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Wilson loop instantons

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Abstract

Wilson Loop symmetry breaking is considered on a spacetime of the form $M_4 \times K$, where M_4 is a four dimensional spacetime and K is an internal space with non-trivial and *finite* fundamental group. We show in a simple model that the different vacua obtained by breaking a non-Abelian gauge group by Wilson loops are separated in the space of gauge potentials by a finite energy barrier. We then construct an interpolating gauge configuration between these vacua and show it to have minimum energy. Finally some implications of this construction are discussed.

Wilson loop symmetry breaking in nonabelian gauge theories defined on spacetimes of the form $M_4 \times K$, where M_4 is a four-dimensional spacetime and K is a compact multiply-connected internal space has been used in Klauza-Klein theory[1] and in models of low-energy superstring theory[2] as an alternative to the conventional Higgs mechanism. The four dimensional gauge bosons of the broken generators eat the gauge fields of internal components and become massive.

In the Higgs mechanism, different broken symmetry vacua are separated in field configuration space by an energy barrier. In Wilson loop symmetry breaking with a circle S^1 as the internal space, the Wilson loop parameterizes the fictitious magnetic flux through the center of the circle. As we can change the flux continuously, the Wilson loop will change continuously, and as the field strength is zero on the circle, there is no energy barrier. When the internal space has *finite* fundamental group however, it is not at all clear whether there is any interpolation in the gauge potential space between the different broken-symmetry vacua with finite energy barrier.

There have been calculations of one-loop effective vacuum energy at zero and finite temperature for Wilson loop symmetry breaking.[3,4] Energy differences between different vacua arising from the one-loop quantum and thermal corrections indicate which vacuum has the lowest energy at a given temperature. However, those calculations have real meaning only when there is an interpolation between vacua so tunnelling between them can be discussed.

In this letter, we present in a simple model such an interpolation between different broken symmetry vacua with nonzero barrier energy. The interpolation in gauge potential space between vacua is shown to have the minimum energy barrier. Just as conventional instantons are interpolations between vacua characterized by winding numbers, the instantons we find are interpolations between vacua characterized by Wilson loops. The two types of instantons are closely related in our example.

Consider a nonabelian gauge theory defined on $M_4 \times K$. Suppose the multiply-connected internal space K has *finite* fundamental homotopy group $\pi_1(K) = C$. Then, there is a universal covering space $\tilde{K}$ on which C acts freely, such that $K = \tilde{K}/C$. The gauge fields and strengths are single-valued on K and so can be lifted to $\tilde{K}$ to be invariant under C.

Vacuum gauge fields $B \equiv -iB^a_m T^a dy^m$, are those fields of zero field strength $F \equiv -\frac{i}{2}F^a_{mn}T^a dy^m dy^n = dB + B^2 = 0$. Generally the gauge field is not globally well-defined because of topological obstructions. However, suppose that the vacuum gauge field B is globally well-defined on K (this is true at least in the simple example we will consider). Then there is a multivalued gauge function h on K such that $B = h^+ dh$ is single-valued on K. It is well known that h can be lifted on $\tilde{K}$ as a single-valued gauge function.

Associated with B and any closed curve γ is the path-ordered Wilson loop

$$U_B = P\exp\left(\int_\gamma B\right). \tag{1}$$

U_B is invariant under continuous deformation of γ. On a simply-connected manifold any loop can be contracted to a point so $U = I$. However, on a multiply-connected manifold there is at least one noncontractible loop, γ, and $U \neq I$ necessarily. If the order of the fundamental homotopy group is finite, say, $\pi_1(K) = Z_n$, then γ^n is homotopic to the identity and so $U^n = I$. Suppose that the generator of π_1 is represented by a loop γ that starts from a point P on K and ends at P. That loop can be lifted to $\tilde{K}$ as a curve from P_1 to P_2, where $P_{1,2}$ are lifted points of P related by the generator of the free group $\pi_1(K)$ of $\tilde{K}$. For vacuum gauge potentials $B = g^+ dg$, and the Wilson loop becomes

$$U_B = h(P_2)^+ h(P_1). \tag{2}$$

A vacuum gauge field $B = h^+ dh$ with a nontrivial Wilson loop cannot be gauge equivalent to the trivial one, $B = 0$, because the gauge function h is multi-valued on K.

