 6$  string.  $p$  and  $q$  correspond to the level numbers of the decay products - the axes are marked off in integers starting at  $p = q = 1$ , corresponding to massless particles.  $p + q = \text{odd}$  decays are forbidden by C conservation. The vertical scale is in units of the height of the largest peaks, which correspond to decay into a photon plus a string at

the next level down.

Figure 8

The decay of an  $N = 20$  string. The axes are as in Figure 7.

$$2 \operatorname{Im} \left( \text{---} \bigcirc \text{---} \right) = \sum_f \left| \text{---} \begin{array}{l} \text{---} \\ \text{---} \end{array} \right|^2$$

Figure 1

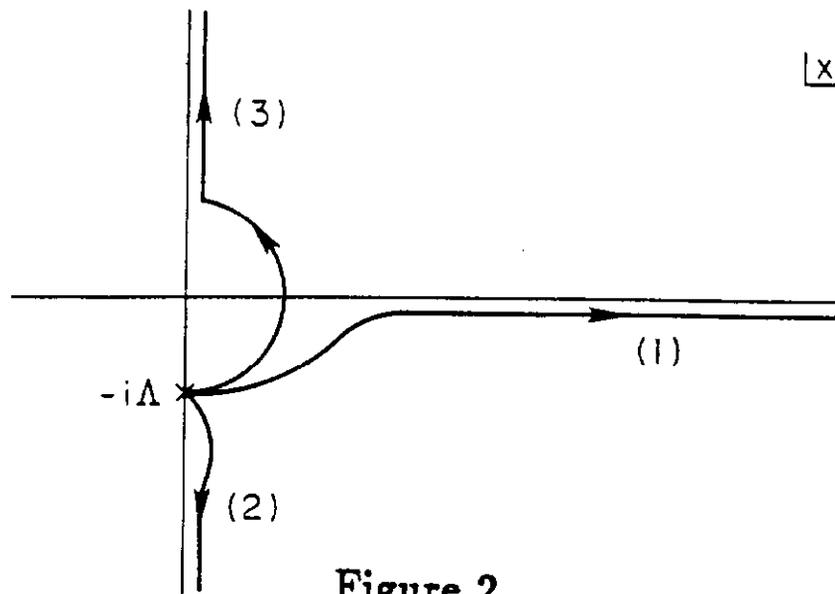


Figure 2

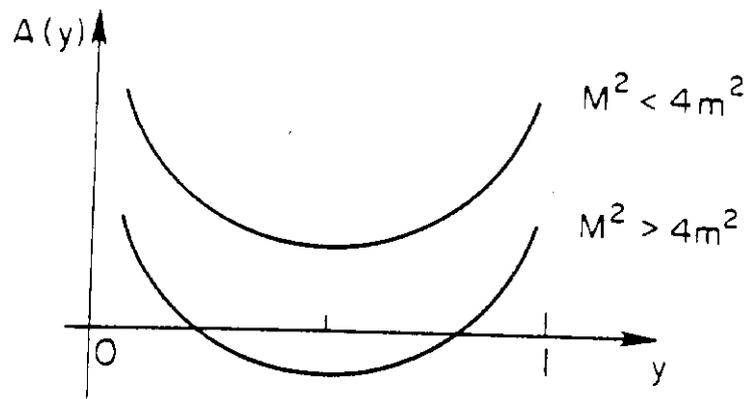


Figure 3

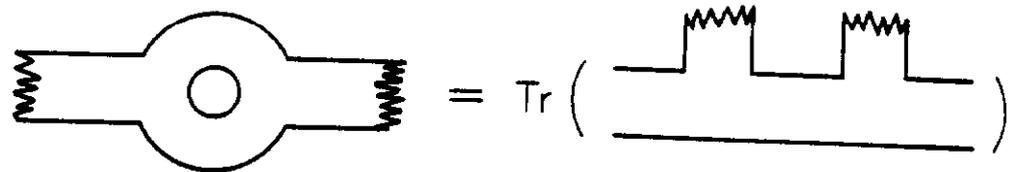


Figure 4

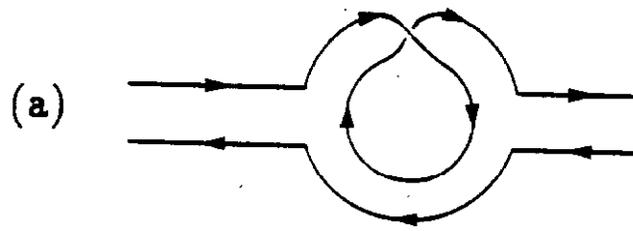
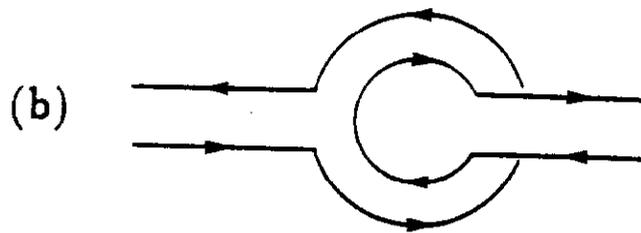


