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THE DIRAC EQUATION IN A NON-RIEMANNIAN MANIFOLD

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Abstract

The Dirac wave equation is obtained in the non-Riemannian manifold of the Einstein-Schroedinger nonsymmetric theory. A new internal connection is determined in terms of complex vierbeins, which shows the coupling of the electromagnetic potential with gravity in the presence of a spin-1/2 field. The consequence is the existence of a virtual particle in high curvature zones.

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I. Introduction

The Einstein-Schroedinger (ES) non-symmetric theory [1] was an attempt to geometrize, in an unitary way, the gravitational and the electromagnetic fields. However, structure problems in the theory have not permitted a coherent interpretation for the field equations. On the other hand, the idea to geometrize fields introduced by Einstein was and continues to be a fascinating tool in classical field theory. Actually there have been some works [2] where the electromagnetic field is taken in a more explicit way in that theory; this is accomplished by first taking the skew-symmetric part of the metric as being proportional to the electromagnetic tensor and then, adding a source-like term to the Lagrangian of the theory, which permits one to re-obtain the Einstein-Maxwell equations through a correspondence principle[1].

When we face the problem to develop a Dirac theory using the space-time manifold of the ES-non-symmetric theory, we use then, complex vierbeins. This is equivalent to introduce an internal **C**-space in the manifold of the GR theory. However we arrive at a problem of how to obtain a coherent mechanism that fits the correct rules for the tranformations group. We can see, for example, that the way the problem was developed by S. Marques and C. G. Oliveira [4] in a study of the geometrical properties of **C**, **Q**, and **O** tangent spaces, does not permit the corresponding Dirac-field equations (considering the complex case, presently). The reason of what happens was born in the fact the internal **C**-connection was ignored when the introduction of complex vierbeins. Instead, they generalize the tangent real-connection (that one originating on the local real-tangent space) to a complex one. This induce them to generalize both, the Lorentz group (to a pseudo-unitary group) and its representation $U(L)$, for the generalized Dirac-field-theory. However, in spite of being possible to show that, on the local tangent space , the trace of the symmetric part of the tangent-connection should correspond to the electromagnetic field, there is no way to obtain the desirable correspondence to a (actual-**R**)

[1]It has been shown recently [3] that the identification of the skew-symmetric metric of the non-symmetric theory, with the electromagnetic tensor is incorrect. The objections were overcome through a suggestion by Moffat in a new interpretation, where the metric is taken, in a general sense, as a non-symmetrical gravitational field. Also, the "source-term" of the Lagrangian was changed by a term proportional to the square of the electromagnetic tensor.

Dirac-field theory. The problem is solved when we consider besides the tangent connection, the connection corresponding to an internal **C**-space. This forces us to maintain the Lorentz group as being that of the (local) space-time transformations on the (local) tangent-space. On the other hand, we must also have an "internal" transformation, corresponding to the internal **C**-space.

In the space-time of General Relativity it is possible to generalize the Dirac field equation by doing the transition:

$$\gamma^\mu \longrightarrow \gamma^\mu(x),$$

$$\psi_{,\mu} \longrightarrow \psi_{\|\mu} = \psi_{,\mu} + \Delta_\mu \psi,$$

where $\psi(x)$ is now the electron field equation in the curved space-time, and Δ_μ is the geometrical connection with relation to the internal space generated by the constant γ-matrices, $\{\Gamma_i\}$. It is easy to show that Δ_μ is given by:

$$\Delta_\mu = \frac{1}{8}([\gamma^\nu, \gamma_{\nu,\mu}] - \{{}_{\mu}{}^{\rho}{}_{\nu}\}[\gamma^\nu, \gamma_\rho]). \tag{1.1}$$

where $\{{}_{\mu}{}^{\rho}{}_{\nu}\}$ are the Christoffel symbols for the space-time connection. In terms of real vierbeins h^ν_a (and its inverse h^a_ν), (1.1)is written as:

$$\Delta_\mu = \frac{-1}{4i}(h^{\nu b} h^a_{\nu,\mu} - \{{}_{\mu}{}^{\rho}{}_{\nu}\} h^a_\rho h^{\nu b})\sigma_{ab}. \tag{1.2}$$

The function $\psi^i(x)$, $i = 1, ..., 4$ above, satisfies a Dirac-equation defined on the curved space-time manifold of GR, which has now the form:

$$\gamma^\mu(x)(\psi_{,\mu} + \Delta_\mu \psi) - \mu\psi = 0, \tag{1.3}$$

where μ is the mass coefficient of the equation. Then, the gravitational field is present in this equation through the connection Δ_μ.

