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Dynamical Instability of Bosonic Stellar Configurations

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Abstract

We study the problem of the dynamical stability of the equilibrium solutions for the bosonic stellar configurations in the framework of general relativity. The time evolution of infinitesimal radial oscillations, which conserve the total number of particles, is analyzed starting from the scalar wave equation coupled to the Einstein's field equations. Following the method developed by Chandrasekhar, one finds a variational principle for determining the eigenfrequencies of the oscillations. Using the variational principle, one can find numerically an upper bound for the central densities where dynamical instability occurs. As examples, we consider the equilibrium configurations, found by Ruffini and Bonazzola, for the non-interacting massive complex scalar fields as well as the quartic self-interacting case, $V(|\phi|) = \frac{\tilde{\lambda}}{4}|\phi|^4$ ($\tilde{\lambda} > 0$), discussed by Colpi, Shapiro and Wasserman. In the non-interacting case, we find that for central densities bigger than $\rho = 2,1 \times 10^{98} m^2\ g/cm^3$ (m is the boson mass in grams) the configuration is dynamically unstable; whereas in the interacting case, with a value $\tilde{\lambda} = 3,8 \times 10^{12} m^2$, the bound is given by $\rho = 1,3 \times 10^{98} m^2\ g/cm^3$.

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I. Introduction

Recent developments in particle physics and cosmology suggest that evolving scalar fields may have played an important role in the evolution of the early universe[1], for instance in primordial phase transitions, and that they act as an important source for the missing mass[2,3]. This latter possibility raises naturally the questions whether cold, gravitational equilibrium configurations of massive scalar fields—"boson stars"—may exist and whether they are dynamically stable[4].

In an early work, Ruffini and Bonazzola[5] found spherically symmetric gravitational equilibrium configurations in asymptotically flat space–times for non–interacting massive complex fields by solving the coupled system of Einstein–Klein–Gordon equations. They used the simplifying assumptions that the radial scalar field solution is nodeless and that the configuration is at zero temperature. The equilibrium configurations are macroscopic quantum states of cold, degenerate bosons held together by gravity with supporting pressure given by Heisenberg's uncertainty principle. For a quantum state confined into a region of size R–the boson star radius–one gets a typical boson momentum $p \sim \frac{1}{R}$ (we use throughout the paper $\hbar \equiv c \equiv 1$), and for a moderately relativistic boson star $p \sim m$, where m is the scalar field mass, so that $R \sim 1/m$. The results obtained in Ref. [5] have been confirmed and extended in subsequent works listed in Refs. [6–8].

In Ref. [5], it has been shown that boson stars have many similarities with their fermionic counterparts, white–dwarfs and neutron–stars, on the other hand, there are also many differences, which would call for a more detailed study of those objects. For the boson stars, with the scalar fields in the ground state, there is also a critical mass and a critical particle number†, above which mass gravitational collapse occurs. These quantities are respectively given by $M_{crit} = 0.633 M^2_{Planck}/m$ and $N_{crit} = 0.653 M^2_{Planck}/m^2$, where $M_{Planck} = (G)^{-1/2}$ is the Planck mass. The order of magnitude of these limits can be easily obtained from Chandrasekhar's dimensional argument presented in Ref. [9], or by using Landau's argument, adapted for a bosonic configuration by minimizing the energy of a scalar particle with above momentum in the background gravitational potential for all self-gravitating particles $E_{pot} \sim -\frac{GMm}{R}$. In the fermionic case, the Chandrasekhar mass is $M_{Chan} \sim M^3_{Planck}/m^2$ and the critical particle number behaves as $N_{Lan} \sim M^3_{Planck}/m^3$. For fields of mass ~ 1 GeV, we obtain a critical mass for the boson star of $10^{14} g$ and 10^{38} particles, which is clearly much lighter than their fermionic counterparts. On the other hand, the critical central density for the boson stars is $\rho_{crit} = 5.26 \times 10^{97} m^2\ g/cm^3$ (the boson mass m is given in grams), which is enormously higher than in the cold, degenerate fermions case.

The extension to the self-interacting case, for a potential $V(|\phi|) = \frac{1}{4}\tilde{\lambda}|\phi|^4$ ($\tilde{\lambda} > 0$), has been considered in the work of Colpi, Shapiro and Wasserman[10]. The consequence of switching on a quartic self-interaction between the scalar particles is to increase the above limits on the critical mass and particle number. In particular, if $m_{boson} \simeq \tilde{\lambda}^{1/4} m_{fermion}$, the critical mass for the boson star becomes comparable to the Chandrasekhar limit. Other models, including soliton-stars[11] and boson stars with non-minimal coupling for the scalar field[12], show the same tendency to increase the critical mass and the particle number.

A crucial difference between the bosonic and the fermionic configurations arises if

†This is still true if one considers radial scalar field solutions with nodes. The critical mass as well as the critical number grow by increasing the number of nodes[8].

one tries to implement a macroscopic description for the bosonic condensate. As it was already shown in Ref. [5], it is not consistent with the Klein–Gordon equation to describe the spherically symmetric boson condensate as a perfect fluid with a given equation of state. In fact, the energy-momentum tensor is not isotropic in the 3 spatial directions with the radial component differing from the 2 angular ones. As a consequence, we cannot apply to the boson stars the theorem on stability of a fluid star[13] (the boundary between stability and instability being given by the conditions: $\frac{\partial M(\rho)}{\partial \rho} = 0$ and $\frac{\partial N(\rho)}{\partial \rho} = 0$; M and N being functions of the central density ρ), which is based upon the assumption of a perfect fluid behaviour. Nevertheless, if we plot the mass for the equilibrium configurations against central density for these stars, we find, except for the scales, a behaviour remarkably similar to those of neutron-stars: the mass quickly raises to a maximum (for $\rho = \rho_{crit}$), drops a little, oscillates and approaches an asymptotic value at large central densities, the same happening for the particle number (see fig. 1 and 2). These similarities lead naturally to the conjecture[6] that the stability behaviour is similar to the fermionic case, namely that the boundary between stable and unstable configurations being given by $\rho = \rho_{crit}$. Note also that the binding energy, $E_b = M_\infty - Nm$, becomes positive for a finite value of the central density, bigger than ρ_{crit}, suggesting the existence of possible configurations with excess energy.

In this paper, we study the problem of the dynamical stability of the bosonic equilibrium configurations in the framework of general relativity. We discuss both the case of a non-interacting massive complex scalar field, as well as the case with a quartic self-interacting potential. However, the analytic part of our results can immediately be extended to any interacting potential for complex scalar fields. (The extension to soliton-stars[11] is not straightforward.) We analyze the time evolution of infinitesimal radial oscillations, which conserve the total number of particles, starting from the system described by the scalar wave equation coupled to the Einstein's field equations. We follow closely the method developed by Chandrasekhar[4]; the main modification being that we cannot use the energy-momentum tensor of a perfect fluid, but instead we use the energy-momentum tensor of a quartic self-interacting massive complex field (the non-interacting case being given by setting $\tilde{\lambda} = 0$). We find an eigenvalue equation which determines the normal modes of the radial oscillations, and as in the perfect fluid case we find a variational principle for determining the eigenfrequencies of the oscillations. This allows then to find numerically, using suitable trial functions, upper bounds to the central density, of the equilibrium configurations, from which on dynamical instability will occur.

