Fermi National Accelerator Laboratory

FERMILAB-Pub-87/154-A
October 1987

INFLATION-PRODUCED, LARGE-SCALE MAGNETIC FIELDS

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Abstract We study the production of large-scale ($\sim$ Mpc) magnetic fields in inflationary Universe models. The magnetic fields produced are uninterestingly small unless the conformal invariance of the electromagnetic field is broken. We consider three ways of breaking the conformal invariance: through gravitational couplings of the photon; through the coupling of the photon to a charged massless, nonconformally-invariant scalar field; and through the anomalous coupling of the photon to axions. The primeval magnetic fields which result can have astrophysically interesting strengths, but are very model-dependent.

Submitted to The Physical Review D.

I. Introduction

Today, magnetic fields are present throughout the Universe and play an important rôle in a multitude of astrophysical situations. Our galaxy and many other spiral galaxies are endowed with coherent magnetic fields (ordered on scales $\gtrsim 10$ kpc) with typical strength[1] $\sim 3 \times 10^{-6}$ G, or energy density relative to the cosmic microwave background radiation (CMBR): $r = (B^2/8\pi)/\rho_\gamma \simeq (B/3.2 \times 10^{-6}\mathrm{G})^2 \sim 1$. The magnetic field of our galaxy plays an important rôle in the dynamics of the galaxy — confining cosmic rays, transferring angular momentum away from protostellar clouds so that they can collapse and become stars (without the loss of angular momentum, protostellar clouds would collapse to a low-density, centrifugally-supported, unstar-like state!); magnetic fields also play an important rôle in the dynamics of pulsars, white dwarfs, and even black holes. Elsewhere in the Universe, magnetic fields are known to exist and be dynamically important — in the intracluster gas of rich clusters of galaxies, in QSOs, and in active galactic nuclei. Finally, we mention a very exotic (but topical) 'use' for primeval magnetic fields: primeval magnetic fields are necessary to initiate substantial currents in superconducting cosmic strings[2] (if such objects exist, they may have important consequences for the Universe — production of UHE cosmic rays[3], and possibly the initiation of structure formation[4]). [The origin and importance of cosmic magnetic fields is discussed in Refs. 5 and 6, and has also recently been reviewed by Rees[7].]

How do these ubiquitous cosmic magnetic fields arise? Many astrophysicists believe that galactic magnetic fields are generated and maintained by dynamo action[8] (whereby the energy associated with the differential rotation of spiral galaxies is converted into magnetic field energy, see Refs. 5 and 6). The dynamo mechanism is an amplification mechanism and requires a seed magnetic field. If it has operated over the entire age of the galaxy (~ 10 Gyr), it could have amplified a seed field by a factor of $\exp(O(30))$, implying that a seed magnetic field of $O\left(3 \times 10^{-19}\mathrm{G}\right)$ is required. Equivalently, a pregalactic cosmic magnetic field strength characterized by $r \simeq 10^{-34}$ is needed. A minority of astrophysicists believe the galactic magnetic field owes it existence to primeval magnetic flux trapped in the gas that collapsed to form the galaxy[9]; in this case the primeval field strength required is $r \simeq 10^{-8}$.

[A pregalactic, cosmic magnetic field which collapses with the gas that forms the galaxy increases in strength as $(\rho_{gal}/\bar{\rho}(t))^{2/3}$, owing to flux conservation (here $\bar{\rho}(t)$ is the average cosmic mass density at time t). Since $\bar{\rho}(t) \propto a^{-3}$ and $\rho_{gal}/\bar{\rho}(t_o) \simeq 10^6$ today ($t = t_o$), it follows that the stength of the magnetic field trapped in the galaxy is: $B_{gal} \simeq 10^4(a(t_{formation})/a(t_o))^2 B_{cosmic}$, or $B_{gal} \simeq 3r^{1/2} \times 10^{-2}G$. From this relationship we obtain $r \simeq 10^{-34}$ to seed the galactic dynamo, and $r \simeq 10^{-8}$ to seed the galactic magnetic field itself.]

Harrison[10] has proposed a mechanism for producing the small seed field ($r \simeq 10^{-34}$) required for the galactic dynamo, wherein the relative motions of protons and electrons induced by vorticity present prior to decoupling produce primeval currents and magnetic fields—of course, this presupposes the existence of primeval vorticity. Other more exotic scenarios have also been suggested (see Refs. 7, 11, 12, and references therein). A fair summary of the present situation is that no compelling mechanism has yet been suggested for the origin of primeval magnetic fields.

Since the Universe through *most* of its history has been a good conductor (see Appendix), any primeval magnetic field present will evolve conserving magnetic flux: $Ba^2 \sim cons't$, or $\rho_B \propto a^{-4}$ (a = the cosmic scale factor), so that the dimensionless ratio $r = (B^2/8\pi)/\rho_\gamma$ remains approximately constant and provides a convenient invariant measure of magnetic field strength[13]. The primeval values of r required for the purposes discussed above are: $r \gtrsim 10^{-34}$ to seed a galactic dynamo; $r \gtrsim 10^{-10}$ to produce astrophysically-interesting currents in superconducting cosmic strings; $r \gtrsim 10^{-8}$ to avoid the necessity of a galactic dynamo altogether.

We believe that inflation[14] is a prime candidate for the production of primeval magnetic fields for three basic reasons: (1) Inflation provides the kinematic means of producing very long wavelength effects at very early times through microphysical processes operating on scales less than the Hubble radius. A given Fourier component (labeled by its comoving wavelength λ or wavenumber $k \equiv 2\pi/\lambda$ and normalized so that λ is the physical wavelength today, i.e., $a_{today} = 1$) crossed outside the Hubble radius $N(= 45 + \ln(\lambda/\mathrm{Mpc}) + 2\ln(M_{14})/3 + \ln(T_{10})/3)$ e-folds before the end of inflation[15] (see Fig. 1). Here M^4 is the vacuum energy density during inflation with $M = M_{14}\ 10^{14} GeV$, and $T_{RH} = T_{10}\ 10^{10} GeV$ is the reheat temperature. Since an electromagnetic wave with $\lambda_{phys} \gtrsim H^{-1}$ has the appearance of static $\vec{E}$ and $\vec{B}$ fields, very long wavelength photons ($\lambda_{phys} \gg H^{-1}$) can lead to large-scale magnetic fields (which become current-supported).

(2) Inflation provides the dynamical means of exciting these long-wavelength electromagnetic waves: de Sitter-produced QM fluctuations excite modes with $\lambda_{phys} \lesssim H^{-1}$; the energy density in the mode with $\lambda_{phys} \simeq H^{-1}$ is: $d\rho/dk \sim H^3$.

(3) During inflation (and perhaps most or all of reheating) the Universe is devoid of charged plasma and is not a good conductor, so that magnetic flux is not necessarily conserved and r can increase.

There are, however, non-trivial obstacles to overcome. A pure U(1) gauge theory with the standard Lagrangian, $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, is conformally invariant, from which it follows that B always decreases as $1/a^2$, irrespective of plasma effects. During the de Sitter phase of inflation, the total energy density in the Universe, ρ_{tot}, is dominated by vacuum energy, $\rho_o \equiv M^4 \propto$ *cons't* and therefore the energy density in any magnetic fields produced during inflation, relative to ρ_{tot}, is greatly suppressed. To be precise, the primeval field energy produced yields a disappointing $r = \rho_B(k)/\rho_\gamma = 10^{-104}\lambda_{Mpc}^{-4}$ independent of T_{RH} and M (here $\rho_B(k) = kd\rho_B/dk$). Thus the conformal invariance of electromagnetism must be broken to produce appreciable primeval magnetic flux. We are quick to point out that nature shows no sign of being conformally invariant! In this paper we study a number of ways of doing this: (i) explicitly break the conformal invariance of U(1) through gravitational couplings, such as $RA_\mu A^\mu$, $R_{\mu\nu}A^\mu A^\nu$; or (ii) $R_{\mu\nu\lambda\kappa}F^{\mu\nu}F^{\lambda\kappa}/m^2$, $R_{\mu\nu}F^{\mu\kappa}F^{\nu}_{\ \kappa}/m^2$, or $RF^{\mu\nu}F_{\mu\nu}/m^2$ (here m^2 is some mass scale squared, as required by dimensional considerations; such terms arise due to 1-loop vacuum-polarization effects in curved spacetime[16]); (iii) couple the photon to a charged field which is not conformally coupled; (iv) through the anomalous coupling of the photon to the axion.

Possibility (iii) in many respects is the most attractive possibility, but is computationally the most challenging. Our preliminary results suggest that it is promising, but we have not completed our analysis.