The non-symmetric manifold of the ES theory will be used in this work in which we define locally the complex vierbeins above referred. These vierbeins define new Fock-Ivanenko coefficients which permits the construction of the corresponding Dirac equations related to the non-Riemannian manifold of the ES theory. In Section II we will present briefly the properties of the complex tangent space as well as the corresponding field equations obtained in the ES non-symmetric theory. In

Section III the generalized Fock-Ivanenko coefficients will be determined, as well as the new Dirac equation. In Section IV we will proceed to their analysis. Throughout, we will use the $\mu, \nu, \ldots,$ indexes, as those on the non-Riemannian manifold; the $a, b, \ldots,$ indexes will be those on the complex tangent space.

II. The complex tangent-space.

According to the correspondence principle there exists, in each point of the curved space-time of General Relativity, a local tangent space [5] with the structure of a flat space-time, with the metric given by the Minkowski tensor η_{ab}. Therefore, we must have the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}dx^a dx^b$, locally, where $g_{\mu\nu} = g_{\nu\mu}$.

In the ES non-symmetric theory, the metric of curved space-time has the symmetry property: $g^*_{\mu\nu} = g_{\nu\mu}$, $g_{\mu\nu} = g_{\underline{\mu\nu}} + ik_{\underset{\vee}{\mu\nu}}$. Defining complex vierbeins e^a_μ (and their inverse e^μ_a), we have that [6]:

$$g_{\mu\nu} = e^{*a}_{\ \nu} e^b_\mu \eta_{ab}, \tag{2.1}$$

$$g^{\mu\nu} = e^{*\mu}_{\ a} e^\nu_b \eta^{ab}, \tag{2.2}$$

where η_{ab} (and its inverse η^{ab}) is the metric of the tangent space which we take here as the Minkowski tensor. The metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are such that: $g^{\mu\nu}g_{\sigma\nu} = \delta^\mu_\sigma$, in this order for the indexes. From there, we obtain the orthogonality conditions for the complex vierbeins:

$$e^{*a}_{\ \mu} e^\mu_b = e^a_\mu e^{*\mu}_{\ b} = \delta^a_b, \tag{2.3}$$

$$e^{*a}_{\ \mu} e^\nu_a = e^a_\mu e^{*\nu}_{\ a} = \delta^\nu_\mu. \tag{2.4}$$

As well known, the transformation law for vectors in the complex tangent space, local to a curved space-time, is defined by:

$$e'^a_\mu(x) = L^a_{\ b}(x) e^b_\mu(x), \tag{2.5}$$

where $L^a_{\ b}$ are the Lorentzian rotation matrices, which follow the property:

$$L^T \eta L = \eta \tag{2.6}$$

and, as $e^a_\mu(x)$ is a complex function, $e^a_\mu = e^a_{\mu R} + ie^a_{\mu I}$. Then

$$\overline{e}^a_\mu(x) = e^a_{\mu R} - ie^a_{\mu I} \tag{2.7}$$

is the conjugate of e^a_μ. This means that we have attached to the Minkowskian tangent space, an "internal space", the **C**-space. The "internal" transformations of an object of the **C**-space, K, is:

$$K' = U(1)K \ , \tag{2.8}$$

where $U(1)$ states for a unitary 1×1 (local) transformation matrix, $U(1) = e^{i\phi(\Gamma)}$, and,

$$\overline{K}' = \overline{U}(1)\overline{K} \ , \tag{2.9}$$

where, $\overline{U}(1) = U^{-1}(1) = e^{-i\phi(\Gamma)}$. A more general transformation law for the complex vierbeins should be now:

$$e'^a_\mu(x) = U(1)L^a{}_b(x)e^b_\mu(x) \ . \tag{2.10}$$

The covariant derivative of the vierbeins e^a_μ and e^{*a}_μ on this tangent space must be now given by:

$$e^a_{\mu|\nu} = e^a_{\mu,\nu} + \Lambda_\nu{}^a{}_b e^a_\mu + C_\nu e^b_\mu \ , \tag{2.11}$$

$$e^{*a}{}_{\mu|\nu} = e^{*a}{}_{\mu,\nu} + \Lambda_\nu{}^a{}_b e^{*b}{}_\nu - C_\nu e^{*a}{}_\mu \ , \tag{2.12}$$

where $\Lambda_\nu{}^a{}_b$ is the tangent connection related to the Minkowskian space and C_ν is the "internal connection". Their transformations law are respectively:

$$\Lambda'_\nu = L\Lambda L^{-1} - L_{,\nu}L^{-1} \ , \quad \text{(space}-\text{time transfs.)} \tag{2.13}$$

$$C'_\nu = U(1)C_\nu U^{-1}(1) - U_{,\nu}(1)U^{-1}(1) \ . \quad \text{(internal transfs.)} \tag{2.14}$$

