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Non-topological cosmic strings

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Abstract

The existence of a class of non-topological "string" soliton solutions is demonstrated. The string consists of a long, thin region of false vacuum supported against collapse by the pressure of massless particles trapped in its interior. It is shown that the string configuration is stable against decay into free particles. The stability of such "strings" to arbitrary small perturbations is analyzed and the solutions are shown to be unstable to the migration of charge along the string, leading to a conversion of the non-topological string into spherical non-topological solitons.

I. INTRODUCTION

Recently there has been a great deal of interest in the possible role of topological solitons (e.g., the monopole solutions of t'Hooft[1] and Polyakov[2] and the string solutions of Nielsen and Olesen[3]) in cosmology. However, solitons can also be non-topological in nature. Friedberg, Lee, and Sirlin[4] [FLS] demonstrated that such a class of complex scalar field soliton solutions in 3 spatial dimensions existed, and for a large enough charge were in fact stable, both classically and quantum mechanically, to arbitrary small perturbations in the fields. These spherical solutions could be thought to correspond to the monopoles of the topological case. There have recently been attempts to study the production of such object in cosmological phase transitions.[5]

In this paper we investigate the possible existence of non-topological string solutions, analogous to the topologically inspired vortex solutions.[3] We demonstrate the existence of such solutions. As in FLS, the necessary conditions for having such solutions are: (1) the conservation of an additive quantum number Q, carried by some complex field ϕ (in our discussion a scalar field), (2) the presence of a neutral ($Q = 0$) scalar field, σ, that acquires a non-zero vacuum expectation value in the classical ground state, and (3) the mass of the ϕ field depends upon the vacuum expectation value of σ.

FLS demonstrated the existence and stability of the finite, spherically-symmetric solutions. For $Q > Q_C$, where Q_C is some critical value for the charge, soliton solutions where $\sigma = 0$ (a local *maximum* of the classical potential) in some localized region of space containing the charge Q will have a lower energy than the mass of Q free massive ϕ's with σ equal to the global *minimum* of the classical potential. In this paper we study infinite, cylindrically-symmetric solutions. We find soliton configurations that have a lower energy than the free particle solutions, hence stable against decay into free particles. However when we allow the charge to migrate along the string by perturbing the scalar field solutions, we find that for long wavelength perturbations, the effect is to cause the string "tension" to vary along its length, causing the string to pinch in regions of lower string tension. Thus the strings appear unstable to forming spheres. If we introduce a current along the string or give it some angular momentum, the effect is to mitigate, but not remove, the instability.

We will initially review FLS for we rely heavily on their analysis in this paper. Assume the system to exist of two scalar fields, ϕ is complex and σ is hermitian. The Lagrangian density must have non-linear couplings in the fields for the existence of non-topological solutions. We assume

$$\mathcal{L}[\phi,\ \sigma] = |\partial_\mu \phi|^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - f^2\sigma^2|\phi|^2 - \frac{g^2}{8}(\sigma^2 - \sigma_0^2)^2 \tag{1}$$

where f and g are dimensionless coupling constants, and σ_0 has dimensions of energy and sets the scale of spontaneous symmetry breaking. We are using a metric signature $(+,\ -,\ -,\ -)$. The theory possesses a discrete symmetry

$$\sigma \rightarrow -\sigma$$

and the $U(1)$ symmetry

$$\phi \rightarrow \exp(-i\theta)\phi.$$

Hence, there exists a conserved current $\partial j_\mu/\partial x_\mu = 0$, where $j_\mu = i(\phi^* \overleftrightarrow{\partial_\mu}\ \phi)$, and the corresponding charge

$$Q \equiv i\int d^3r\, j_0 \tag{2}$$

is the constant of the motion. Now, since Q is a linear function of $\dot{\phi}$, the classical solution for $Q \neq 0$ must be time dependent, and for the lowest energy state $\phi \propto \exp(-i\omega t)$. Scaling away the physical dimension and the coupling g, we introduce field variables A and B defined by

$$\sigma(\mathbf{r}, t) \equiv \left(\frac{\mu}{g}\right) A(\rho) \tag{3}$$

$$\phi(\mathbf{r}, t) \equiv \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\mu}{g}\right) B(\rho)\exp(-i\omega t) \tag{4}$$

where A is real, B can be complex, $\rho \equiv \mu \mathbf{r}$, $\mu \equiv g\sigma_0$, and $m \equiv f\sigma_0$ is the mass of the neutral σ at the minimum of the classical potential.

The equations of motion for A and B are

$$\nabla_i\nabla^i A - \kappa^2 B^2 A - \frac{1}{2}A(A^2 - 1) = 0 \tag{5}$$

$$\nabla_i\nabla^i B + \kappa^2 A^2 B - \nu^2 B = 0 \tag{6}$$

where $\kappa \equiv m/\mu$, $\nu \equiv \omega/\mu$, and ∇_i is the gradient with respect to the dimensionless parameter ρ. The charge Q is found from Eq. (2):

$$Q = \frac{\nu}{g^2} \int d^3\rho \, B^2 \tag{7}$$

and the energy of the system is:

$$E = \left(\frac{\mu}{g^2}\right) \int d^3\rho \, \varepsilon$$

where

$$\varepsilon = \frac{1}{2}(\nabla_i A)^2 + \frac{1}{2}(\nabla_i B)^2 + \frac{1}{2}(\nu^2 + K^2 A^2) B^2 + \frac{1}{8}(A^2 - 1)^2. \tag{8}$$

Note how ν is to be interpreted as a function of Q and a functional of $B(\rho)$.

By using a variational approach FLS show the existence of the soliton solution. The free particle solution for Q charges, each of mass m, is $E = Qm$, where $\omega = m$ and the corresponding infinite volume limits on the fields are $A = 1$, $B = 0$. Bearing in mind these boundary conditions on A and B, FLS assume, using spherically-symmetric trial functions

$$A = \begin{cases} 0 & r \leq R \\ 1 - \exp(-(r - R)/L) & r \geq R \end{cases} \tag{9}$$

$$B = \begin{cases} (B_0/r) \ \sin(\omega r) & r \leq R \\ 0 & r \geq R \end{cases} \tag{10}$$

where $r = |\mathbf{r}|$, R and L are two length parameters. R is the radius of the soliton, given by the first zero of B: $B(R) = 0$ implies $R = \pi/\omega$. L is the thickness of the domain wall separating the $\sigma = 0$ interior region of false vacuum from the $\sigma = \sigma_0$ exterior region of true vacuum. The thickness of the wall is expected to be $L = O(1/g\sigma_0)$. B_0 can be related to the charge Q through Eq. (7): $B_0 = (\pi\mu^{-1}) g (Q/2)^{1/2}$. The trial functions satisfy the equations of motion when $r \leq R$ and approach the correct boundary condition at infinity, although the derivatives don't match at $r = R$.

