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EINSTEIN METRICS AND BRANS-DICKE SUPERFIELDS

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Abstract

It is obtained here a space conformal to the Einstein space-time, making the transition from an internal bosonic space, constructed with the Majorana constant spinors in the Majorana representation, to a bosonic "superspace", through the use of Einstein vierbeins. These spaces are related to a Grassmann space constructed with the Majorana spinors referred to above, where the "metric" is a function of internal bosonic coordinates. The conformal function is a scale factor in the zone of gravitational radiation. A conformal function dependent on space-time coordinates can be constructed in that region when we introduce Majorana spinors which are functions of those coordinates. With this we obtain a scalar field of Brans-Dicke type.

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I. Introduction

Due to the frequency with which symplectic spaces have been used in theoretical physics, there has been a big development in the techniques of the treatment of these spaces. The most well known example of such a structure in physical theory is the phase space. Another example is the vectorial (internal) space of two component spinors in special and general relativity. A current example is the Fermi space, which is used as a half part of supersymmetry space[1].

From 1943 to 1945, H.C. Lee[2] developed the analogue of a local Riemannian geometry in a real symplectic manifold, represented by the phase space of Analytical Mechanics. However, his theory is different than the usual Riemannian geometry in spaces with symmetric metrics. First, it does not include affinities and second, the curvature tensor is an object with three indices which are completely skew-symmetric. Finally, the Killing equation for the "metric" involves the curvature tensor. Fronsdal[3] proved that these difficulties disappear if the manifold is replaced by a Grassmann space[4].

In this work we use the Majorana spinors for the realization of the Grassmann algebra in the Majorana representation. A complexification of this basis, in such a way that it remains complex even in the Majorana representation, permits us to realize the Grassmann algebra by means of Dirac spinors (in the form of $\psi + i\chi$, where ψ and χ are Majorana spinors). By using Majorana spinors we will introduce a local, internal, flat bosonic space whose coordinates $x^{(\alpha)}$ are formed through the combination of the two components of Majorana spinors in such a way that we have an even element in the Grassmann algebra where the metric is conformal to the Minkowski metric. It is also necessary that the "metric" of the Grassmann space generated by the Majorana spinors, in the Majorana representation, be a function of the coordinates $x^{(\alpha)}$. Next, the gravitational field is introduced in this space by means of a transition from $x^{(\alpha)}$ to y^{α} through local vierbeins. This makes the $x^{(\alpha)}$ transform into the y^{α} variables, which are superfields with an even character in the Grassmann algebra. The metric in this new space is conformal to the Einstein metric and it is shown that the conformal function is a constant scale factor in the zone of gravitational field. A Brans-Dicke scalar field is obtained in this region, when the Grassmann variables θ^{i} transform themselves into spin $\frac{1}{2}$ fields, which are now functions of space-time coordinates.

In this work the greek indices are bosonic indices and the roman are fermionic (or Grassmann) indices. The associated Grassmann parity is zero for the bosonic indices and one for each fermionic (or Grassmann) index. The Einstein summation convention is also used here for all types of indices. The signature of any bosonic space defined in this work will be +2.

II. The Grassmann Algebra and the Grassmann Space.

Let $\{1, \theta^i\}$, $i = 1, ..., n$, be the generators of a Grassmann algebra [4] over $\mathbf{C}$, the set of complex numbers. In this set, 1 is the identity element of the algebra, and the product of two elements, θ^i, θ^j, satisfies the properties of the exterior product of Cartan for differential forms, i.e.[2],

$$\theta^i\theta^j + \theta^j\theta^i = 0 \ . \tag{2.1}$$

The Grassmann algebra will be denoted here by $\mathcal{A}$. One realization of $\mathcal{A}$ takes place when we take spin 1/2 Majorana spinors (or a combination of them that form Dirac spinors). In this case, θ^i, $i = 1...,4$, represents the four components of a constant Majorana spinor. Another realization of $\mathcal{A}$ happens when we take the exterior algebra of Cartan for differential forms [6] over a space of usual coordinates x^i, $i = 1,2,3$.

Due to the property (2.1), we observe that:

1. Every θ^i satisfies $(\theta^i)^2 = 0$.
2. For a product of generators $\{\theta^i\}$ to be non-zero, all indices must different.
3. Every element of $\mathcal{A}$ can be obtained as a polynomial of the generators with coefficients in $\mathbf{C}$.

Every element of $\mathcal{A}$ which is formed as a sum of monomials of odd (even) degree of the generators is an odd (even) element of $\mathcal{A}$. A "mixed" element of $\mathcal{A}$ will be then formed with odd and even elements of $\mathcal{A}$. Every odd (even) element of $\mathcal{A}$ has parity +1 (0), which is the so called "Grassmann parity" of the element. The "mixed" element does not have definite parity.

It is easy to see, then, that the product of any two elements A and B of $\mathcal{A}$ which have definite parities, i.e., excluding the mixed elements, follows the "graded rule" of commutation:

$$AB = (-1)^{p_A p_B} BA \ , \tag{2.2}$$

[2]Historically, Grassmann was the first to introduce the concept of exterior algebra; the algebra of differential forms of Cartan is a rediscovery of Grassmann's works in projective geometry [5].

where p_A is the parity of A, and p_B is the parity of B, and where A and B are each polynomials in the generators (see ref. [3]). A transformation of the coordinates in $\mathcal{A}$ is defined as

$$\theta'^i = f^i(\theta) \quad , \tag{2.3}$$

in such a way that the new coordinates are odd functions of the old ones, i.e., $f^i(\theta)$ is an odd function of θ. This is clearly an automorphism of the algebra. In the case of infinitesimal transformation we must have

$$\theta'^i = \theta^i + \epsilon\psi^i(\theta) \quad , \tag{2.4}$$

where $\psi^i(\theta)$ must be an odd element, function of θ in $\mathcal{A}$, and where ϵ is an infinitesimal element of first order in **C**. The relation (2.4) permits non-linear transformations $\theta'^i = \theta'^i(\theta)$.