C_ν transforms as a vector under space-time transformations. Considering the particular case where we have only the internal transformations represented by the matrices $U(1) = \mathbf{1} + i\phi$, the internal connection C_ν transforms in first order, as:

$$C'_\nu = C_\nu + i\phi_{,\nu} \ , \tag{2.15}$$

which is of the same kind of the gauge transformations of an electromagnetic potential. We also have the relation:

$$R^{\rho}{}_{\mu\nu\gamma}e^{a}_{\rho} - S_{\nu\gamma}{}^{a}{}_{b}e^{b}_{\mu} = 0, \tag{2.16}$$

where $R^{\rho}_{\mu\nu\gamma}$ is the curvature in the non-Riemannian space-time written in terms of the non-symmetric affinity, and $S_{\nu\gamma}$ is the curvature over the complex tangent space:

$$S_{\nu\gamma} = \Lambda_{\nu,\gamma} - \Lambda_{\gamma,\nu} - [\Lambda_{\nu}, \Lambda_{\gamma}], \tag{2.17}$$

which is skew-symmetric with relation to the curved-space indexes and anti-Hermitian in the tangent-space indexes. The internal curvature can be also obtained, which is:

$$P_{\nu\gamma} = C_{\nu,\gamma} - C_{\gamma,\nu}\ . \tag{2.18}$$

Then, in the particular case of (2.15), the internal curvature can be considered to correspond to the Maxwell eletromagnetic tensor.

One of the field equations of the ES non-symmetric theory, obtained through a variational principle, is[1]: $g_{\underset{+-}{\mu\nu};\alpha} = 0$, where the symbol (;) means that the connection used in this equation is the Schroedinger connection [1], $\theta^{\rho}{}_{\mu\nu}$, $\theta_{\mu} = \theta^{\rho}{}_{\mu\rho} = 0$. (In general, $\theta^{\rho}{}_{\mu\alpha}$ is a non-symmetric connection such that $\theta^{*\rho}_{\mu\alpha} = \theta^{\rho}{}_{\alpha\mu}$.) This equation corresponds to the following vierbeins equations:

$$e^{a}_{\underset{+}{\mu}|\alpha} = (e^{*a}_{\underset{-}{\mu}|\alpha})^{*} = e^{a}_{\mu,\alpha} - \theta^{\rho}{}_{\mu\alpha}e^{a}_{\rho} + \Lambda_{\alpha}{}^{a}{}_{a}e^{*\mu}_{b} + C_{\alpha}e^{*\mu}_{b} = 0. \tag{2.19}$$

where (2.1) was used. Taking the inverse equation: $g^{\underset{+-}{\mu\nu}}{}_{;\alpha} = 0$, and (2.2), we have the corresponding equations for the inverse vierbeins:

$$e^{*\overset{\mu}{+}}_{a|\alpha} = (e^{\overset{\mu}{-}}_{a|\alpha})^{*} = e^{*\mu}_{a,\alpha} + \theta^{\mu}{}_{\rho\alpha}e^{*\rho}_{a} - \Lambda_{\alpha}{}^{b}{}_{a}e^{*\mu}_{b} - C_{\alpha}e^{*\mu}_{b} = 0. \tag{2.20}$$

We can rewrite equations (2.13) and (2.14) as:

$$e^{a}_{\underset{+}{\mu}|\alpha} = e^{a}{}_{\mu,\alpha} - \theta^{\mu}{}_{\rho\alpha}e^{\rho}{}_{a} - \Lambda_{\alpha}{}^{b}{}_{a}e^{\mu}{}_{b} = 0, \tag{2.21}$$

[1]The notation used in this work is the same used by S. Marques and C.G. Oliveira in ref. [4]. See also, M.A. Tonellat, ref. [9].

$$e^{*\overset{\mu}{+}}{}_{a|\alpha} = e^{*\mu}{}_{a,\alpha} + \theta^{\mu}{}_{\rho\alpha} e^{*\rho}_{a} - \mathbf{\Lambda}_{\alpha}{}^{b}{}_{a} e^{*\mu}_{b} = 0, \tag{2.22}$$

where:

$$\mathbf{\Lambda}_{\alpha}{}^{a}{}_{b} = \Lambda_{\alpha}{}^{a}{}_{b} + \delta^{a}_{b} C_{\alpha} \ ,$$

$$\mathbf{\Lambda}^{*}{}_{\alpha}{}^{a}{}_{b} = \Lambda_{\alpha}{}^{a}{}_{b} - \delta^{a}_{b} C_{\alpha}. \tag{2.23}$$

