

The Tetrahedron Equation and the Four Simplex Equation

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Abstract

The tetrahedron equation and the four-simplex equation are multidimensional generalizations of the Yang-Baxter or triangle equations. We discuss common features of these members of the family of "simplex-equations." Zamolodchikov solution of the tetrahedron equation is rewritten in an algebraic form and a generalization of it to the four-simplex case is proposed. Relevance of the simplex equation for the understanding of multidimensional integrability is briefly discussed.

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1. It is well known that the concept of quantum and classical integrability in one- and two-dimensional systems is intimately linked to the notion of triangle or Yang-Baxter equations[1-4]. They emerged on the one hand as the conditions for the commutation of transfer matrices in statistical mechanical spin systems[5,6], and on the other hand, as the consistency condition of factorizability for the S -matrix in two-dimensional quantum field theory[7]. Thus, the triangle equations

$$\begin{aligned}
 & S \begin{matrix} k_1 & k_2 \\ i_1 & i_2 \end{matrix} (\theta) S \begin{matrix} j_1 & k_3 \\ k_1 & i_3 \end{matrix} (\theta + \theta') S \begin{matrix} j_2 & j_3 \\ k_2 & k_3 \end{matrix} (\theta') = \\
 & = S \begin{matrix} k_2 & k_3 \\ i_2 & i_3 \end{matrix} (\theta') S \begin{matrix} k_1 & j_3 \\ i_1 & k_3 \end{matrix} (\theta + \theta') S \begin{matrix} j_1 & j_2 \\ k_1 & k_2 \end{matrix} (\theta) \quad (.1)
 \end{aligned}$$

is the compatibility condition for the factorization of the three-particle S -matrix in terms of the two-particle scattering amplitudes, $(\theta, \theta', \theta + \theta')$, are the difference of rapidity between the three incident particles and the indices $i_1 \dots$, stand for the internal quantum numbers of the particles, $(\alpha = 1, \dots, N)$ pictorially depicted in fig. 1. In eq. (1), we have taken into account the constraint on the, in general, three parameters $\theta_1, \theta_2, \theta_3$ entering the equation, coming from the well-known relation of the three angles of a triangle in a plane.

Nowadays, there exists a great deal of insight into the algebraic structure of the Yang-Baxter equations[?], as well as an almost exhaustive classification of solutions within the context of simple Lie algebras at the classical level[?] and to some extent also at the quantum level[?].

An equivalent way of writing eq. (1) is in terms of Boltzmann weights $w(a|b, c|d)$, depending on the values of spins around a plaquette on a square two dimensional lattice denoted here by a, b, c, d which may take values \pm , or, more generally, any

integer number. In terms of these objects, the Yang-Baxter equation read [?]:

$$\begin{aligned} & \sum_c w(a|b_1, b_2|c) w'(b_2|c, b'_1|a') w''(c|b_1, a'|b'_2) \\ = & \sum_c w''(b_2|a, b'_1|c) w'(a|b_1, c|b'_2) w(c|b'_2, b'_1|a') \end{aligned} \quad (.2)$$

The sum being performed over an internal spin value c . Eq. (2) which is also referred to as generalized star-triangle relation, implying the commutation of transfer matrices for the associated Lattice spin model, is pictorially given in fig. 2. The prime attached to the w in eq. (2) stands for different values of internal continuous parameters.

A multidimensional generalization of the triangle equations was proposed in 1980 by Zamolodchikov, who wrote down the conditions for the factorizability of the scattering amplitudes of straight strings in a plane[]. He also proposed, what is up to now, the one and only non-trivial solution of these equations[]. We shall not write down here the original Zamolodchikov's equations because he used a rather complicated representation. A more convenient representation is obtained by generalizing eq. (2), cf. [], in terms of Boltzmann weights which no longer live on faces of a two-dimensional square lattice, but on cubes of a three-dimensional cubic lattice, depending hence on eight spins values, i.e., $w(a|efg|bcd|h)$. In terms of this description, Baxter was able to prove that Zamolodchikov proposal indeed solves the so-called tetrahedron equation, which we now state in full:

$$\begin{aligned} & \sum_d w(a|b_1 b_2 b_3 | c_1 c_2 c_3 | d) . w'(b_3 | c_2 c_1 c'_3 | b'_1 b'_2 d | a') \\ & . w''(c_1 | d b_2 b'_1 | c'_1 a' c_3 | b'_3) . w'''(d | c_2 c_3 a' | b'_3 b'_2 b_1 | c'_1) \\ = & \sum_d w''(c_1 | b_3 b_2 b'_1 | c'_2 c'_3 a | d) . w''(b_3 | c_2 a c'_3 | d b'_2 b_1 | c'_1) \\ & . w'(a | b_1 b_2 d | c'_2 c'_1 c_3 | b'_3) . w(d | c'_1 c'_2 c'_3 | b'_1 b'_2 b'_3 | a') \end{aligned} \quad (.3)$$

The sum being performed over a central spin value d . The eq. (3) which describes the conditions for commutativity of the layer-to-layer transfer matrices of a three-dimensional lattice model is given pictorially in fig. 3. Again, the primes attached to the w in eq. (3) denote the dependence on (continuous) parameters, on which we will comment below.