The paper is organized as follows: in Section II, we present the known results for the equilibrium configurations for the boson stars. In Section III, we derive the equations governing to first order the infinitesimal radial perturbations. In particular, since we describe the complex scalar field by two independent real fields, we obtain a system of two coupled eigenvalue equations, for which we get a variational principle for determining the eigenfrequencies. In Section IV, we present the numerical results for the upper bounds of the eigenfrequencies, for different values of the central density. We conclude in Section V with a summary of our results as well as some comments on possible developments.

II. Equilibrium Configurations

The many particle system is described by a second-quantized complex scalar field coupled to gravity. The action for this system is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} + \frac{g^{\mu\nu}}{2}\phi^*_{;\mu}\phi_{;\nu} - \frac{m^2}{2}|\phi|^2 - \frac{\tilde{\lambda}}{4}|\phi|^4\right] \tag{2.1}$$

This action is invariant under a global $U(1)$ phase transformation, $\phi \to e^{i\theta}\phi$, which implies the conservation of its generator Q, the number of particles minus the number of antiparticles. By varying the action with respect to $g^{\mu\nu}, \phi$ and ϕ^* we obtain the Einstein's field equations

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = -8\pi G\, T^\mu_\nu \tag{2.2}$$

with the energy–momentum tensor

$$\begin{aligned} T^\mu_\nu &= \frac{1}{2}g^{\mu\sigma}\left(\phi^*_{;\sigma}\phi_{;\nu} + \phi_{;\sigma}\phi^*_{;\nu}\right) \\ &- \frac{1}{2}\delta^\mu_\nu\left[g^{\lambda\sigma}\phi^*_{;\lambda}\phi_{;\sigma} + m^2|\phi|^2 + \frac{\tilde{\lambda}}{2}|\phi|^4\right] \end{aligned} \tag{2.3}$$

and the scalar wave equation in curved–space

$$g^{\mu\nu}\phi_{;\mu\nu} + m^2\phi + \tilde{\lambda}|\phi|^2\phi = 0 \tag{2.4}$$

and its complex conjugate. The field ϕ can be expanded in creation and annihilation operators

$$\phi(r,t) = \sum_n \left(a_n\phi_n\ (r,t) + a_n^+\phi_n^+\ (r,t)\right) \tag{2.5a}$$

where

$$\phi_n\ (r,t) = \left(\phi_{1n}(r,t) + i\phi_{2n}(r,t)\right)e^{-i\omega_n t} \tag{2.5b}$$

$\phi_{1n}(r,t)$ and $\phi_{2n}(r,t)$ are real functions. For our considerations it is more convenient to write the complex scalar field in term of two real fields. ϕ as well as ϕ_{1n} and ϕ_{2n} are functions of r and t alone since we consider only spherically symmetric equilibrium configurations. In fact, we expect them to correspond to solutions with minimal energy. We express the metric in Schwarzschild coordinates

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.6}$$

where ν and λ are functions of r and t only ($g^{00} = -e^{-\nu}$ and $g^{rr} = e^{-\lambda}$). For $r \to \infty$, we require $e^{\nu(r,t)} \to 1$ as well as $e^{\lambda(r,t)} \to 1$. The creation and annihilation operators satisfy the usual commutation relations

$$\left[a_n, a^+_{n'}\right] = \delta_{nn'} 2w_n \tag{2.7}$$

and the ground state with N particles is given by

$$|N;0> = \prod_1^N a_0^+|0> \tag{2.8}$$

where $|0>$ is the vacuum state in the background metric and a_0 is the operator associated with a nodeless $\phi_0(r,t)$. In the following, we will drop the subscript 0, since we consider only the nodeless solution. The mean values of the components of the energy–momentum tensor T^μ_ν are given by

$$T^\mu_\nu = < N;0| : T^\mu_\nu : |N;0> \tag{2.9}$$

where :: denotes normal ordering and the operator T^μ_ν is defined in eq. (2.3).

With the above metric, eq. (2.6), the Einstein's equations are, where in the following T^μ_ν denotes the mean value as defined in eq. (2.9).

$$R^0_0 - \frac{1}{2}R = -8\pi G T^0_0 = -\frac{1}{r^2}\left(re^{-\lambda}\right)' + \frac{1}{r^2} \tag{2.10}$$

$$R^1_1 - \frac{1}{2}R = -8\pi G T^1_1 = -e^{-\lambda}\left(\frac{1}{r}\nu' + \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{2.11}$$

$$\begin{aligned} R^2_2 &- \frac{1}{2}R = R^3_3 - \frac{1}{2}R = -8\pi G T^2_2 = -8\pi G T^3_3 \\ &- e^{-\lambda}\left[\frac{1}{2}\nu'' - \frac{1}{4}\nu'\lambda' + \frac{1}{4}(\nu')^2 + \frac{1}{2r}(\nu' - \lambda')\right] \\ &+ e^{-\nu}\left[-\frac{1}{4}\dot{\lambda}\dot{\nu} + \frac{1}{2}\ddot{\lambda} + \frac{1}{4}\left(\dot{\lambda}\right)^2\right] \end{aligned} \tag{2.12}$$

and

$$R^1_0 = -8\pi G T^1_0 = -\frac{e^{-\lambda}}{r}\dot{\lambda} \tag{2.13}$$

where the prime and the dot denotes differentiation with respect to r and t respectively.

Combining eq. (2.10) and (2.11) we obtain the useful relation

$$\frac{e^{-\lambda}}{r}(\lambda' + \nu') = 8\pi G\left(T^1_1 - T^0_0\right) \tag{2.14}$$

Equations (2.10)-(2.13) are not all independent due to the fact that the covariant divergence of $R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R$ vanishes identically, which implies $(T^\mu_\nu)_{;\mu} = 0$. This leads to the following two relations

$$\dot{T}^0_0 + T^{1\prime}_0 + \frac{1}{2}\left(T^0_0 - T^1_1\right)\dot{\lambda} + T^1_0\left[\frac{1}{2}(\lambda' + \nu') + \frac{2}{r}\right] = 0 \tag{2.15}$$

and

$$\dot{T}^0_1 + T^{1\prime}_1 + \frac{1}{2}T^0_1\left(\dot{\lambda} + \dot{\nu}\right) + \frac{1}{2}\left(T^1_1 - T^0_0\right)\nu' + \frac{2}{r}\left(T^1_1 - T^2_2\right) = 0\,. \tag{2.16}$$

Notice that eq. (2.16) is equivalent to the scalar wave equation, as can be seen when T^{μ}_{ν}, given in eq. (2.3), is explicitly substituted into it.