Figure 5



# Decay Rate

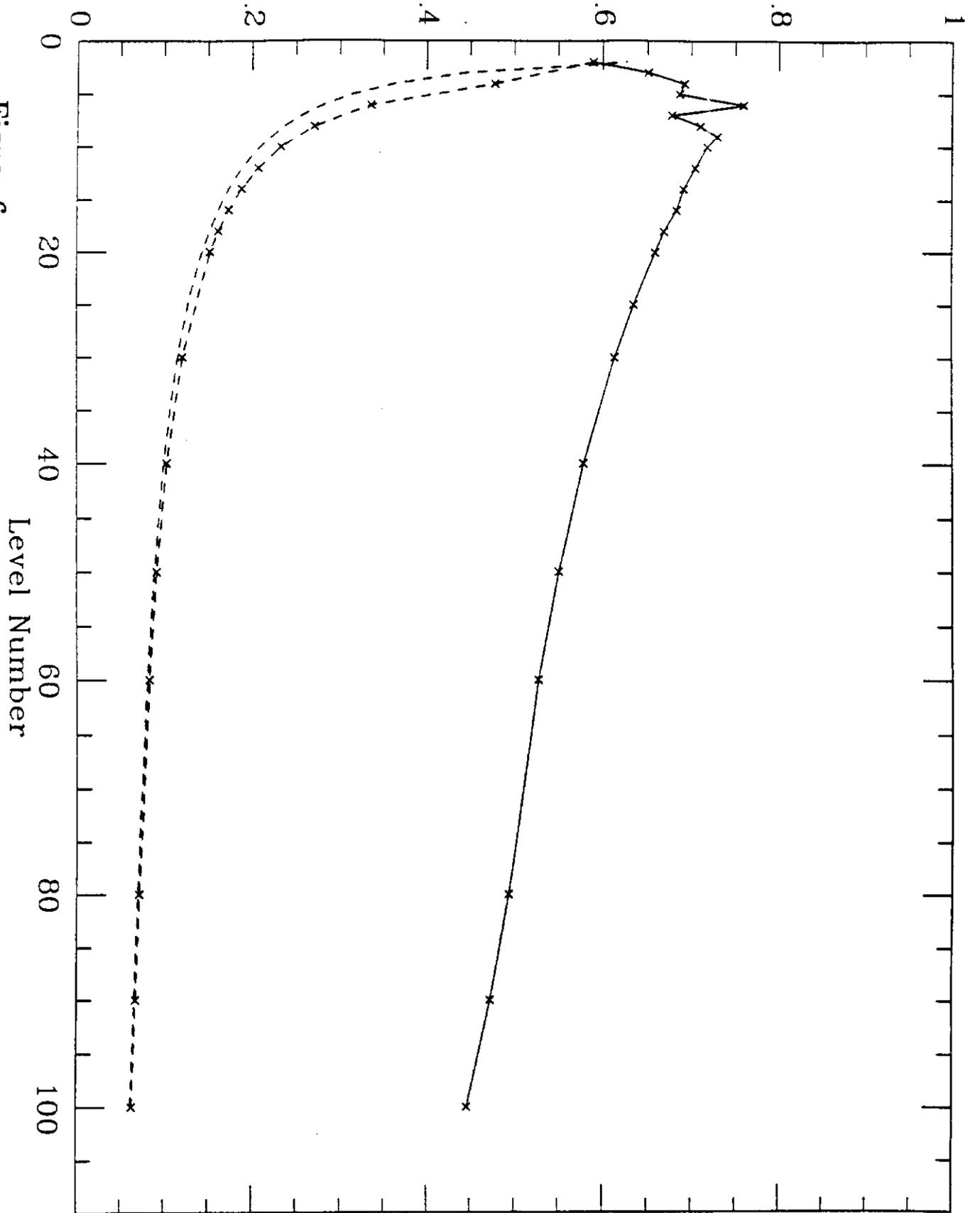


Figure 6