Possibility (i) is perhaps the least attractive possibility, as such terms explicitly break U(1) gauge invariance by giving the photon a mass squared of the order of H^2 (although at a level which is far below detectability); here $H \equiv (da/dt)/a$ is the expansion rate of the Universe. Computationally it is the most tractable, and with such a term primeval fields with strength as large as $r \sim 10^{-8}$ can be generated(!), with a spectrum, $r(\lambda) \propto \lambda^{-n}$, where n is model-dependent and can be either positive or negative.

There is good theoretical motivation[16] for (ii), and with a mass scale as small as the electron mass. Unfortunately, for the scales of astrophysical interest, $\lambda \sim$ Mpc, the primeval fields produced are typically small, $r \lesssim 10^{-68}$.

The outline of our paper is as follows: in Sec. II we consider possible gravitational couplings of the photon and compute the primeval fields which result; in Sec. III we consider the coupling of the photon to a nonconformally-invariant, massless charged scalar field and to axions; we summarize our work in Sec. IV. In the Appendix we discuss the effects of the conductivity of the Universe.

II. Gravitational Couplings in Electrodynamics

Here we study the production of large-scale magnetic fields in inflationary Universe models due to the direct coupling of the gravitational and electromagnetic fields. We first consider additional terms in the Lagrangian of the form RA^2 and $R_{\mu\nu}A^\mu A^\nu$ (where R is the curvature scalar and A^μ is the electromagnetic potential). These terms give the photon an effective, time-dependent mass. At first sight, this is quite repulsive: gauge invariance (or equivalently charge conservation) is broken. Yet these terms do not lead to any effects which contradict present day observations or experiments. The photon mass that arises due to these terms is: $m_\gamma \sim R^{-1}$ where $R \sim H^{-1}$ is the curvature scale. The photon mass today would be: $m_\gamma \sim H_{today} \sim 10^{-33} eV$, well below present limits to the photon mass: $m_\gamma < 3 \times 10^{-27} eV$ (Ref. 17). Charge nonconservation would only manifest itself on scales of the horizon or larger ($\gtrsim H^{-1} \sim 10^{28}$ cm), but again, this is an effect which has no observable consequences (that we can think of!). Of course, neither the RA^2 nor $R_{\mu\nu}A^\mu A^\nu$ terms would affect the propagation of photons outside massive bodies as both of these terms vanish in vacuum. Finally, one might worry about corrections these terms would introduce to the equation of state in a radiation-dominated Universe. These corrections are of order $H^2/T^2 \sim T^2/m_{pl}^2$, and are negligible for temperatures where the evolution of the Universe is relatively well understood (e.g., nucleosynthesis, recombination, etc.). Thus these terms cannot spoil successful predictions made using the standard Maxwell equations. However, during the de Sitter and reheating phases in an inflationary Universe, the RA^2 terms have a dramatic effect on photons whose wavelengths are greater than the horizon. If certain conditions on the coefficients of these terms are met, then the amplitude of the fluctuations in the A^μ field can grow while outside the horizon, leading to significant large-scale magnetic fields.

We also consider terms of the form RF^2 ($F^{\mu\nu}$ is the electromagnetic field strength tensor) in all possible invariant combinations of R, $R_{\mu\nu}$, and $R_{\mu\nu\lambda\kappa}$ with $F_{\mu\nu}$. The coefficients of these terms must have dimension $(mass)^{-2}$. Such terms have the virtue of being explicitly gauge invariant and thus are far more palatable. Furthermore, there is some indication that these terms are present in the complete theory of quantum electrodynamics in curved space. Drummond and Hathrell[16], for example, have calculated an effective Lagrangian for QED in curved space to one loop. Their expression for the Lagrangian contains all possible RF^2 terms, and the coefficients of these terms are all $O(m_e^{-2})$, where $m_e \simeq 0.511 MeV$ is the electron mass (the electron being the lightest charged particle). [It is not clear, however, whether their work is directly applicable to the problem at hand.] For definiteness though, we take the coefficients of the additional terms to be of the form $cons't/m_e^2$ where the dimensionless constants are of order unity. At early times, when $R^{1/2} \sim H \sim \rho_{tot}^{1/2}/m_{pl} \gg 10^{-11} m_{pl}$ (in a radiation-dominated Universe this corresponds to $T \gg 10^8$ GeV), these terms dominate the usual $F_{\mu\nu}F^{\mu\nu}$ term, while at late times, when $R^{1/2} \ll 10^{-11} m_{pl}$, these terms are negligible. In a model Universe filled with a perfect fluid having an equation of state $p = \gamma\rho$, with $-7/9 < \gamma \lesssim -1/2$ (i.e., so-called power-law inflation[18]), the amplitude of fluctuations in the A^μ field outside the horizon grows. But as will be discussed below, it is difficult to find a scenario using power-law inflation in which the amplitude of the large-scale fields is large enough to be astrophysically interesting.

a) Preliminaries

Before discussing the production of large-scale magnetic fields we review some properties common to both inflation and power-law (or generalized) inflation (PI). We consider spatially flat Friedmann-Robertson-Walker (FRW) cosmologies where the stress energy is described by a perfect fluid with an equation of state, $p = \gamma\rho$. We take the line element to be given by

$$ds^2 = \begin{cases} -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right] \\ a^2(\eta)\left[-d\eta^2 + dx^2 + dy^2 + dz^2\right] \end{cases} \tag{2.1}$$

where t (η) is the clock (conformal) time. In what follows, overdot will always indicate a derivative with respect to conformal time, and ρ_{tot} will always refer to the total energy density of the Universe. We use units where $k_B = c = \hbar = 1$ and $G = m_{pl}^{-2}$, where the planck mass $m_{pl} = 1.22 \times 10^{19} GeV$. Physical length scales (those measured by meter sticks) are related to comoving length scales by: (physical length scale) $= a(t) \times$ (comoving length scale); in addition, we normalize our comoving scales such that today: (physical scale) = (comoving scale), i.e., $a_{today} = 1$. A given Fourier component (or 'scale') will be labeled by its comoving wavelength λ or its comoving wavenumber $k = 2\pi/\lambda$.

The physical size of the presently-observed Universe ($H^{-1} \simeq 10^{28} h^{-1}$ cm $\simeq 3000 h^{-1}$ Mpc; where $H = 100\ h\ km\ sec^{-1} Mpc^{-1}$ is the present value of the Hubble parameter) scales as a, whereas the size of a 'causal domain' (i.e., Hubble sized region, size $\sim H^{-1}$) scales as : $H^{-1} \propto a^{3(1+\gamma)/2}$. During either the radiation- or matter-dominated phases of the standard big bang model, the Hubble radius, H^{-1}, grows faster than the size of the presently-observed Universe (as a^2 and $a^{3/2}$ respectively). Put another way, when we consider early times, the comoving volume which contains the presently-observed Universe was comprised of many causally-distinct regions. This is the celebrated 'horizon problem'. In order to arrange that the region corresponding to the present Hubble volume was once subhorizon-sized, we require that for some period of time in the early Universe, the Hubble radius grows more slowly than a, i.e., $\gamma < -1/3$. It is then possible that the region corresponding to the Universe today was, at some time in the past, contained within a causal region. Moreover, other astrophysical scales λ also begin subhorizon sized, exit the horizon during inflation (first horizon crossing), and then later reënter the horizon (second horizon crossing) during the radiation- or matter-dominated phases. Second horizon crossing for the region corresponding to the presently-observed Universe occurs, by definition, today (see Fig. 1). [Though not always technically correct, we will use interchangably the terms 'Hubble radius' and 'horizon'; we will always precisely mean Hubble radius.]

We need to specify the epoch (time and temperature) at which a given length scale crosses back inside the horizon (i.e., has a physical length $\sim H^{-1}$) during the post-inflation era. It is straightforward to calculate that:

$$T_{hor}(\lambda) = \begin{cases} 73\ eV/\lambda_{Mpc} & \lambda \lesssim 12h^{-2}Mpc \\ 860\ h^{-2}eV/\lambda_{Mpc}^2 & \lambda \gtrsim 12h^{-2}Mpc \end{cases} \tag{2.2}$$

where $\lambda_{Mpc} \equiv \lambda/Mpc$. The two regimes correspond to scales which reenter the horizon before and after the epoch of equal matter and radiation densities ($T_{eq} = 6h^2 eV$ and $t_{eq} \simeq 3 \times 10^{10} h^{-4}$ sec). During the radiation-dominated epoch $T \simeq 1$ MeV$(t/\text{sec})^{-1/2}$, and it also follows that for those scales

$$t_{hor} \simeq 2 \times 10^8 sec \lambda_{Mpc}^2 \tag{2.3}$$

Let us also review some pertinent aspects of inflation[15]. During inflation the Universe is in a nearly de Sitter phase (dS) during which the total energy density, $\rho_{tot} \simeq \rho_o \equiv M^4 \simeq H_o^2 m_{pl}^2$, is approximately constant. After the de Sitter phase follows the reheating epoch (RH) in which the energy density is dominated by the coherent oscillations of the scalar field responsible for inflation and $\rho_{tot} \propto a^{-3}$. During reheating, the temperature T ($\simeq \rho_\gamma^{1/4}$, where ρ_γ is the energy density in light particles produced by the decay of the coherent oscillations) decreases from $(T_{RH}M)^{1/2}$ at the beginning of reheating to T_{RH} at the end of the reheating, when $\rho_{tot} \simeq \rho_\gamma$ and the energy density in coherent oscillations begins to decrease exponentially[19]. Reheating is followed by the usual radiation-dominated (RD) and matter-dominated (MD) phases of the standard big bang model. It is straightforward to show that the comoving length scale λ crossed outside the Hubble radius during inflation (i.e., $a\lambda \simeq H^{-1}$) $N(\lambda)$ e-folds before the end of inflation, where[15]

$$N(\lambda) = 45 + \ln \lambda_{Mpc} + \frac{2}{3}\ln(M_{14}) + \frac{1}{3}\ln(T_{10}) \tag{2.4}$$

and $M = M_{14}10^{14}$ GeV, $T_{RH} = T_{10}10^{10}$ GeV. Setting $\lambda \simeq 3000$ Mpc, it follows that N must be greater than about 53 + 'ln terms' in order that a subhorizon-sized region will grow to a size larger than the presently-observable Universe by the present epoch.