Writing ϕ, and ϕ^*, as given in eq. (2.5b), we get for the scalar wave equation a set of two coupled equations for ϕ_1 and ϕ_2

$$\begin{aligned} \phi_1'' &+ \left(\frac{2}{r}+\frac{\nu'}{2}-\frac{\lambda'}{2}\right)\phi_1' + e^{\lambda}\left(w^2 e^{-\nu} - m^2 - \tilde{\lambda}\phi_1^2\right)\phi_1 \\ &- e^{\lambda-\nu}\ddot{\phi}_1 + \frac{1}{2}e^{\lambda-\nu}\left(\dot{\nu}-\dot{\lambda}\right)\dot{\phi}_1 \mp \frac{1}{2}e^{\lambda-\nu}\left(\dot{\nu}-\dot{\lambda}\right)w\phi_2 \\ &\pm 2e^{\lambda-\nu}w\dot{\phi}_2 - e^{\lambda}\tilde{\lambda}\phi_1\phi_2^2 = 0 \end{aligned} \tag{2.17}$$

where the equation for ϕ_2 is obtained by interchanging in above equation the subscript $1 \to 2$ and by taking the lower signs.

The global invariance in the action (2.1) leads to the current conservation equation

$$J^{\mu}{}_{;\mu} = 0 \tag{2.18}$$

where the current vector J^{μ} is

$$J^{\mu} = ig^{\mu\nu}\left(\phi_{;\nu}\phi^* - \phi^*_{;\nu}\phi\right) \quad . \tag{2.19}$$

As a consequence of eq. (2.18) the total charge (the number of particles minus the number of antiparticles) given by

$$Q = \int dx^3\sqrt{-g}\, J^0 \tag{2.20}$$

is a conserved quantity.

At the equilibrium configuration, the functions ν, λ, ϕ_1 and ϕ_2 are all time independent. In order to recover the solutions found in Refs. [5],[6],[8] and [10], we set

$\phi_2 = 0$ at the equilibrium. In this way, we have only particles and the charge is then the total number of particles N_0. We will denote all the equilibrium quantities by a subscript o. We set $\phi_{1_0} \equiv \phi_0(r)$, thus at the equilibrium $\phi_0(r,t) = \phi_0(r)e^{-i\omega t}$. It has been proven by Friedberg, Lee and Pang, see Ref. [8], that the minimum energy solution requires $\phi_0(r,t)$ to have a harmonic time dependence.

As independent equations, we can take eqs. (2.10), (2.11), (2.13) and (2.17). As mentioned above, eq. (2.16) is the scalar wave equation and, thus, equivalent to (2.17), whereas eqs. (2.13) and (2.15) vanish identically at the equilibrium since all quantities are time independent.

We are left with the following three independent equations (2.10, 2.11) and (2.17)

$$\left(re^{-\lambda_0}\right)' = 1 - 4\pi G r^2 \left[\left(m^2 + e^{-\nu_0}w^2 + \frac{\tilde{\lambda}}{2}\phi_0^2\right)\phi_0^2 + e^{-\lambda_0}\phi_0'^2\right] \tag{2.21}$$

$$\frac{e^{-\lambda_0}}{r}\nu_0' = \frac{1}{r^2}\left(1 - e^{-\lambda_0}\right) + 4\pi G\left[\left(e^{-\nu_0}w^2 - m^2 - \frac{\tilde{\lambda}}{2}\phi_0^2\right)\phi_0^2 + e^{-\lambda_0}\phi_0'^2\right] \tag{2.22}$$

$$\phi_0'' + \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{2}\right)\phi_0' - e^{\lambda_0}\left(m^2 - e^{-\nu_0}w^2 + \tilde{\lambda}\phi_0^2\right)\phi_0 = 0 \tag{2.23}$$

and equation (2.14) becomes

$$(\lambda_0' + \nu_0') = 8\pi G r\left(e^{\lambda_0 - \nu_0}\phi_0^2 w^2 + \phi_0'^2\right) \quad . \tag{2.24}$$

In order to get the mass, we use the relation

$$e^{\lambda} = \left(1 - \frac{2M(r)}{r}\right)^{-1} \quad . \tag{2.25}$$

Inserting (2.25) into eq. (2.21), we get

$$M'(r) = r^2\left[\frac{1}{2}\left(e^{-\nu_0}\frac{w^2}{m^2} + m^2\right)\phi_0^2 + \frac{\tilde{\lambda}}{4}\phi_0^4 + \frac{\phi_0'^2}{2}e^{-\lambda_0}\right] \quad . \tag{2.26}$$

The total mass is then given by the value of $M(r)$ at infinity.

For the numerical integration of the set of equations (2.21), (2.22) and (2.23), it is convenient to introduce the new variables $x = mr, \sigma = (4\pi G)^{1/2}\phi_0$, with the factor w^2/m^2 being absorbed into the definition of the metric function e^{ν_0}. Note that the coupling constant $\tilde{\lambda}$ is also replaced by $\Lambda = \frac{\tilde{\lambda} M_{Planck}^2}{4\pi m^2}$.

The appropriate initial conditions and boundary conditions are: $\sigma(0) = const$, $\sigma'(0) = 0, e^{\lambda_0(0)} = 1, e^{\nu_0(\infty)} = \frac{w^2}{m^2}$ (since we absorbed w^2/m^2 in e^{ν_0}). One finds then the solution of the system of equations, which constitute an eigenvalue problem for $e^{\nu_0(0)}$ and $e^{\nu_0(\infty)}$ with the constraint that $\sigma(x)$ has no nodes. For more details, see References [5] and [10]. In Figs. 1 and 2, we plot the total mass and particle number against $\sigma(0)$, which is directly related to the value of the central density, for the equilibrium configurations obtained for $\Lambda = 0$ (non-interacting case) and $\Lambda = 30$. Note the existence of a critical mass and a critical particle number and how their extrema overlap.

For $\Lambda \neq 0$, the critical mass is bigger, whereas the corresponding critical central density is smaller than in the $\Lambda = 0$ case [10]. The central density corresponding to the critical mass in the $\Lambda = 0$ case is $\rho_{crit} \sim 5.3 \times 10^{97} m^2\ g/cm^3$, where the boson mass m is in grams, and for the $\Lambda = 30$ case, $\rho_{crit} \sim 3 \times 10^{97} m^2\ g/cm^3$.

III. Eigenvalue Equation and Variational Principle

We consider now the situation where the equilibrium configuration is perturbed in a way such that the spherical symmetry is still preserved. These perturbations will give rise to motions in the radial direction. Taking only radial perturbations on one hand simplifies a lot the calculations, and on the other hand, one would expect that non-radial perturbations give a higher value for the limiting central density above which the star is unstable against gravitational collapse, due to the angular momentum.