Note that the tetrahedron equations are invariant under a multiplication of w by:

$$\phi = \frac{y(e|c, d|h)y(f|b, d|h)y(g|b, c|h)}{y(a|e, f|d)y(a|e, g|c)y(a|f, g|b)}$$

in which $y(a|b, c|d) = y(a|c, b|d) = y(d|b, c|a) = y(c|a, d|b)$, is independent of the parameters.

2. In [] it was argued that both the triangle equations as well as the tetrahedron equations are part of a larger family of equations corresponding to increasing dimensionalities, the so-called d -simplex equations. We give here the first member of this family (summation over repeated indices is understood):

$$d = 1 : A_i^k \cdot B_k^j = B_i^k \cdot A_k^j \tag{.4}$$

(commutativity equation)

$$d = 2 : \begin{matrix} A & \begin{matrix} k_1 & k_2 \\ i_1 & i_2 \end{matrix} & \cdot & B & \begin{matrix} j_1 & k_3 \\ k_1 & i_3 \end{matrix} & \cdot & C & \begin{matrix} j^2 & j_3 \\ k_2 & k_3 \end{matrix} & = & C & \begin{matrix} k_2 & k_3 \\ i_2 & i_3 \end{matrix} & \cdot & B & \begin{matrix} k_1 & j_3 \\ i_1 & k_3 \end{matrix} & \cdot & A & \begin{matrix} j_1 & j_2 \\ k_1 & k_2 \end{matrix} \end{matrix} \tag{.5}$$

(Yang-Baxter equation)

$$\begin{aligned}
 d = 3 : \quad & A \begin{matrix} k_1 & k_2 & k_3 \\ i_1 & i_2 & i_3 \end{matrix} \cdot B \begin{matrix} j_1 & k_4 & k_5 \\ k_1 & i_4 & i_5 \end{matrix} \cdot C \begin{matrix} j_2 & j_4 & k_6 \\ k_2 & k_4 & i_6 \end{matrix} \cdot D \begin{matrix} j_3 & j_5 & j_6 \\ k_3 & k_5 & k_6 \end{matrix} = \\
 & = D \begin{matrix} k_3 & k_5 & k_6 \\ i_3 & i_5 & i_6 \end{matrix} \cdot C \begin{matrix} k_2 & k_4 & j_6 \\ i_2 & i_4 & k_6 \end{matrix} \cdot B \begin{matrix} k_1 & j_4 & j_5 \\ i_1 & k_4 & k_5 \end{matrix} \cdot A \begin{matrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{matrix} \quad (.6)
 \end{aligned}$$

(Zamolodchikov equation)

$$\begin{aligned}
 d = 4 : \quad & A \begin{matrix} k_1 & k_2 & k_3 & k_4 \\ i_1 & i_2 & i_3 & i_4 \end{matrix} \cdot B \begin{matrix} j_1 & k_5 & k_6 & k_7 \\ k_1 & i_5 & i_6 & i_7 \end{matrix} \cdot C \begin{matrix} j_2 & j_5 & k_8 & k_9 \\ k_2 & k_5 & i_8 & i_9 \end{matrix} \cdot \\
 & \cdot D \begin{matrix} j_3 & j_6 & j_8 & k_{10} \\ k_3 & k_6 & k_8 & i_{10} \end{matrix} \cdot E \begin{matrix} j_4 & j_7 & j_9 & j_{10} \\ k_4 & k_7 & k_9 & k_{10} \end{matrix} = E \begin{matrix} k_4 & k_7 & k_9 & k_{10} \\ i_4 & i_7 & i_9 & i_{10} \end{matrix} \quad (.7) \\
 & \cdot D \begin{matrix} k_3 & k_6 & k_8 & j_{10} \\ i_3 & i_6 & i_8 & k_{10} \end{matrix} \cdot C \begin{matrix} k_2 & k_5 & j_8 & j_9 \\ i_2 & i_5 & k_8 & k_9 \end{matrix} \cdot B \begin{matrix} k_1 & j_5 & j_6 & j_7 \\ i_1 & k_5 & k_6 & k_7 \end{matrix} \cdot \\
 & \cdot A \begin{matrix} j_1 & j_2 & j_3 & j_4 \\ k_1 & k_2 & k_3 & k_4 \end{matrix}
 \end{aligned}$$

(Bazhanov-Stroganov equation)

It is now easy to see from eqs. (4-7) how to generalize them to an arbitrary dimension d leading to the d -simplex equations. These equations can be considered to be generalizations of matrix commutativity conditions for objects depending on multiple ($2d$) indices. The $(d + 1)$ quantities A, B, C, \dots entering the d -simplex equation are assumed to be respectively equal to the different values of one object,