In constructing an acceptable inflationary model one must satisfy two basic constraints on M and T_{RH}. First graviton production leads to the constraint that[20]: $H/m_{pl} < 10^{-4}$, or equivalently that $\rho_o = M^4 < 10^{-8} m_{pl}^4$. This is necessary so that long wavelength gravitational waves produced during inflation and just entering the horizon today do not distort the microwave background beyond the present limits of isotropy[21]. Next, we require that

$M, T_{RH} \gtrsim 1GeV$, which ensures that the Universe is RD by the epoch of primordial nucleosynthesis, so that the successful predictions of primordial nucleosynthesis are not upset. Baryogenesis probably provides an even more stringent constraint on M and T_{RH}; however, at present it is not possible to be more quantitative. The production of adiabatic[22] (and perhaps even isoçurvature[23]) density perturbations also provides a very important and stringent constraint on inflationary scenarios. Although ensuring that adiabatic density perturbations are consistent with the isotropy of the microwave background and/or galaxy formation tends to lead to models with low RH temperatures, this consideration does not directly constrain M and T_{RH} in a simple way.

For generalized inflation, the equation of state during the period of quasi-inflation is $p = \gamma\rho$ with $-1 \leq \gamma < -1/3$, and the total energy density varies as $a^{-3(1+\gamma)}$. A comoving length scale λ crosses outside the Hubble radius when the energy density is

$$\rho_{tot}(\lambda)/m_{pl}^4 = (3.9 \times 10^{-53})^x \lambda_{Mpc}^{-2x} (M/m_{pl})^{4-4x/3} (T_{RH}/m_{pl})^{-2x/3} \tag{2.5}$$

where $x = 3(1+\gamma)/(1+3\gamma)$, M^4 is the energy density at the end of power-law inflation, and T_{RH}^4 is the energy density at the beginning of the usual radiation-dominated epoch. Note that $x \leq 0$ for $-1 \leq \gamma < -1/3$. Consideration of graviton production requires $\rho_{tot}(\lambda)$ to be less than about $10^{-8} m_{pl}^4$ on the scale of the present Hubble radius (the graviton constraint also applies to power-law inflation[18]), and the following constraint follows:

$$x \geq x_{min}, \quad \text{or} \tag{2.6a}$$

$$\gamma \leq \gamma_{max} \equiv (x_{min} - 3)/(3 - 3x_{min}) \simeq -0.86 \tag{2.6b}$$

where $x_{min} = -0.27(1 - 0.14\ln(M_{14}))/(1 + 0.012\ln(M_{14}) + 0.0062\ln(T_{10}))$. Note that the upper bound to γ decreases with increasing x_{min} and approaches -1 for $x_{min} \to 0$ and approaches -1/3 for $x_{min} \to -\infty$. The largest plausible upper limit to γ obtains for $M = T_{RH} \simeq 1$ GeV: $\gamma_{max} \simeq -0.50$.

In this paper, we will be concerned, for the most part, with the evolution of fluctuations in the electromagnetic field whose wavelengths are much greater than the horizon. Recall that a given mode starts with subhorizon size ($a\lambda \lesssim H^{-1}$ or $k \gtrsim aH$), crosses outside the horizon during inflation, and during the subsequent RD or MD phase crosses back inside the horizon (see Fig. 1). It is well known that for a minimally-coupled scalar field in de Sitter space, there are fluctuations in that field whose energy density corresponds to that of a thermal bath at the Gibbons-Hawking temperature, $H/2\pi$ (Refs. 24, 25). We will make the seemingly *reasonable* assumption that this result holds for all massless fields during an inflationary phase and in particular, for the electromagnetic field (although to our knowledge, this is an unverified assumption[26]). Thus a given mode will be initially excited when it is subhorizon-sized during the de Sitter epoch associated with inflation. In particular, this implies that at first horizon crossing, $\rho(k = aH)/\rho_{tot} \simeq (H/m_{pl})^2 \simeq (M/m_{pl})^4$. Here $\rho(k)$ is the energy density in the kth mode: $\rho(k) \simeq kd\rho/dk$. Further, we assume that after a given mode crosses outside the horizon, it can be treated classically, i.e., obeys its classical equations of motion. In essence, we are assuming that a given mode is excited quantum-mechanically while it is subhorizon-sized and then as it crosses outside the horizon 'freezes in' as a classical fluctuation. [We use the same strategy for power-law inflation; see Ref. 18.]

b) RA^2 Terms

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{b}{2}RA^2 - \frac{c}{2}R_{\mu\nu}A^\mu A^\nu \tag{2.7}$$

Where A^μ and $F^{\mu\nu}$ are the electromagnetic potential and field strength tensor. The equations of motion for the photon field are

$$\nabla^\mu F_{\mu\nu} - bRA_\nu - cR^\mu{}_\nu A_\mu = 0 \tag{2.8}$$

$$\partial_\mu F_{\lambda\kappa} + \partial_\kappa F_{\mu\lambda} + \partial_\lambda F_{\kappa\mu} = 0 \tag{2.9}$$

where all spatial derivatives are with respect to comoving coordinates. To study these equations we write them in terms of the electric and magnetic fields where

$$F_{\mu\nu} = a^2 \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{2.10}$$

Using the fact that $R^i{}_i = \ddot{a}/a^3 + (\dot{a}/a^2)^2$ (no sum on i) and $R = 6\ddot{a}/a^3$, we can write, for Eqns. (2.7) and (2.8)

$$\frac{1}{a^2}\frac{\partial}{\partial\eta}a^2\vec{E} - \vec{\nabla}\times\vec{B} - \frac{n}{\eta^2}\frac{\vec{A}}{a^2} = 0 \tag{2.11}$$

$$\frac{1}{a^2}\frac{\partial}{\partial\eta}a^2\vec{B} + \vec{\nabla}\times\vec{E} = 0 \tag{2.12}$$

where

$$n \equiv \eta^2\left(6b\frac{\ddot{a}}{a} + c\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)\right) \tag{2.13}$$

Note that n is a constant whenever $a(\eta)$ varies as a power of η, which occurs in all cases of interest to us. Taking the curl of Eqn. (2.11) and using Eqn. (2.12) to eliminate $\vec{E}$ we find

$$\frac{1}{a^2}\frac{\partial^2}{\partial\eta^2}a^2\vec{B} - \nabla^2\vec{B} + \frac{n}{\eta^2}\vec{B} = 0. \tag{2.14}$$

This equation is linear in $\vec{B}$ and can easily be expanded in terms of its Fourier components. With the definition $\vec{F}_k(\eta) \equiv a^2 \int d^3x e^{i\vec{k}\cdot\vec{x}}\vec{B}(\vec{x},\eta)$ we have

$$\ddot{\vec{F}}_k + k^2\vec{F}_k + \frac{n}{\eta^2}\vec{F}_k = 0. \tag{2.15}$$

The quantity $\vec{F}_k$ is a measure of the magnetic flux associated with the comoving scale $\lambda \sim k^{-1}$. The energy density in the kth mode of the magnetic field is: $\rho_B(k) \propto |\vec{F}_k|^2/a^4$, where as usual $\rho_B(k) = k d\rho_B/dk$.