From (2.19) and (2.20) we obtain the relation:

$$\mathbf{\Lambda}_{\alpha}{}^{a}{}_{b} = e^{a}_{\mu} e^{*\overset{\mu}{+}}{}_{b;\alpha} = -e^{a}_{\underset{+}{\mu};\alpha} e^{*\mu}_{b}, \tag{2.24}$$

and from (2.23),

$$\Lambda_{\alpha} = \mathrm{Re}[e^{a}_{\mu} e^{*\overset{\mu}{+}}{}_{a;\alpha}] = \mathrm{Re}[-e^{a}_{\underset{+}{\mu};\alpha} e^{*\mu}_{b}], \tag{2.25}$$

$$C_{\alpha} = i\,\mathrm{Im}[-e^{a}_{\mu} e^{*\overset{\mu}{+}}{}_{a;\alpha}] = i\,\mathrm{Im}[-e^{a}_{\underset{+}{\mu};\alpha} e^{*\mu}_{b}], \tag{2.26}$$

Taking (2.25), we can expand it in terms of real and imaginary parts. We then obtain for Λ_{ν}:

$$\Lambda_{\nu} = \mathrm{Re}[e^{a}_{\mu} e^{*\overset{\mu}{+}}{}_{a;\nu}]$$

$$= e^{a}_{\mu R} e^{\mu b}{}_{R,\nu} + e^{a}_{\mu I} e^{\mu b}{}_{I,\nu} + e^{a}_{\mu R} \theta^{\mu}_{\underline{\rho\nu}} e^{\rho b}{}_{R} + e^{a}_{\mu I} \theta^{\mu}_{\underline{\rho\nu}} e^{\rho b}{}_{I} + - e^{a}_{\mu I} \theta^{\mu}_{\underset{\vee}{\rho\nu}} e^{\rho b}{}_{R} + e^{a}_{\mu R} \theta^{\mu}_{\underset{\vee}{\rho\nu}} e^{\rho b}{}_{I} \tag{2.27}$$

or,

$$\Lambda_{\alpha} = \mathrm{Re}[-e^{a}_{\underset{+}{\mu};\alpha}\ e^{*\mu}_{b}]$$

$$= -e^{a}_{\mu R,\alpha} e^{\mu}_{b\,R} - e^{a}_{\mu I,\alpha} e^{\mu}_{b\,I} + \theta^{\rho}_{\underline{\mu\alpha}} e^{a}_{\rho R} e^{\mu}_{b\,R} + \theta^{\rho}_{\underline{\mu\alpha}} e^{a}_{\rho I} e^{\mu}_{b\,I} + \theta^{\rho}_{\underset{\vee}{\mu\alpha}} e^{a}_{\rho R} e^{\mu}_{b\,I} - \theta^{\rho}_{\underset{\vee}{\mu\alpha}} e^{a}_{\rho I} e^{\mu}_{b\,R} \tag{2.28}$$

Analogously, from (2.26), we obtain for C_{α} :

$$C_{\alpha} = i Im[e^{a}_{\mu} e^{*\overset{\mu}{+}}_{a}{}_{;\alpha}]$$

$$= e^{a}_{\mu I} e^{\mu}_{b\,R,\alpha} - e^{a}_{\mu R} e^{\mu}_{b\,I,\alpha} + e^{a}_{\mu R} \theta^{\mu}_{\underset{\vee}{\rho\alpha}} e^{\rho}_{b\,R} + e^{a}_{\mu I} \theta^{\mu}_{\underset{\vee}{\rho\alpha}} e^{\rho}_{b\,I} + e^{a}_{\mu I} \theta^{\mu}_{\underline{\rho\alpha}} e^{\rho}_{b\,R} - e^{a}_{\mu R} \theta^{\mu}_{\underline{\rho\alpha}} e^{\rho}_{b\,I} \tag{2.29}$$

or,

$$C_{\alpha} = i\,\mathrm{Im}[-e^{a}_{\underset{+}{\mu};\alpha}\ e^{*\mu}_{b}]$$

$$= e^{a}_{\mu R,\alpha} e^{\mu}_{b\,I} - e^{a}_{\mu I,\alpha} e^{\mu}_{b\,R} - \theta^{\rho}_{\underline{\mu\alpha}} e^{a}_{\rho R} e^{\mu}_{b\,I} + \theta^{\rho}_{\underline{\mu\alpha}} e^{a}_{\rho I} e^{\mu}_{b\,R} + \theta^{\rho}_{\underset{\vee}{\mu\alpha}} e^{a}_{\rho R} e^{\mu}_{b\,R} + \theta^{\rho}_{\underset{\vee}{\mu\alpha}} e^{a}_{\rho I} e^{\mu}_{b\,I} \tag{2.30}$$