say, $S^{i_1, \dots, i_d} (D_{ij})$ corresponding to $(d+1)$ choices of the set of parameters j_1, \dots, j_d $D_{ij}, i, j = 1, \dots, d$. The matrix of parameters D_{ij} is a $d \times d$ symmetric matrix that we construct as follows. Let us first consider a d -simplex defined in a d -dimensional alline euclidean space by giving $(d+1)$ points, p_1, \dots, P_{d+1} which form a complete coordinate system for this space. The $(d+1)$ faces of the d -simplex are respectively the $(d+1)$ alline hyper-planes H_j of codimension one constructed out of the d points $P_{i_k}, k = 1, \dots, d, i_k \neq j$. To each face H_j we associate a unit vector \vec{u}_j orthogonal to it and pointing outside the d -simplex. Now, each point P_i belongs to the d faces $H_j, j \neq i$. Hence, to each vertex P_i of the d -simplex we can associate the d unit vectors $\vec{u}_j, j \neq i$, and the matrix $D_{kl}^{(i)}$ of their scalar products, i.e., $D_{kl}^{(i)} = \vec{u}_k \cdot \vec{u}_l$. We also associate to each vertex P_i the tensor $S^{i_1, \dots, i_d} (D_{ke}^{(i)})$ where the indices ik and jk are, for each k , respectively characterizing a line joining the point P_i to another point $P_e, e \neq i$, of the d -simplex. If now, we consider this tensor as a statistical weight attached to the vertex P_i , we can compute the position function associated to the d -simplex; it is just the product of the tensors $S^{(i)}, i = 1, \dots, d+1$, the sum being performed on the indices ik or jk attached to the internal edges of the d -simplex. The d -simplex equation express now that this partition function is the same for the d -simplex we started with (for which all the unit vectors \vec{u}_j are pointing outside of it) and for the d -simplex obtained by a translation of the hyper-planes H_j (conserving their orientation) in such a way that all the proceedings \vec{u}_j are now pointing inside the new d -simplex. The first examples of these equations for $d = 1, 2, 3, 4$ are given in eqs. (4-7).

Let us note that an equivalent description of the set of parameters $D_{kl}^{(i)}$ is given in

terms of the $\frac{d(d+1)}{2}$ relative angles θ_a between the $(d+1)$ unit vectors \vec{u}_j . However, not all these angles are independent, since $(d+1)$ vectors in d dimensions do not form a free vector system. If we write $\vec{u}_i \cdot \vec{u}_j = \cos(\pi - \theta_{ij})$, we obtain one relation between these $\frac{d(d+1)}{2}$ angles, given by the vanishing of the determinant of the $(d+1) \times (d+1)$ matrix $\Omega_{ij} = \vec{u}_i \cdot \vec{u}_j$.

In the case of the tetrahedron equation ($d = 3$), the tensor S depends on three parameters, e.g., the respective angles between the three planes defining each vertex $P_i, i = 1, \dots, 4$. Hence we have:

$$\begin{aligned}
 & S \begin{matrix} k_1 & k_2 & k_3 \\ i_1 & i_2 & i_3 \end{matrix} (\theta_1, \theta_2, \theta_3) \cdot S \begin{matrix} j_1 & k_4 & k_5 \\ k_1 & i_4 & i_5 \end{matrix} (\theta_1, \theta_4, \theta_5) \\
 & S \begin{matrix} j_1 & j_4 & k_6 \\ k_2 & k_4 & i_6 \end{matrix} (\theta_2, \theta_4, \theta_6) \cdot S \begin{matrix} j_3 & j_5 & j_6 \\ k_3 & k_5 & k_6 \end{matrix} (\theta_3, \theta_5, \theta_6) = \\
 = & S \begin{matrix} k_3 & k_5 & k_6 \\ i_3 & i_5 & i_6 \end{matrix} (\theta_3, \theta_5, \theta_6) S \begin{matrix} k_2 & k_4 & j_6 \\ i_2 & i_4 & k_6 \end{matrix} (\theta_2, \theta_4, \theta_6) \quad (.8) \\
 & S \begin{matrix} k_1 & j_4 & j_5 \\ i_1 & k_4 & k_5 \end{matrix} (\theta_1, \theta_4, \theta_5) \cdot S \begin{matrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{matrix} (\theta_1, \theta_2, \theta_3)
 \end{aligned}$$

where the six parameters $\theta_k, k = 1, \dots, 6$ are not all independent as we have seen before. In fact, if $\cos(\pi - \theta_i) = C_i, i = 1, \dots, 6$, we have the relation:

$$\det \begin{pmatrix} 1 & c_1 & c_2 & c_4 \\ c_1 & 1 & c_3 & c_5 \\ c_2 & c_3 & 1 & c_6 \\ c_4 & c_5 & c_6 & 1 \end{pmatrix} = 0 \quad (.9)$$

Note that the angles θ_i and the angles α_j between the lines in the tetrahedron of fig.

4 are related by spherical trigonometry[].

The relation between the tetrahedron equations (6,8) and the equation (3) in terms of Boltzmann weights is given by the following correspondence:

$$w(a|efg|bcd|h) \leftrightarrow S \begin{array}{ccc} gbch, & fbdh, & ecdh \\ aefd, & aegc, & afgb \end{array}$$

assuming that any one of the indices of S in eq. (8) correspond to a set of four spins on any one of the faces of the cube with which we associate the Boltzmann weight w .

In a similar fashion as before, the 4-simplex equation is given by eq. (7) where the tensors A, B, \dots are given by $S \begin{array}{cccc} i_1 & i_2 & i_3 & i_4 \\ j_1 & j_2 & j_3 & j_4 \end{array} (\theta_1, \dots, \theta_8)$ for various values of the angles θ_k . The 4-simplex is described by the 10 relative angles between its 5 faces, 9 of them being independent, the last one being constrained by the vanishing of the matrix determinant, $\det(\Omega_{ij}) = 0, i, j = 1, \dots, 5$.