The equations governing the small perturbations are obtained by expanding all functions to first order and then by linearizing the equations. We, therefore, write

$$\lambda(r,t) = \lambda_0(r) + \delta\lambda(r,t) \tag{3.1a}$$

$$\nu(r,t) = \nu_0(r) + \delta\nu(r,t) \tag{3.1b}$$

$$\phi_1(r,t) = \phi_0(r) + \delta\phi_1(r,t) \tag{3.1c}$$

$$\phi_2(r,t) = \delta\phi_2(r,t) \tag{3.1d}$$

for the various perturbed quantities.

The corresponding linearized equations for the small perturbations, derived from eq. (2.10), (2.11) and (2.13) are

$$\left(re^{-\lambda_0}\delta\lambda\right)' = -8\pi G r^2 \delta T_0^0 \tag{3.2}$$

with

$$\begin{aligned} \delta T_0^0 &= \frac{1}{2}\delta\nu e^{-\nu_0} w^2 \phi_0^2 + \frac{1}{2}\delta\lambda e^{-\lambda_0}(\phi_0')^2 \\ &- e^{-\nu_0}\left(w^2\phi_0\delta\phi_1 + w\phi_0\delta\dot{\phi}_2\right) \\ &- m^2\phi_0\delta\phi_1 - \tilde{\lambda}\phi_0^3\delta\phi_1 - e^{-\lambda_0}\phi_0'\delta\phi_1' \end{aligned} \tag{3.3}$$

$$\frac{e^{-\lambda_0}}{r}\left(\delta\nu' - \nu_0'\delta\lambda\right) = \frac{e^{-\lambda_0}}{r^2}\delta\lambda + 8\pi G\delta T_1^1 \tag{3.4}$$

with

$$\delta T_1^1 = -\delta T_0^0 - 2m^2\phi_0\delta\phi_1 - 2\tilde{\lambda}\phi_0^3\delta\phi_1 \tag{3.5}$$

and

$$\delta\dot{\lambda} = 8\pi G r \left(\phi_0'\delta\dot{\phi}_1 + w\phi_0\delta\phi_2' - w\phi_0'\delta\phi_2\right) \tag{3.6}$$

The linearized equations for $\delta\phi_1$ and $\delta\phi_2$ are obtained from eq. (2.17)

$$\begin{aligned}
\delta\phi_1'' &+ \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{2}\right)\delta\phi_1' + \frac{1}{2}\phi_0'\left(\delta\nu' - \delta\lambda'\right) \\
&+ \delta\lambda e^{\lambda_0}\left[w^2 e^{-\nu_0}\phi_0 - m^2\phi_0 - \tilde{\lambda}\phi_0^3\right] + e^{\lambda_0}\left(w^2 e^{-\nu_0} - m^2 - 3\tilde{\lambda}\phi_0^2\right)\delta\phi_1 \\
&- e^{-\lambda_0-\nu_0}\phi_0 w^2\delta\nu - e^{\lambda_0-\nu_0}\delta\ddot{\phi}_1 + 2e^{\lambda_0-\nu_0}w\delta\dot{\phi}_2 = 0
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\delta\phi_2'' &+ \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{2}\right)\delta\phi_2' + e^{\lambda_0}\left(w^2 e^{-\nu_0} - m^2 - \tilde{\lambda}\phi_0^2\right)\delta\phi_2 \\
&- e^{\lambda_0-\nu_0}\delta\ddot{\phi}_2 + \frac{1}{2}e^{\lambda_0-\nu_0}w\left[\phi_0\left(\delta\dot{\nu} - \delta\dot{\lambda}\right) - 4\delta\dot{\phi}_1\right] = 0 \ .
\end{aligned} \tag{3.8}$$

The corresponding linearized equation coming from the current conservation equation (2.18)

$$\delta J^{\mu}{}_{;\mu} = 0 \tag{3.9}$$

turns out to be identical to the above eq. (3.8) for $\delta\phi_2$, and, therefore, gives no new condition. Similarly, one can compute also δQ from eq. (2.20)

$$\begin{aligned}
\delta Q &= 4\pi w^2 \int_0^\infty dr\, r^2 e^{(\lambda_0-\nu_0)/2}\phi_0 [\, \phi_0\left(\delta\nu - \delta\lambda\right) - \\
&- 2\left(2w\delta\phi_1 + \delta\dot{\phi}_2\right)] \ .
\end{aligned} \tag{3.10}$$

Due to eq. (3.9), it follows immediately that $\delta\dot{Q} = 0$. We will see later on that the requirement $\delta Q = 0$ imposes conditions on the boundary values of $\delta\phi_2$. Notice that the linearized equation derived from eq. (2.15) vanish identically, as was the case at zero–order. This fact is independent from the particular choice of the energy–momentum tensor T^μ_ν.

It might seem that we have a set of five equations ((3.2), (3.4), (3.6), (3.7) and (3.8)) for four unknown functions $(\delta\nu, \delta\lambda, \delta\phi_1, \delta\phi_2)$. It turns out, in fact, that these equations are not all independent. More precisely one can, for instance, show that eq. (3.8) is a consequence of the other equations. To see this one can proceed as follows: from eq. (3.2), with definition (3.3) for δT^0_0, one can solve for $\delta\nu$. Then one takes the time derivative of the so obtained equation and subtracts next eq. (3.6), thus obtaining an expression for $(\delta\dot{\nu} - \delta\dot{\lambda})$, which is then inserted into eq. (3.8). After rearranging all the terms and using the zero–order scalar wave eq. (2.23) and eq. (2.24), one finds that equation (3.8) is identically satisfied.

We proceed now by showing how we can reduce the basic system of four equations ((3.2), (3.4), (3.6) and (3.7)) to a system of two coupled equations involving only the real components of the perturbation of the scalar field. Notice that in the case of a fluid star[4], the radial velocity of the perturbation is related to a single "Lagrangian displacement," thus yielding only one final equation. Here, as we are dealing with a complex scalar field, we will naturally get two coupled equations, making the problem of obtaining a variational principle more subtle.