It is useful at this stage, to compare this equation with the equation of motion for a massless scalar field which is coupled to gravity through the usual $\xi R\phi^2$ term ($\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\xi R\phi^2$). In terms of $\omega = a\phi$ the equation of motion for the kth Fourier component is

$$\ddot{\omega}_k + k^2\omega_k + \frac{n_\phi}{\eta^2}\omega_k = 0 \tag{2.16}$$

where $n_\phi \equiv \eta^2(6\xi - 1)\ddot{a}/a$, and as usual $\phi_k(\eta) = \int d^3x e^{i\vec{k}\cdot\vec{x}}\phi(\vec{x},\eta)$. Note that $\rho_\phi(k) \propto |\omega_k|^2/a^4$ so that there is a direct correspondence between ω_k and $\vec{F}_k$. Moreover, if $n = n_\phi$ then Eqns. (2.15) and (2.16) are equivalent. (A distinction one must keep in mind is that $\vec{F}_k$ carries a vector index while ω_k is a scalar quantity.) The condition $n = n_\phi$ implies a relation between b, c, and ξ (though this relationship can be different during different phases in the evolution of the Universe). We note that for $b = -1/6$ and $c = 0$, A^μ behaves like a minimally-coupled scalar field ($\xi = 0$), while for $n = 0$ (the usual gauge-invariant Maxwell theory), A^μ behaves as a conformally-coupled scalar field ($\xi = 1/6$).

For modes well outside the horizon, $a\lambda \gg H^{-1}$ or $|k\eta| \ll 1$, and we have that $|\vec{F}_k| \propto \eta^{m_\pm}$ where $m_\pm = 1/2\left(1 \pm \sqrt{1-4n}\right)$. During dS, $a \propto -1/H_o\eta$ so that $\ddot{a}/a = 2(\dot{a}/a)^2 = 2/\eta^2$, $n = 12b + 3c$, and $|\vec{F}_k| \propto a^{-m_\pm}$. During either RH or MD, when $\rho \propto a^{-3}$, we have that $a \propto \eta^2$, $n = 12b + 6c$, and $|\vec{F}_k| \propto a^{m_\pm/2}$. During the RD epoch, when $\rho \propto a^{-4}$, we have that $a \propto \eta$, $n = c$, and $|\vec{F}_k| \propto a^{m_\pm}$. Again we note that for $b = -1/6$ and $c = 0$, A^μ behaves precisely like a minimally-coupled scalar field; $|\vec{F}_k| \propto a$ and $\rho_B \propto a^{-2}$.

For $n = 0$ (standard electromagnetic theory), or $n_\phi = 0$ in the case of a scalar field, the energy density associated with a given mode always decreases as a^{-4}, just as one would expect for a conformally-coupled field. However, in the minimally-coupled case, $\xi = 0$ for the scalar field or $b = -1/6, c = 0$ for the electromagnetic field, the energy density in a given mode only decreases as a^{-2} when the mode is well outside the horizon. In the case of the graviton or a minimally-coupled scalar field this is known as 'superadiabatic amplification' (Refs. 27). [The equation of motion for gravitons, the tensor perturbations of $g_{\mu\nu}$, is precisely that of a minimally-coupled scalar field.] For $\xi < 0$, the energy density decreases even more slowly. Physically this occurs because for $\xi < 0$, the field has a negative effective mass squared term which indicates an instability.

Once b and c are specified, it is easy to compute the behavior of $\vec{F}_k$ as a function of $a(\eta)$. It is then straightforward to compute the amplitude of a given fluctuation and the energy density associated with that fluctuation. Before doing so, we must consider the effects of the conducting plasma in the Universe.

In a highly conducting plasma, one expects the magnetic flux through an arbitrary comoving loop to remain constant. In the expanding Universe, this implies that $\rho_B \propto a^{-4}$. This well-known result can be seen directly from the Maxwell equations by including a current source term $\vec{J} = \sigma_c \vec{E}$ and letting $\sigma_c \to \infty$. We show this for our modified Maxwell equations in the Appendix.

Here we simply summarize the conclusions reached in the Appendix. The conductivity of the Universe is proportional to the temperature of the charged particles present. During the de Sitter phase, the temperature is exponentially small and the conductivity is negligible. During reheating the coherent oscillations of the inflating field are converted into relativistic particles. Assuming that a reasonable fraction of the particles produced are charged, by the end of RH, the conductivity is very high and, in fact, is the dominant effect for determining the evolution of the magnetic field. We conclude that there is some temperature T_*, $(MT_{RH})^{1/2} \lesssim T_* \lesssim T_{RH}$, at which the plasma effects become dominant, and that for $T \lesssim T_*$, ρ_B is necessarily $\propto a^{-4}$. An exact calculation of T_* depends on the details of the reheating process. In particular, one must track the number density of light, charged particles during RH. In the Appendix, we estimate the value of T_*. We find that $T_* \sim min\left((T_{RH}M)^{1/2},\ (T_{RH}^2 m_{pl})^{1/3}\right)$ though we must stress that this is only an order of magnitude calculation. Although the details of reheating (e.g., what types of particles are created as the coherent oscillations decay, etc.) are model-dependent and hence uncertain, it seems very certain that by the time the Universe becomes radiation-dominated ($T = T_{RH}$) the conductivity will be very high.

We are now ready to calculate the energy density in the kth mode of the magnetic field. Let $p \equiv m_- = 1/2\left(1 - \sqrt{1 - 48b - 12c}\right)$ (m_- being evaluated in dS) and $q \equiv m_+ = 1/2\left(1 + \sqrt{1 - 48b - 24c}\right)$ (m_+ being evaluated in RH). The exponents p and q correspond to the fastest growing solutions for $\vec{F}_k$ in dS and RH respectively. As discussed earlier, we assume that quantum fluctuations in the electromagnetic field are excited during the de Sitter expansion and that once these fluctuations cross outside the horizon, they can be treated as classical fluctuations in the electromagnetic field. At first horizon crossing ($a = a_1$), the ratio of the energy density stored in the kth mode magnetic field fluctuation, $\rho_B(k)$, to the total energy density in the Universe, ρ_{tot}, is given by

$$\left.\frac{\rho_B(k)}{\rho_{tot}}\right|_{a=a_1} \simeq \left(\frac{M}{m_{pl}}\right)^4 \tag{2.17}$$

During the rest of the dS, $\rho_B(k) \propto a^{-2(p+2)}$ while the total energy density of the Universe $\rho_{tot} \simeq \rho_o \equiv M^4 \propto cons't$. During RH, for $T \gtrsim T_*$, $\rho_B(k) \propto a^{(q-4)}$, while the energy density of the Universe, $\rho_{tot} \propto a^{-3}$. For $T \lesssim T_*$, $\rho_B \propto a^{-4}$, due to the high conductivity of the Universe. The invariant measurement of the magnetic flux on the scale λ, $r{=}\rho_B(k)/\rho_\gamma$, is therefore

$$r \;\simeq\; e^{-2N(p+2)} \left(\frac{M}{m_{pl}}\right)^{4(q+2)/3} \left(\frac{T_{RH}}{m_{pl}}\right)^{4(q+1)/3} \left(\frac{T_*}{m_{pl}}\right)^{-8q/3} \tag{2.18}$$

where $N(\lambda)$ is the number of e-folds the Universe expands between first horizon crossing and the end of inflation. The wavelength dependence of r in Eqn.(2.18) enters through $N(\lambda)$. Using Eqn.(2.4) for $N(\lambda)$, we find

$$r \simeq (7\times 10^{25})^{-2(p+2)} \left(\frac{M}{m_{pl}}\right)^{4(q-p)/3} \left(\frac{T_{RH}}{m_{pl}}\right)^{2(2q-p)/3} \left(\frac{T_*}{m_{pl}}\right)^{-8q/3} \lambda_{Mpc}^{-2(p+2)} \tag{2.19}$$

irrespective of whether T_{hor} occurs during RD or MD.

Before evaluating r for different inflationary scenarios (i.e., different choices of M and T_{RH}) it is important to recall the two constraints to M and T_{RH} discussed earlier. First, production of very long wavelength gravitons during dS leads to distortions in the microwave background; the requirement that these distortions not exceed the present upper limits to the microwave anisotropy leads to the constraint that $M^4 < 10^{-8} m_{pl}^4$. Secondly, we require that M, $T_{RH} \gtrsim 1GeV$ so the Universe becomes RD before nucleosynthesis.

Using Eqn. (2.19) one can determine the energy density in the magnetic field on the comoving scale λ. To do so, one must specify M, T_{RH}, and p and q (or equivalently b and c). We illustrate, in Table I, the range of results possible with a few examples. The measure of the magnetic flux $r = (\rho_B/\rho_\gamma)|_{1\,Mpc}$, as well as p, q, M, T_{RH}, and the λ–dependence of r are tabulated in Table I.