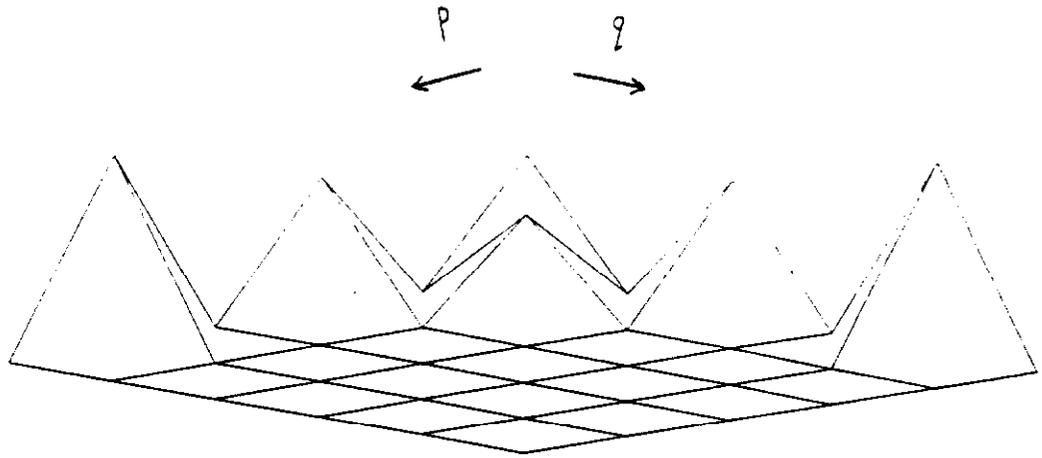


Figure 7

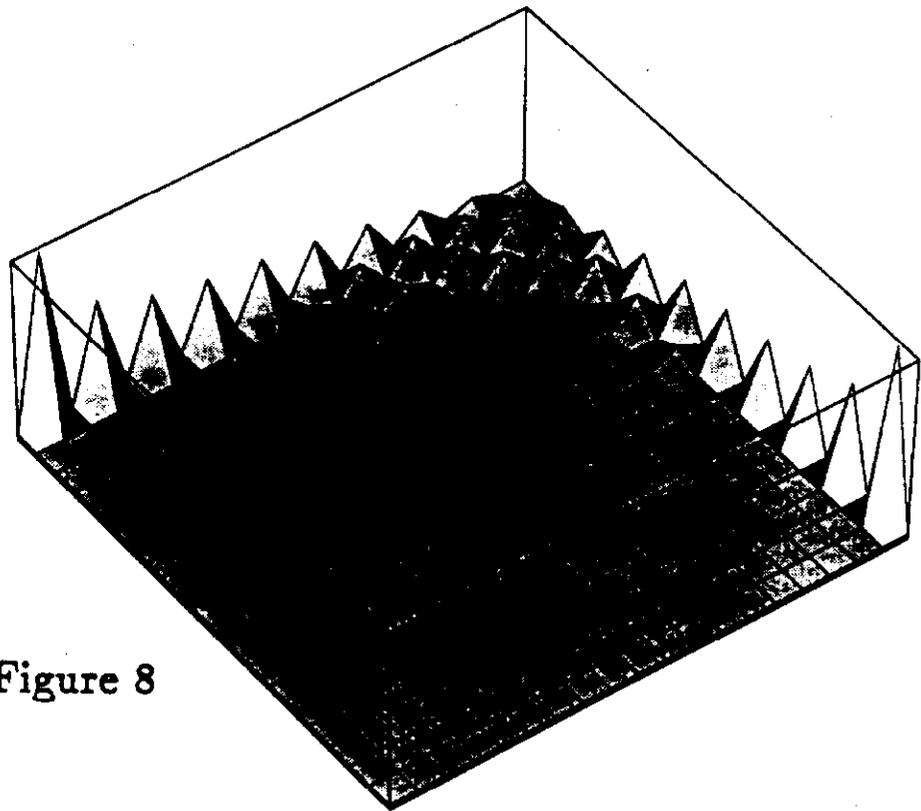


Figure 8