Again, we can associate, in a similar manner as before, with eq. (7) a representation of the 4-simplex equation in terms of Boltzmann weights depending on spins on a hyper cubic 4-dimensional lattice. Let $a, b_1, \dots, b_4, c_1, \dots, c_8, d_1, \dots, d_4, e$ denote the 16 spin values corresponding to the 16 vertices of an elementary hypercube of such a lattice (see fig. 5), then we can introduce the following correspondence:

$$w(a|b_1, \dots, b_4|c_1, \dots, c_8|d_1, \dots, d_4|e) \leftrightarrow S \begin{array}{cccc} i_1 & i_2 & i_3 & i_4 \\ j_1 & j_2 & j_3 & j_4 \end{array}$$

where every index of S corresponds to a face of the hypercube, i.e., a cube with 8 spin values that determine the respective values of the indices i_k and $j_k, u = 1, \dots, 4$

in the following way:

$$i_1 \leftrightarrow (b_1 c_4 c_5 c_6 d_2 d_3 d_4 e) \quad j_1 \leftrightarrow (a b_2 b_3 b_4 c_1 c_2 c_3 d_1)$$

$$i_2 \leftrightarrow (b_2 c_2 c_3 c_6 d_1 d_3 d_4 e) \quad j_2 \leftrightarrow (a b_1 b_3 b_4 c_1 c_2 c_5 d_2)$$

$$i_3 \leftrightarrow (b_3 c_1 c_3 c_5 d_1 d_2 d_4 e) \quad j_3 \leftrightarrow (a b_1 b_2 b_4 c_2 c_6 d_3)$$

$$i_4 \leftrightarrow (b_4 c_1 c_2 c_3 d_1 d_2 d_3 e) \quad j_4 \leftrightarrow (a b_1 b_2 b_3 c_4 c_5 c_6 d_4)$$

Using this correspondence we can write the following representation of the 4-simplex equation in terms of Boltzmann weights w :

$$\begin{aligned}
& \sum_e w \quad (a|b_1 b_2 b_3 b_4|c_1 c_2 c_3 c_4 c_5 c_6|d_4 d_3 d_2 d_1|e) \cdot \\
& \quad .w' \quad (b_1|d'_1 c_6 c_5 c_4|d_2 d_3 d_4 c'_3 c'_2 c'_1|b'_4 b'_3 b'_2 e|a') \\
& \quad .w'' \quad (c_6|c'_1 b_2 d_4 d_3|e c_2 c_3 b'_3 b'_4 d'_2|c'_4 c'_5 a' d_1|b'_1) \\
& \quad .w''' \quad (d_4|b'_4 c_3 c_5 e|d_2 d_1 b_3 a' c'_2 c'_4|d'_3 b'_1 b'_2 c_1|c'_6) \\
& \quad .w'''' \quad (e|a' d_1 d_2 d_3|c_4 c_2 c_1 b'_3 b'_2 b'_1|c'_6 c'_5 c'_3 b_4|d'_4) \\
= & \sum_e w'''' \quad (d_4|b'_4 c_3 c_5 c_6|b_1 b_2 b_3 c'_1 c'_2 c'_4|d'_3 d'_2 d'_1 a|e) \\
& \quad .w'''' \quad (c_6|c'_1 b_2 b_1 d_3|c_4 c_2 a b'_3 d'_1 d'_2|e c'_5 c'_3 b_4|d'_4) \\
& \quad .w'' \quad (b_1|d'_1 a c_5 c_4|d_2 b_4 b_3 c'_3 c'_2 e|b'_3 b'_4 b'_2 c_1|c'_6) \\
& \quad .w' \quad (a|e b_2 b_3 b_4|c_1 c_2 c_3 d'_4 d'_3 d'_2|c'_4 c'_5 c'_6 d_1|b'_1) \\
& \quad .w \quad (e|d'_1 d'_2 d'_3 d'_4|c'_6 c'_5 c'_4 c'_3 c'_2 c'_1|b'_4 b'_3 b'_2 b'_1|a') \tag{.10}
\end{aligned}$$

where the summation is over a center spin e , and the primes attached to the w stand for different values of the parameters.

As for the tetrahedron equation, the 4-simplex equation is invariant under the multiplication of w^e by the quantity:

$$\begin{aligned} & y(a|b_1b_2b_3|c_4c_5c_6|d_4).y(a|b_1b_2b_4|c_2c_4c_6|d_3). \\ & .y(a|b_1b_3b_4|c_1c_4c_5|d_2).y(a|b_2b_3b_4|c_1c_2c_3|d_1). \\ & .y^{-1}(b_4|c_1c_2c_3|d_3d_2d_1|e).y^{-1}(b_3|c_1c_4c_5|d_4d_2d_1|e) \\ & .y^{-1}(b_2|c_2c_3c_6|d_4d_3d_1|e).y^{-1}(b_1|c_4c_5c_6|d_4d_3d_2|e) \end{aligned}$$

where $y(a|b_1b_2b_3|c_1c_2c_3|d)$ is a function, depending on the cube of spin values $(a_1b_1b_2b_3c_1c_2c_3d)$ and which is invariant by the various permutations of the spins values describing geometrically the same cube. In addition, it is independent of the parameters.