For a given Q, FLS obtain the energy corresponding to the trial functions in Eqs. (9) and (10)

$$E = \frac{\pi Q}{R} + \frac{\pi\mu^4}{6g^2} R^3 + \frac{\pi\mu^4}{6g^2} \left[\frac{11}{4} R^2 L + \frac{84}{24} R L^2 + \frac{635}{288} L^3\right]$$

$$+\frac{\pi\mu^2}{g^2 L}\left[R^2 + RL + \frac{1}{2}L^2\right].$$

The first term in the energy is the kinetic energy of Q massless ϕ particles confined to a region of size R. The second term is the false-vacuum potential energy of the σ field in the interior region of the soliton. The other terms depending on L are contributions from the wall separating the two regions.

Assuming Q is large, surface terms depending on L can be neglected and the trial functions result in an energy $E = \pi Q/R + \pi\mu^4 R^3/6g^2$. Minimizing the energy with respect to R gives $R_{min} \simeq (2Qg^2)^{1/4}/\mu$, and the trial functions result in a 'ground state' energy $E_{trial} = E(R_{min})$. Since the true energy must be less than the energy found with the trial functions

$$E_{min} \leq E(R_{min}) \simeq \frac{4\pi}{3}\mu(2g^2)^{-1/4}Q^{3/4}. \tag{11}$$

This is less than the plane wave solution ($E = Qm$) when $Q > Q_S \sim (4\pi\mu/3m)^4/2g^2$. Hence, when $Q > Q_S$ the soliton solution exists and is absolutely stable. The stability of the solitons is demonstrated in an extremely elegant proof. We will simply sketch the idea here, presenting a more complete analysis in Section 3. Under arbitrary variations δA and δB from the solutions for A and B of Eqs. (5) and (6), the first order variation in E is zero, as A and B are solutions to the equations of motion. The sign of the second order variation in E for fixed Q determines whether the solutions are a local maxima or minima in the energy. If $(\delta^2 E)_Q > 0$, the perturbation leads to a higher energy state and the system will be perturbatively stable against such a perturbation. If $(\delta^2 E)_Q < 0$ however, the perturbation will lead to a new configuration of lower energy, so the original solution is unstable. FLS were able to demonstrate that any perturbation in A and B always lead to a higher energy state. The crucial reason for this was that although the perturbed soliton solution itself possessed negative eigenvalues when acted on by the Hamiltonian (which normally results in a growing unstable mode), this was always cancelled by a positive definite eigenvalue term arising from the condition of charge conservation. Not allowing charge to leave the spherically symmetric soliton enforces it to remain in its lowest energy configuration. We will see in Section 3 that when the charge is allowed to migrate along the string, this second term vanishes, leaving us with a growing mode in the perturbation.

The application of the above techniques to "string" configurations is given below. In Section 2, trial functions are obtained to demonstrate the possible existence of string solutions. In Section 3 we present a stability analysis of these configurations and discuss the effect angular momentum and current have on the string's stability. Finally in the conclusions we discuss some of the implications of our results for the formation of such objects in the early Universe and address the issue of the formation of non-topological solitons in general.

II. NON-TOPOLOGICAL SOLITON STRINGS

The possible existence of string configurations can be seen by assuming cylindrically symmetric solutions in Eqs. (5) and (6). Again, using a variational approach we have trial solutions with cylindrical symmetry (here $r = \sqrt{x^2 + y^2}$)

$$A = \begin{cases} 0 & r \leq R \\ 1 - e^{-(r-R)/L} & r \geq R \end{cases} \tag{12}$$

$$B = \begin{cases} B(r,\theta,z) & r \leq R \\ 0 & r \geq R. \end{cases} \tag{13}$$

The solution for $B(r,\theta,z)$ is obtained by demanding that B remains finite as $\rho \to 0$. Then we obtain for a string with angular momentum n, carrying current $j_z = i(\phi^* \overleftrightarrow{\partial_z} \phi) = \mu^2 k/g^2$ along the z-axis, the trial function

$$B(\rho,\theta,z) = B_0 J_n(\gamma\rho)\exp(in\theta)\exp(ikz) \tag{14}$$

where $\gamma^2 = \nu^2 - k^2/\mu^2 > 0$.

$J_n(\gamma\rho)$ is the nth order Bessel function. The radius of the cylinder, ρ_0, is determined by the first zero of the Bessel function, denoted as α_n:

$$J_n(\alpha_n) = 0 \Longrightarrow \gamma\rho_0 = \alpha_n. \tag{15}$$

Note that as n increases the size of the soliton solution increases (see Fig. 1).[6]

Now we can substitute these trial solutions into Eqs. (7) and (8) to obtain the charge and energy of such configurations. However as we are using an infinite string, we must evaluate quantities like the charge per unit length, (dQ/dz), the energy per unit length, (dE/dz), etc., and compare these with the corresponding quantities for the free particle solution. Later we shall be investigating closed loops of string where the radius of curvature is large compared with the string width. In this regime the infinite string solution is a good approximation to the loop and we can deal in terms of total energy.

For simplicity we initially investigate the case of no angular or z dependence on B. We then write

$$B(\rho) = B_0 J_0(\nu\rho). \tag{16}$$

Now the trial solutions in Eqs. (12) and (13) match at the boundary $\alpha_0 = \nu\rho_0 = 2.405$ - the first zero of $J_0(\nu\rho)$ - although the derivatives are discontinuous at the boundary.

Substituting Eq. (16) into Eq. (7) we can write B_0 in terms of Q, for a given charge Q, in dimensionless scale η and ρ ($\eta \equiv z\mu$; $\rho \equiv r\mu$)

$$\frac{dQ}{d\eta} = \frac{2\pi\nu}{g^2}\int_0^{\rho_0} \rho d\rho B^2 = \frac{\pi}{g^2\nu} B_0^2 \alpha_0^2 J_1^2(\alpha_0). \tag{17}$$

Using the trial functions of Eqs. (12) and (13), the energy per unit length of the string is

$$\begin{aligned}\left(\frac{dE}{d\eta}\right)_{trial} &= \left(\frac{dQ}{d\eta}\right)\frac{\alpha_0}{2R}\left[1 + \frac{J_2^2(\alpha_0)}{J_1^2(\alpha_0)}\right] \\ &\quad + \frac{\pi\mu^3}{8g^2}\left[R^2 + \frac{2}{\mu^2} + \frac{4R}{L\mu^2} + \frac{11}{6}LR + \frac{89}{72}L^2\right].\end{aligned} \tag{18}$$

The larger the charge Q in the soliton, the larger the radius R, so for $R \gg \mu^{-1} \simeq L$ we can ignore the surface terms in Eq. (18) and write an upper bound for the energy density:

$$\left(\frac{dE}{d\eta}\right)_{true} \leq \left(\frac{dE}{d\eta}\right)_{trial} \simeq \left(\frac{dQ}{d\eta}\right)\frac{\alpha_0}{2R}\left[1 + \frac{J_2^2(\alpha_0)}{J_1^2(\alpha_0)}\right] + \frac{\pi\mu^3}{8g^2}R^2. \tag{19}$$