Eq. (3.6) integrates directly in time if we write $\delta\phi_2$ as

$$\delta\phi_2(r,t) \equiv \phi_0(r)\dot{g}(r,t) \tag{3.11}$$

we get

$$\delta\lambda = 8\pi G r \left(\phi_0' \delta\phi_1 + w\phi_0^2 g'\right) \quad . \tag{3.12}$$

Instead of taking the equations (3.2) and (3.4), it turns out to be more convenient to take as independent equations (3.2) and the linear combination (3.2)-(3.4),

$$\delta\nu' - \delta\lambda' = \left(\nu_0' - \lambda_0' + \frac{2}{r}\right)\delta\lambda - 16\pi G r e^{\lambda_0}\left(m^2\phi_0 + \tilde{\lambda}\phi_0^3\right)\delta\phi_1 \tag{3.13}$$

As done before, we solve eq. (3.2) with respect to $\delta\nu$ and we insert definition (3.11) for $\delta\phi_2$ and substitute $\delta\lambda$ (as well as $\delta\lambda'$) from eq. (3.12). We also make use of the zero-order scalar wave eq. (2.23) to get

$$\begin{aligned}\delta\nu &= \delta\phi_1\left(\frac{4}{\phi_0} + 8\pi G r\phi_0'\right) + \frac{2}{w}\ddot{g} - \frac{2e^{\nu_0-\lambda_0}}{w}g'' \\ &+ g'\frac{e^{\nu_0-\lambda_0}}{w}\left(-8\pi G r\phi_0'^2 - \frac{4}{r} + 2\lambda_0' - 4\frac{\phi_0'}{\phi_0}\right)\end{aligned} \tag{3.14}$$

Next we differentiate (3.14) with respect to r and substitute the so obtained result for $\delta\nu'$ into eq. (3.13), where we also substitute $\delta\lambda$ (and $\delta\lambda'$) using (3.12). This way we find an equation for the function $g'(r,t)$, which from now on we write as $g'(r,t) \equiv f_2(r,t)$. In order to keep the notation transparent, we also introduce $\delta\phi_1(r,t) \equiv f_1(r,t)$. Thus, for f_2 we obtain the following equation

$$\begin{aligned}f_2'' &+ f_2'\left(\frac{3}{2}(\nu_0' - \lambda_0') + \frac{2}{r} + \frac{2\phi_0'}{\phi_0}\right) - e^{\lambda_0-\nu_0}\ddot{f}_2 \\ &+ f_2\Big[\frac{2(\nu_0' - \lambda_0')}{r} - \frac{2}{r^2} + \frac{(\nu_0'' - \lambda_0'')}{2} + 2\frac{\phi_0''}{\phi_0} - 2\left(\frac{\phi_0'}{\phi_0}\right)^2 \\ &+ \frac{(\nu_0' - \lambda_0')}{2}\left((\nu_0' - \lambda_0') + 4\frac{\phi_0'}{\phi_0}\right) + 4\pi G r w^2\phi_0^2 e^{\lambda_0-\nu_0}\left(\nu_0' - \lambda_0' + \frac{2}{r}\right)\Big] \\ &- f_1'\frac{2}{\phi_0}e^{\lambda_0-\nu_0}w + f_1\Big[\frac{2}{\phi_0^2}\phi_0' w e^{\lambda_0-\nu_0} - 8\pi G r e^{\lambda_0}\left(m^2\phi_0 + \tilde{\lambda}\phi_0^3\right)w e^{\lambda_0-\nu_0} \\ &+ 4\pi G r\phi_0'\left(\nu_0' - \lambda_0' + \frac{2}{r}\right)w e^{\lambda_0-\nu_0}\Big] = 0 \ .\end{aligned} \tag{3.15}$$

The analogous equation for f_1 is obtained by substituting $(\delta\nu' - \delta\lambda'), (\delta\lambda - \delta\nu)$ and $\delta\phi_2$ into eq. (3.7) by using the above equations (3.11), (3.12), (3.13) and (3.14). The resulting equation is

$$
\begin{aligned}
f_1'' &+ f_1'\left(\frac{2}{r}+\frac{1}{2}\left(\nu_0'-\lambda_0'\right)\right)+f_1\Big[e^{\lambda_0}\left(-3w^2e^{-\nu_0}-m^2-3\bar{\lambda}\phi_0^2\right)\\
&- 16\pi Gre^{\lambda_0}\phi_0'\left(m^2\phi_0+\bar{\lambda}\phi_0^3\right)+4\pi Gr\left(\phi_0'\right)^2\left(\nu_0'-\lambda_0'+\frac{2}{r}\right)\Big]\\
&- \ddot{f}_1e^{\lambda_0-\nu_0}+\dot{f}_2'2w\phi_0+\dot{f}_2\Big[4\pi Gr\phi_0'\left(\nu_0'-\lambda_0'+\frac{2}{r}\right)w\phi_0^2\\
&+ 8\pi Gre^{\lambda_0}\left(-m^2\phi_0-\bar{\lambda}\phi_0^3\right)w\phi_0^2+4w\phi_0'\\
&+ 4\frac{\phi_0 w}{r}+\left(\nu_0'-\lambda_0'\right)w\phi_0\Big]=0 \ .
\end{aligned}
\tag{3.16}
$$

Before going on with the analysis of equations (3.15) and (3.16), we will address the question of the particle number conservation. We expect the radial perturbations not to change the total particle number N_0. N_0 is given at the equilibrium configuration, using eq. (2.20), by

$$N_0 = 4\pi w\int_0^\infty dr\, r^2\phi_0^2 e^{\frac{1}{2}(\lambda_0-\nu_0)} \ . \tag{3.17}$$

After perturbing we will have $Q = N_0+\delta Q, \delta Q$ given by eq. (3.10). We can now easily compute δQ, by substituting $\delta\phi_2, \delta\nu$ and $\delta\lambda$, as given by eq. (3.11), (3.12) and (3.14), into eq. (3.10). Using the zero–order relation (2.24), we find (with the above definition for f_1 and f_2) that the integrand of δQ is a total derivative with respect of r, thus getting

$$\delta Q = 8\pi\left(f_2 r^2\phi_0^2 e^{(\nu_0-\lambda_0)/2}\right)\Big|_0^\infty \tag{3.18}$$

Therefore, the requirement $\delta Q = 0$ translates into conditions on the boundary values for f_2. For $r\to\infty$ we require $(f_2r^2\phi_0^2)\to 0$ $(e^{\lambda_0-\nu_0}\to 1)$. We know that $r^2\phi_0^2\to 0$ (for $r\to\infty$) such that N_0 (see eq. (3.17)) is finite. This imposes restrictions on the behaviour of f_2 at $r\to\infty$. For $r\to 0$, since $\phi_0(0) = const$ and $e^{\lambda_0-\nu_0}\to const$, we require $r^2f_2\to 0$ which is automatically satisfied as long as f_2 does not diverge faster than $r^{-2+\epsilon}(\epsilon > 0)$ at the origin. An inspection of eq. (3.15), inserting for f_2 the function $r^{-\alpha}$ in order to see the behaviour of the solution near the origin, shows that $\alpha > 2$ is not allowed; the limiting case being $\alpha = 2$, which has thus to be excluded by appropriately choosing the boundary condition at $r = 0$.