In the Grassmann algebra, the concept of derivation, which will here be called "antiderivation" to distinguish it from the more commonly known derivation, must be introduced in two forms: the operator ∂^L_a defined by:

$$\partial^L_a P = A_a \quad , \tag{2.5}$$

where $P = \theta^a A_a + Q$, with Q independent of θ^a and fixed "a" indice, and the operator ∂^R_a defined by

$$\partial^R_a P = B_a \quad , \tag{2.6}$$

where, for the same Q above, $P = B_a\theta^a + Q$. (In general, $A_a = (-1)^{p+1}B_a$ and $B_a = (-1)^{p+1}A_a$.) The relation (2.5) is called a "left antiderivation over $\mathcal{A}$", and the relation (2.6) is called a "right antiderivation over $\mathcal{A}$". The differential $d\theta^i$ can be written then, in terms of right and left derivatives, as

$$d\theta^i = d\theta^k \frac{\partial^L\theta^i}{\partial\theta^k} = \frac{\partial^R\theta^i}{\partial\theta^k}\ d\theta^k = \delta^i_k d\theta^k \quad . \tag{2.7}$$

It is easy to show that taking two polynomials P_1 and P_2 of definite parity, p_1 and p_2, respectively, we have:

$$\partial^L_a (P_1P_2) = \left(\partial^L_a P_1\right) P_2 + (-1)^{p_1} P_1 \left(\partial^L_a P_2\right) \tag{2.8 - a}$$

$$\partial^R_a (P_1P_2) = P_1 \left(\partial^R P_2\right) + (-1)^{p_2} \left(\partial^R P_1\right) P_2 \quad . \tag{2.8 - b}$$

In the treatment of the Grassmann algebra we can use the operator ∂_a^L as well as the operator ∂_a^L, as long as we keep in mind the relations:

$$\partial_a^R P = (-1)^{p+1}\partial_a^L P \quad , \tag{2.9 - a}$$

$$\partial_a^L P = (-1)^{p+1}\partial_a^R P \quad , \tag{2.9 - b}$$

which gives the relation between them. In (2.9) P is any polinomial in $\mathcal{A}$ of definite parity p. The operator ∂_a^L will always be used here indicated simply by ∂_a, remembering that it is now always a left antiderivative.

If we now introduce a non-singular skew-symmetric matrix in $\mathcal{A}$, $\omega = (\omega_{ij})$, we can define the 2-form

$$ds^2 = d\theta^i \omega_{ij} d\theta^j \quad . \tag{2.10}$$

(This expression has all the properties of a a 2-form of Cartan.) It is invariant under a general coordinate transformation. The quantities $(\mathcal{A}, ds^2)$ compose the so called "Grassmann symplectic space", $\mathcal{G}$. In (2.10), the matrix $\omega = (\omega_{ij})$ can be proportional to the identity of the algebra (constant in the θ-space) which corresponds to a flat space, here indicated as $\mathcal{G}_f$. On the other hand, if the matrix ω is a function of the θ-coordinates, it should correspond to a "curved" space and it will then be indicated as $\mathcal{G}_c$. In both cases, obviously, we assume that there exists curvature in $\mathcal{G}_c$, and that it is zero in $\mathcal{G}_f$.

The 2-form defined in (2.10) corresponds to the line element in a "Riemannian space" $\mathcal{G}_c$, for $\omega = \omega_{ij}(\theta)$, in general. We saw that $d\theta^i$ and $d\theta^j$ in (2.10) are odd elements in $\mathcal{A}$. So, the matrix ω_{ij} has a definite even parity[3]. In this case, ds^2 is even and corresponds to a complex number in C, the algebra of complex numbers, and this is the nearest to the usual Riemannian line-element. In $\mathcal{G}_f$ there exists a canonical coordinate system where ω_{ij} goes to a constant proportional to the identity element of the algebra, and where the affinity and the curvature go to zero globally. This coordinate system is the analogue of cartesian coordinates in symmetrical spaces. Performing non-linear transformations over $\mathcal{G}_f$ we can construct the non-zero affinity and metric, since the curvature remains globally zero.

The coordinate transformations in the flat Grassmann space, $\mathcal{G}_f$, which are symmetry transformations, are linear, i.e.,

$$\theta'^i = L^i{}_k \theta^k \quad , \tag{2.11}$$

[3]The vectors and tensors used in this work are functions of θ and respect the graduation of parities. For example, v^i odd, v^{ij} even, etc. See the ref. [3].

and for ds^2 to be invariant under these transformations, we must have the relation

$$L^T \omega L = \omega \quad . \tag{2.12}$$

where L and ω are the matrices $L^i{}_k$ and ω^{ij}, respectively, and L^T is the transpose of L. For infinitesimal transformations, we have,

$$L = \mathbf{1}_n + N \quad , \qquad N^T = N \quad , \tag{2.13}$$

where N is an infinitesimal matrix and $\mathbf{1_n}$ is the unitary matrix in this n-dimensional space. Therefore, for the case of linear transformations, the relation (2.11) must be, for L of the type above:

$$\theta'^i = \theta^i + \epsilon^i{}_j \theta^j \quad , \tag{2.14}$$

where the infinitesimal constant coefficients $\epsilon = (\epsilon_{ij})$ are symmetric $(\epsilon_{ik} = \epsilon_{ki})$. The set of linear transformations satisfying (2.12), makes up the n-dimensional symplectic group, denoted here by $Sp(n)$. This group has $\frac{n(n+1)}{2}$ parameters which are in general complex numbers.

We choose here, the relation:

$$v_i = \omega_{ij} v^j \tag{2.15}$$

as the process of lowering indexes of the vectors v^j. From eq. (2.5) it follows that the scalar quantity $v^2 = v^i v_i$ is positive. The indices of an arbitrary tensor $T^{ijk\dots}$, are lowered as: $\omega_{ij} T^{jkl\dots} = T_i{}^{kl\dots}$. In the same way, we define the process of raising indexes of an arbitrary tensor, for example $T_{klj\dots}$, as: $\omega^{ij} T_{jkl\dots} = T^i{}_{kl\dots}$, where ω^{ij} is the inverse "metric", such that

$$\omega^{ij} \omega_{jk} = \omega_{jk} \omega^{ji} = \delta^i_k \quad . \tag{2.16}$$

The inverse matrix ω^{ij} is also, a skew-symmetric matrix of even parity in $\mathcal{A}$.

The "line element" defined in (2.10), can then be written alternatively, as:

$$ds^2 = -d\theta_i \omega^{ij} d\theta_j = d\theta^i d\theta_j \quad , \tag{2.17}$$

where the minus sign is due to the convention of "positive contraction" for vectors defined above.