Suppose that there is a non-zero vacuum gauge potential B, leading to a non-trivial Wilson loop U_B. If H is the maximal subgroup of the gauge group G which commutes with U_B, then G is spontaneously broken to H. The gauge bosons of the broken generators will become massive after eating the gauge bosons of the internal components.

Suppose now that B_1 and B_2 are the background gauge potentials of two different Wilson loops, corresponding to two different broken-symmetry vacua. Then we can define a parameterized potential

$$A(f) = (1-f)B_1 + fB_2, \tag{3}$$

which is well defined and single valued on K because B_1 and B_2 are. Thus, $A(f)$ is a perfectly acceptable gauge field. Now the field strength related to $A(f)$ is

$$F(f) = dA(f) + A^2(f) = f(f-1)(B_1 - B_2)^2. \tag{4}$$

This is nonzero for f in the interval $(0,1)$ only if $(B_1 - B_2)^2$ is nonzero, which will be proven now.

For the nontrivial vacuum gauge potential B, $B^2 \neq 0$, which means that $B_m^a T^a$'s are not commuting, i.e., B is nonabelian. To show this, let us assume otherwise. If B is abelian, the path ordered integral becomes an ordinary integral. Because γ^n is contractible, it is the boundary of a disc D. Then,

$$\log U^n = \int_{\gamma^n} B = \int_D dB. \tag{5}$$

As U is nontrivial, the second term is nonzero. However, if B is abelian with zero field strength, the last term is zero, which leads to a contradiction. (This is not true for $K = S^1$ with $\pi_1(S^1) = Z$, because γ^n cannot be contractible for any n.)

The four dimensional potential energy density for the path (4) is

$$\begin{aligned} V(f) &= \int_K d^N y \sqrt{g(y)} F_{mn}^2(f) \\ &= (f-f^2)^2 \int_K d^N y \sqrt{g(y)} [f^{abc}(B_1-B_2)_m^b (B_1-B_2)_n^c]^2, \end{aligned} \tag{6}$$

where $g(y)$ is the metric of K. As f changes from zero to one, the gauge potential changes from one vacuum to the other vacuum. Between the two vacua there is an energy barrier with the non-zero field strength. This "ϕ^4" potential is well known in the Higgs mechanism. When the field strength is not zero, the Wilson loop is path dependent and the condition $U^n = I$ need not be satisfied. So far the crucial point in the argument is that the vacuum gauge potential is single valued and nonabelian.

Now, let us present a specific model. The gauge group is chosen to be $SU(3)$ and the internal space K to be the projective three sphere $P^3 = S^3/Z_2$ of unit radius. As S^3 is simply-connected, $\pi_1(P^3) = Z_2$. $\{x^\mu | \mu = 0,1,2,3,\}$ and $\{y^m | m = 1,2,3\}$ are coordinates of M^4 and P^3, respectively. With the metric $g_{mn}(y)$ of P^3, the action is

$$S = \int_{M_4 \times P^3} d^4x d^3y \sqrt{g} \{ -\frac{1}{4e^2} F_{MN}^a F^{aMN} \}, \tag{7}$$

where M, N run over all 7 coordinates. The three sphere, S^3, of unit radius can be represented by the coordinates $\{z^m \mid m = 1,2,3,4\}$ with condition $\Sigma_m z^m z^m = 1$. P^3 is defined by identifying $\{z^m\}$ with $\{-z^m\}$. In spherical coordinates ψ, θ and ϕ of S^3, the metric is

$$ds^2 = d\psi^2 + \sin^2\psi d\theta^2 + \sin^2\psi \sin^2\theta d\phi^2, \tag{8}$$

where $0 \le \psi, \theta \le \pi$ and $0 \le \phi \le 2\pi$. The transformation $\{z^m\} \to \{-z^m\}$ becomes $\psi \to \pi - \psi, \quad \theta \to \pi - \theta, \quad \phi \to \pi + \phi$. The theory is defined on the space of single-valued gauge fields and field strengths on P^3. Note that length and mass dimensions are fixed by the radius of P^3, which is taken to be unity.