Supposing that we have a theory where the antisymmetrical part of the space-time connection is zero, $\theta^{\rho}{}_{\underset{\vee}{\mu\nu}} = 0$, but still with complex vierbeins, i.e., a theory where we have a complex, antisymmetrical part for the metric, we now obtain for the tangent and internal connections,

$$\begin{aligned}\Lambda_\nu &= e^a_{\mu R} e^{\mu b}{}_{R,\nu} + e^a_{\mu I} e^{\mu b}{}_{I,\nu} + e^a_{\mu R}\Gamma^\mu_{\rho\nu} e^{\rho b}{}_R + e^a_{\mu I}\Gamma^\mu_{\rho\nu} e^{\rho b}{}_I \\ &= -e^a_{\mu R,\nu} e^{\mu b}{}_R - e^a_{\mu I,\nu} e^{\mu b}{}_I + e^a_{\rho R}\Gamma^\rho_{\mu\nu} e^{\mu b}{}_R + e^a_{\rho I}\Gamma^\rho_{\mu\nu} e^{\mu b}{}_I \ , \end{aligned} \tag{2.31}$$

and

$$\begin{aligned}C_\nu &= e^a_{\mu I} e^{\mu b}{}_{R,\nu} - e^a_{\mu R} e^{\mu b}{}_{I,\nu} + e^a_{\mu I}\Gamma^\mu_{\rho\nu} e^{\rho b}{}_R - e^a_{\mu R}\Gamma^\mu_{\rho\nu} e^{\rho b}{}_I \\ &= e^a_{\mu R,\nu} e^{\mu b}{}_I - e^a_{\mu I,\nu} e^{\mu b}{}_R - e^a_{\rho R}\Gamma^\rho_{\mu\nu} e^{\mu b}{}_I + e^a_{\rho I}\Gamma^\rho_{\mu\nu} e^{\mu b}{}_R \ , \end{aligned} \tag{2.32}$$

where we used the notation $\Gamma^\rho{}_{\mu\nu}$ for the symmetrical connection.

We can see that the relation of the (complex) metric, with the new complex vierbeins, adds new extra terms to the tangent connection. Also, the internal connection has a relation with the vierbeins, which would not exist if the vierbeins are real. It is noticeable, from (2.31) and (2.32), that the same happens in a "complex theory" without a complex torsion term. It easy to conclude, as in ref.[2], that the Einstein-Maxwell theory is reached in a convenient limit such as to eliminate the complex part of the metric and therefore, the corresponding complex ones for the vierbeins. However, some years ago, this fact was criticised by theoretical analysis [3] which does not change the power of a geometrical analysis. Thinking from a geometrical point of wiew, we will go forward, obtaining of field (Dirac) equations and see what we can get in this "complex theory".

III. The generalization of the Fock-Ivanenko coefficients.

The Dirac constant γ-matrices satisfy the anticommutation relations:

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbf{1_4}, \tag{3.1}$$

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1_4}, \tag{3.2}$$

where η_{ab} (and its inverse η^{ab}) is the Minkowsky tensor with signature +2, and to the relation: $\gamma^a{}_{,b} = 0$. The set formed with combinations of γ-matrices, $\{\Gamma_i\} = \{\mathbf{1_4}, \gamma_a, \sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b], \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3, \gamma_5\gamma_a\}$, composes a linearly independent set in the internal-space of the Dirac wave functions ψ.

Now, multiplying (3.1) by e^{*a}_{ν} and e^{b}_{μ}, and using (2.1), we obtain:

$$\{\gamma_\mu, \dot{\gamma}_\nu\} = 2g_{\mu\nu}\mathbf{1_4} \ , \tag{3.3}$$

where $g_{\mu\nu}$ is now, the ES non-symmetric metric. In (3.4) we have defined γ_μ and $\dot{\gamma}_\mu$ by:

$$e^{a}_{\mu}\gamma_a = \gamma_\mu \ ; \qquad e^{*a}_{\mu}\gamma_a = \dot{\gamma}_\mu \ . \tag{3.4}$$