III. Realization of Grassmann Algebra through Majorana Spinors.

A Dirac spinor satisfies the condition,

$$\psi^{(c)i} = \left(C\overline{\psi}^T\right)^i = \left(C\gamma_o^T\psi^*\right)^i \quad , \qquad i = 1, \cdots, 4 \ , \tag{3.1}$$

where $\overline{\psi} = \psi^\dagger\gamma_o$ and C is the charge conjugation matrix, which satisfies the conditions $C^T = -C$ and $C^\dagger = C^{-1}$. ($\psi^{(c)}$ denotes the charge conjugate of ψ.) The Dirac matrices (see Pauli[7]), are such that $\gamma_\mu^T = -C^{-1}\gamma_\mu C$, which gives for $\psi^{(c)}$:

$$\psi^{(c)} = -\gamma_o C\psi^* \ . \tag{3.2}$$

A Majorana spinor satisfies then, the condition $\psi^{(c)} = \psi$. From now on we will represent it as θ. Also, it will be useful to consider a coordinate independent spinor. We will represent it as :

$$\theta^{(c)} = \theta = C\overline{\theta}^T = -\gamma_0 C\theta^* \ . \tag{3.3}$$

Taking now the Majorana representation, we have $C = \gamma_o$. Then, from (3.3) and remembering that $\gamma_o^2 = -1$, we have $\theta^* = \theta$. This means that the Majorana spinor is real in the Majorana representation[4]. θ^i, $i = 1 \cdots 4$, represents then, the four components of a Majorana constant spinor and the charge conjugation matrix satisfies, besides the conditions above described, the condition[5] $C^* = C$. Therefore, for a Majorana spinor we have the anticommutation relations:

$$\overline{\theta}_i\theta^j + \theta^j\overline{\theta}_i = 0 \ , \tag{3.4 - a}$$

$$\theta^i\theta^j + \theta^j\theta^i = 0 \ , \tag{3.4 - b}$$

$$\overline{\theta}_i\overline{\theta}_j + \overline{\theta}_j\overline{\theta}_i = 0 \ . \tag{3.4 - c}$$

The relations (3.3) and (3.4) suggest that the charge conjugation matrix may be used as the metric of a Grassmann space in four dimensions, where the "line element" is defined as the 2-form:

$$ds^2 = d\theta^i C^{-1}_{ik} d\theta^k = d\theta^i d\overline{\theta}_i \tag{3.5}$$

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[4] Inversely, if we impose $\theta^* = \theta$, we obtain $C = \gamma_o$, the Majorana representation.

[5] The Dirac matrices are real in the Majorana representation and space-time signature $+2$.

or,

$$ds^2 = d\bar{\theta}_i C^{ik} d\bar{\theta}_k = d\theta^i d\bar{\theta}_i \ . \tag{3.6}$$

(The minus sign in (3.6) is due to the convention of "positive contraction operation" for tensors made in the last section.) Therefore, in this case, the "metric" of the Grassmann space is

$$\omega_{ij} = C^{-1}_{ij} \ , \qquad \omega^{ij} = C^{ij} \ . \tag{3.7}$$

As the charge conjugation matrix is a constant matrix, of this four-dimensional Grassmann space is flat.

The Grassmann space generated by the Majorana constant spinors is frequently called a "Fermi space", since it refers to objects of spin 1/2 (fermions), where θ^i are the "Fermi coordinates". Due to the relations (3.4), the algebra generated by the θ^i is a four-dimensional Grassmann algebra, $\mathcal{A}_4$, and every element of $\mathcal{A}_4$ will have then, the properties defined in the last section. The symmetry group which describe them is the $Sp(C)$ in four dimensions. This group has 10 (ten) parameters which are real quantities in the Majorana representation[6]. For a generalization of the Grassmann space generated by the Majorana spinors, to an analogous Riemannian local space, see C.G. Oliveira in reference [3]. In this reference, a treatment of the curved Grassmann space $\mathcal{G}_c$, in four dimensions, was made using an analogue of the tetrad formalism. (The existence of unique tetrads is one of the fundamental reasons for the choice of ω_{ij} as an even object, made in the last section.)

IV. An internal bosonic space associated to the Grassmann variables.

Let the zero-parity variables be

$$x^{(\alpha)} = \bar{\theta}\gamma_5\gamma^{(\alpha)}\theta \ \ , \tag{4.1}$$

in such way that the metric in $\mathcal{G}_c$ is of the form

$$\omega_{ij} = \omega_{ij}\left(x^{(\alpha)}\right) \ \ . \tag{4.2}$$

[6]Another example of a space where the symmetry group has 10 (ten) parameters is the 5-dimensional flat space which contains the De Sitter space. We can choose this space in such way that three dimensions be space-like and two be time-like, and so, its symmetry group will be $SO(3,2)$. In this space, the coordinates are (x^A), $A = 1 \cdots 5$, and the signature of the "flat-metric," η_{AB}, is $(+++--)$.

Derivatives of even functions, depending on the $x^{(\alpha)}$, are of the form

$$\partial_i = \frac{\partial x^{(\alpha)}}{\partial \theta^i}\frac{\partial}{\partial x^{(\alpha)}} = \varepsilon_i^{(\alpha)}\frac{\partial}{\partial x^{(\alpha)}} \quad . \tag{4.3}$$

This process of derivation obeys the properties of antiderivative defined in the Grassmann space:

$$\text{a)} \quad \partial_i\partial_j + \partial_j\partial_i = 0 \quad , \tag{4.4 - a}$$

$$\begin{aligned} \text{b)} \quad \partial_i(\Phi_1\Phi_2) &= (\partial_i\Phi_1)\Phi_2 + (-1)^{p_1p_2}(\partial_i\Phi_2)\Phi_1 \\ &= (\partial_i\Phi_1)\Phi_2 + (-1)^{p_1}\Phi_1(\partial_i\Phi_2) \quad , \end{aligned} \tag{4.4 - b}$$

where, in (4.4-b), p_1 and p_2 are the Grassmann parities of Φ_1 and Φ_2, respectively. If ∂_i operates on tensors of $\mathcal{G}_c$, (even functions of $x^{(\alpha)}$), we must correct the antiderivative by the covariant antiderivative:$\partial_i \longrightarrow \nabla_i$,

$$\nabla_i\Phi^{cr} = \partial_i\Phi^{cr} + \Gamma^c_{bi}\Phi^{br} + \Gamma^r_{bi}\Phi^{cb} \quad . \tag{4.5}$$

where the objects Γ^c_{ab} are connections on $\mathcal{G}_c$.