The Wilson loop in this theory is defined along the line from $\{z^m\}$ to $\{-z^m\}$ and satisfies $U^2 = I$. After diagonalizing U, there are only two possibilities, $U = I$ and $U = \mathrm{diag}(-1,-1,1)$, corresponding to gauge groups $SU(3)$ and $SU(2)\times U(1)$, respectively. An acceptable background vector potential for the $U = I$ case is $A = 0$.

For the $U = \mathrm{diag}(-1,-1,1)$ case, let us introduce a gauge function $h(z)$ on S^3, $h(z) = z^4 + iz^i\sigma^i$, where σ^i, $i = 1,2,3$, are the Pauli matrices. $h(z)$ belongs to an $SU(2)$ subgroup imbedded in the upper left 2×2 matrix of $SU(3)$. $h(z)$ is also double-valued on P^3. However, the vacuum gauge potential $B = h^+dh$ is even under $\{z^m\}\to\{-z^m\}$ and so single-valued on P^3. In spherical coordinates, $B = h^+dh$ becomes

$$\begin{aligned} B &= i\vec{\sigma}\cdot\{\hat{\psi}d\psi + \hat{\theta}(\cos\psi\sin\psi d\theta - \sin^2\psi\sin\theta d\phi) \\ &\quad +\hat{\phi}(\sin^2\psi d\theta + \cos\psi\sin\psi\sin\theta d\phi)\}, \end{aligned} \tag{9}$$

where $\hat{\psi},\hat{\theta}$, and $\hat{\phi}$ form the radial and spherical unit vectors of R^3. A noncontractible curve γ on P^3 is a curve from $\{z^m\}$ to $\{-z^m\}$ on S^3. The Wilson loop for this background gauge field from Eq.2 is $U = \mathrm{diag}(-1,-1,1)$. Thus, B in Eq.9 is an acceptable background gauge field for the $SU(2)\times U(1)$ vacuum.

Consider now the interpolation using B of Eq.9, $A(f) = fB$, between two vacua with symmetry $SU(2)\times U(1)$ and $SU(3)$. As B is well defined on P^3, so is $A(f)$. Assume that f is a function of four dimensional space-time, whose coordinates are x^μ. Then, the four-dimensional effective lagrangian for $f(x)$ is

$$\mathcal{L} = (\partial_\mu f)^2\frac{1}{2e^2}\int_{P^3}\sqrt{g}B^a_mB^a_ng^{mn} - (f^2-f)^2\frac{1}{4e^2}\int_{P^3}\sqrt{g}(\partial_mB^a_n - \partial_nB^a_m)^2. \tag{10}$$

From Eq.9, $B^a_mB^a_ng^{mn} = 12$, $(\partial_mB^a_n - \partial_nB^a_m)^2 = 96$. After integration, the effective lagrangian becomes

$$\mathcal{L} = \frac{6\pi^2}{e^2}(\partial_\mu f)^2 - \frac{24\pi^2}{e^2}(f^2-f)^2. \tag{11}$$

This effective lagrangian has two vacua, one at $f = 0$ with the symmetry $SU(3)$ and another at $f = 1$ with the symmetry $SU(2) \times U(1)$. Both vacua have zero energy so there can be a domain wall which separate them.

Take the domain wall to lie in the $x - y$ plane. The field f will then depend only on the z coordinate. The solution with minimum energy per unit area for the action can be easily obtained, $f(z) = (1 + \tanh z)/2$. The energy per unit area is $\mathcal{E} = 4\pi^2/e^2$.