Minimizing Eq. (19) as a function of R, the minimum energy configuration is achieved when

$$R_{min} = \left[\left(\frac{dQ}{d\eta}\right)\frac{2\alpha_0 g^2}{\pi\mu^3}\left(1 + \frac{J_2^2(\alpha_0)}{J_1^2(\alpha_0)}\right)\right]^{1/3} \tag{20}$$

which when substituted into Eq. (19) gives (again, ignoring the surface energy terms depending on L)

$$\begin{aligned}\left(\frac{dE}{d\eta}\right)_{trial} &\simeq \mu\frac{3}{2}\left\{\left(\frac{\pi}{16g^2}\right)^{1/2}\alpha_0\left[1+\left(\frac{J_2^2(\alpha_0)}{J_1^2(\alpha_0)}\right)\right]\left(\frac{dQ}{d\eta}\right)\right\}^{2/3} \\ &\equiv \mu K\left(\frac{dQ}{d\eta}\right)^{2/3}. \end{aligned} \tag{21}$$

Note that the effective string tension $(dE/d\eta)$ now depends on the string charge density $(dQ/d\eta)$, so in principle it can vary along the length of the string. This is unlike the case of usual cosmic string solutions. Eq. (21) should be compared with the energy density of the plane wave solution for Q free particles

$$\left(\frac{dE}{d\eta}\right)_{free} = m\frac{dQ}{d\eta} \tag{22}$$

where $m = f\mu/g$ from Eqs. (3) and (4). The string-like soliton solution is stable against decay into free particles when it is formed with a lower energy density than the free-particle distribution. Comparing Eqs. (21) and (22), this occurs when

$$\frac{dQ}{d\eta} \geq \frac{27}{128}\frac{g\pi}{f^3}\alpha_0^2\left[1+\frac{J_2^2(\alpha_0)}{J_1^2(\alpha_0)}\right]^2 = \left(\frac{dQ}{d\eta}\right)_C. \tag{23}$$

Thus if the charge density at formation is larger than $(dQ/d\eta)_C$, then it is energetically favorable to form soliton-strings.

It is possible to obtain constraints on the coupling constants which allow string solutions. From Eq. (15) the radius of the cylinder is $\rho_0 = \alpha_0/\nu$, and from Eq. (17) we have $dQ/d\eta$ in terms of B_0^2, g^2 and ν. Substituting Eq. (17) into Eq. (20) and equating R_{min} to ρ_0 we obtain

$$B_0 = \left[2\nu^2(J_1^2(\alpha_0)+J_2^2(\alpha_0))\right]^{-1/2}. \tag{24}$$

This is the value of B_0 when the string is in its minimum energy configuration for a given frequency ν.

Substituting back into Eq. (17) for B_0^2 and into Eq. (23) for $dQ/d\eta$ we obtain the condition:

$$\frac{\nu}{K} \leq \frac{2^{1/3}\cdot 4}{3}\left(1+\frac{J_2^2(\alpha_0)}{J_1^2(\alpha_0)}\right)^{-1}. \tag{25}$$

Equation (14) allows for both a θ and z dependence to the string solution. The effect of giving the string some angular momentum (n is of course an integer in Eq. (14) in order that ϕ is single valued) is to increase the radius of the cylinder for a given charge (see Fig. 1). This is just because the radius is proportional to the value of the first zero of the Bessel function $J_n(\alpha_n)$, and this increases with n. The physical picture follows basically that of $n = 0$. The charge density is modified from Eq. (17) to

$$\frac{dQ}{d\eta} = \frac{\pi}{g^2\nu} B_0^2 \alpha_n^2 J_{n+1}^2(\alpha_n) \tag{26}$$

where α_n is the first zero of the nth Bessel function. The energy density of Eq. (18) becomes (again, ignoring the surface contribution)

$$\frac{dE}{d\eta} = \left(\frac{dQ}{d\eta}\right) \frac{\alpha_n}{2R} \left[1 + \frac{J_{n+2}^2(\alpha_n)}{J_{n+1}^2(\alpha_n)}\right] \tag{27}$$

due to the presence of the θ dependence in B the condition for soliton strings becomes

$$\frac{dQ}{d\eta} \geq \frac{27}{128} \frac{g\pi}{f^3} \alpha_n^2 \left[1 + \frac{J_{n+2}^2(\alpha_n)}{J_{n+1}^2(\alpha_n)}\right] \tag{28}$$

with

$$R_{min} = \left[\left(\frac{dQ}{d\eta}\right) \frac{2\alpha_n g^2}{\pi\mu^3} \left(1 + \frac{J_{n+2}^2(\alpha_n)}{J_{n+1}^2(\alpha_n)}\right)\right]^{1/3}. \tag{29}$$

Including a z-dependence in the solution, i.e. giving it a current, j_3, is in fact equivalent to Lorentz boosting the z-independent string solutions to a velocity (k/ω). Hence the physics of the non-soliton strings with current can be related to the z-independent solutions. This fact is easy to see. Under a Lorentz transformation along the z-axis we have the relation between the original z-independent string in a frame (z', t') and the Lorentz boosted coordinate frame (z, t)

$$\begin{aligned} z' &= \gamma(z - vt) \\ t' &= \gamma(t - vz) \end{aligned} \tag{30}$$

where $\gamma = (1 - v^2)^{-1/2}$, $v = dz/dt$. The energies in the two frames are also related by

$$E' = \gamma(E - vp_z) \tag{31}$$

where p_z is the particle momentum in the z-direction. Equivalence of z-dependent and z-independent solutions is obtained by taking $v = k/\omega$, in Eq. (30). Taking E as in Eq. (8) with the appropriate k dependence, we see from Eq. (31) that this corresponds to an inertial frame energy E' which corresponds to Eq. (8) without the k dependence. In the rest frame there is no current and so no z-dependence in Eq. (14). Thus we can always Lorentz boost our k-dependent solution to an inertial frame where there is no such dependence, and use our results of Eqs. (27) - (29).

A useful exercise is to evaluate the stress-tensor $T_{\mu\nu}$ for the string solutions. Using $S = \int \sqrt{-g} d^4x \mathcal{L}$ where $\mathcal{L}$ is given by Eq. (1), we have

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \\ &= 2(\partial_\mu \phi)(\partial_\nu \phi^*) + (\partial_\mu \sigma)(\partial_\nu \sigma) - g_{\mu\nu} \Big[|\partial_\alpha \phi|^2 + \frac{1}{2}(\partial_\alpha \sigma)^2 - f^2 \sigma^2 |\phi|^2 \\ &\quad - \frac{g^2}{8}(\sigma^2 - \sigma_0^2)^2 \Big] . \end{aligned} \tag{32}$$

Assuming the trial functions in Eqs. (12) and (13) we obtain for the region inside the string:

$$\begin{aligned} \rho = T_{00} &= (\partial_0 \phi)(\partial_0 \phi^*) - (\partial_r \phi)(\partial_r \phi^*) - \frac{1}{r^2}(\partial_\theta \phi)(\partial_\theta \phi^*) \\ &\quad -(\partial_z \phi)(\partial_z \phi^*) + (g^2/8)\sigma_0^4 \\ p_r = T_{rr} &= (\partial_0 \phi)(\partial_0 \phi^*) + (\partial_r \phi)(\partial_r \phi^*) - \frac{1}{r^2}(\partial_\theta \phi)(\partial_\theta \phi^*) \\ &\quad -(\partial_z \phi)(\partial_z \phi^*) - (g^2/8)\sigma_0^4 \\ p_\theta = T_{\theta\theta} &= (\partial_0 \phi)(\partial_0 \phi^*) - (\partial_r \phi)(\partial_r \phi^*) + \frac{1}{r^2}(\partial_\theta \phi)(\partial_\theta \phi^*) \\ &\quad -(\partial_z \phi)(\partial_z \phi^*) - (g^2/8)\sigma_0^4 \\ p_z = T_{zz} &= (\partial_0 \phi)(\partial_0 \phi^*) - (\partial_r \phi)(\partial_r \phi^*) - \frac{1}{r^2}(\partial_\theta \phi)(\partial_\theta \phi^*) \\ &\quad +(\partial_z \phi)(\partial_z \phi^*) - (g^2/8)\sigma_0^4. \end{aligned} \tag{33}$$