Clearly, there is a wide range of choices for p and q (or equivalently, b, c), and M and T_{RH} such that the strength of the large-scale magnetic field generated could be astrophysically-interesting.

c) RF^2 Terms

We now consider the coupling of gravitational and electromagnetic fields through terms in the Lagrangian of the form RF^2. The most general Lagrangian containing such terms can be written

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \mathcal{L}_{\mathcal{G}} \tag{2.20}$$

$$\mathcal{L}_{\mathcal{G}} \;=\; -\frac{1}{4m_e^2}\left[bRF_{\mu\nu}F^{\mu\nu} \;+\; cR_{\mu\nu}F^{\mu\kappa}F^{\nu}{}_{\kappa} \;+\; dR_{\mu\nu\lambda\kappa}F^{\mu\nu}F^{\lambda\kappa}\right] \tag{2.21}$$

where, as previously discussed, we take the dimensionful quantity in the coupling constant of the additional terms to be the electron mass. The equations of motion, found by varying the action with respect to A^μ, are

$$\nabla^\mu F_{\mu\nu} \;+\; \frac{1}{m_e^2}\nabla^\mu\left[bRF_{\mu\nu} \;+\; c/2\left(R^\lambda{}_\mu F_{\lambda\nu} \;-\; R^\lambda{}_\nu F_{\lambda\mu}\right) \;+\; dR^{\lambda\kappa}{}_{\mu\nu}F_{\lambda\kappa}\right] \;=\; 0. \tag{2.22}$$

These equations were studied by Drummond and Hathrell[16] who computed the coefficients b, c, and d by calculating one-loop vacuum polarization diagrams in curved space. Since their results are probable not applicable for the case of interest to us: $R/m_e^2 \gg 1$, we will leave b, c, and d as arbitrary parameters.

We will study Eqn.(2.22) in an FRW background spacetime with $p = \gamma\rho$ for which it follows that $\rho_{tot} \propto a^{-3(1+\gamma)} \propto \eta^{-6(1+\gamma)/(1+3\gamma)}$ and $H^{-1} \propto \rho_{tot}^{-1/2} \propto a^{3(1+\gamma)/2}$. The curvature tensors are functions of time only and the non-zero components are given here in terms of ρ_{tot} and γ:

$$R \;=\; 8\pi\rho_{tot}(1-3\gamma)/m_{pl}^2 \tag{2.23}$$

$$R^0{}_0 \;=\; -4\pi\rho_{tot}(1+3\gamma)/m_{pl}^2 \quad R^i{}_i \;=\; 4\pi\rho_{tot}(1-\gamma)/m_{pl}^2 \tag{2.24}$$

$$R^{0i}{}_{0i} \;=\; -4\pi\rho_{tot}(1+3\gamma)/3m_{pl}^2 \quad R^{ij}{}_{ij} \;=\; 8\pi\rho_{tot}/3m_{pl}^2 \qquad i \neq j \tag{2.25}$$

where $i, j = 1, 2, 3$ are spatial indices. Here and throughout this subsection, there are *no* implicit sums over spatial indices.

The non-standard terms in Eqn.(2.22) are formally of the order of R/m_e^2 relative to the usual $\nabla^\mu F_{\mu\nu}$ term. Since $R \sim O(\rho_{tot}/m_{pl}^2)$, they dominate (are smaller than) the usual $\nabla^\mu F_{\mu\nu}$ term for $\rho_{tot} \gtrsim m_{pl}^2 m_e^2 \simeq (10^8 GeV)^4$ $(\rho_{tot} \lesssim (10^8 GeV)^4)$. For $R/m_e^2 \gg 1$, we can neglect the usual $\nabla^\mu F_{\mu\nu}$ term in Eqn.(2.22). Furthermore, in evaluating the covariant derivatives, one finds that the terms involving Christoffel symbols drop out and so we have

$$\partial^\mu\left[bRF_{\mu\nu} \;+\; c/2\left(R_\mu{}^\lambda F_{\lambda\nu} \;-\; R_\nu{}^\lambda F_{\lambda\mu}\right) \;+\; dR_{\mu\nu}{}^{\lambda\kappa}F_{\lambda\kappa}\right] \;=\; 0. \tag{2.26}$$

It also follows that the equations for $A_i(k,\eta) = \int d^3k e^{i\vec{k}\cdot\vec{x}} A_i(\vec{x},\eta)$ in the Coulomb gauge, $A_0 = \sum_{i=1}^{3} \partial_i A_i = 0$, are

$$\begin{aligned} b\Big[R\Big(\ddot{A}_i \,+\, k^2 A_i\Big) \,+\, \dot{R}\dot{A}_i\Big] \\ + \frac{c}{2}\Big[\Big(R^0{}_0 \,+\, R^i{}_i\Big)\ddot{A}_i \,+\, 2R^i{}_i k^2 A_i \,+\, \Big(\dot{R}^0{}_0 \,+\, \dot{R}^i{}_i\Big)\dot{A}_i\Big] \\ + \, 2d\Big[R^{0i}{}_{0i}\ddot{A}_i \,+\, R^{ij}{}_{ij} k^2 A_i \,+\, \dot{R}^{0i}{}_{0i}\dot{A}_i\Big] \;=\; 0. \end{aligned} \tag{2.27}$$

We note that in de Sitter space ($\gamma = -1$), Eqn.(2.26) reduces to

$$\rho_o\,(12b \,+\, 3c \,+\, 2d)\,\partial^\mu F_{\mu\nu} \;=\; 0 \tag{2.28}$$

where, because H_o is a constant, the solutions are the same as those found using the usual Maxwell equations in flat space. This point was discussed by Drummond and Hathrell[16].

We are interested in modes well outside the horizon, $|k\eta| \ll 1$, and so we can neglect terms of order $k^2 RA$. Using Eqns.(2.23-2.25) to evaluate the various components of the curvature tensors in Eqn.(2.27) we find

$$\Big(6b(1-3\gamma) \;-\; 6c\gamma \;-\; 2d(1+3\gamma)\Big)\,\Big[\ddot{A}_i \,+\, \frac{s}{\eta}\dot{A}_i\Big] \;=\; 0 \tag{2.29}$$

where $s \equiv d\ln\rho/d\ln\eta \;=\; -6(1+\gamma)/(1+3\gamma)$. The solutions to this equation, which are *independent* of b, c, and d, are $A_i = cons't$ and

$$\dot{A}_i \;=\; \dot{A}_i(\lambda)\frac{\rho_{tot}(\lambda)}{\rho_{tot}} \tag{2.30a}$$

$$A_i \;=\; A_i(\lambda)\left(\frac{a}{a_1}\right)^{(7+9\gamma)/2} \;+\; cons't \tag{2.30b}$$

where $\rho_{tot}(\lambda)$ $(A_i(\lambda))$ is defined to be the total energy density (electromagnetic potential) when the scale λ crosses outside the horizon during inflation (when $a = a_1$). The constant term in Eqn.(2.30b) leads to $\rho_B \propto a^{-4}$ and is therefore uninteresting for our purposes.

It is useful at this point, to compare these results with those for a free Maxwell field (i.e., $b = c = d = 0$). For the free field, $A^\mu \propto (\cos k\eta,\ \sin k\eta)$ irrespective of the value of $k\eta$, and which for $|k\eta| \ll 1$, reduces to $A^\mu \propto (cons't,\ \eta)$. [That the form of the solution is independent of whether $k\eta < 1$ or $k\eta > 1$ reflects the conformal invariance of a free Maxwell field.] During the de Sitter expansion, $\eta \propto -1/H_o a$ and $\gamma = -1$, and it is easy to see that the results for the usual free Maxwell field and for the RF^2-coupled Maxwell field coincide, as was noted above. One can see that for $\gamma > -7/9$, there is a growing mode solution for A^μ which can quickly come to dominate over the $A^\mu = cons't$ solution.

Thus, if we are to obtain significant production of primordial magnetic flux we are lead to consider power-law inflation[28].

As discussed earlier, for acceptable power-law inflation we must have

$$-1 < \gamma < \gamma_{max} \tag{2.31a}$$

$$\gamma_{max} = (x_{min} - 3)/(3 - 3x_{min}) \tag{2.31b}$$

where $x_{min} \simeq 0.135(1 + 0.5\log(M/m_{pl}))/(1 + 0.022\log(M/m_{pl}) + 0.011\log(T_{RH}/m_{pl}))$, and the plausible upper limit to γ_{max} is ~ -0.5. We also need to have $-7/9 < \gamma < \gamma_{max}$ to take advantage of the growing mode solution.