To further deal with the partial differential equations (3.15) and (3.16) we use the standard method of the Fourier decomposition for the time variable, all coefficients entering the equations being time independent. Thus

$$f_i(r,t) = \sum_n f_{in}(r) e^{i\sigma_n t} \quad i = 1,2 \tag{3.19}$$

where σ_n are the characteristic frequencies of the perturbations, which will determine the stability behaviour. Equations (3.15) and (3.16) constitute thus an eigenvalue problem, with eigenvalue σ_n^2 and corresponding eigenfunctions f_{1n} and f_{2n}. For notational simplicity, we will drop from now on the subscript n in σ_n^2 as well in f_{1n} and f_{2n}. In the following, we are interested only in the value of the lowest eigenvalue σ_0^2. As next we multiply equation (3.15) by the function $G_2(r) = r^2\phi_0^2 e^{3/2(\nu_0-\lambda_0)}$ and equation (3.16) by $G_1(r) = r^2 e^{1/2(\nu_0-\lambda_0)}$. This will allow to find more easily the variational principle for which eq. (3.15) and (3.16) are then the corresponding Euler-Lagrange equations. Finally we get as eigenvalue equation

$$L_{ij} f_j = \sigma^2 M_{ij} f_j \quad i,j = 1,2 \tag{3.20}$$

where

$$M_{ij} = e^{\nu_0-\lambda_0} \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \tag{3.21}$$

$$L_{ij} = \left[\begin{array}{c|c} -\frac{\partial}{\partial r}G_1\frac{\partial}{\partial r} + G_1C_1(r) & -2w\frac{\partial}{\partial r}(G_1\phi_0) + G_1C_3(r) \\ \hline 2wG_1\phi_0\frac{\partial}{\partial r} + G_1C_3(r) & -\frac{\partial}{\partial r}G_2\frac{\partial}{\partial r} + G_2C_2(r) \end{array}\right] \tag{3.22}$$

$$C_1(r) = e^{\lambda_0}\Big[\left(3w^2e^{-\nu_0} + m^2 + 3\tilde{\lambda}\phi_0^2\right) + 16\pi G e^{\lambda_0} r\phi_0'\left(m^2\phi_0 + \tilde{\lambda}\phi_0^3\right)$$
$$-4\pi Gr\,(\phi_0')^2\left(\nu_0' - \lambda_0' + \frac{2}{r}\right)\Big] \tag{3.23a}$$

$$C_2(r) = \Big[\frac{2(\lambda_0' - \nu_0')}{r} + \frac{2}{r^2} - \frac{(\nu_0'' - \lambda_0'')}{2} - 2\frac{\phi_0''}{\phi_0} + 2\left(\frac{\phi_0'}{\phi_0}\right)^2$$
$$+\frac{(\lambda_0' - \nu_0')}{2}\left[(\nu_0' - \lambda_0') + 4\frac{\phi_0'}{\phi_0}\right] - 4\pi Grw^2\phi_0^2 e^{\lambda_0 - \nu_0}\left(\nu_0' - \lambda_0' + \frac{2}{r}\right)\Big] \tag{3.23b}$$

$$C_3(r) = -2\phi_0'w + 8\pi G r e^{\lambda_0}\left(m^2\phi_0 + \tilde{\lambda}\phi_0^3\right)w\phi_0^2$$
$$-4\pi Gr\phi_0'\left(\nu_0' - \lambda_0' + \frac{2}{r}\right)w\phi_0^2 \tag{3.23c}$$

Equation (3.20) is the required "pulsation equation"[4].

The appropriate boundary conditions coming from the requirement that $\delta\phi_1$ and $\delta\phi_2 \to 0$ for $r \to \infty$ and $\delta\phi_1 = const, \delta\phi_2 = const$ for $r \to 0$, along with $\delta Q = 0$ are

$$r \to \infty \;:\; f_1 \to 0, \quad r^2\phi_0^2 f_2 \to 0$$
$$r \to 0 \;:\; f_1 = const, \quad r^2 f_2 \to 0\,. \tag{3.24}$$

The operators L_{ij} as well as M_{ij} are both symmetric given the above boundary conditions (3.24), which means that they satisfy

$$\int_0^\infty dr\ \eta_i L_{ij} \xi_j = \int_0^\infty dr\ \xi_i L_{ij} \eta_j \tag{3.25}$$

and similarly for M_j. In order to show eq. (3.25), one uses partial integrations. The vanishing of the boundary terms is related to the fact that we require $\delta Q = 0$. Since L_{ij} and M_{ij} are symmetric and also real, it follows that the eigenvalues σ_n^2 are real. Dynamical instability will occur whenever $\sigma_0^2 \leq 0$.

From the form of the equations, as given in (3.20), with definitions (3.21)-(3.23), one can find the following variational principle

$$\begin{aligned}
&\sigma^2 \int_0^\infty \frac{1}{2} e^{(\lambda_0 - \nu_0)} \left(G_1 f_1^2 + G_2 f_2^2 \right) dr = \\
&\int_0^\infty \left[\frac{1}{2} G_1 f_1'^2 + \frac{1}{2} G_1 f_1^2 C_1(r) + \frac{1}{2} G_2 f_2'^2 + \frac{1}{2} G_2 f_2^2 C_2(r) + \right. \\
&\left. f_2 f_1' \left(2\phi_0 w G_1 \right) + f_2 f_1 G_1 C_3(r) \right] dr
\end{aligned} \tag{3.26}$$

Eigenfunctions related to different eigenvalues satisfy the following orthogonality relation

$$\int_0^\infty \frac{1}{2} e^{\lambda_0 - \nu_0} \left(G_1 f_{1i} f_{1j} + G_2 f_{2i} f_{2j} \right) dr = \delta_{ij} \tag{3.27}$$

Equation (3.26) is a minimal (and not just an extremal) principle, in fact, the integral on the right hand side of eq. (3.26) is bounded from below. From this, it is clear that a sufficient condition for dynamical instability to occur is that the right hand side of eq. (3.26) vanishes (or becomes negative) for some chosen pair of trial functions f_1, f_2, which satisfy the appropriate boundary conditions (3.24).

In the next section, we will apply these results to the particular cases $\Lambda = 0$ (non-interacting) and $\Lambda = 30$ ($\Lambda = \tilde{\lambda} M_{Planck}^2 / 4\pi m^2$) for the quartic self-interacting case. The generalization of the above results to other potentials is straightforward.

IV. Upper Bounds for Dynamical Instability

As a first step we solve numerically the equilibrium equations, in order to determine all coefficients, which enter in eq. (3.26). Each equilibrium configuration is parameterized by a given value of $\phi_0(0)$ or the corresponding value of the central density, which is used an an initial condition for the integration of equations (2.21)-(2.23). Once we have the equilibrium solutions, we can evaluate numerically the integrals in (3.26) for a given pair of trial functions, obeying the boundary conditions (3.24) with different values of the central density. Dynamical instability will be present, whenever the right-hand side of eq. (3.26) is ≤ 0. The integral on the left-hand side of eq. (3.26) is always positive and is only a normalization condition.