Infinitesimal transformations in Grassmann coordinates generate infinitesimal transformations in the variables $x^{(\alpha)}$. From the identity

$$dx^{(\mu)} = d\theta^i\frac{\partial x^{(\mu)}}{\partial\theta^i} = d\theta^i e_i^{(\mu)} \quad , \tag{4.6}$$

it follows that the $dx^{(\mu)}$ are even quantities. If we then form the product

$$\begin{aligned} dx^{(\mu)}dx^{(\nu)} &= d\theta^i e_i^{(\mu)} d\theta^j e_j^{(\nu)} \quad , \\ &= -d\theta^i e_i^{(\mu)} e_j^{(\nu)} d\theta^j \quad , \end{aligned} \tag{4.7}$$

and observe that $dx^{(\mu)}$ and $dx^{(\nu)}$ are quantities with even Grassmann parity, we can define an "internal flat space", of coordinates $x^{(\mu)}$ and symmetric metric $\eta_{\mu\nu}$(bosonic space). In this space a "line element" can be defined as:

$$-d\sigma^2 = \eta_{\mu\nu}dx^{(\mu)}dx^{(\nu)} = -d\theta^i e_i^{(\mu)} e_j^{(\nu)} d\theta^j \eta_{\mu\nu} \quad . \tag{4.8}$$

From this relation, we can write that

$$e_i^{(\mu)} e_j^{(\nu)} \eta_{\mu\nu} = -\omega_{ij} \tag{4.9}$$

$$-d\sigma^2 = \eta_{\mu\nu} dx^{(\mu)} dx^{(\nu)} = d\theta^i \omega_{ij} d\theta^j = d\lambda^2 \quad , \tag{4.10}$$

where $d\lambda^2$ indicates here the 2-form associated with the Grassmann space with metric $\omega_{ij}(x^{(\alpha)})$. Note from (4.9) that $\omega_{ij} = -\omega_{ij}$, since $dx^{(\mu)}dx^{(\nu)} = dx^{(\nu)}dx^{(\mu)}$ (remembering that $dx^{(\mu)}$ are even variables), which implies that $\eta_{\mu\nu} = \eta_{\nu\mu}$. In the following, we will be more interested in the situation in which instead of (4.9), we have

$$e_i^{(\mu)} e_j^{(\nu)} \eta_{\mu\nu} = -\phi(\theta^k)\omega_{ij} = -\overline{\omega}_{ij} \quad , \tag{4.11}$$

$$-d\sigma^2 = \overline{d\lambda^2} = \phi(\theta^k) d\lambda^2 \quad , \tag{4.12}$$

where $\phi(\theta^k)$ is an even element of $\mathcal{A}$. Multiplying both members of (4.11) by ω^{ij} we have

$$e_i^{(\mu)} e_j^{(\nu)} \omega^{ij} \eta_{\mu\nu} = -\phi(\theta)\omega_{ij}\omega^{ij} \quad .$$

As $-\omega_{ij}\omega^{ij} = -\omega^{ij}\omega_{ij} = 4$, we have:

$$e_i^{(\mu)} e_j^{(\nu)} \omega^{ij} \eta_{\mu\nu} = 4\phi(\theta) \quad . \tag{4.13}$$

Then

$$e_i^{(\mu)} e_j^{(\nu)} \omega^{ij} = \phi(\theta)\eta^{\mu\nu} \quad . \tag{4.14}$$

Therefore, the introduction of even (bosonic) variables $x^{(\alpha)}$ with the definition (4.8), and the relation (4.9), permits us to define an internal flat space of bosonic structure constructed over the Grassmann variables θ^k. Due to the definition (4.8), the signature of this new bosonic space is opposite to the signature of Minkowski space. However, this does not affect the properties associated with signatures, such as the properties of Dirac constant matrices in the Majorana representation. Also, it does not affect any of the above equations.

The variables $x^{(\alpha)}$ were chosen as canonical coordinates of this internal space, *i.e.*, in these coordinates the metric assumes the canonical form $\eta_{\mu\nu}$ = diag. (-1,+1,+1,+1). Some transformations of the form $\delta\theta^i = \zeta^i(\theta)$ permitted in the space $\mathcal{G}_c$ will preserve this canonical form. In fact, in general

$$\begin{aligned} \delta x^{(\alpha)} &= \delta\overline{\theta}\gamma_5\gamma^{(\alpha)}\theta + \overline{\theta}\gamma_5\gamma^{(\alpha)}\delta\theta \quad , \\ &= \overline{\zeta}\gamma_5\gamma^{(\alpha)}\theta + \overline{\theta}_5\gamma^{(\alpha)}\zeta \quad . \end{aligned} \tag{4.15}$$

The variables $x^{(\alpha)}$ can be written in the form

$$x^{(\alpha)} = M^{(\alpha)}_{ij}\theta^i\theta^i \quad , \tag{4.16}$$

where,

$$M^{(\alpha)} = -\left[\gamma_1\gamma_2\gamma_3\gamma^{(\alpha)}\right] = -\left[C^{-1}\gamma_5\gamma^{(\alpha)}\right] \quad . \tag{4.17}$$

So,

$$\delta x^{(\alpha)} = M^{(\alpha)}_{ij}\delta\theta^i\theta^j + M^{(\alpha)}_{ij}\theta^i\delta\theta^j \quad . \tag{4.18}$$

Taking $\delta\theta^i = a^i{}_k\theta^k$ with $a^i{}_k$ constant, and observing that $M^{(\alpha)}_{ij}$ is skew-symmetric in (ij) we can rewrite (4.18) as[7]

$$\begin{aligned} \delta x^{(\alpha)} &= M^{(\alpha)}_{ij}a^i{}_k\theta^k\theta^j + M^{(\alpha)}_{ij}a^i{}_k\theta^k\theta^i \quad , \\ &= 2M^{(\alpha)}_{ij}a^i{}_k\theta^k\theta^j \quad . \end{aligned} \tag{4.19}$$

For the case of transformations with infinitesimal coefficients $a^i{}_k$ such that

$$M^{(\alpha)}_{ij}a^i{}_k = \frac{1}{2}\epsilon^{(\alpha)}{}_{(\beta)}M^{(\beta)}_{kj} \quad , \tag{4.20}$$

we will have

$$\delta x^{(\alpha)} = \epsilon^{(\alpha)}{}_{(\beta)}M^{(\beta)}_{kj}\theta^k\theta^j = \epsilon^{(\alpha)}{}_{(\beta)}x^{(\beta)} \quad , \tag{4.21}$$

which is a transformation of the "Lorentz infinitesimal-type" under which $\eta_{\mu\nu} \longrightarrow \eta_{\mu\nu}$ in (4.8). The other transformations in the θ-space which do not satisfy this property will be equivalent to the introduction of new non-canonical coordinates $x'^{(\alpha)}$ (as if they were internal curved coordinates).