We now calculate the energy density in large scale magnetic fields relative to the CMBR. For simplicity, we take $T_{RH} = M$. Relaxing this assumption does not qualitatively change our results. At first horizon crossing we have $\rho_B/\rho_{tot}(\lambda) = \rho_{tot}(\lambda)/m_{pl}^4$ for power-law inflation. This is just the analogue of Eqn. (2.17). During PI, $\rho_B/\rho_{tot} \propto a^{6(1+2\gamma)} \propto \rho_{tot}^{-2(1+2\gamma)/(1+\gamma)}$, where we assume that $\gamma > -7/9$ and consider the fastest growing mode of A. This behavior continues until $\rho_{tot} = max(Cm_e^2 m_{pl}^2, \rho_*)$ where $Cm_e^2 m_{pl}^2$ is the energy density at the time when the RF^2 terms become subdominant and $\rho_*(\leq M^4)$ is the energy density when the plasma effects become important and freeze in the magnetic flux present. Here $C = [6b(1-3\gamma) - 6c\gamma - 2d(1+3\gamma)]^{-1}$ (see Eqn. (2.29)). First, consider the case: $Cm_e^2 m_{pl}^2 \gtrsim M^4$ (i.e., growth ceases before plasma effects become important). Again, we compute $r = \rho_B/\rho_\gamma$, the energy density in the magnetic field fluctuation with comoving wavelength λ relative to the CMBR:

$$r = \frac{\rho_{tot}(\lambda)}{m_{pl}^4}\left(\frac{\rho_{tot}(\lambda)}{Cm_e^2 m_{pl}^2}\right)^{2(1+2\gamma)/(1+\gamma)} \left(\frac{M^4}{Cm_e^2 m_{pl}^2}\right)^{(1-3\gamma)/3(1+\gamma)} \tag{2.32}$$

Of course, γ must satisfy $\gamma < \gamma_{max}(M, T_{RH})$. $\rho_{tot}(\lambda)$ is given in terms of $\rho_{tot}(3000)$, the total energy density when the present Hubble volume crossed outside the horizon during power law inflation, by the expression

$$\rho_{tot}(\lambda)/\rho_{tot}(3000) = (3.8 \times 10^6)^x \lambda^{-2x}. \tag{2.33}$$

Bringing everything together, we have

$$\begin{aligned}\log r = &\left(\frac{3+5\gamma}{1+\gamma}\right)\log(\rho_{tot}(3000)/m_{pl}^4) + 15\left(\frac{7+9\gamma}{1+\gamma}\right) + 1.3\left(\frac{1-3\gamma}{1+\gamma}\right)\log(M/m_{pl}) \\ &+ 6.6y - 2y\log\lambda_{Mpc} - C'\end{aligned} \tag{2.34}$$

where $y \equiv 3(3+5\gamma)/(1+3\gamma)$ and $C' \equiv (7+9\gamma)/3(1+\gamma)\log C$.

We will now evaluate the energy density in large-scale magnetic fields for different power law inflationary scenarios (i.e., different choices of M and γ). It is both difficult and cumbersome to analyze the above expressions in general and so we search for the 'best case scenario'. To begin we saturate the graviton constraint (Eqn.(2.6)): $\rho_{tot}(3000)/m_{pl}^4 = 10^{-8}$ so that

$$\gamma \simeq \gamma_{max} \simeq (x_{min} - 3)/(3 - 3x_{min}) \tag{2.35}$$

where $x_{min} \simeq 0.135(1 + 0.5\log(M/m_{pl}))/(1 + 0.033\log(M/m_{pl}))$. We then evaluate $r = \rho_B/\rho_\gamma$ for values of M and γ satisfying Eqn. (2.35). Our results for r and the scaling of r with λ are given in Table II.

The best case scenario gives $r(1\ \mathrm{Mpc}) \sim 10^{-68}$, apparently far too small to be important for galactic dynamos or other astrophysical effects. This failure illustrates a problem generic to the RF^2 terms. On the one hand, it is clear from Eqns. (2.30) that growth in the electromagnetic field requires ρ_{tot} to be changing rapidly (i.e., $\gamma \gg -1$). On the other hand, we want fields to exit the horizon during PI and reenter the horizon during RD or MD and this requires that during PI, $H \propto \rho_{tot}^{1/2}$ is not changing too rapidly (i.e.,$\gamma \lesssim -1/2$). Finally, there is the fact that the RF^2 terms are negligible for $T < 10^8 GeV$. The end result, when all of the various constraints are taken into account, is that the energy density in large scale magnetic fields arising from the RF^2 terms is too small.

III. Scalar and Axion Electrodynamics

In this section, we discuss preliminary work on models in which the electromagnetic field is coupled to other non-conformal, matter fields. In particular, we consider a massless, charged scalar field, minimally-coupled to both gravity and the electromagnetic field. Scalar electrodynamics in very special cosmological models has been considered by a number of authors[29]. Ford[30] has studied the stability of a charged scalar field in de Sitter space to determine if an instability of the coupled system might render de Sitter space unstable, and perhaps provide a mechanism for cancelling off any cosmological constant. Here, our hope is that the charged scalar field will act as a source for the electromagnetic field. We also consider the axion, which, through the anomaly couples to $\vec{E}\cdot\vec{B}$. Thus the axion field too could provide a source term for large scale magnetic fields.

The Lagrangian for massless scalar electrodynamics is

$$\mathcal{L} = -D_\mu\phi(D^\mu\phi)^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3.1}$$

where for simplicity we are neglecting the ϕ field's coupling to other fields. We note that it is not necessary that the ϕ field be exactly massless; only that its mass be $\ll H$ during the epochs of interest ($\rho_{tot} \gtrsim T_{RH}^4$). The complex scalar field ϕ couples to electromagnetism through the usual gauge covariant derivative, $D_\mu = \partial_\mu - ieA_\mu$.

The equations of motion in an FRW background are

$$\ddot{A}_i \;-\; \nabla^2 A_i \;=\; iea^2\left(\phi\partial_i\phi^* \;-\; \phi^*\partial_i\phi\right) \;-\; 2e^2a^2A_i|\phi|^2 \tag{3.2}$$

$$\ddot{\phi}_i \;+\; 2\frac{\dot{a}}{a}\dot{\phi} \;-\; \nabla^2\phi \;=\; -\sum_i\left(2ieA_i\partial_i\phi \;+\; e^2A_iA_i\phi\right) \tag{3.3}$$

where, as before, we work in the Coulomb gauge, $A_o = \partial_i A_i = 0$. The source term on the right hand side of Eqn. (3.2) contains two terms; a source term involving only the ϕ-field and a charge density term which gives an effective mass to the photon.

The coupled equations are difficult to solve as they are non-linear. We are interested in the evolution of a particular Fourier mode A_k of the electromagnetic field. The kth Fourier component of a term such as $\phi^*\partial_i\phi$ is

$$\int d^3x\; e^{ik\cdot x}\phi^*\partial_i\phi \;=\; i\int d^3q\; q_i\phi^*_{k-q}\phi_q \tag{3.4}$$

As a reasonable first approximation, we replace this expression by $k^3|\phi_k|^2\; k_i$.

The equation of motion, for A_k with $|k\eta| \ll 1$ can then be written

$$\ddot{A}_k \;\sim\; 2ea^2k^4|\phi_k|^2\left(1 \;+\; ek^2A_k\right) \tag{3.5}$$

where in the above expression, we have dropped the index i. Neglecting the backreaction of the electromagnetic field on the scalar field, for $|k\eta| \ll 1$ we have: $|\phi_k|^2k^3 \simeq H_o^2$, and it follows that

$$\ddot{A}_k \sim 2ea^2kH_o^2\left(1 \;+\; ek^2A_k\right) \tag{3.6}$$

We first study this equation to lowest order in e though we will show in a moment that such an analysis is fundamentally flawed. Keeping only the lowest order term in e, the current term, one finds that in dS, $A_k \propto \mathit{lna} \;+\; \mathit{cons't}$ and in RH, $A_k \propto a^3$. This would imply that during dS magnetic flux undergoes slow growth ($\rho_B/\rho_{tot} \;\propto\; (\ln a)^2a^{-4}$) and that during RH it undergoes rapid growth: $\rho_B/\rho_{tot} \propto a^5$. One might suppose that this indicates an efficient transfer of energy from the scalar field to the electromagnetic field during RH. However, if one takes into account the second order term in e, then the solution is apparently $A_k = -1/ek^2 + (\mathit{decaying\ terms})$. We do not claim that this is indeed the correct solution or even that it displays the gross features of the correct solution. The appearance of the coupling constant in the denominator suggests that non-perturbative effects are important and that a perturbative analysis may be flawed. We do believe that the scalar electrodynamic system is potentially very interesting and we are currently studying the full coupled equations of motion.