In the case where $\phi(r,t)$ only these simplify to

$$\begin{aligned} \rho &= |\partial_0 \phi|^2 - (\partial_r \phi)(\partial_r \phi^*) + g^2/8 \ \sigma_0^4 \\ p_r &= |\partial_0 \phi|^2 + (\partial_r \phi)(\partial_r \phi^*) - g^2/8 \ \sigma_0^4 \\ p_\theta &= p_z = |\partial_0 \phi|^2 - (\partial_r \phi)(\partial_r \phi^*) - g^2/8 \ \sigma_0^4. \end{aligned} \tag{34}$$

Note that because of the radial and time dependence resulting from the charge of the ϕ, the stress-energy tensor does not have the familiar cosmic string form.

The analysis so far has concentrated on the non-topological string structures. As FLS demonstrated, spherically symmetric solutions also exist. For a given charge Q how does the energy of the spherical object of size R compare with that string loop of length P?

Assuming an even distribution of charge along the loop we write

$$\frac{dQ}{dz} = \frac{Q}{P}.$$

Now from Eq. (21), for a loop with $\mu L \gg 1$,

$$\frac{dE}{d\eta} \propto \frac{\mu}{g^{2/3}} \left(\frac{dQ}{d\eta}\right)^{2/3}$$

so using $\eta = \mu z$ we have upon integrating along the loop

$$E_{loop} \propto \frac{(\mu P)^{1/3} Q^{2/3}}{g^{2/3}} \mu. \tag{35}$$

Similarly for the spherical solution of charge Q, the energy is (neglecting surface terms) [FLS]:

$$E_{sphere} \propto \frac{\mu Q^{3/4}}{g^{1/2}} \tag{36}$$

where R, the radius of the sphere, is

$$R = \frac{(Qg^2)^{1/4}}{\mu}.$$

Hence we obtain the ratio

$$\frac{E_{loop}}{E_{sphere}} \propto \left(\frac{P}{R}\right)^{1/3}. \tag{37}$$

This is only valid for large P, i.e. $P \gg R$, so as expected for a given charge Q, it is energetically more favorable for the charge to be distributed in a sphere rather than in a string configuration. This doesn't necessarily imply that a string configuration as described in this section will decay into spherical objects. We must investigate the stability of these solutions to arbitrarily small perturbations in the fields. This is what we do in the following section.

III. STABILITY OF STRING SOLUTIONS

In an extremely elegant proof FLS were able to demonstrate the classical stability of their spherical soliton configurations to arbitrary perturbations in the fields. We will follow their arguments here, showing where necessary how it changes when string configurations are allowed.

From Eq. (1) the energy of the system is

$$E = \int d^3x \left\{ |\partial_0\phi|^2 - |\nabla_i\phi|^2 + \frac{1}{2}(\partial_0\sigma)^2 - \frac{1}{2}(\nabla_i\sigma)^2 + V(|\phi|,\sigma) \right\} \tag{38}$$

where $V(|\phi|,\sigma) = f^2\sigma^2|\phi|^2 + (g^2/8)(\sigma^2 - \sigma_0^2)^2$.

Under arbitrary perturbations

$$\begin{aligned} \phi &\rightarrow \tilde{\phi} = \phi + \delta\phi(\mathbf{x},t) \\ \sigma &\rightarrow \tilde{\sigma} = \sigma + \delta\sigma(\mathbf{x},t) \end{aligned} \tag{39}$$

where $\delta\phi$ can be complex but $\delta\sigma$ must be real, we have

$$\begin{aligned} \delta E &= \int d^3x[(\partial_0\delta\phi)(\partial^0\phi^*) + (\partial_0\phi)(\partial^0\delta\phi^*) - (\nabla_i\delta\phi)(\nabla^i\phi^*) - (\nabla_i\phi)(\nabla^i\delta\phi^*) \\ &\quad +(\partial_0\delta\sigma)(\partial_0\sigma) - (\nabla_i\sigma)(\nabla^i\delta\sigma) + V'_\phi\delta\phi + V'_{\phi^*}\delta\phi^* + V'_\sigma\delta\sigma]. \end{aligned} \tag{40}$$

Here $V'_\phi \equiv dV/d\phi$, $V'_{\phi^*} \equiv dV/d\phi^*$, and $V'_\sigma \equiv dV/d\sigma$.

We also know from Eq. (2) that

$$Q = i\int d^3x(\phi^* \overleftrightarrow{\partial_0} \phi).$$

The requirement of charge conservation, $\delta Q = 0$, becomes

$$i\int d^3x[\delta\phi^* \overleftrightarrow{\partial_0} \phi + \phi^* \overleftrightarrow{\partial_0} \delta\phi] = 0. \tag{41}$$

Using the ansatz

$$\begin{aligned} \phi &= \phi_0(r,\theta,z)\exp(-i\omega t) \\ \sigma &= \sigma_0(r,\theta,z) \end{aligned} \tag{42}$$

charge conservation implies

$$i\int d^3x\,[-(\phi_0\partial_0\delta\phi^*) + (\phi_0^*\partial_0\delta\phi) - i\omega(\phi_0\delta\phi^* + \phi_0^*\delta\phi)] = 0.$$

Upon substitution into Eq. (40) and after integrating by parts

$$\delta E = \int d^3x \left[(\nabla_i \nabla^i)\phi_0^* - \omega^2 \phi_0^* + V'_\phi\right] \delta\phi + \left[(\nabla_i \nabla^i)\phi_0 - \omega^2 \phi_0 + V'_{\phi^*}\right] \delta\phi^* + \left[(\nabla_i \nabla^i)\sigma_0 + V'_\sigma\right] \delta\sigma, \tag{43}$$

where now $V'_\phi \equiv dV/d\phi_0$, $V'_{\phi^*} \equiv dV/d\phi_0^*$, and $V'_\sigma \equiv dV/d\sigma_0$. The first-order variation in E vanishes from the equations of motion as expected. We also expect momentum to be conserved when we perturb the fields; writing

$$p_i = \int d^3x T_{0i}$$

we have from Eq. (32)

$$p_i = \int d^3x \left[(\partial_0\phi)(\nabla_i\phi^*) + (\partial_0\phi^*)(\nabla_i\phi) + (\partial_0\sigma)(\nabla_i\sigma)\right]. \tag{44}$$

Then we obtain

$$\delta p_i = \int d^3x \left[-i\delta\omega(\phi_0 \overleftrightarrow{\nabla}_i \phi_0^*) - 2i\omega(\delta\phi(\nabla_i\phi_0^*) - \delta\phi^*(\nabla_i\phi_0))\right] \tag{45}$$

with $\partial_0\sigma = 0$. Here we have used the fact that under a perturbation in ϕ we expect the frequency ω to also change: $\omega \to \omega_0 + \delta\omega$. In fact from Eq. (7) we see how ω is to be interpreted as a function of Q and B. From $Q = (\omega/\mu g^2) \int d^3\rho |B|^2$

$$\delta Q = 0 \quad \Rightarrow \quad \delta\omega Q = -\frac{2\omega^2}{\mu g^2} \int d^3\rho (B\delta B^* + B^*\delta B). \tag{46}$$

We will investigate the case when all the perturbations are real and the A and B fields are real. This is sufficient to see where the instability arises in the case of string configurations. Equation (45) clearly vanishes in this case; momentum will automatically be conserved.