Analogously, multiplying (3.2) by $e^{*\mu}_{a}$ and e^{ν}_{b} , we obtain:

$$\{\dot{\gamma}^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1_4} \ , \tag{3.5}$$

where:

$$e^{\mu}_{a}\gamma^a = \gamma^\mu \ , \qquad e^{*\mu}_{a}\gamma^a = \dot{\gamma}^\mu \ , \tag{3.6}$$

and the relation (2.2) was used. The covariant derivative of $\gamma^\mu(x)$ over the non-Riemannian manifold of non-symmetric theory, is given by:

$$\gamma_{\underset{+}{\mu}|\nu} = \gamma_{\mu,\nu} - \Omega^{\rho}{}_{\mu\nu}\gamma_\rho + [\Delta_\nu, \gamma_\mu], \tag{3.7}$$

where Δ_μ is the internal connection, corresponding to the space of generalized γ-matrices (or, also, of Dirac wave functions space), and $\Omega^{\rho}{}_{\mu\nu}$ is a more general space-time affinity (that at least in principle, includes the internal connection C_μ). Taking now, the equality (3.5) and the equations (2.19), we have that:

$$\gamma_{\underset{+}{\mu}|\nu} = (e^{a}_{\underset{+}{\mu}}\gamma_a)_{|\nu} = (e^{a}_{\underset{+}{\mu}|\nu})\gamma_a = 0 \ , \tag{3.8}$$

since γ_a is a constant matrix. In the same way, we obtain:

$$\dot{\gamma}_{\underset{-}{\mu}|\nu} = (e^{*a}_{\underset{-}{\mu}|\nu})\gamma_a = 0. \tag{3.9}$$

Expanding (3.8) and (3.9) we have:

$$\gamma_{\underset{+}{\mu}} = \gamma_{\mu,\nu} - \theta^{\rho}{}_{\mu\nu}\gamma_\rho + [\Delta_\nu, \gamma_\mu] + C_\nu\gamma_\mu = 0, \tag{3.10}$$

$$\dot{\gamma}_{\underset{-}{\mu}|\nu} = \dot{\gamma}_{\mu,\nu} - \theta^{\rho}{}_{\nu\mu}\dot{\gamma}_\rho + [\Delta_\nu, \dot{\gamma}_\mu] - C_\nu\dot{\gamma}_\mu = 0. \tag{3.11}$$

We can observe then, from (3.10) and (3.11), that we obtain a relation similar of that of General Relativity, i.e.:

$$\Delta_\nu = \frac{1}{4i} \Lambda_\nu{}^{\overset{ab}{\vee}} \sigma_{ab} = \frac{1}{4i} \operatorname{Re}[e^a_\mu e^{\overset{*\mu b}{+}}{}_{;\nu}] \sigma_{ab} \ , \tag{3.12}$$

or,

$$\Delta_\nu = \frac{1}{4i} \operatorname{Re}[-e^a_{\underset{+}{\mu};\alpha} e^{*\mu b}] \sigma_{ab} \ , \tag{3.13}$$

where it was used (2.21) for Λ_ν.

If we consider now, $\psi(x)$ as the wave function of a spin 1/2 particle of mass m, placed in a non-Riemmanian manifold of ES non-symmetric theory, $\overline{\psi}(x) = \psi^\dagger \gamma_o$ will be the wave function of its antiparticle, and the corresponding Dirac wave equations are, respectively:

$$\gamma^\mu (\overrightarrow{\partial}_\mu + \Delta_\mu + C_\mu) \psi - \mu \psi = 0 \ , \tag{3.14}$$

$$- \overline{\psi} (\overleftarrow{\partial}_\mu + \Delta_\mu - C_\mu) \dot{\gamma}^\mu - \mu \overline{\psi} = 0 \ , \tag{3.15}$$

where $\mu = mc/\hbar$.

The new operator $(\partial_\mu + \Delta_\mu + C_\mu)$ comes from the covariant derivation of the function $\psi(x)$, which besides being an object which transforms, locally, under the representation of Lorentz Group ($U(L)$), also transforms under the (internal) $U(1)$ group. The equations (3.14), and (3.15) describes particles placed in a curved non-Riemannian space-time of the ES non-symmetric theory, since the connections Δ_μ and C_μ are now related to complex vierbeins, as well as to the complex space-time connection.

Another way to obtain the equations (3.14) and (3.15) is through a Minimal Action Principle. In this case the Action is:

$$A = \int \mathcal{L} d^4 x \ ,$$

where the Lagrangian is given by:

$$\mathcal{L} = \sqrt{-g} [\overline{\psi} \gamma^\mu (\psi_{,\mu} + \Delta_\mu \psi + C\mu) + (\overline{\psi}_{,\mu} + \overline{\psi} \Delta_\mu - C_\mu) \dot{\gamma}^\mu - \mu \overline{\psi} \psi] \ . \tag{3.16}$$

From (3.15), the wave equation for the charge conjugate function, ψ^c, is:

$$\dot{\gamma}^\mu (\psi^c_{,\mu} + \Delta_\mu \psi^c - C_\mu \psi^c) - \mu \psi^c = 0 \ , \tag{3.17}$$

where $\psi^c = C\overline{\psi}^T$, and C is the charge conjugate matrix. Therefore, if the wave equation of a particle is constructed with the set γ^μ and $(\Delta_\mu + C_\mu)$, the wave equation for its "charge conjugate" will be constructed with the set $\dot{\gamma}^\mu$ and $(\Delta_\mu - C_\mu)$.