3. We believe that the representation of the simplex equation in terms of the quantities S , such as in eq. (1) and eqs. (4-8) is the most simple one to get an insight into their algebraic structure and also for studying their solutions. Using now the correspondence between the Boltzmann weights w and the tensor S we may obtain an explicit form of Zamolodchikov's solution of the tetrahedron equations, which has the advantage above previous descriptions[] that it is more algebraic in nature. Zamolodchikov's solution corresponds to the case where the spins values in eq. (3) and consequently the indices in eqs. (6,8)—which are the products of 4 spins values along a face of a cube of the 3-dimensional lattice—take only two values \pm . Furthermore, we impose the symmetry conditions

$$(i) S \begin{matrix} j_1 & j_2 & j_3 \\ i_1 & i_2 & i_3 \end{matrix} (\theta_1, \theta_2, \theta_3) = S \begin{matrix} j_2 & j_3 & j_1 \\ i_2 & i_3 & i_1 \end{matrix} (\theta_2, \theta_3, \theta_1) \quad (11a)$$

$$(ii) S \begin{matrix} j_1 & j_2 & j_3 \\ i_1 & i_2 & i_3 \end{matrix} (\theta_1, \theta_2, \theta_3) = S \begin{matrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \end{matrix} (\theta_1, \theta_2, \theta_3) \quad (11b)$$

Then, the Zamolodchikov's solution is given by the following choice of matrices $\begin{pmatrix} S & j_1 & j_2 \\ i_1 & i_2 & i_3 \end{pmatrix}$ with matrix elements $\begin{pmatrix} S & j_1 & j_2 \\ i_1 & i_2 & i_3 \end{pmatrix} \begin{matrix} j_3 \\ i_3 \end{matrix} \equiv S \begin{matrix} j_1 & j_2 & j_3 \\ i_1 & i_2 & i_3 \end{matrix}, i\alpha, j\alpha = \pm :$

$$\begin{aligned} S_{++}^{++} &= \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix}, & S_{+-}^{+-} &= \begin{pmatrix} x_3 & 0 \\ 0 & y_2 \end{pmatrix} \\ S_{-+}^{-+} &= \begin{pmatrix} x_2 & 0 \\ 0 & y_3 \end{pmatrix}, & S_{--}^{--} &= \begin{pmatrix} y_1 & 0 \\ 0 & y_0 \end{pmatrix} \\ S_{--}^{++} &= \begin{pmatrix} 0 & R_1 \\ R_0 & 0 \end{pmatrix}, & S_{++}^{--} &= \begin{pmatrix} 0 & R_0 \\ R_1 & 0 \end{pmatrix} \\ S_{+-}^{-+} &= \begin{pmatrix} 0 & R_3 \\ R_2 & 0 \end{pmatrix}, & S_{-+}^{+-} &= \begin{pmatrix} 0 & R_2 \\ R_3 & 0 \end{pmatrix} \end{aligned} \quad (.12)$$

in which $X_i = P_i + Q_i, Y_i = P_i - Q_i$ and R_i are functions of the angles $\theta_1, \theta_2, \theta_3$.

Inserting (12) into eq. (6) we essentially get two types of equations for these functions:

$$X_1 X_1' X_1'' R_0''' + R_1 R_1' R_1'' X_0''' = R_0''' X_0'' X_0' X_0 + X_0''' R_1'' R_1' R_0 \quad (13a)$$

$$X_0 X_0' R_0'' R_1''' + R_0 R_0' Y_1'' Y_1''' = X_0''' X_0'' R_0' R_2 + R_0''' R_1'' Y_2' Y_2 \quad (13b)$$

Equations (13a, 13b) and similar equations which are obtained either by inverting all spin, leading to $X \leftrightarrow Y$, or by permutating the indices 1, 2, 3, allow for a parametrization in terms of functions of spherical angles, as was shown by Baxter[]. In fact, let $\theta_1, \theta_2, \theta_3$ be the interior angles of a spherical triangle and $\alpha_{23}, \alpha_{13}, \alpha_{12}$ the respective

opposite sides, then we have:

$$\begin{aligned} P_0 &= 1, \quad Q_0 = t_0 t_1 t_2 t_3, \quad P_i = t_j t_k \text{ (cyclic)} \\ R_0 &= \frac{S_0}{c_1 c_2 c_3}, \quad R_i = \frac{S_1}{c_0 c_j c_k} \text{ (cyclic)} \quad Q_i = t_0 t_i \end{aligned} \quad (.14)$$

where $S_i = (\sin \frac{\alpha_i}{2})^{1/2}$, $c_1 = (\cos \frac{\alpha_1}{2})^{1/2}$, $t_i = (\tan \frac{\alpha_i}{2})^{1/2}$ and $2\alpha_0 = \theta_1 + \theta_2 + \theta + 3 - \pi$, $\alpha_i = \alpha_0 - \theta_i + \pi$. Another way of writing (12) in a more algebraic and compact way, is in terms of a threefold tensor product of SL_2 . Namely, let us define the following tensors:

$$\Omega_0 = \frac{1}{4} \{ \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_x \otimes \sigma_x + \sigma_x \otimes \mathbf{1} \otimes \sigma_x + \sigma_x \otimes \sigma_x \otimes \mathbf{1} \}$$

$$\Omega_1 = (\mathbf{1} \otimes \mathbf{1} \otimes \sigma_x) \Omega_0 (\mathbf{1} \otimes \mathbf{1} \otimes \sigma_x)$$

$$\Omega_2 = (\sigma_x \otimes \mathbf{1} \otimes \mathbf{1}) \Omega_0 (\sigma_x \otimes \mathbf{1} \otimes \mathbf{1})$$

$$\Omega_3 = (\mathbf{1} \otimes \sigma_x \otimes \mathbf{1}) \Omega_0 (\mathbf{1} \otimes \sigma_x \otimes \mathbf{1})$$

with the property that $\Omega_0 + \Omega_1 + \Omega_2 + \Omega_3 = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$, and where $\sigma_x, \sigma_y, \sigma_z$ are the usual Pauli matrices. Then the Zamolodchikov's solution (12) can be written as:

$$S = \Omega_\alpha P_\alpha - (\sigma_z \otimes \sigma_z \otimes \sigma_z) \Omega_\alpha Q_\alpha + (\sigma_x \otimes \sigma_x \otimes \sigma_x) \Omega_\alpha R_\alpha \quad (.15)$$

where the sum over $\alpha = 0, 1, 2, 3$ is understood.

4. Rewriting Zamolodchikov's solution of the tetrahedron equation in terms of the quantities S as in eq. (12) immediately suggest a possible form of the solution of the 4-simplex equation, which should have been hard to guess from the representation

of it in terms of Boltzmann weights as in eq. (10). In fact, let S $\begin{matrix} J_1 & j_2 & j_3 \\ i_1 & i_2 & i_3 \end{matrix}$ denote

the matrices with components $\begin{pmatrix} S & j_1 & j_2 & j_3 \\ & i_1 & i_2 & i_3 \end{pmatrix} \begin{matrix} j_4 \\ i_4 \end{matrix} \equiv S \begin{matrix} j_1 & j_2 & j_3 & j_4 \\ i_1 & i_2 & i_3 & i_4 \end{matrix}$, we wish to suggest the following form of a possible solution of the 4-simplex equation (7):

5. In this letter, we have tried to put forward the idea that the triangle equations and the tetrahedron equations are members of a hierarchy of equations, called the simplex equations, and that it is useful to consider these equations as such. An insight on how this hierarchy develops as we increase the dimensionality might lead to a systematic understanding of the algebraic structure of its individual members and hopefully to the development of solution methods. The recent interest in the theory of quantum groups[], which is until now linked solely to the quantum Yang-Baxter equations, might also gain from a broadening of horizon by taking other members of the hierarchy of simplex equations into consideration. Another field of application is the study of multidimensional integrability. It is well known that the classical and quantum Yang-Baxter equations are one of the fundamental building blocks in the theory of two-dimensional integrability[]. The development of a genuinely multidimensional notion of integrability is one of the major problems in the theory of integrable systems of these days. In a previous publication[] we have addressed ourselves more explicitly to these problems. The main idea is that the notion of integrability is intimately linked to the question of the possibility of posing an overdetermined, but solvable, system of equations which does not trivially reduce to a dimensionally smaller system. The simplex equations can be of help in the search of such systems, because these equations themselves can be shown to emerge as the consistency conditions of a related system of equations, namely those involving the permutations of transfer matrices. In a future publication[], we shall treat these results in more detail and show that by analogy with the two-dimensional case, this leads in a natural way to a notion of

integrability on multidimensional lattices.

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Figure Captions

- Fig. 1: Condition for factorization of 3-particle scattering in terms of 2-particle scattering amplitudes.
- Fig. 2: Generalized star-triangle equation for Boltzmann weights W, W', W'' of a lattice spin model.
- Fig. 3: Pictorial representation of the tetrahedron equation.
- Fig. 4: The tetrahedron equation in the representation of eq. (4).
- Fig. 5: Boltzmann weight $w(a|b_1b_2b_3b_4|c_1c_2c_3c_4c_5c_6|d_4d_3d_2d_1|e)$ for a 4-dimensional lattice spin model.