To find the interpolation with the minimum barrier energy density, the correct quantity to compare is the total energy density per unit area of the domain wall rather than the z-integration of the potential energy density. The question is then whether the interpolation we use has the lowest energy density. There is also another question of whether there are topologically inequivalent background gauge potentials leading to the same Wilson loop. Both questions can be answered rather easily by the general understanding of instanton physics.[5]

Let us start with a mapping h from S^3 to $SU(2)$, characterized by the winding number

$$n = \frac{1}{24\pi^2} \int_{S^3} tr(h^+ dh)^3. \tag{12}$$

For example, h in the above example gives $n = 1$. The gauge theory defined on $R \times S^3$ has vacua characterized by the winding number. Any interpolation $A(z)$ across the domain wall of winding number difference n satisfies an inequality,

$$2\mathcal{E} = \frac{1}{4e^2} \int_{R\times S^3} dz d^3y \sqrt{g} F^2 \geq \pm \frac{1}{4e^2} \int_{R\times S^3} dz d^3y F\tilde{F} = \frac{8\pi^2}{e^2} n, \tag{13}$$

where $\tilde{F}^a_{MN} = (\sqrt{g}/2)\epsilon_{MNKL}F^{KL}$ and the factor of two in $2\mathcal{E}$ comes from the integration over S^3 rather than P^3. The equality holds when the (anti-)self duality condition $F^a_{MN} = \pm \tilde{F}^a_{MN}$ is satisfied. The metric of $R \times P^3$ is $ds^2 = dz^2 + d\psi^2 + \sin^2\psi d\theta^2 + \sin^2\psi \sin^2\theta d\phi^2$.

$A(z) = fB$ with $f(z) = (1 + \tanh z)/2$ and B in Eq.9 is an interpolation between two vacua of $n = 0$ and $n = 1$. The single-valued fluctuations $\delta A(z)$ around fB on P^3 should be such that $A = fB + \delta A(z)$ become vacua of winding number 0, 1 at $z = -\infty, \infty$, respectively. The self duality condition becomes an equation $\partial_z f = -2(f^2 - f)$, which is satisfied by the solution for f. As any fluctuations on P^3 is acceptable on S^3, the inequality Eq.(13) is saturated, and the interpolation we found has the minimum energy density per unit area. As expected, the energy density from the last term of Eq.13 with $n = 1$ coincides with $\mathcal{E} = 4\pi^2/e^2$.

Consider vacua for a given Wilson loop. On the three space of M^3, one can consider vacua characterized by the winding number and connected by the standard instantons. Here, the basic unit of the winding number is one. This is ordinary instanton physics and will not be considered here.

There is additional degeneracy for a given Wilson loop. It is easy to see that $B = h^{+n} dh^n$ with $h(z) = z^4 + iz^i\sigma^i$ leads to $U = I$ for even n and $U = \mathrm{diag}(-1, -1, 1)$ for odd n. There is no quantum mixing between these vacua in a given symmetry breaking, because the instanton between two vacua, which is independent of three space, will have infinite Euclidean action. There would be however domain walls connecting these separate sectors. There will be no lifting of this degeneracy by quantum corrections.

In conclusion, we have found in a simple model an interpolation between different vacua resulting from Wilson loop symmetry breaking. Usually, the one-loop quantum correction will lift the degeneracy of different symmetry vacua. The lower energy vacuum has been claimed to be the one with $SU(2) \times U(1)$.[3] Thus, the domain wall solution is not stable. Rather, the tunnelling between two vacua would be interesting. Whether the symmetry is restored at high temperature is addressed in Ref. 4. However, that paper claims that at zero temperature the unbroken $SU(3)$ has lower energy density than the broken one. It would be interesting to settle this question. In addition, to generalize our result it is necessary to know whether any vacuum can be represented by a single valued gauge field. This would require the understanding of the classification of flat bundles (the vacuum gauge fields) on arbitrary manifolds.

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