Let us write now the internal connection C_μ as:

$$C_\mu = leA_\mu(x) \ . \tag{3.18}$$

Then, after (2.15), we can interpret e as the eletric charge for the electron, $A_\mu(x)$ as the electromagnetic potential, and l will be a constant such that it balance units. The equations (3.14) and(3.17) can be written now as:

$$\gamma^\mu(\partial_\mu + \Delta_\mu + elA_\mu)\psi - \mu\psi = 0 \ , \tag{3.19}$$

$$\dot{\gamma}^\mu(\partial_\mu + \Delta_\mu - elA_\mu)\psi^c - \mu\psi^c = 0 \ . \tag{3.20}$$

IV. Conclusion

We have learned that complexifying the space-time manifold of General Relativity is equivalent to attaching it an internal **C**-space. The new metric is no more symmetric and its antisymmetric part should be proportional to the electromagnetic tensor [2]. Through complex vierbeins, it is possible to obtain a local (complex)tangent space. Then using these concepts, we obtained here Dirac field equations for a spin 1/2 particle, locally to this non-symmetric curved space of ES type. The internal complex connection corresponds proportionally, to the electromangetic potential.

We can observe that it is possiple define the complex vierbeins as:

$$e^a_\mu = e^a_{\mu R} + i\kappa\lambda n^a_\mu \tag{4.1}$$

where κ is considered now as a parameter, and λ is a constant. We use here, as in ref. [2],

$$\lambda \propto \frac{e}{L^2} \sim \frac{c^3 e}{\hbar G} = 1.82 \times 10^{56} \frac{statvolt}{cm^2} \ .$$

[2] See A. Einstein, ref.[1], equations (11) to (17) in Section 3.

In a limit where the parameter $\kappa \to 0$, we obtain from (2.27) and (2.28) the real conection Λ_ν from General Relativity. Using the above expression for the vierbeins, we can display an interesting behavior of the new Dirac equations which appears when we split up $\gamma^\mu(x)$ in terms of its real and imaginary parts, and also suppose a complex mass term: $\mu = \mu_R + i\mu_I$, where we again can take $\mu_I = \kappa\lambda m$. Then, from (3.19),

$$[e^\mu_{aR}\gamma^a(\partial_\mu + \Delta_\mu + elA_\mu)\psi + \mu_R\psi]$$
$$+ i\kappa\lambda[n^\mu_a\gamma^a(\partial_\mu + \Delta_\mu + elA_\mu)\psi + m\psi] = 0 \quad , \qquad (4.2)$$

In the limit of the parameter $\kappa \to 0$, we should get the normal Dirac equations in the presence of gravitational and electromagnetic fields. Therefore, it means that we can get another identical set of Dirac equations if we take $n^\mu_a \equiv e^\mu_{aR} \sim h^\mu_a$, and $m \equiv \mu_R$, where h^μ_a and μ_R will be vierbeins and the mass term of General Relativity theory.

This could suggest that if we consider for example, the κ-parameter as something ruling out the influence of gravitational field with electromagnetic fields, in "high curvature zones", this would produce an (imaginary) spin $\frac{1}{2}$ particle with the same wave function. The intensity of this "imaginary" part, depends of $\kappa \to \lambda^{-1}$.

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APPENDIX A: Comments on a more general transformation law in a tangent space associated to a complex internal space.

Let us consider instead of (2.10), a more general transformation law for objects in the complex tangent space. Considering, for instance, the vectors e^a_μ, its transformation law can be defined as:

$$e'^a_\mu(x) = \mathbf{L}^a{}_b(x)e^b_\mu(x) \ , \qquad (A.1)$$

$$e'^{*a}_{\mu}(x) = \mathbf{L}^{*a}{}_{b} e^{*b}_{\mu}(x) \ . \qquad \mathbf{(A.2)}$$