The second variation of E is easily obtained from Eq. (40). Again we must use Eq. (41) to substitute for the $\delta\omega$ terms. As in FLS we obtain for $(\delta^2 E)_Q$:

$$(\delta^2 E)_Q = \int d^3x \psi^T H \psi + \frac{4\omega^3}{Q} \left[\int d^3x \psi^T b\right]^2 \tag{47}$$

where

$$\psi = \begin{pmatrix} \delta\phi \\ \delta\sigma \end{pmatrix}, \quad b = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \tag{48}$$

and

$$
\begin{aligned}
H &= (\nabla_i \nabla^i) \\
&\quad + \begin{pmatrix} f^2\sigma - \omega^2 & 2f^2\phi^*\sigma \\ 2f^2\sigma\phi & -\partial_0\partial^0 + 2f^2|\phi|^2 + (g^2/2)(3\sigma^2 - \sigma_0^2) \end{pmatrix} .
\end{aligned} \tag{49}
$$

Stability of a particular configuration is established by evaluating Eq. (47) or at least the sign of Eq. (47). If $(\delta^2 E)_Q > 0$ this implies that the perturbation has produced a new configuration of higher energy than the original string configuration, and so the original is stable. If however $(\delta^2 E)_Q < 0$ then the string configuration is unstable to such perturbations. FLS evaluated this by investigating the corresponding eigenvalue equation

$$
H\psi = -d^2\psi/dt^2 \tag{50}
$$

or

$$
H\psi_i = \lambda_i \psi_i \text{ for } \psi_i \sim \exp(i\sqrt{\lambda_i} t)\psi_i(r) \tag{51}
$$

where ψ_i satisfies the usual boundary conditions of a Schrödinger wave function.

FLS were able to use translational invariance, to show $A(\rho+\varepsilon)$ and $B(\rho+\varepsilon)$ also satisfied the equations of motion for A and B (Eqs. (5) and (6)). For an infinitesimal ε, the deviations from the original soliton solution $A(\rho)$ and $B(\rho)$ meant it was possible to construct three p-state eigenfunctions of H, all with zero eigenvalues. However the lowest s-state eigenvalue of H must be lower than the lowest p-state eigenvalue, hence H has at least one negative eigenvalue. They went on to demonstrate that for the particular cases they were interested in, the soliton solution indeed only had one negative eigenvalue. The proof that under arbitrary variations δA and δB, this meant that $(\delta^2 E)_Q \geq 0$ is extremely elegant. The crucial equations for our purpose are their Eqs. (3.22) and (3.23)

$$
\left(\frac{2g^2}{\mu}\right)(\delta^2 E)_Q = {\sum_i}' {\sum_j}' (c_i M_{ij} c_j) \tag{52}
$$

where the sums exclude zero λ_i's, and

$$
M_{ij} = \lambda_i \delta_{ij} + 4\nu^3 (Q g^2)^{-1} b_i b_j . \tag{53}
$$

The c_i are defined for an arbitrary eigenfunction ψ by $\psi(\rho) = \sum_i c_i \psi_i(\rho)$ where ψ_i are a complete orthonormal set of real functions, b_i is defined by $b(\rho) = \sum_i b_i \psi(\rho)$ where $b(\rho)$ is given in Eq. (48), and λ_i is the set of eigenvalues satisfying Eq. (50), hence one of them is negative. The eigenvalues of M_{ij} are then evaluated and shown to be always positive semi-definite. Physically what is occurring is that the first term in Eq. (52), which is negative for the 'i' corresponding to the s-state eigenfunction, is compensated for by the positive definite second term leading to a total $(\delta^2 E)_Q > 0$. The latter term comes from demanding charge conservation, and in the case of a spherically symmetric solution it can never be zero.

The case for the non-topological string solutions is different. Looking at Eq. (49) we can rewrite the Laplacian as: $(\nabla_{(\rho,\theta)}\nabla^{(\rho,\theta)} + \nabla_z \nabla^z)$ where we are separating the (ρ, θ) components from the z-component. Now for a string along the z-axis there exists a two-dimensional translational invariance of A and B in the $(\rho - \theta)$ plane (see Fig. 2). Again these 'zero modes' can be used to construct a p-state with zero eigenvalue. We can then again use an analogous argument to demonstrate that there then exists at least one negative eigenvalue for the $(\rho - \theta)$ component of H since the s-state must have lower energy than the p-state. Without going into a type of analysis that FLS performed we cannot say whether there exist more than one negative eigenvalue of H, however if there are at least 2 eigenfunctions of H that have negative eigenvalues, a suitable linear combination of these two eigenfunctions would allow the construction of an ψ orthogonal to b, and from Eq. (47) the corresponding $(\delta^2 E)_Q$ would be less than 0.

The next step in the analysis is to investigate possible perturbations allowed by the equations of motion. One such perturbation arises from the following. If we have azimuthal symmetry in the B field, and the solution for B is independent of z, then we can have a perturbation of the form:

$$\delta B(\rho, \theta, z) = \varepsilon \Omega(\rho, \theta) \cos(kz) \tag{54}$$

where $|\varepsilon| \ll 1$ and $\Omega(\rho, \theta)$ is the s-state eigenfunction that has a negative eigenvalue when acted upon by H. This type of perturbation is orthogonal to b, so the charge conservation term in Eq. (47) vanishes, leaving in Eq. (52)

$$M_{ij} = \lambda_i \delta_{ij} + k^2 \tag{55}$$

where k^2 comes from the $\nabla_z \nabla^z$ term in the laplacian of Eq. (49) acting on Eq. (54). We can immediately see that for sufficiently small k^2, i.e., long wavelength perturbations, the overall eigenvalue of M_{ij} for the λ_i corresponding to the s-state will remain negative, hence $(\delta^2 E)_Q < 0$ for some perturbation. Only for small wavelength perturbations along the z-axis is the string solution stable.

What is happening physically is that under perturbations of the fields along the z-axis, the charge is allowed to migrate along the string (Fig. 3). From Eq. (21) this means that the effective string tension, proportional to $(dQ/d\eta)^{2/3}$, now varies along the string. Regions with a very small string tension which have lost charge become surrounded by regions with a high tension which have gained the charge. The result is that the low tension regions become pinched off and spherical solitons form with an intrinsic size $\sim k^{-1}$ (see Fig. 4). Now, for large k, the charge barely migrates along the string, there is hardly any change in the string tension so the string configuration then remains stable.