Next consider axion electrodynamics. For energies well below the Peccei-Quinn symmetry breaking scale f_a, the effective Lagrangian for axion electrodynamics is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\theta\partial^\mu\theta - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g_a\theta F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{3.7}$$

where g_a is a coupling constant of the order α, and the vacuum angle $\theta = \phi_a/f_a$ (ϕ_a = axion field). The equations of motion are

$$-\frac{1}{a^2}\frac{\partial}{\partial\eta}a^2\vec{E} + \vec{\nabla}\times\vec{B} = g_a\left(\dot{\theta}\vec{B} + \vec{\nabla}\theta\times\vec{E}\right) \tag{3.8}$$

$$\frac{1}{a^2}\frac{\partial}{\partial\eta}a^2\vec{B} + \vec{\nabla}\times\vec{E} = 0 \tag{3.9}$$

$$\ddot{\theta} + 2\frac{\dot{a}}{a}\dot{\theta} + k^2\theta + g_a a^2\vec{E}\cdot\vec{B} = 0. \tag{3.10}$$

The axion field, like other scalar fields, will be excited in de Sitter space, giving rise to $<\theta^2>\sim(H_o/f_a)^2$, which in principle can act as a source term for the electromagnetic field A^μ. The coupled equations are difficult to solve and at present, we have not completed our analysis. We note however that the model is similar to the RF^2 models. The current on the right hand side of Eqn. (3.8) which could potentially be a source term for large scale magnetic fields, depends on derivatives of the axion field and we must look for models in which θ is rapidly changing.

IV. Summary

The origin of the primeval magnetic flux required to seed the magnetic fields which are so ubiquitous and so important in the Universe today is still uncertain. A primordial seed field on the scale of $\sim$ Mpc as small as $r\simeq 10^{-34}$ might be sufficient, and primordial fields as large as $r\simeq 10^{-8}$ could be required, and could have other interesting cosmological consequences, e.g., initiating currents in superconducting cosmic string loops (if they are indeed present). For three reasons inflation seems like an ideal candidate mechanism for generating such large-scale, primeval fields very early on. Again these reasons are: (i) inflation provides the kinematic means of producing very large scale phenomena via microphysics operating on subhorizon scales (see Fig. 1); (ii) inflation, through de Sitter-space-produced quantum fluctuations, provides the means of exciting quantum fields, including the electromagnetic field; (iii) inflation takes place before the Universe is filled with a highly-conducting, charged plasma and so it is possible for the magnetic flux (equivalently r) to increase.

The fundamental obstacle that one must face is the conformal invariance of the free field Maxwell theory (pure $U(1)$ gauge theory). Conformal invariance insures that $\rho_B \propto a^{-4}$, irrespective of wavelength. In this case the primeval flux produced is a disappointing $r = \rho_B(k)/\rho_\gamma \simeq 10^{-104}\lambda_{Mpc}^{-4}$, independent of M and T_{RH}. Such a primeval magnetic flux in apparently very far from being of astrophysical or cosmological interest.

The key then is to break the conformal invariance of electromagnetism. We have considered two mechanisms in detail: (1) 'RA^2 terms' which explicitly break conformal as well as gauge invariance; and (2) gauge-invariant 'RF^2 terms', which arise in any case from one-loop gravitational corrections. Both possibilities are computationally straightforward to analyze. In case (1) the field equations can be recast in a form analogous to those for a massless, free scalar field, and the primeval fields which can be generated are substantial, easily as large as $r \simeq 10^{-8}$. In the second case, amplification above the 'conformal result' only occurs for power-law inflation, and the largest primeval magnetic flux produced is only of order 10^{-68} (on the scale of 1 Mpc).

We also briefly discussed two other possibilities for breaking the conformal invariance of electromagnetism: (1) the addition of a massless, charged and non-conformally-coupled scalar field; (2) the coupling of the electromagnetic field to an axion (via the anomaly). While both of these possibilities seem on the face of it more natural, the analysis is more difficult. Possibility (1) seems very promising, and work is still in progress.

In sum, the generation of seed, primordial magnetic fields through inflation still appears to be a very attractive possibility. However, because the seed fields which result depend upon breaking conformal invariance and, moreover, the details of how it is broken, at this time a clean, definitive prediction does not seem possible. This, of course, is in stark contrast to the predictions for the resulting adiabatic density perturbations and gravitaitional wave perturbations and, is understandably somewhat disappointing. There is the hope that perhaps in the not too distant future a unified theory of nature (e.g., superstrings) could rectify this situation by making definite predictions about the gravitational couplings of all the quantum fields, thereby eliminating the arbitrariness in the present calculation and making a definitive prediction possible.

Acknowledgments

We gratefully acknowledge stimulating discussions with F. Accetta, G. Field, C. Hill and J. Silk. This work was supported in part by the DoE (at Chicago and Fermilab), the NASA (at Fermilab), MST's Alfred P. Sloan Fellowship, and LMW's NASA GSRP

fellowship.

Appendix

In this Appendix, we discuss in more detail, the effects of the conducting plasma in the Universe on the evolution of cosmological magnetic fields. These effects are take into account by including, on the right hand side of Eqn. (2.11), a current source term $\vec{J} = \sigma_c \vec{E}$, where σ_c is the conductivity of the plasma and $\vec{J}$ is the ordinary current (that measured in an orthonormal coordinate system). Proceeding as we did before, we find, instead of Eqn.(2.15),

$$\ddot{\vec{F}}_k \; + \; k^2 \vec{F}_k \; + \; \frac{n}{\eta^2}\vec{F}_k \; = \; -\sigma_c a \dot{\vec{F}}_k. \tag{A.1}$$

For $\sigma_c \gg 1/\eta a \sim H$ we find that $\partial \vec{F}/\partial \eta \to 0$ and $\vec{F} \sim cons't$. As discussed in the text, this implies that $\rho_B \propto a^{-4}$ (conservation of magnetic flux).

First we calculate the conductivity, σ_c, during RH. As noted, σ_c depends on the number density of charged particles, n, and therefore on the details of the reheating process. In what follows, we will make the seemingly reasonable assumption that the number density of charged particles $n \simeq \rho_\gamma / T$, i.e., there are about as many charged particles around as there are other types of relativistic particles.

In an electrically conducting plasma, $\vec{J} = ne\vec{v}$ where $\vec{v}$ is the mean velocity of the charged particles. Due to the electric field $\vec{E}$, the typical particle drift velocity is: $\vec{v} \simeq e\vec{E}\tau/m$, where m is the inertia of the particles (rest mass for a NR particle; energy for an ultrarelativistic particle). The quantity τ is the average time between particle interactions, $\tau \simeq 1/n\sigma$, where σ is the interaction cross section. It is important to note that if $1/n\sigma$ is greater than the age of the Universe, t, then we should take $\tau \simeq t$. We have that $\vec{J} \simeq ne^2\vec{E}\tau/m$ with $\tau \simeq min(1/n\sigma,\ t)$ giving

$$\sigma_c \simeq min\left[\frac{e^2}{m\sigma}, \frac{ne^2 t}{m}\right]. \tag{A.2}$$

During RH, $\rho_{tot} \propto a^{-3}$ and decreases from M^4 to T_{RH}^4 whereas the energy density in radiation, $\rho_\gamma \sim T^4$ decreases from $(MT_{RH})^2$ to T_{RH}^4, and varies as t^{-1}, from which it follows that $\rho_{tot} \sim T_{RH}^{-4} T^8$. During RH, $n \simeq T^3$, $\sigma \simeq e^4/T^2$ (the cross section for interactions mediated by a massless gauge boson, in which significant momentum ($\sim T$) is transferred) and the effective particle mass is $m \simeq T$. Putting all this together, we find

$$\sigma_c \simeq min\left[\frac{T}{e^2}\ ,\ \frac{e^2 m_{pl} T_{RH}^2}{T^2}\right]. \tag{A.3}$$

We define T_* to be the temperature when the Universe first becomes a good conductor, i.e., $\sigma_c \simeq H$. Using $H \simeq T^4/T_{RH}^2 m_{pl}^2$ and neglecting numerical factors, we find that $T_* \simeq (m_{pl}T_{RH}^2)^{1/3}$. We note that $T_* > T_{RH}$ so that plasma effects should become important during RH. Also, if $T_* \gtrsim (T_{RH}M)^{1/2}$ (i.e., $M^3 \gtrsim m_{pl}T_{RH}^2$), plasma effects should be important throughout RH. In summary, we have that

$$T_* \simeq min\left[(T_{RH}M)^{1/2}\ ,\ \left(T_{RH}^2 m_{pl}\right)^{1/3}\right] \tag{A.4}$$

Once again we caution that our *estimate* for T_* is necessarily dependent upon the details of RH; in particular, upon the fraction of charged particles produced by the decay products of the coherent oscillations.

Next consider RD. In the usual radiation-dominated regime which follows reheating we can be fairly certain that a fraction of order unity of the relativistic particles present are charged. If not present initially, they will quickly be produced by particle interactions. The conductivity then will be very large: $\sigma_c \sim T/e^2$ and $\sigma_c(\eta a) \sim \sigma_c/H \sim e^{-2}(m_{pl}/T) \gg 1$.