The metric in $\mathcal{G}_c$ can be expanded as a power series in the variables $x^{(\alpha)}$ of the form

$$\omega^{ij}(x^{(\alpha)}) = \overset{\circ}{\omega}{}^{ij} + A^{ij}_{(\alpha)}x^{(\alpha)} + B^{ij}_{(\alpha)(\beta)}x^{(\alpha)}x^{(\beta)} \quad , \tag{4.22}$$

where $\overset{\circ}{\omega} = C^{ij}$ is the charge conjugation matrix in the Majorana representation. We have, by (4.16) and (4.22), that

$$\begin{aligned} \omega^{ij} &= C^{ij} + A^{ij}_{(\alpha)}M^{(\alpha)}_{ks}\theta^k\theta^s + \\ &+ B^{ij}_{(\alpha)(\beta)}M^{(\alpha)}_{ks}M^{(\beta)}_{rn}\theta^k\theta^s\theta^r\theta^n \quad . \end{aligned} \tag{4.23}$$

[7]See Appendix

Also, from (4.16) we have,

$$e_l^{(\alpha)} = \frac{\partial x^{(\alpha)}}{\partial \theta^l} = \left(M_{li}^{(\alpha)} - M_{il}^{(\alpha)}\right)\theta^i = 2M_{li}^{(\alpha)}\theta^i \quad . \tag{4.24}$$

Given $\omega^{ij}(x^{(\alpha)})$ from equation (4.23) and the $e_l^{(\alpha)}$ from equation (4.24) we can determine $\phi(\theta^i)$ using (4.14):

$$\phi(\theta^i) = \overset{\circ}{\phi} + K_{ij}\theta^i\theta^i + L_{ijkl}\theta^i\theta^j\theta^k\theta^l \quad , \tag{4.25}$$

with

$$\overset{\circ}{\phi} \;=\; 0 \tag{4.26}$$

$$K_{ij} \;=\; \eta_{\alpha\beta}M_{ik}^{(\alpha)}M_{jl}^{(\beta)}\,\overset{\circ}{\omega}^{kl} \quad , \tag{4.27}$$

$$L_{ijkl} \;=\; \left[\left(\eta_{\alpha\beta}M_{ni}^{(\alpha)}M_{mj}^{(\beta)}\right)A_{(\gamma)}^{nm}\right]M_{kl}^{(\gamma)} \quad . \tag{4.28}$$

Straightforward but tedious calculation, using equation (4.23), gives the following result for the explicit value of the coefficients K_{ij} in the Majorana representation,

$$K_{ij} = -4C_{ij}^{-1} \quad . \tag{4.29}$$

Similarly, we can determine the explicit value of the coefficients L_{ijkl} in this representation. (Notice that there exists a flat limit for the function $\phi(\theta^i)$ which occurs when $L_{ijkl} \longrightarrow 0$, giving:

$$\phi(\theta^i) = K_{ij}\theta^i\theta^j \quad , \tag{4.30}$$

where K_{ij} is given by (4.29).)

Another quantity which can be explicitly calculated in the Majorana representation is

$$-\sigma^2 = \eta_{\alpha\beta}x^{(\alpha)}x^{(\beta)} = 2^5\theta^1\theta^2\theta^3\theta^4 \quad . \tag{4.31}$$

In this way, the internal space of the variables $x^{(\alpha)}$, with metric $\eta_{\alpha\beta}$ in the interpretation of "bosonic space-like", does not have an interpretation similar to that of Minkowski space; the position $x^{(\alpha)}$ of a point in this space assumes always the above form and does not split into "time-like" and "space-like" sections. Therefore, in this internal space, the light cone does not exist. From the algebraic point of view, $(-\sigma^2)$ is an even element of the Grassmann algebra, such that $(\sigma^2)^2 = 0$.

V. The Einstein space associated to a bosonic internal space.

Let x^μ be the coordinates of an Einstein space with local tetrads $h^\mu_{(\alpha)}(x)$ and metric

$$g^{\mu\nu}(x) = h^\mu_{(\alpha)}(x) h^\nu_{(\beta)}(x) \eta^{\alpha\beta} \quad . \tag{5.1}$$

Let us also consider the linear functions of the internal variables $x^{(\alpha)}$ of the form

$$y^\mu = h^\mu_{(\alpha)}(x) x^{(\alpha)} = y^\mu \left(x^\lambda, \theta^i \right) \quad , \tag{5.2}$$

where y^μ are bosonic "superfields" with even character in the Grassmann algebra and x^λ are the coordinates of the curved space-time. From (5.1) and (5.2) it follows that

$$\frac{\partial y^\mu}{\partial x^{(\alpha)}} \frac{\partial y^\lambda}{\partial x^{(\beta)}} \eta^{\alpha\beta} = g^{\mu\lambda}(x) \quad . \tag{5.3}$$

Let us define now the quantity $\psi^{\mu\lambda}$ through:

$$\frac{\partial y^\mu}{\partial \theta^i} \frac{\partial y^\lambda}{\partial \theta^i} \omega^{ij} = \psi^{\mu\lambda} \quad . \tag{5.4}$$

As we can define the vierbein E^μ_i through the identity:

$$E^\mu_i = \frac{\partial y^\mu}{\partial \theta^i} = e^{(\alpha)}_i \frac{\partial y^\mu}{\partial x^{(\alpha)}} = h^\mu_{(\alpha)}(x) e^{(\alpha)}_i \quad , \tag{5.5}$$

it follows that

$$\psi^{\mu\lambda} = e^{(\alpha)}_i e^{(\beta)}_j \omega^{ij} \frac{\partial y^\mu}{\partial x^{(\alpha)}} \frac{\partial y^\lambda}{\partial x^{(\beta)}} \quad . \tag{5.6}$$

Using (4.12) and (5.3), we can rewrite this relation in the form

$$\psi^{\mu\lambda} = \phi(\theta) g^{\mu\lambda}(x) \quad . \tag{5.7}$$

Therefore, the object $\psi^{\mu\lambda}$ is a bosonic superfield, with even algebraic character in $\mathcal{A}$ and with the space-temporal character of a field which is conformal to the Einstein metric. Its explicit form follows from the expression for $\phi(\theta)$ obtained through equations (4.25) and (4.28). Notice that the quantities

$$e^{(\mu)}_i = \frac{\partial x^{(\mu)}}{\partial \theta^i} \quad , \quad E^\mu_i = \frac{\partial y^\mu}{\partial \theta^i} \quad ,$$

are fields with spin 3/2 of Rarita-Schwinger type. From this point of view, equations (4.14) and (5.4) are similar to the conditions proposed in the literature for these kind of fields. (See for example, the papers of I. Bars and S. MacDowell, reference [8].)