The right hand-side of eq. (3.26) (call it I) can be written as a sum of three integrals, which we shall call I_1, for the piece containing f_1 alone, I_2, for the piece containing f_2 alone, and I_3, for the piece containing the mixed terms $f_1 f_2$ and $f_2 f_1'$. In defining the trial functions f_1 and f_2, we are free to multiply f_1 by a constant a_1 and f_2 by another constant a_2(a_1 and a_2 being real). Only the ratio, for instance $x = a_2/a_1$, will be relevant, therefore I will be a function of x

$$I = I_1 + x^2 I_2 + x I_3 \tag{4.1}$$

Whenever $I \leq 0$ also $\sigma^2 \leq 0$. The condition that $I \leq 0$ is given by

$$I_3^2 - 4 I_1 I_2 \geq 0 \tag{4.2}$$

Thus, for a given pair of trial functions, we will compute the integrals I_1, I_2 and I_3 and verify if the condition (4.2) is satisfied.

Since we parameterize the equilibrium configurations by the value of the central density, what we will find is an upper bound for the central density from which on dynamical instability will occur.

We tried for both cases we considered, namely $\Lambda = 0$ and $\Lambda = 30$, with different pairs of trial functions. We found, so far, that the lowest bound are obtained by using as trial functions

$$f_1 = f_2 = \phi_0' \tag{4.3}$$

ϕ_0' being the derivative of the equilibrium solution $\phi_0(r)$ for a given central density. We found in the case $\Lambda = 0$, that for all values of $\sigma(0) \geq 1.09$ ($\sigma(0) = (4\pi G)^{1/2}\phi_0(0)$) the condition (4.2) is satisfied (we checked this till $\sigma(0) = 2$). Therefore, an upper bound for the occurrence of dynamical instability in the case $\Lambda = 0$ is given by $\sigma(0) = 1.09$ corresponding to a central density $\rho = 2.1 \times 10^{98} m^2\ g/cm^3$($m$ is the boson mass in grams).

Similarly we analyzed the interacting-case, where we took as an example $\Lambda = 30$ ($\Lambda = \tilde{\lambda} M_{Planck}^2/4\pi m^2$, corresponding to $\tilde{\lambda} = 3,8 \times 10^{12} m^2, m$ boson mass in g). We found that for $\sigma(0) \geq 0.69$, corresponding to a central density $\rho = 1.3 \times 10^{98} m^2\ g/cm^3$, the condition (4.2) is satisfied. We notice that in the interacting case the critical central density corresponding to the maximal mass is smaller than in the non-interacting case[10].

In the range of central densities, we were looking at, we found numerically that the function $C_1(r)$ (eq. (3.23a)) is always positive for all values of r, giving thus a positive value for I_1. On the other hand for the function $C_2(r)$ (eq. (3.23b)) there are ranges of r, where it is negative. In particular in the case $\Lambda = 0$, for $\sigma(0) > 0.8$ $C_2(r)$ is negative in the region nearby the origin ($r = 0$). The trial function ϕ_0' has its maximum value also near the origin (at $r = 0$ $\phi_0'(0) = 0$) and afterwards decays exponentially. This leads, using ϕ_0' as trial function, to a negative value for I_2 for $\sigma(0) > 1.2$, making thus the condition (4.2) automatically satisfied. We have marked with an arrow in figures 1 and 2 the upper bounds we found for the two cases respectively. It is immediately apparent that the upper bounds, we found, are above the critical central density, corresponding to the maximal mass, the point one would conjecture to be the boundary between stable and unstable configurations.

V. Conclusions

We have addressed, in the framework of general relativity, the problem of the dynamical instability against small perturbations for bosonic stellar configurations made up from massive complex scalar fields, with or without a quartic self-interaction.

Starting from the coupled system of scalar wave and Einstein's field equations we derived, following Chandrasekhar's method[4], an eigenvalue equation (eq. (3.20)), describing the perturbations, and subsequently found a variational principle (eq. (3.26)) for the characteristic eigenfrequencies of the radial oscillations.

We then found numerically an upper bound for the central density above which the bosonic stellar configuration becomes dynamically unstable. In both cases, we studied numerically, the bounds are above the critical densities corresponding to the maximum masses. Our limits are also above the point where the binding energy becomes positive. It is well possible that with more sophisticated trial functions one could find lower bounds, compared to ours.

A weak point of the variational method is that we cannot easily estimate how far our bounds are from the actual values. Our results can thus not be conclusive about the conjecture whether the border between stability and instability is given by the central density corresponding to the maximum mass.

On the other hand, given the complexity of the problem, the fact that we could derive a variational principle for the eigenfrequencies of the radial oscillations starting from the "microscopic" scalar wave equation, rather than using an effective equation of state, is certainly already a step forward towards a complete understanding of the problem of the stability.

As a next step, it would be interesting to apply the method we developed to configurations where the radial scalar field function has nodes, corresponding to excited states, as well as to the soliton-star case.

The study of the bosonic stellar configurations, from their formation and subsequent evolution and stability, may be of relevance in order to understand the role scalar fields may have played in the early universe or may still play as candidates for the missing mass. These facts, together with the wide range of masses boson stars may have, according to different models, suggest the richness of the phenomena involved.

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Figure Captions

Fig. 1: Boson star mass in units of M^2_{Planck}/m (continuous line) and particle number n units of M^2_{Planck}/m^2 (dotted line) as a function of $\sigma_c(= (4\pi G)^{1/2}\phi_0(0))$ for $\Lambda = 0$. The arrow indicates the upper bound for the occurrence of dynamical instability.

Fig. 2: Boson star mass in units of M^2_{Planck}/m (continuous line) and particle number in units of M^2_{Planck}/m^2 (dotted line) as a function of σ_c for $\Lambda = 30$. The arrow indicates the upper bound for the occurrence of dynamical instability.