The complex matrix $\mathrm{L}^{a}{}_{b}$, now, is a kind of pseudo-Lorentz matrix such that it follows the relation:

$$\mathbf{L}^{\dagger}\eta\mathbf{L} = \eta \ . \qquad \mathbf{(A.3)}$$

The covariant derivative of e^{a}_{μ} on this complex tangent space is then defined as:

$$e^{a}_{\mu|\nu} = e^{a}_{\mu,\nu} + \mathbf{A}_{\nu}{}^{a}{}_{b} e^{b}_{\mu} \ , \qquad \mathbf{(A.4)}$$

$$e^{*a}_{\ \mu|\nu} = e^{*a}_{\ \mu,\nu} + \mathbf{A}^{*}{}_{\nu}{}^{a}{}_{b} e^{*b}_{\ \mu} \ , \qquad \mathbf{(A.5)}$$

where the affinity is complex. Its transformation law is:

$$\mathbf{A}'_{\mu} = \mathbf{L}\mathbf{A}_{\mu}\mathbf{L}^{-1} - \mathbf{L}_{,\mu}\mathbf{L}^{-1} \ , \quad \mathbf{A}'^{*}_{\ \mu} = \mathbf{L}^{*}\mathbf{A}^{*}\mathbf{L}^{*-1} - \mathbf{L}^{*}_{,\mu}\mathbf{L}^{*-1} \ . \qquad \mathbf{(A.6)}$$

It is directly shown (see ref.[4]), that through the Einstein field equations for the non-symmetric theory (a complex theory), $g_{\underset{+-}{\mu\nu};\alpha} = 0$, we obtain the same corresponding field equations for the vierbeins described in equations (2.21) and (2.22). However, we must also have $\eta_{\underset{+-}{ab|\mu}} = 0$, where the "minus" sign corresponds to the complex conjugate of the affinity $\mathbf{A}_{\mu}$:

$$\eta_{\underset{+-}{ab|\mu}} = \eta_{ab,\mu} - \mathbf{A}_{\mu}{}^{c}{}_{a}\eta_{cb} - \mathbf{A}^{*}{}_{\mu}{}^{c}{}_{b}\eta_{ac} = 0 \ . \qquad \mathbf{(A.7)}$$

As η_{ab} lowers indexes, we have that $\mathbf{A}_{\mu}$ is anti-hermitian with respect to the index of the tangent space. Then, we have:

$$\mathbf{A}_{\mu ab} = \mathbf{A}_{\underset{\vee}{\mu ab}} + i\mathbf{A}_{\mu\underline{ab}} \ . \qquad \mathbf{(A.8)}$$

The expansion of $\mathbf{L}$ in first order is, from (A.1) to (A.3):

$$\mathbf{L} \cong \mathbf{1} + \varepsilon + i\mu \ , \quad \mathbf{L}^{-1} \cong \mathbf{1} - \varepsilon - i\mu \ , \qquad \mathbf{(A.9)}$$

where again, $\varepsilon = \varepsilon(x)$ are infinitesimal rotation matrices as before and $\mu = \mu(x)$ are symmetric infinitesimal matrices. We can write these last ones as:

$$\mu_{ab} = (a + \mathbf{1}\frac{1}{4}Tr\mu)_{ab} \ , \qquad \mathbf{(A.10)}$$

where a is a symmetric trace-free matrix. Considering then, a particular transformation such that

$$\mathbf{L} \cong \mathbf{1} + \frac{1}{4}K \ , \qquad K = Tr\mu \ , \tag{A.11}$$

the affinity Λ_α of this complex theory is transformed as:

$$\Lambda'_\alpha = \Lambda_\alpha - \frac{i}{4}K_{,\alpha}$$

$$Tr\Lambda'_\alpha = Tr\Lambda_\alpha - iK_\alpha \ , \tag{A.12}$$

which is similar to the gauge transformation of an electromagnetic potential. (In the same way, we can show that the complex part of a non-symmetric tangent curvature obtained with the above Λ_α will be related to the Maxwell electromagnetic tensor.)

Now, from (A.6,7) and (3.8,9), we can easily obtain a relation between Λ_α and the connection Δ_α :

$$\Lambda_\alpha{}^{ab} e_{\mu b}\gamma_a = \eta^{ac} e_{\mu c}[\Delta^{(1)}_\alpha, \gamma_a] \ ,$$

$$\Lambda^*{}_\alpha{}^{ab} e^*_{\mu b}\gamma_a = \eta^{ab} e^*_{\mu c}[\Delta^{(2)}_\alpha, \gamma_a] \ , \tag{A.13}$$

where $\Delta^{(1)}_\alpha$ and $\Delta^{(2)}_\alpha$ are now, (general) Dirac-connections corresponding to the Fock-Ivanenko coefficients. However, expanding $\Delta^{(1)}_\alpha$ and $\Delta^{(2)}_\alpha$ in terms of the set $\{\Gamma_i\}$, we can see that the real part of Λ_α is of the same form of General Relativity, but that there is no way to relate the complex symmetrical part of Λ_α in terms of that expansion, since the only one symmetrical term there, which is proportional to unit element of set (the unit 4×4 matrix), is eliminated through the commutator in (A.15). This prooves that this is not the correct choice for the transformation matrix **L**. As we saw the correct one is the product expressed in (2.10).

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