VI. Spaces conformal to the Einstein space-time generated by the Grassmann coordinates

We saw from (5.4) that $\psi^{\mu\lambda}$ is a field which is conformal to the Einstein metric through the factor $\phi(\theta)$, equation (5.7). The functions $y^{\mu}(x,\theta)$ can be interpreted in the following way: given the $x^{(\alpha)}$ associated to an internal bosonic flat space, obtained in Section IV, we can transform these variables to a curved space through the vierbein process, by means of:

$$\gamma^{(\alpha)}\begin{pmatrix}\text{constant Dirac}\\ \text{matrices}\end{pmatrix} \longrightarrow \gamma^{\alpha}(x) = h^{\alpha}_{(\tau)}(x)\gamma^{(\tau)} \quad . \tag{6.1}$$

In this transformation (which is convensionaly used in vierbein theory in curved spaces, and also known as the Bargmann theory[9]) we will have, obviously: $x^{(\alpha)} \longrightarrow y^{\alpha}$ (defined by the equation (5.2)), since, in the Bargmann theory, $\gamma_5 \rightarrow \gamma_5$. Therefore, the y^{α} are locally the variables $x^{(\alpha)}$ when they are seen by an observer in curved space. This can also be seen through the equation (5.3), which shows that all happens as if a "coordinate transformation" $x^{(\alpha)} \longrightarrow y^{\alpha}\left(x^{(\lambda)}\right)$ is performed, such that $\eta^{\alpha\beta} \rightarrow g^{\alpha\beta}$.

On the other hand, the quantities $\psi^{\mu\nu}$, being superfields, also depend on the θ^i and are more general than the $g^{\mu\lambda}$. From (5.4), the $\psi^{\mu\lambda}$ can be imagined as coming from the "coordinate transformations" $\theta^i \rightarrow y^{\mu}, \omega^{ij} \rightarrow \psi^{\mu\lambda}$. (It is important to notice that the terminology "coordinate transformation" must be thought of as a transition between variables that are non-homogeneous, such as $z^{(\alpha)} \rightarrow y^{\alpha} \quad , \theta^i \rightarrow y^{\mu}$, and therefore, distinguishable from the usual coordinate transformation, which are different mappings in the same manifold. Here we have transitions between different spaces.)

We saw in the previous section that, according to the analogy with "mixed vierbeins" of Rarita-Schwinger-type, we have the equation (5.5):

$$E^{\mu}_{i} = \frac{\partial y^{\mu}}{\partial \theta^{i}} = h^{\mu}_{(\alpha)} e^{(\alpha)}_{i} \quad .$$

In this notation the equations (4.14) and (5.4) are written as:

$$\psi^{\mu\lambda} = E_i^{\mu} E_j^{\mu} \omega^{ij} \ , \tag{6.2}$$

$$\phi\eta^{\alpha\beta} = e_i^{(\alpha)} e_j^{(\beta)} \omega^{ij} \ . \tag{6.3}$$

We see then, clearly, that in the local transition of gravitational vierbeins we have

$$\eta^{\alpha\beta} \to g^{\alpha\beta} \ , e_i^{(\alpha)} \to E_i^{\alpha} \tag{6.4}$$

and, therefore,

$$\phi\eta^{\alpha\beta} \longrightarrow \phi g^{\alpha\beta} = E_i^{\alpha} E_j^{\beta} \omega^{ij} = \psi^{\alpha\beta} \ , \tag{6.5}$$

which is again equation (5.7).

It is convenient to notice the fundamental fact that the existence of variables y^{μ} presupposes the existence of gravitation (by means of Einstein vierbeins). It follows then, that all the relations previously involving Dirac constant matrices (in Sections 4 and 5) must be corrected. We have from equations (4.25), (4.26), (4.27) and (4.28),

$$\begin{aligned} \phi &= \eta_{\alpha\beta} M_{ik}^{(\alpha)} M_{jl}^{(\beta)} C^{kl} \theta^i \theta^j + \\ &+ \eta_{\alpha\beta} M_{ni}^{(\alpha)} M_{mj}^{(\beta)} A_{(\gamma)}^{nm} M_{kl}^{(\gamma)} \theta^i \theta^j \theta^k \theta^l \ , \end{aligned} \tag{6.6}$$

where, by equation (4.17),

$$M^{(\alpha)} = -\left(C^{-1} \gamma_5 \gamma^{(\alpha)}\right) \ , \quad C^{-1} = -\gamma_{(0)} \ . \tag{6.7}$$

In the transition to the vierbein theory we will have

$$\begin{aligned} &\eta_{\alpha\beta} \longrightarrow g_{\alpha\beta}(x) \ ; \ \gamma_5 \longrightarrow \gamma_5(x) = \gamma_5 \ ; \\ - \ &C^{-1} = \gamma_{(0)} = -\gamma^{(0)} \longrightarrow C^{-1}(x) = \gamma^0(x) = h_{(\tau)}^0 \gamma^{(\tau)} \ ; \\ &M^{(\alpha)} \longrightarrow \mathcal{M}^{\alpha}(x) = -h_{(\tau)}^0 h_{(\beta)}^{\alpha} \left(\gamma^{(\tau)} \gamma_5 \gamma^{(\beta)}\right) \ , \end{aligned}$$

where we have used (6.7).