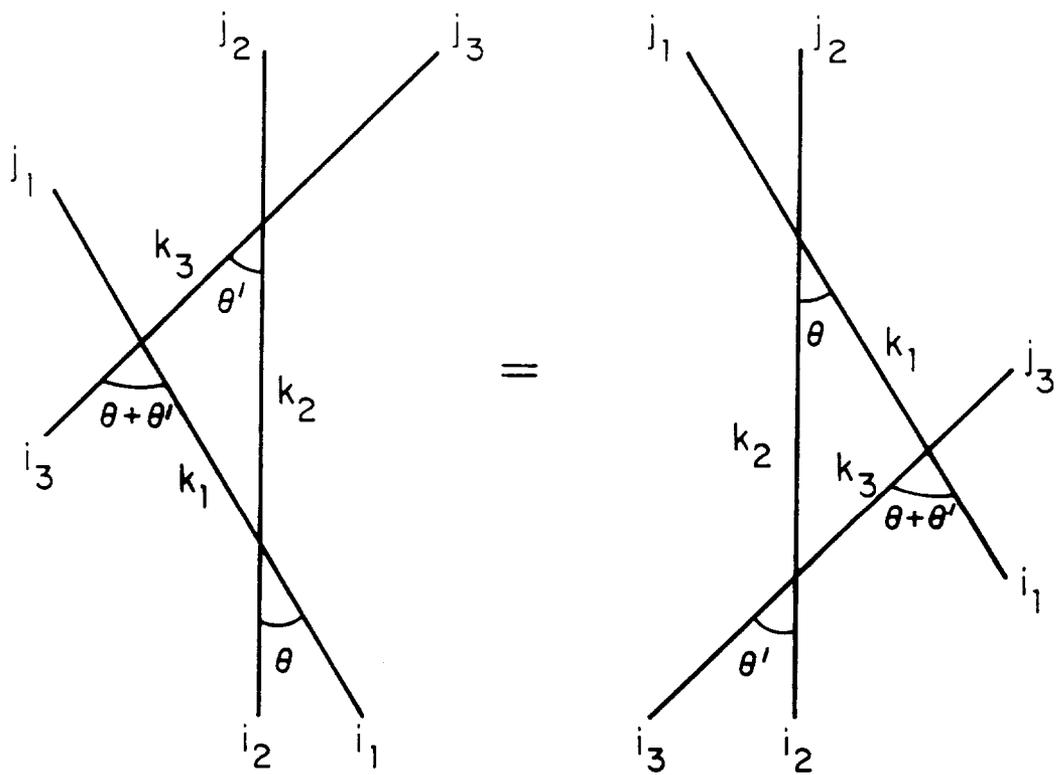


Fig. 1

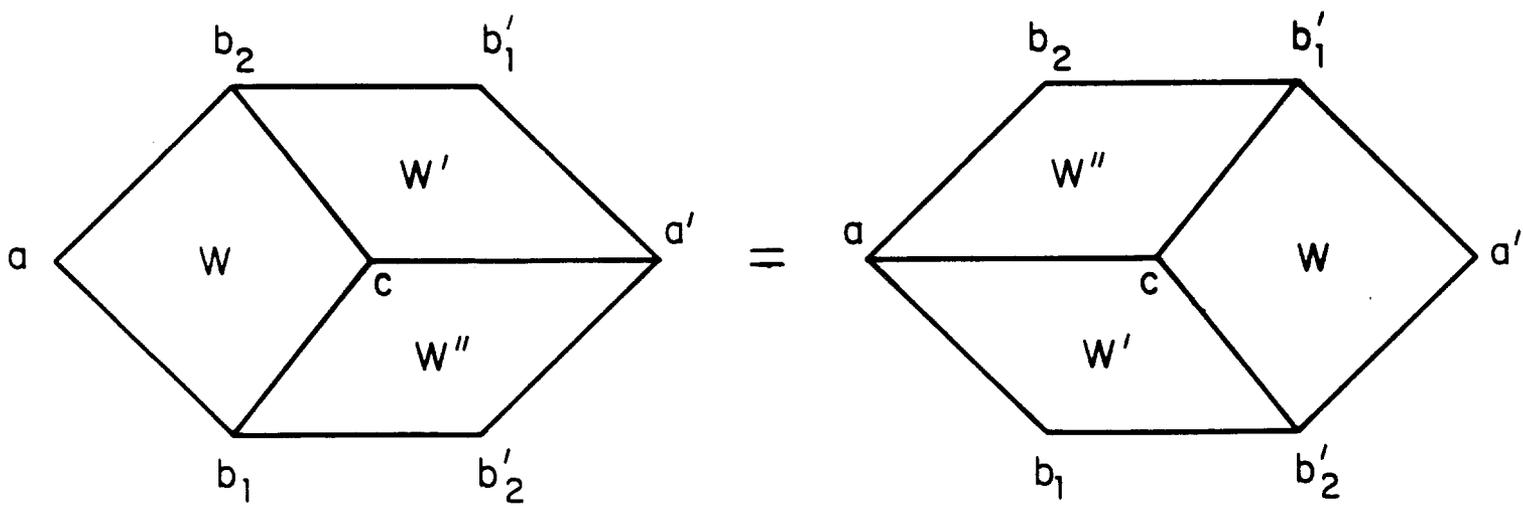


Fig. 2