The case of complex B and δB is more complicated, although the basic results go through as described above. For example if we allow solutions of the form

$$B(\rho, \theta) \propto J_n(\nu\rho) \exp(in\theta) \tag{56}$$

the result is to increase the radius of the string (Fig. 1). Now the negative eigenvalue still persists in that there is still a translational invariance in the $\rho - \theta$ plane. However the magnitude of the negative eigenvalue decreases, because of the θ-dependence in the solution, and the "decay" rate of the string decreases, i.e. for a given k, as n increases it takes longer to "pinch" the string. In fact the decay time goes roughly as $t_{dec} \sim 1/|\lambda| + k^2$ where λ is the negative eigenvalue. So for decreasing $|\lambda|$, the decay time increases.

It has previously been demonstrated that putting a current along the string is equivalent to Lorentz boosting a string in an inertial frame with no current, to a frame moving with relative velocity k/ω. Hence we expect the physics of the perturbation analysis when there is no z-dependence on B to follow through even when there is initially a current present.

4. CONCLUSIONS

In this paper we have discussed the possible existence of non-topological string structures. Although the energetics and equations of motion allow for them to be formed, it appears that they would be unstable to the formation of the FLS type solitons. There is no topological reason for their stability unlike the case of cosmic strings. One could be tempted to ask why there are stable flux tubes of liquid He, yet our solutions, even with angular momentum, appear unstable to perturbations in the scalar fields. The former are topologically stable in that they have a winding number that it is impossible to get rid of.[7] This is similar to the case of the $U(1)$ cosmic string. The Higgs field which gives rise to the string is defined out to infinity, whereas as we have seen the scalar field which gives the non-topological string its shape is only defined out to the boundary of the string, going to zero past that (Fig. 5). The energy required to unwind a topological vortex is effectively infinite, whereas only a finite amount of energy is required to unwind a non-topological string. However, the wider the string, or the more angular momentum it has, the less unstable it becomes. It requires more energy to unwind such a string configuration.

An interesting question which so far has only been partially addressed is the mechanism for the formation of non-topological solitons in the early Universe. A model has recently been proposed[5] where the solitons can be thought of as regions of false vacuum surrounded by true vacuum, formed as the Universe cools below the Ginzburg temperature T_G. The structure and distribution of the vacuum domains below T_G is estimated using percolation studies.

It would be interesting to try and place soliton stars in some kind of standard electroweak model, as that would then increase the motivation for studying such objects. This work is currently in progress.[8]

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FIGURE CAPTIONS:

1. Fig. 1: Trial functions $B(\rho\ ,\theta) = B_0 J_n(\gamma\rho)\exp(in\theta)$ as a function of n.
2. Fig. 2: String solutions are invariant with respect to a two-dimensional translation in the '$x - y$' plane.
3. Fig. 3: Unstable perturbations of non-topological cosmic strings. The size of the $\times$'s are meant to indicate the magnitude of the charge per unit length.
4. Fig. 4: Non-topological cosmic strings pinch off into spherical non-topological solitons.
5. Fig. 5: Comparison of topological cosmic strings and non-topological cosmic strings.