Consequently, ϕ will transform into a function of coordinates $x^{\mu} : \phi \longrightarrow \phi(\theta^i, x^{\mu})$. For $\phi(\theta^i, x^{\mu})$ to be an Einstein scalar, we must restrict this transition. We see, from (6.6), that

ϕ will be an Einstein scalar under this transition if $M_{ik}^{(\alpha)} \longrightarrow \mathcal{M}_{ik}^{\alpha}$, and $A_{(\gamma)}^{nm} \longrightarrow \mathcal{A}_{\gamma}^{nm}$, as long as $\mathcal{M}_{ik}^{\alpha}$ is an Einstein covector in the index γ. We have that

$$M^{(\alpha)} \longrightarrow \mathcal{M}^{\alpha} = -h_{(\tau)}^{0} h_{(\beta)}^{\alpha} \left(\gamma^{(\tau)} \gamma_5 \gamma^{(\beta)}\right) \quad , \tag{6.8}$$

$$C^{-1} \longrightarrow C^{-1}(x) = h_{(\tau)}^{0} \gamma^{(\tau)} \ . \tag{6.9}$$

We will consider the class of gravitational radiation fields, with the coordinate conditions in the radiation zone:

$$g^{00} = -1, \ \ g^{0i} = 0, \ \ i = 1,2,3 \ . \tag{6.10}$$

Under these conditions:

$$g^{00} = \frac{1}{g^{00}} = -1 \ , \tag{6.11}$$

which implies that it is neglecting, asymptotically, the static sources of Newtonian type ($\Phi = 0$). This is consistent with the conditions of fixing the frame of vierbeins (see, for example, ref. [10]), $h_{(i)}^{0} = 0$ and with the condition $h_{(0)}^{0} = \sqrt{-g^{00}} = 1$. Given these conditions, it follows from equations (6.8) and (6.9) that

$$\begin{aligned} \mathcal{M}^{\alpha} &= -h_{(\beta)}^{\alpha} \left(\gamma^{(0)} \gamma_5 \gamma^{(\beta)}\right) = h_{(\beta)}^{\alpha} \left(\gamma_{(0)} \gamma_5 \gamma^{(\beta)}\right) && (6.12) \\ &= -h_{(\beta)}^{\alpha} \left(C^{-1} \gamma_5 \gamma^{(\beta)}\right) = h_{(\beta)}^{\alpha} M_{ij}^{(\beta)} \ , \\ C^{-1}(x) &= \gamma^{(0)} = -\gamma_{(0)} = C^{-1} \ , && (6.13) \end{aligned}$$

which implies that $\mathcal{M}_{ij}^{\alpha}$ is a contravector and that $C(x) = C$ is constant for the class of coordinate transformations which maintain the conditions (4.25). From now to the end of this section we will restrict ourselves to this class of space-time transformations.

For the coefficients $A_{(\gamma)}^{nm}$ which transform in $\mathcal{A}_{\gamma}^{nm}(x)$, similar conditions are imposed. As they appear in the expression of ω^{nm}, it follows that they are skew-symmetric in the spinorial (Grassmann) indices and therefore take, in general, the form

$$A_{(\alpha)} = \left(A_{(\alpha)}^{ij}\right) = a \left(\gamma_{(\alpha)} \gamma_5 C\right) \quad , a = cte \ . \tag{6.14}$$

Then for the above conditions, $\mathcal{A}_{\alpha}^{ij}(x)$ is obviously a covetor in the α indice. In this way, $\phi(x^{\mu}, \theta^{i})$ is a space-time scalar field and an even element in $\mathcal{A}$. However,in these conditions, and from equation (6.6), $\phi(\theta^{i}, x^{\mu})$ degenerates in a constant in x^{μ}, identical in value to the $\phi(\theta^{i})$ original (see the equations (6.12, (6.13), and (6.6)). This is the only way to obtain an Einstein scalar by means of the transition of local vierbeins; this scalar is, necessarily, a constant scale factor in the gravitational radiation zone.

VII. Conclusion.

By the use of "internal coordinates" $x^{(\alpha)}$, we define an internal bosonic locally flat space. This space is obtained when we combine two Majorana components $(\overline{\theta}, \theta)$ to form an even element in $\mathcal{A}$. The metric of this internal bosonic space is related to the "Grassmann metric" of the form $\omega^{ij}(x^{(\alpha)})$ through equations (4.11) and (4.14). These equations permit us to determine the element $\phi(\theta)$, an even element in $\mathcal{A}$, through the equations (4.25) to (4.28). In the transition of local vierbeins , which define the gravitational field, the variables $x^{(\alpha)}$ transform to variables y^α which are superfields with an even algebraic character in $\mathcal{A}$. They define the Einstein metric through (5.3) in the canonical coordinates $x^{(\alpha)}$. The variables $\psi^{\mu\lambda}$ are defined through (5.4). These variables are superfields of the form (5.7), conformal to the Einstein metric. In the transition of local vierbeins for gravitational radiation in the gauge given by (6.10) the conformal factor ϕ is a scale in the space-time.

To obtain a Brans-Dicke field, ref.[11], in the gravitational radiation zone we must consider a new contribution that is not included in the transition of local vierbeins. This can be obtained if we define the transition:

$$\theta^i \longrightarrow \theta^i(x) = f(x)\theta^i \quad , \tag{7.1}$$

were x^α are coordinates in the Einstein space-time and $f(x)$ is an Einstein scalar. Then,

$$\phi(\theta^i) \longrightarrow \phi(\theta^i, x^\mu) \quad . \tag{7.2}$$

The field equation of the scalar $f(x)$ follows from the Brans-Dicke equation for the radiation metric $g_{\mu\nu}$, in the gauge (6.10). By (6.6),

$$\phi(\theta) \longrightarrow (f(x))^2 \cdot a(\theta) + (f(x))^4 \cdot b(\theta) = \phi\left(\theta^i, x^\mu\right) \quad . \tag{7.3}$$

This is the conformal factor in the Einstein metric.

As a final observation we note that the contribution of the scalar field in the radiation zone is of non-gravitational origin, because it does not come from the choice of the metric, but appears by means of (7.2), independently of geometrical considerations. However, it has its origin in Grassmann variables.

The treatment here presupposes that it is possible to define locally "internal axes" $x^{(\alpha)}$ of a flat space with signature $+2$ in the "canonical coordinates" $x^{(\alpha)}$. Such an internal space would be placed in the gravitational radiation zone. This hypothesis involves, at the same time, the use of "mixed vierbeins" of Rarita- Schwinger-type, which were also proposed in supergravity theories.