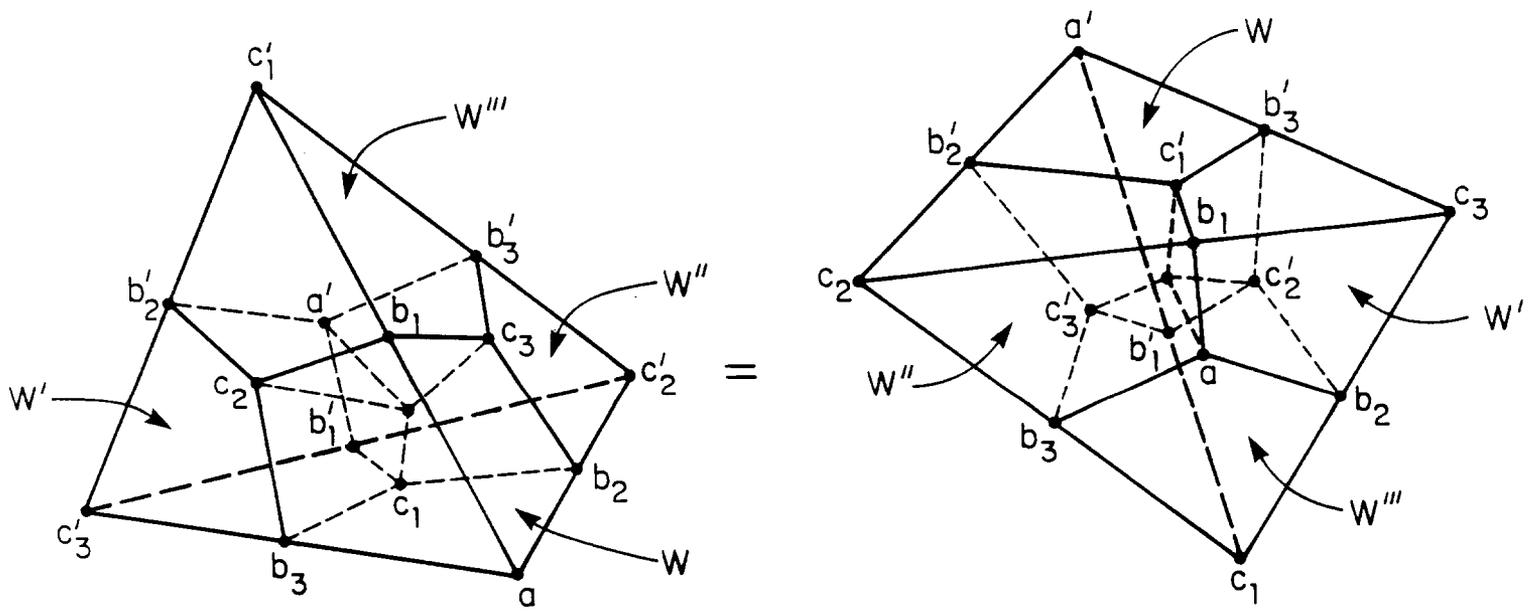


Fig. 3

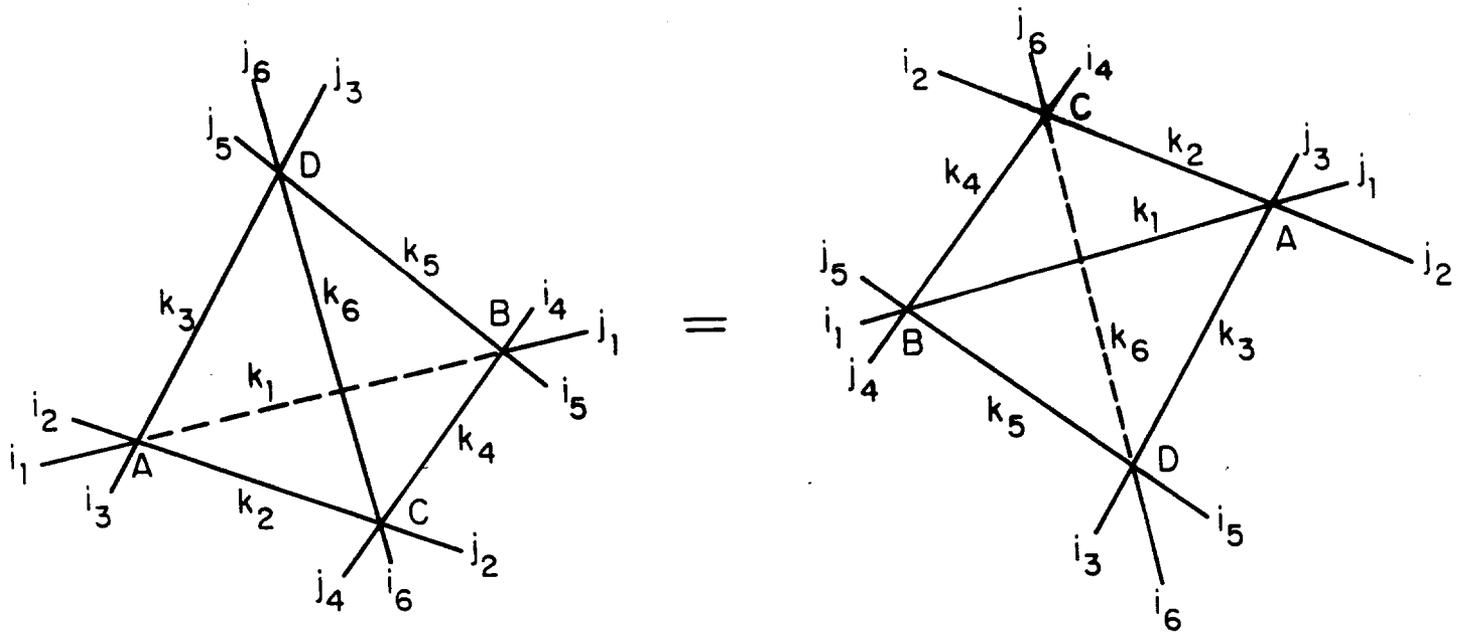


Fig. 4