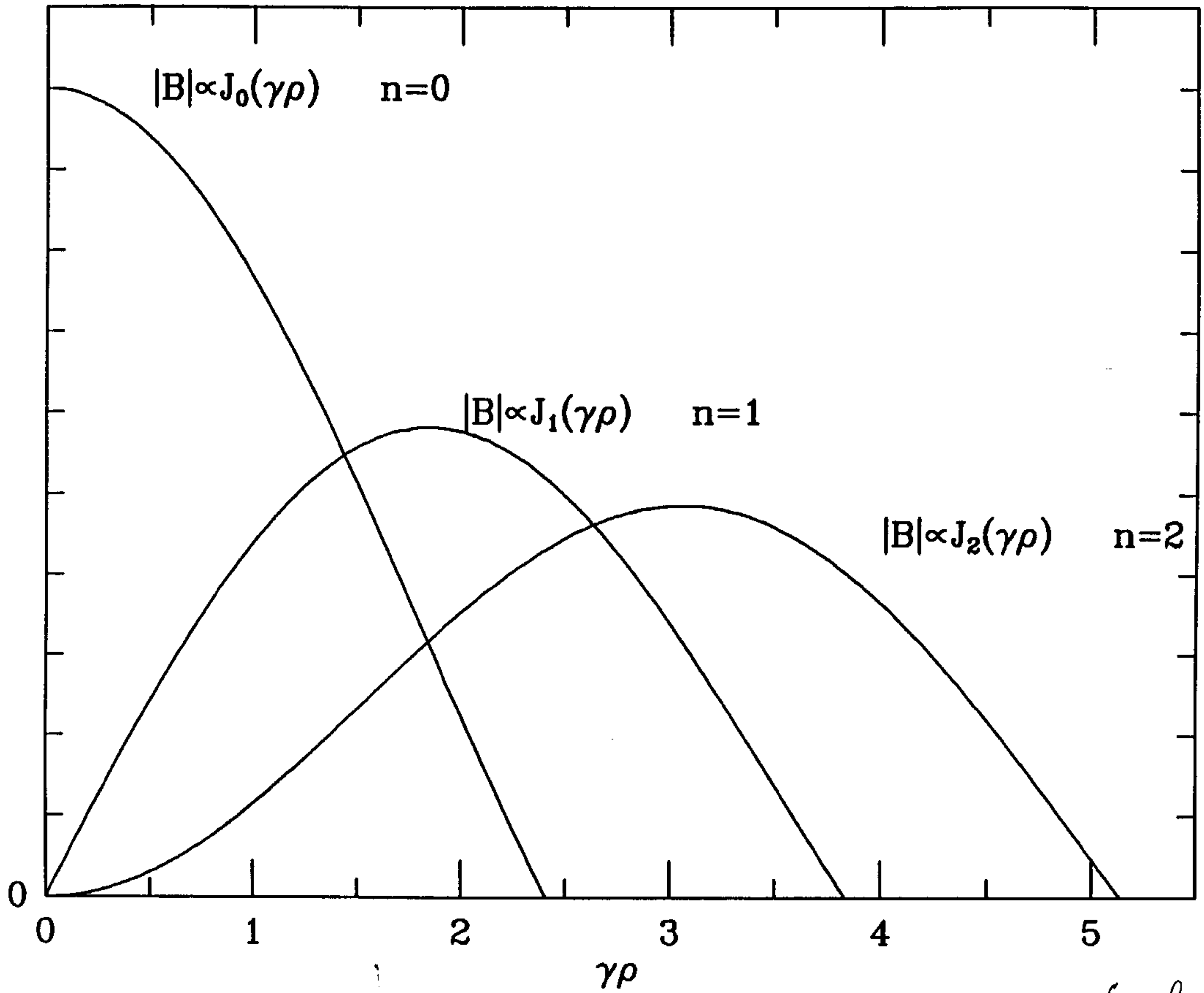

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Fig 1

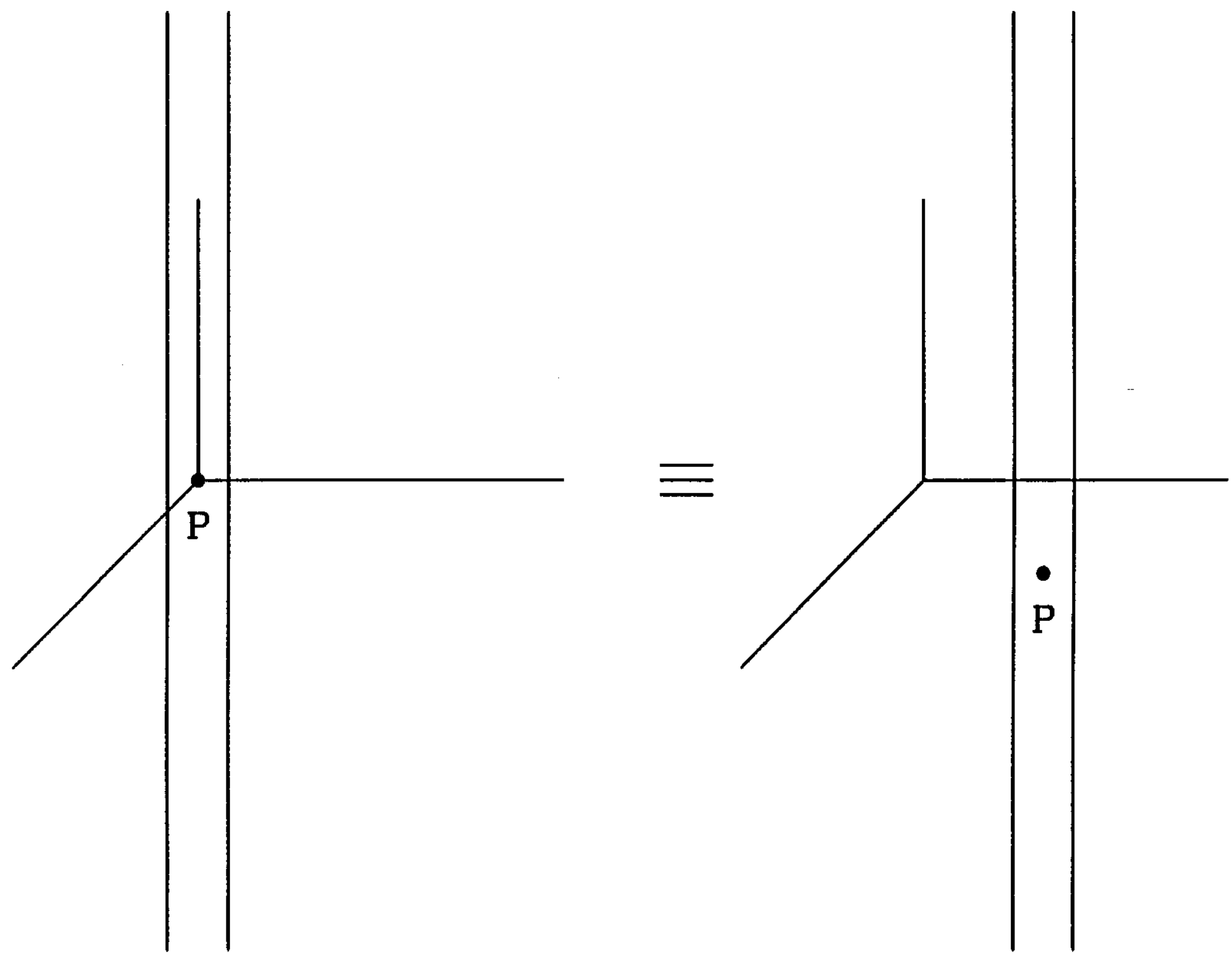

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Fig 7

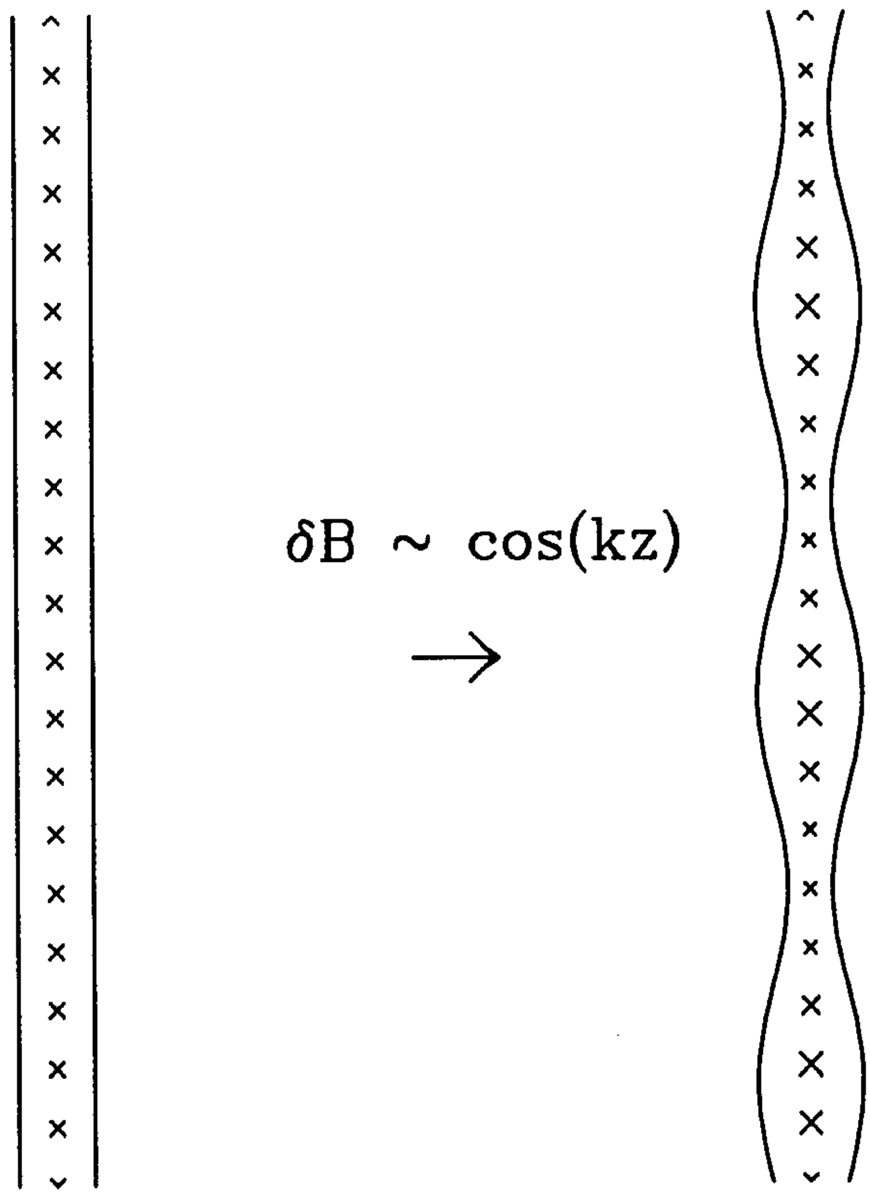