APPENDIX A:

Dirac constant matrices satisfy the relation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} 1_4 \tag{A.1}$$

where $\gamma_\mu = \left(\gamma_\mu{}^i{}_j\right), i, j = 1 \cdots 4$. The signature of the Minkowski space used here is $+2$, *i.e.*, $\eta_{\mu\nu} = \text{diag.}(-1, +1, +1, +1)$. Therefore, $\gamma_0^2 = -1$, $\gamma_i^2 = +1$, $i = 1, 2, 3$. In the Majorana representation (used in this work), the Dirac matrices are real. So, in this representation,

$$\gamma_0^\dagger = \gamma_0^T = -\gamma_0 \ ,$$

$$\gamma_i^\dagger = \gamma_i^T = +\gamma_i \ , \ i = 1, 2, 3 \ . \tag{A.2}$$

A set of $\gamma_\mu = \left(\gamma^i_{\mu\ j}\right)$ which satisfies these conditions is:

$$\gamma_0 = \left(\begin{array}{c|c} 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & i\sigma_2 \\ \hline i\sigma_2 & 0 \end{array}\right) \ , \tag{A.3}$$

$$\gamma_1 = \left(\begin{array}{c|c} 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & \sigma_1 \\ \hline \sigma_1 & 0 \end{array}\right) \ ,$$

$$\gamma_2 = \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \\ \hline \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & \sigma_3 \\ \hline \sigma_3 & 0 \end{array}\right) \ ,$$

$$\gamma_3 = \left(\begin{array}{c|c} 1_2 & 0 \\ \hline 0 & -1_2 \end{array}\right) = \left(\begin{array}{c|c} \sigma_0 & 0 \\ \hline 0 & -\sigma_0 \end{array}\right) \ .$$

where $\sigma_1, \sigma_2, \sigma_3$, are the Pauli matrices and $\sigma_0 = 1_2$. In the Majorana representation $\gamma_0 \equiv C$, the charge conjugation matrix.

With these four Dirac matrices we can define 10 (ten) symmetrical matrices $\gamma_\mu C, \sigma_{\mu\nu} C$ and 6 (six) skew-symmetrical matrices $C, \gamma_5 C, \gamma_5\gamma_\mu C$, where $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$. Then, considering the expression with Majorana spinors, $\overline{\theta}_i \overline{\theta}_k (\gamma_\mu C)^{ik} = Tr\left(\Theta (\gamma_\mu C)^T\right)$, where $\Theta = \left(\overline{\theta}_i \overline{\theta}_k\right)$, we have:

$$\overline{\theta}_i \overline{\theta}_k (\gamma_\mu C)^{ik} = \overline{\theta}\gamma_\mu \theta \quad , \tag{A.4}$$

which must be zero, because the majorana spinors do not have electric charge.

$$\overline{\theta}_i \overline{\theta}_k (\gamma_\mu C)^{ik} = 0 \quad , \tag{A.5}$$

and then, Θ is a skew-symmetric matrix, *i.e.*,

$$\overline{\theta}_i \overline{\theta}_k + \overline{\theta}_k \overline{\theta}_i = 0 \tag{A.6-a}$$

and, as $\overline{\theta}_i = C^{-1}_{ik}\theta^k$, we also have

$$\overline{\theta}_i \theta^l + \theta^l \overline{\theta}_i = 0 \quad , \tag{A.6-b}$$

$$\theta^i \theta^j + \theta^j \theta^i = 0 \quad . \tag{A.6-c}$$

Analogously, we have that

$$\overline{\theta}_i \overline{\theta}_k (\sigma_{\mu\nu} C)^{ik} = \overline{\theta}\sigma_{\mu\nu}\theta = 0 \quad , \tag{A.7}$$

because the Majorana spinors do not have magnetic moment. Still, we have for the skew-symmetric matrices:

$$Tr(\ \Theta\ C) \ = \ -\overline{\theta}\theta \neq 0 \quad , \tag{A.8}$$

$$Tr(\ \Theta\ \gamma_5 C) \ = \ -\overline{\theta}\gamma_5\theta \neq 0 \quad , \tag{A.9}$$

$$Tr(\ \Theta\ \gamma_5 \gamma\mu C) \ = \ -\overline{\theta}\gamma_5\gamma_\mu\theta \neq 0 \tag{A.10}$$

where the expression (A.8) is the analogue of the "line element", written for the θ^i (see Section III).

In the section IV, we saw that the bosonic field $x^{(\alpha)}$,

$$x^{(\alpha)} = \overline{\theta}\gamma_5\gamma^{(\alpha)}\theta \tag{A.11}$$

where $\gamma^{(\alpha)}$ are the Dirac constant matrices, can be written as

$$x^{(\alpha)} = M_{ij}^{(\alpha)}\theta^i\theta_j \ , \tag{A.12}$$

$$M^{(\alpha)} = \left(C^{-1}\gamma_5\gamma^{(\alpha)}\right) = -\left(\gamma_1\gamma_2\gamma_3\gamma^{(\alpha)}\right) \ . \tag{A.13}$$

Also, we have in the Majorana representation that,

$$M^{T(\alpha)} = \left(C^{-1}\gamma_5\gamma^{(\alpha)}\right)^T \tag{A.14}$$

i.e., the $M^{(\alpha)}$ are skew-symmetric. Using the set of Dirac matrices (3) we obtain for $M_{(\alpha)} = \eta_{\alpha\beta}M^{(\beta)}$:

$$M_{(\alpha)} = -(-\chi_1, \chi_2, -\chi_3, \chi_4) \ , \tag{A.15}$$

where,

$$\chi_1 = \gamma_5; \ \ \chi_2 = \gamma_2\gamma_3; \ \ \chi_3 = \gamma_1\gamma_3; \ \ \chi_4 = \gamma_1\gamma_2 \ . \tag{A.16}$$

Then, doing the calculation, we have the set:

$$\chi_1 = \left(\begin{array}{c|c} 0 & \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \\ \hline \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \end{array}\right), \quad \chi_2 = \left(\begin{array}{c|c} 0 & \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \\ \hline \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} & 0 \end{array}\right),$$

$$\chi_3 = \left(\begin{array}{c|c} 0 & \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \end{array}\right), \quad \chi_4 = \left(\begin{array}{c|c} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{array}\right). \tag{A.17}$$

Analogously, for $M^{(\alpha)}$,

$$M^{(\alpha)} = -\begin{pmatrix} \gamma_5 \\ \gamma_2\gamma_3 \\ -\gamma_1\gamma_3 \\ \gamma_1\gamma_2 \end{pmatrix} = -\begin{pmatrix} \chi_1 \\ \chi_2 \\ -\chi_3 \\ \chi_4 \end{pmatrix}, \tag{A.18}$$

where we used the fact that, numerically $\gamma^0 = -\gamma_0$, $\gamma^i = +\gamma_i$, $i = 1,2,3$, for signature $+2$.

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