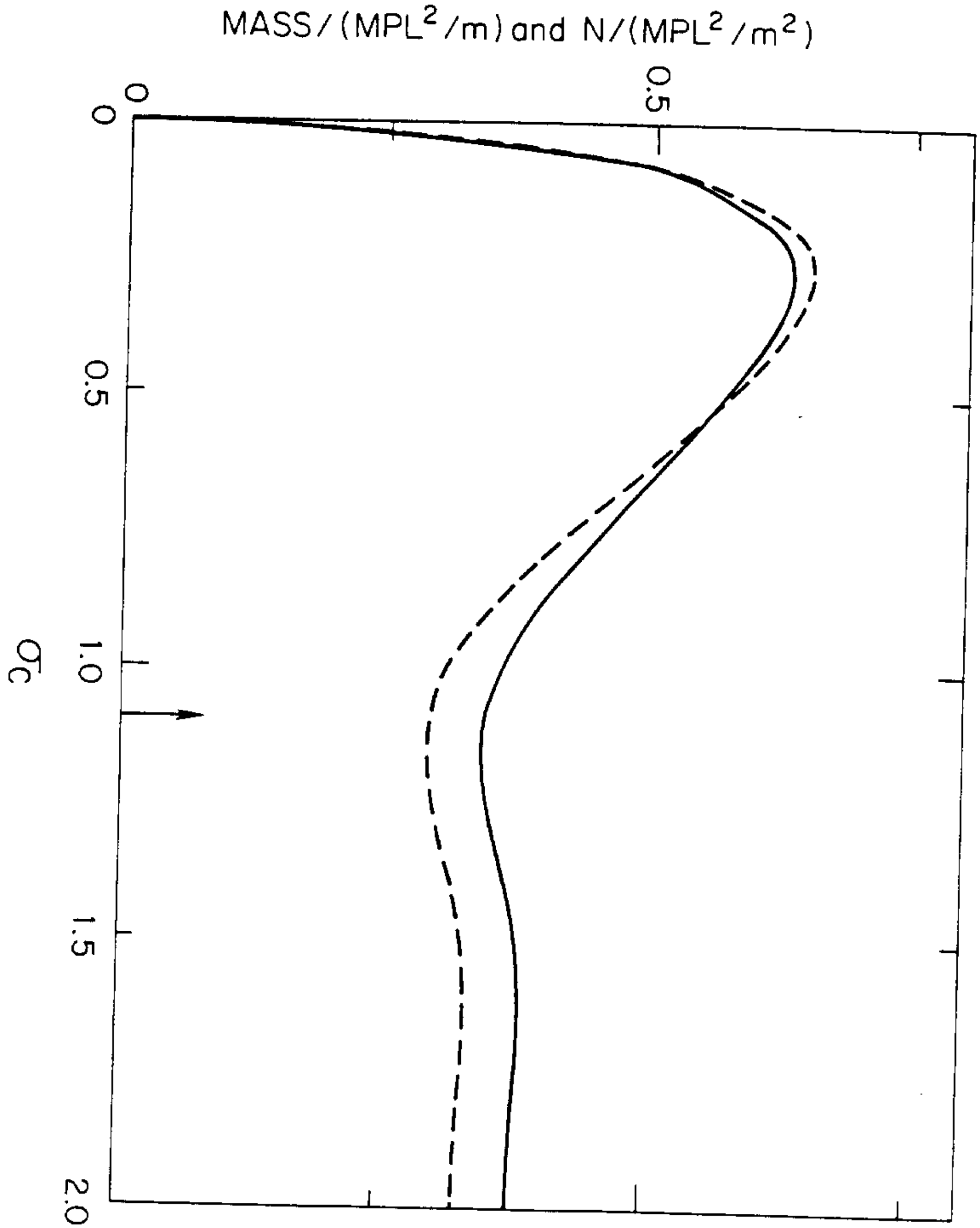

FIGURE 1

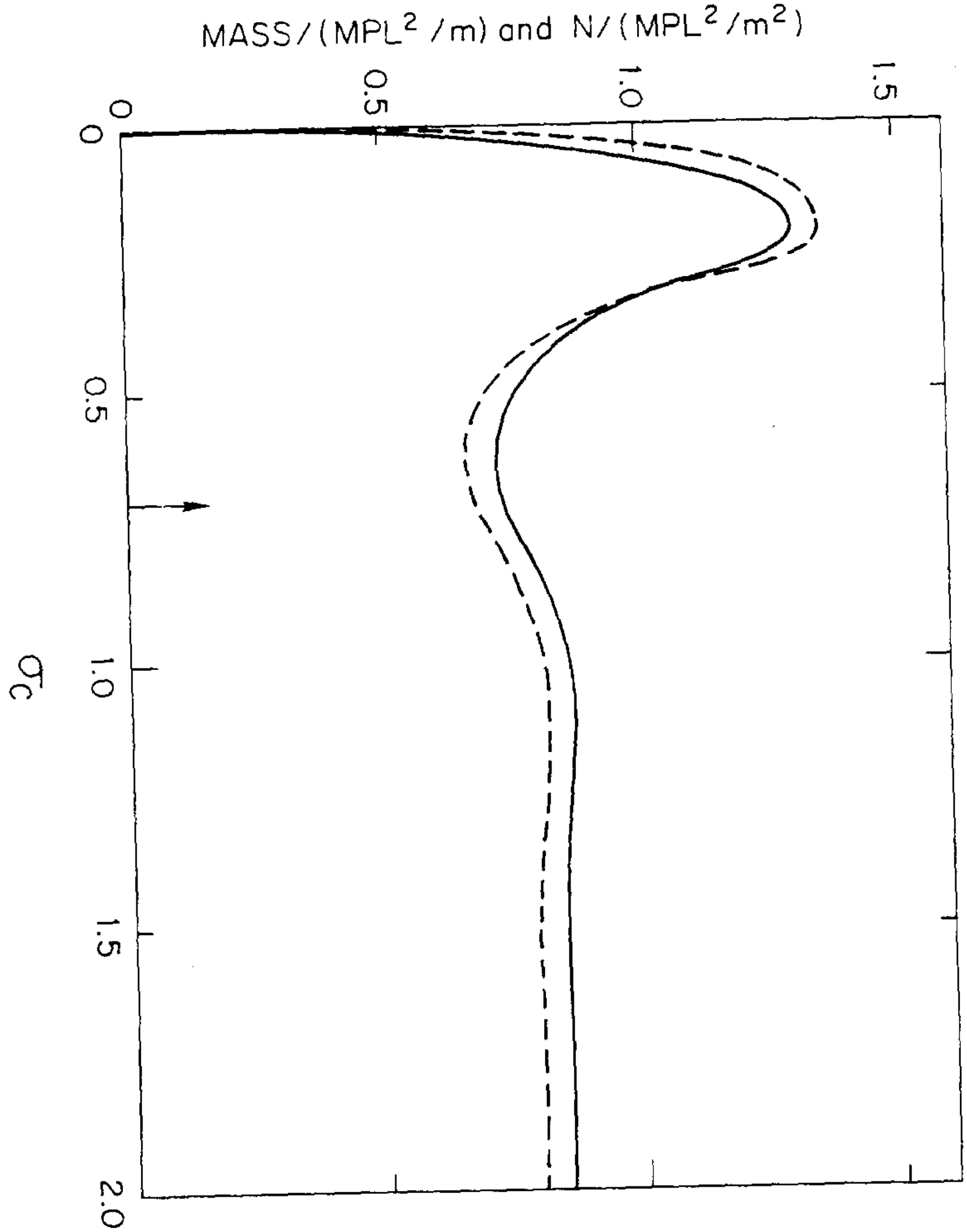

FIGURE 2