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Fig 3

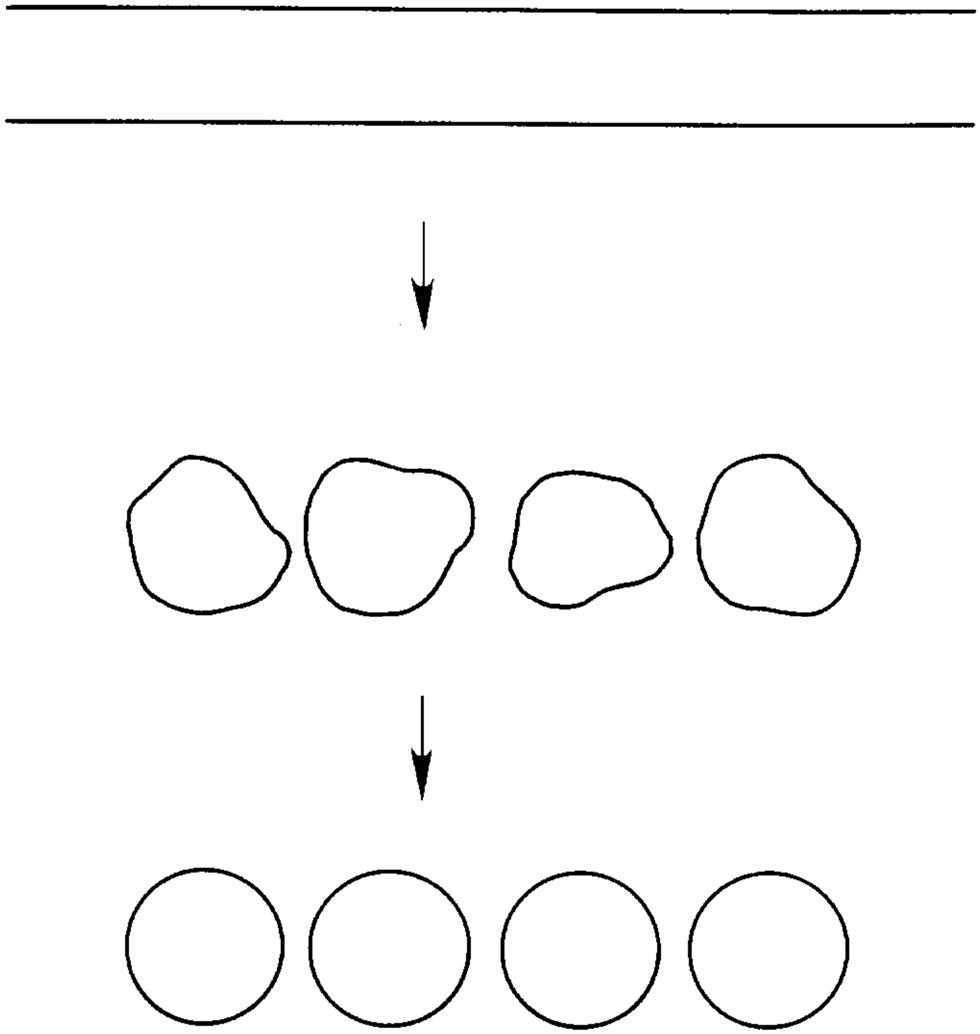

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Fig 4

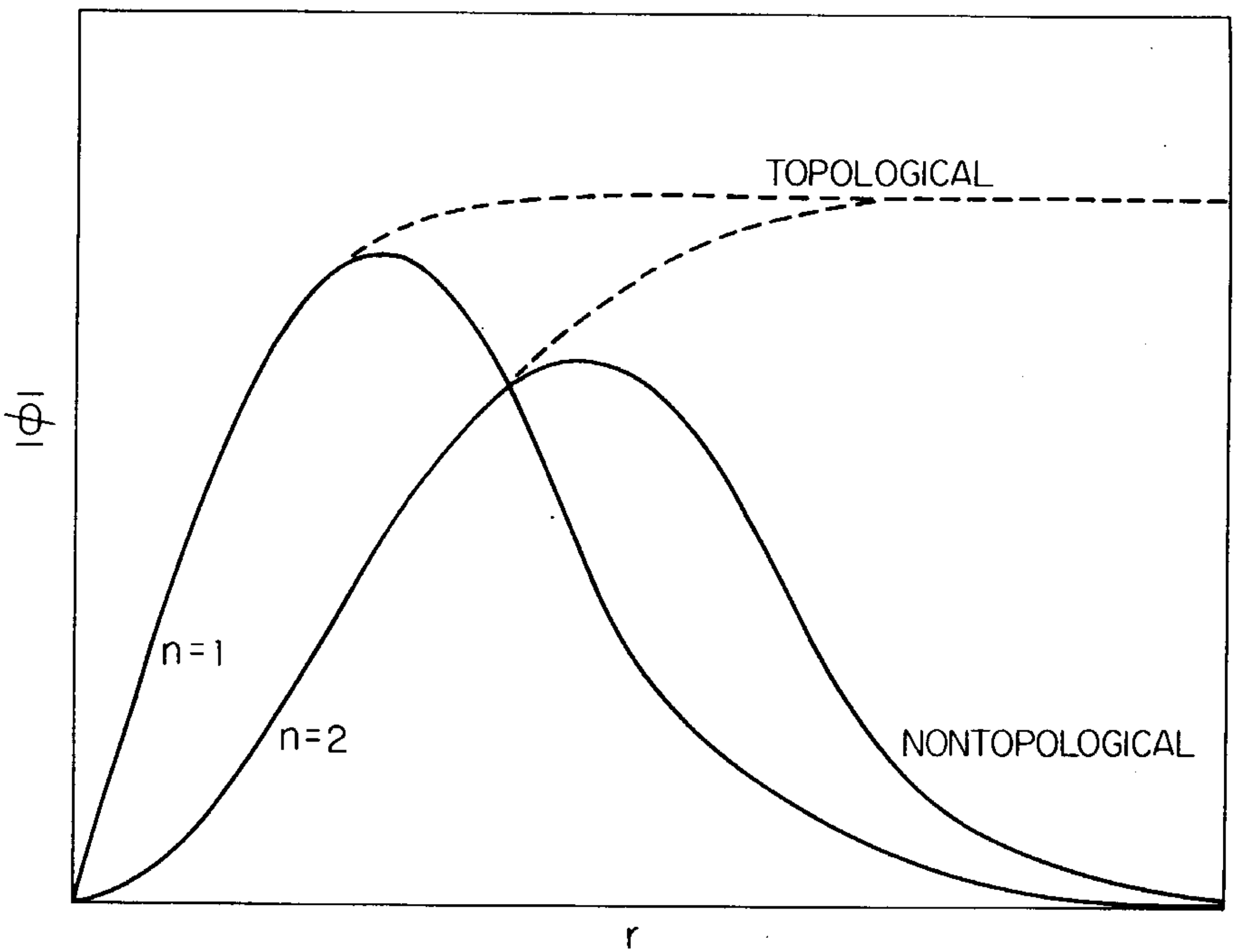

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Fig 5