After the epoch of $e^{\pm}$ annihilation ($T \sim 0.1MeV$), the only charged particles present are the $\sim 10^{-10}$ electrons and ions (p, D, ^3He, ^4He, ^7Li) per photon. The number density of charged particles is: $n \sim 10^{-10}T^3$, and $\sigma \sim \sigma_{Thomson} \sim e^4/m_e^2/T^3$. The conductivity is then $\sigma_c \simeq 10^{-10}m_e/e^2$. The measure of the conductivity, $\sigma_c\eta a \sim \sigma_c/H$, is still very large: $\sigma_c t \simeq 10^{-10}(m_e m_{pl}/e^2T^2) \gg 1$.

Finally, when the Universe becomes matter-dominated ($T \simeq T_{eq} \sim 6eV$) and when the electrons and ions recombine ($T \sim 1/3eV$), the residual ionization ($n_{free\ e^-}/n_B \sim 2\times 10^{-5}(\Omega_B h)^{-1} \simeq 5\times 10^{-4}$) is sufficient to keep the conductivity high. To be more specific: $n_{free\ e^-} \simeq 10^{-13}T^3$; $\tau \simeq 1/(n_\gamma \sigma_T) \simeq e^{-4}m_e^2/T^3$; and $H^{-1} \sim 10^{12}sec(T/eV)^{-3/2}$. From this it follows that: $\sigma_c \sim 10^{-13}(m_e/e^2)$ and $\sigma_c\eta a \simeq \sigma_c/H \sim 10^{22}(T/eV)^{-3/2} \gg 1$.

In sum, from a temperature of T_{RH} (and probably as high as T_*) there is every reason to believe that the Universe was a highly-conducting plasma so that $\rho_B \propto a^{-4}$ for all modes irrespective of $k\eta$. From this it follows that once the Universe first became highly-conducting, the magnetic flux which existed then is frozen in, and the ratio $r = \rho_B(k)/\rho_\gamma$ remains constant thereafter, providing an invariant measure for any magnetic flux created. [Of course, if the onset of high conductivity occurs during RH, while the entropy is still increasing and $\rho_\gamma \propto a^{-3/2}$, r will decrease until the end of RH when entropy production ceases and $\rho_\gamma \propto a^{-4}$. Where necessary, we have taken this into account.]

Table I: Results for $r = (\rho_B/\rho_\gamma)|_{1\ Mpc}$, the magnetic field energy density on a comoving scale of 1 Mpc, relative to the CMBR for the 'RA^2' model. The dependence of r upon comoving scale λ is: $r \propto \lambda^{-n}$.

Table I						
p	q	$T_{RH}(GeV)$	$M(GeV)$	$T_*(GeV)$	$\log(r)\|_{1Mpc}$	n
−1	2	10^9	10^{17}	$10^{12.3}$	−57	2.0
−1	2	10^{17}	10^{17}	10^{17}	−56	2.0
−2	3	10^9	10^{17}	$10^{12.3}$	−13	0.0
−2	3	10^{17}	10^{17}	10^{17}	−8	0.0

Table II: Results for $r = (\rho_B/\rho_\gamma)|_{1\ Mpc}$ for the 'RF^2' model. Again, $r \propto \lambda^{-n}$.

Table II			
γ	$M(GeV)$	$\log(r)\|_{1Mpc}$	n
−0.7	10^9	−80	2.7
−0.65	$10^{7.1}$	−71	1.6
−0.6	10^5	−68	0.0
−0.55	500	−69	−2.3
−0.5	1	−76	−6.0

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27. Superadiabatic amplification seems to have been first discussed in the context of gravitons (see Ref. 20). The equations of motion for the graviton in a flat FRW cosmology are identical to that for a minimally-coupled scalar field, so it applies there as well. Superadiabatic amplification as a means of particle production in the context of inflation is discussed in detail by M.S. Turner and L.M. Widrow, Fermilab preprint (1987).

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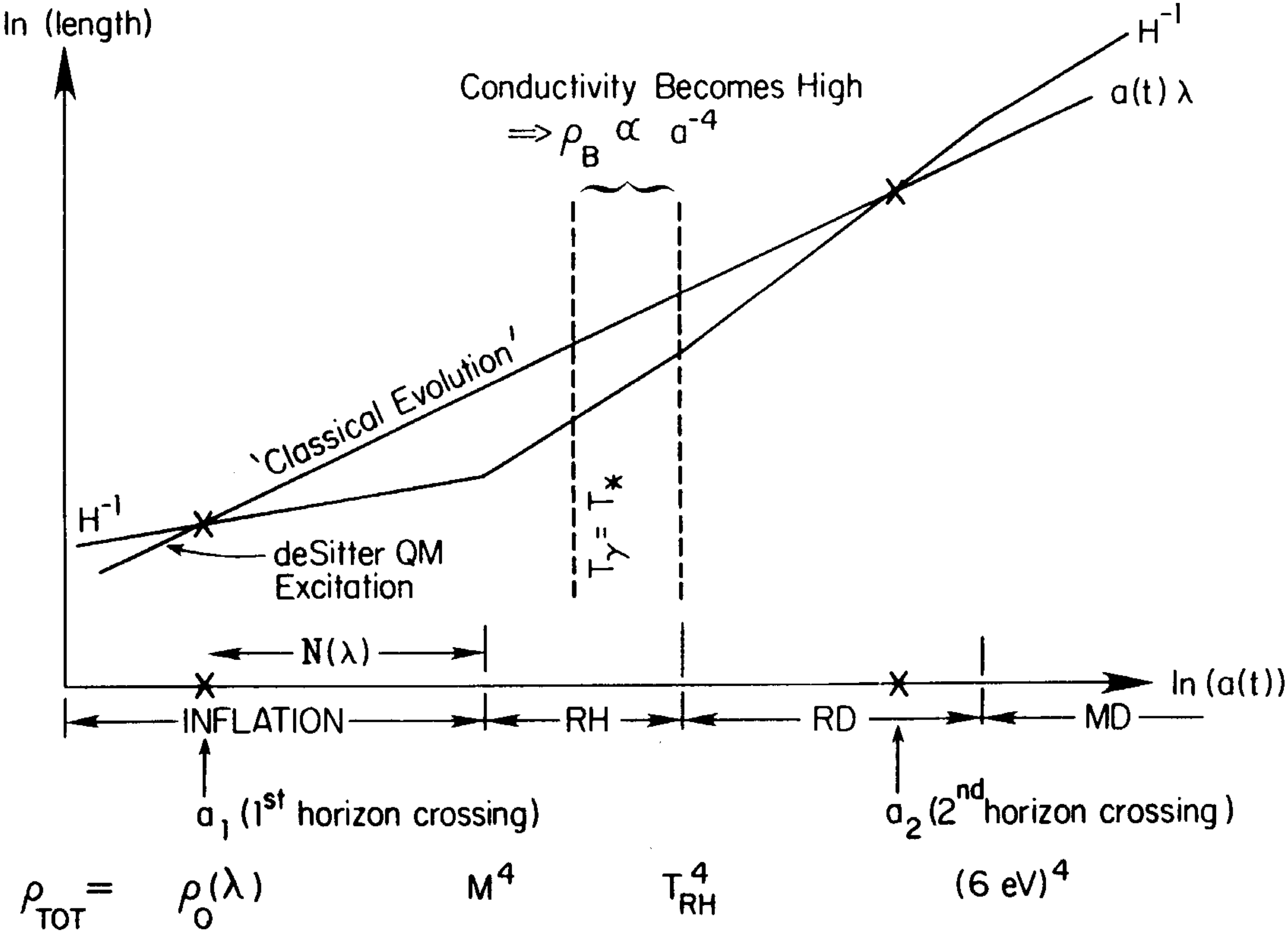
Figure Caption

Figure 1–Schematic summary of the evolution of the mode with wavelength λ. The Universe is assumed to proceed through inflation, reheating (RH), radiation domination (RD), and matter domination (MD), during which $H^{-1} \propto cons't$ (inflation), $a^{3/2}$ (RH), a^2 (RD), and $a^{3/2}$ (MD). The physical wavelength ($\equiv a(t)\lambda$) starts out subhorizon-sized, crosses outside the Hubble radius $N(\lambda)$ e-folds before the end of inflation, at which time it is assumed to freeze in as a classical fluctuation. The conductivity of the Universe becomes high ($\sigma_c/H \gtrsim 1$) some time during reheating, after which $B\,a^2 \sim cons't$ (or $\rho_B \propto a^{-4}$). The fluctuation reenters the horizon when $a(t) = a_2$.

- FIGURE 1 -



ln (length)
Conductivity Becomes High
$\Rightarrow \rho_B \propto a^{-4}$
H^{-1}
$a(t)\lambda$
'Classical Evolution'
$T_\gamma = T^*$
H^{-1}
deSitter QM Excitation
$N(\lambda)$
ln (a(t))
INFLATION
RH
RD
MD
a_1 (1st horizon crossing)
a_2 (2nd horizon crossing)
$\rho_{TOT} =$
$\rho_0(\lambda)$
M^4
T_{RH}^4
$(6\ eV)^4$